COMPUTATIONAL PHYSICS

A Practical Introduction to Computational Physics
and Scientific Computing (using C++)

Athens, 2016

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COMPUTATIONAL PHYSICS
A Practical Introduction to Computational Physics and Scientific Computing (C++ version)

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[1]The first number is the major version, corresponding to an "edition" of a conventional book. Versions differing by major numbers have been altered substantially. Chapter numbers and page references are not guaranteed to match between different versions. The second number is the minor version. Versions differing by a minor version may have serious errors/typos corrected and/or substantial text modifications. Versions differing by only the last number may have minor typos corrected, added references etc. When reporting errors, please mention the version number you are referring to.

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This book has been written assuming that the reader **executes all the commands presented in the text and follows all the instructions at the same time**. If this advice is neglected, then the book will be of little help and some parts of the text may seem incomprehensible.

The book's website is at
`http://www.physics.ntua.gr/~konstant/ComputationalPhysics/`
From there, you can can download the *accompanying software*, which contains, among other things, *all* the programs presented in the book.

Some conventions: Text using the font shown below refers to commands given using a shell (the "command line"), input and output of programs, code written in Fortran (or any other programming language), as well as to names of files and programs:

```
> echo Hello world
Hello world
```

When a line *starts* with the prompt

```
>
```

then the text that follows is a command, which can be given from the command line of a terminal. The second line, `Hello World`, is the *output* of the command.

The contents of a file with C++ code is listed below:

```
int main(){
  double x = 0.0;
  for(int i=0;i<10;i++){
    x += i;
```

```
  }
}
```

What you need in order to work on your PC:

- An operating system of the GNU/Linux family and its basic tools.
- A Fortran compiler. The `gfortran` compiler is freely available for all major operating systems under an open source license at `http://www.gfortran.org`.
- An advanced text editor, suitable for editing code in several programming languages, like Emacs[2].
- A good plotting program, suitable for data analysis, like `gnuplot`[3].
- The shell `tcsh`[4].
- The programs `awk`[5], `grep`, `sort`, `cat`, `head`, `tail`, `less`. Make sure that they are available in your computer environment.

If you have installed a GNU/Linux distribution on your computer, all of the above can be installed easily. For example, in a Debian like distribution (Ubuntu, ...) the commands

```
> sudo apt-get install tcsh emacs gnuplot gnuplot-doc
> sudo apt-get install gfortran gawk gawk-doc binutils
> sudo apt-get install manpages-dev coreutils liblapack3
```

install all the necessary tools.

If you don't wish to install GNU/Linux on your computer, you can try the following:

- Boot your computer using a usb/DVD live GNU/Linux, like Ubuntu[6]. This will not make any permanent changes in your hard drive but it will start and run slower. On the other hand, you may save all

[2] `http://www.gnu.org/software/emacs/`
[3] `http://www.gnuplot.info`
[4] `http://www.tcsh.org`
[5] `http://www.gnu.org/software/gawk`
[6] `http://www.ubuntu.com`

your computing environment and documents and use it on any computer you like.

- Install `Cygwin`[7] in your Microsoft Windows. It is a very good solution for Microsoft-addicted users. If you choose the *full* installation, then you will find all the tools needed in this book.

- Mac OS X is based on Unix. It is possible to install all the software needed in this book and follow the material as presented. Search the internet for instructions, e.g. google “gfortran for Mac”, “emacs for Mac”, “tcsh for Mac”, etc.

[7] `http://www.cygwin.com`

Foreword to the Second Edition

This book has been out "in the wild" for more than two years. Since then, its pdf version has been downloaded 2-5000 times/month from the main server and has a few thousand hits from sites that offer science e-books for free. I have also received positive feedback from students and colleagues from all over the world and that gave me the encouragement to devote some time to create a C++ version of the book. As far as scientific programming is concerned, the material has not changed apart from some typo and error corrections[8].

I have to make it clear that by using this book you will not learn much on the advanced features of C++. Scientific computing is usually simple at its core and, since it must be made efficient and accurate, it needs to go down to the lowest levels of programming. This also partly the reason of why I chose to use Fortran for the core programming in the first edition of the book: It is a language designed for numerical programming and high performance computing in mind. It is simple and a scientist or engineer can go directly into programming her code. C++ is not designed for scientific applications[9] in mind and this reflects on some trivial omissions in its standard. Still, many scientific groups are now using C++ for programming and the C++ compilers have improved quite a lot. There is still an advantage in performance using a Fortran compiler on a supercomputer, but this is not going to last for much longer.

Still, for a scientist, the programming language is a *tool* to solve her *scientific* problems. One should not bind herself to a specific language. The treasures of today are the garbage of tomorrow, and the time scale for this happening is small in today's computing environments. What

[8]Check the *errata* section at the book's homepage.

[9]Object oriented languages' aim is to improve modularity, maintenability and flexibility of programs.

has really lasting value is the ability to solve problems using a computer and this is what needs to be emphasized. Consistent with this idea is that, in the course of reading this book, you will also learn how to make your C++ code interact with code written in Fortran, like in the case of the popular library Lapack. This will improve your "multilingual skills" and flexibility with interacting with legacy code.

The good news for us scientists is that numerical code usually needs simple data structures and programming is similar in *any* language. It was simple for me to "translate" my book from Fortran to C++. Unfortunately I will not touch on all this great stuff in true object oriented programming but you may be happy to know that you will most likely not need it[10].

So, I hope that you will enjoy using my book and I remind you that I love fan mail and I appreciate comments/corrections/suggestions sent to me. Now, if you want to learn about the structure and educational procedure in this book, read the foreword to the first edition, otherwise skip to the real fun of solving scientific problems numerically.

Athens, 2016.

[10]A lot of C++ code out there is realizing procedural and not true object oriented programming.

Foreword to the First Edition

This book is the culmination of my ten years' experience in teaching three introductory, *undergraduate level*, scientific computing/computational physics classes at the National Technical University of Athens. It is suitable mostly for junior or senior level science courses, but I am currently teaching its first chapters to sophomores without a problem. A two semester course can easily cover all the material in the book, including lab sessions for practicing.

Why another book in computational physics? Well, when I started teaching those classes there was no bibliography available in Greek, so I was compelled to write lecture notes for my students. Soon, I realized that my students, majoring in physics or applied mathematics, were having a hard time with the technical details of programming and computing, rather than with the physics concepts. I had to take them slowly by the hand through the "howto" of computing, something that is reflected in the philosophy of this book. Hoping that this could be useful to a wider audience, I decided to translate these notes in English and put them in an order and structure that would turn them into "a book".

I also decided to make the book freely available on the web. I was partly motivated by my anger caused by the increase of academic (e)book prices to ridiculous levels during times of plummeting publishing costs. Publishers play a diminishing role in academic publishing. They get an almost ready-made manuscript in electronic form by the author. They need to take no serious investment risk on an edition, thanks to print-on-demand capabilities. They have virtually zero cost ebook publishing. Moreover, online bookstores have decreased costs quite a lot. Academic books need no advertisement budget, their success is due to their academic reputation. I don't see all of these reflected on reduced book prices, quite the contrary, I'm afraid.

My main motivation, however, is the freedom that independent publishing would give me in improving, expanding and changing the book in the future. It is great to have no length restrictions for the presentation of the material, as well as not having to report to a publisher. The reader/instructor that finds the book long, can read/print the portion of the book that she finds useful for her.

This is *not* a reference book. It uses some interesting, I hope, physics problems in order to introduce the student to the fundamentals of solving a scientific problem numerically. At the same time, it keeps an eye in the direction of advanced and high performance scientific computing. The reader should follow the instructions given in each chapter, since the book *teaches by example*. Several skills are taught through the solution of a particular problem. My lectures take place in a (large) computer lab, where the students are simultaneously doing what I am doing (and more). The program that I am editing and the commands that I am executing are shown on a large screen, displaying my computer monitor and actions live. The book provides no systematic teaching of a programming language or a particular tool. A very basic introduction is given in the first chapter and then the reader learns whatever is necessary for the solution of her problem. There is more than one way to do it[11] and the problems can be solved by following a basic or a fancy way, depending on the student's computational literacy. The book provides the necessary tools for both. A bibliography is provided at the end of the book, so that the missing pieces of a puzzle can be sought in the literature.

This is also *not* a computational physics playground. Of course I hope that the reader will have fun *doing* what is in the book, but my goal is to provide an experience that will set the solid foundation for her becoming a high performance computing, number crunching, heavy duty data analysis expert in the future. This is why the programming language of the core numerical algorithms has been chosen to be Fortran, a highly optimized, scientifically oriented, programming language. The computer environment is set in a Unix family operating system, enriched by all the powerful GNU tools provided by the FSF[12]. These tools are indispensable in the complicated data manipulation needed in scientific research, which requires flexibility and imagination. Of course, Fortran

[11] A Perl moto!

[12] Free Software Foundation, `www.fsf.org`.

is not the best choice for heavy duty object oriented programming, and is not optimal for interacting with the operating system. The philosophy[13] is to let Fortran do what is best for, number crunching, and leave data manipulation and file administration to external, powerful tools. Tools, like `awk`, shell scripting, `gnuplot`, Perl and others, are quite powerful and complement all the weaknesses of Fortran mentioned before. The plotting program is chosen to be `gnuplot`, which provides very powerful tools to manipulate the data and create massive and complicated plots. It can also create publication quality plots and contribute to the "fun part" of the learning experience by creating animations, interactive 3d plots etc. All the tools used in the book are open source software and they are accessible to everyone for free. They can be used in a Linux environment, but they can also be installed and used in Microsoft Windows and Mac OS X.

The other hard part in teaching computational physics to scientists and engineers is to explain that the approach of solving a problem numerically is quite different from solving it analytically. Usually, students of this level are coming with a background in analysis and fundamental physics. It is hard to put them into the mode of thinking about solving a problem using only additions, multiplications and some logical operations. The hardest part is to explain the discretization of a model defined analytically, which can be done in many ways, depending on the accuracy of the approximation. Then, one has to extrapolate the numerical solution, in order to obtain a good approximation of the analytic one. This is done step by step in the book, starting with problems in simple motion and ending with discussing finite size scaling in statistical physics models in the vicinity of a continuous phase transition.

The book comes together with additional material which can be found at the web page of the book[14]. The accompanying software contains all the computer programs presented in the book, together with useful tools and programs solving some of the exercises of each chapter. Each chapter has problems complementing the material covered in the text. The student

---

[13] Java and C++ have been popular choices in computational physics courses. But object oriented programming is usually avoided in the high performance part of a computation. So, one usually uses those languages in a procedural style of programming, cheating herself that she is actually learning the advantages of object oriented programming.

[14] `www.physics.ntua.gr/~konstant/ComputationalPhysics/`

needs to solve them in order to obtain hands on experience in scientific computing. I hope that I have already stressed enough that, in order for this book to be useful, it is not enough to be read in a café or in a living room, but one needs to *do* what it says.

Hoping that this book will be useful to you as a student or as an instructor, I would like to ask you to take some time to send me feedback for improving and/or correcting it. I would also appreciate fan mail or, if you are an expert, a review of the book. If you use the book in a class, as a main textbook or as supplementary material, I would also be thrilled to know about it. Send me email at `konstantmail.ntua.gr` and let me know if I can publish, anonymously or not, (part of) what you say on the web page (otherwise I will only use it privately for my personal ego-boost). Well, nothing is given for free: As one of my friends says, some people are payed in dollars and some others in ego-dollars!

Have fun computing scientifically!

Athens, 2014.

Chapter 1

The Computer

The aim of this chapter is to lay the grounds for the development of the computational skills which are necessary in the following chapters. It is not an in depth exposition but a practical training by example. For a more systematic study of the topics discussed, we refer to the bibliography. Many of the references are freely available on the web.

The are many choices that one has to make when designing a computer project. These depend on the needs for numerical efficiency, on available programming hours, on the needs for extensibility and upgradability and so on. In this book we will get the flavor of a project that is mostly scientifically and number crunching oriented. One has to make the best of the available computing resources and have powerful tools available for a productive analysis of the data. Such an environment, found in most of today's supercomputers, that offers flexibility, dependability, simplicity, powerful tools for data analysis and effective compilers is provided by the family of the Unix operating systems. The GNU/Linux operating system is a Unix variant that is freely available and most of its utilities are open source software. The voluntary work of millions of excellent programmers worldwide has built the most stable, fastest and highest quality software available for scientific computing today. Thanks to the idea of the open source software pioneered by Richard Stallman[1] this giant collaboration has been made possible.

Another choice that we have to make is the programming language. In this edition of the book we will be programming in C++. C++ is a

[1] `www.stallman.org`

language with very high level of abstraction designed for projects where modular programming and the use of complicated data structures is of very high priority. A large and complicated project should be divided into independent programming tasks (modules), where each task contains everything that it needs and does not interfere with the functionality of other modules. Although it has not been designed for high performance numerical applications, it is becoming more and more popular in the recent years.

C++, as well as other languages like C, Java and Fortran, is a language that needs to be *compiled* by a compiler. Other languages, like python, perl, awk, shell scripting, Macsyma, Mathematica, Octave, Matlab, ..., are *interpreted* line by line. These languages can be simple in their use, but they can be prohibitively slow when it comes to a numerically demanding program. A compiler is a tool that analyzes *the whole program* and optimizes the computer instructions executed by the computer. But if programming time is more valuable, then a simple, interpreted language can lead to faster results.

Another choice that we make in this book, and we mention it because it is not the default in most Linux distributions, is the choice of shell. The shell is a program that "connects" the user to the operating system. In this book, we will teach how to use a shell[2] to "send" commands to the operating system, which is the most effective way to perform complicated tasks. We will use the shell `tcsh`, although most of the commands can be interpreted by most popular shells. Shell scripting is simpler in this shell, although shells like `bash` provide more powerful tools, mostly needed for complicated system administration tasks. That may cause a small inconvenience to some readers, since `tcsh` is not preinstalled in Linux distributions[3].

1.1 The Operating System

The Unix family of operating systems offer an environment where complicated tasks can be accomplished by *combining* many different tools,

[2]It is more popular to be called "the command line", or the "terminal", or the "console", but in fact the user interaction is through a shell.

[3]See www.tcsh.org. On Debian like systems, like Ubuntu, installation is very simple through the software center or by the command `sudo apt-get install tcsh`.

each of which performs a distinct task. This way, one can use the power of each tool, so that trivial but complicated parts of a calculation don't have to be programmed. This makes the life of a researcher much easier and much more productive, since research requires from us to try many things before we understand how to compute what we are looking for.

In the Unix operating system everything is a *file*, and files are organized in a unique and unified *filesystem*. Documents, pictures, music, movies, executable programs are files. But also directories or devices, like hard disks, monitors, mice, sound cards etc, are, from the point of view of the operating system, files. In order for a music file to be played by your computer, the music data needs to be written to a device file, connected by the operating system to the sound card. The characters you type in a terminal are read from a file "the keyboard", and written to a file "the monitor" in order to be displayed. Therefore, the first thing that we need to understand is the structure of the Unix filesystem.

1.1.1 Filesystem

There is at least one *path* in the filesystem associated with each file. There are two types of paths, *relative paths* and *absolute paths*. These are two examples:

```
bin/RungeKutta/rk.exe
/home/george/bin/RungeKutta/rk.exe
```

The paths shown above may refer to the same or a different file. This depends on "where we are". If "we are" in the directory `/home/george`, then both paths refer to the same file. If on the other way "we are" in a directory `/home/john` or `/home/george/CompPhys`, then the paths refer[4] to two different files. In the last two cases, the paths refer to the files

```
/home/john/bin/RungeKutta/rk.exe
/home/george/CompPhys/bin/RungeKutta/rk.exe
```

respectively. How can we tell the difference? An absolute path always begins with the / character, whereas a relative path does not. When we

[4] Some times two or more paths refer to the same file, or as we say, a file has two or more "links" in the same filesystem, but let's keep it simple for the moment.

say that "we are in a directory", we refer to a position in the filesystem called the *current directory*, or *working directory*. Every process in the operating system has a unique current directory associated with it.

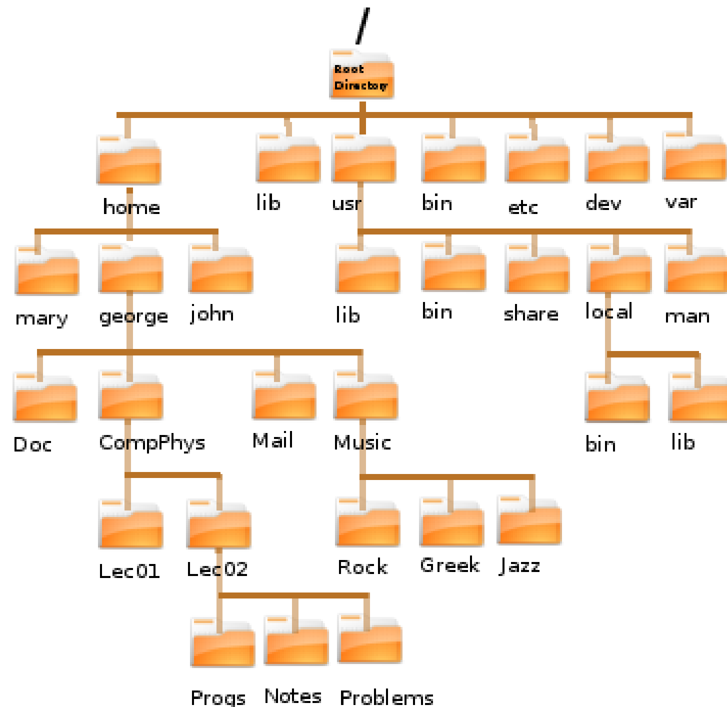

Figure 1.1: The Unix filesystem. It looks like a tree, with the *root directory* / at the top and branches that connect directories with their parents. Every directory contains files, among them other directories called its *subdirectories*. Every directory has a unique *parent directory*, noted by .. (double dots). The parent of the root directory is itself.

The filesystem is built on its *root* and looks like a tree positioned upside down. The symbol of the root is the character / The root is a *directory*. Every directory is a file that contains a list of files, and it is connected to a unique directory, its *parent directory* . Its list of files contains other directories, called its *subdirectories*, which all have it as their parent directory. All these files are the *contents* of the directory. Therefore, the filesystem is a tree of directories with the root directory at its top which branch to its subdirectories, which in their turn branch

into other subdirectories and so on. There is practically no limit to how large this tree can become, even for a quite demanding environment[5].

A path consists of a string of characters, with the characters `/` separating its *components*, and refers to a unique location in the filesystem. Every component refers to a file. All, but the last one, must be directories in a hierarchy, from parent directory to subdirectory. The only exception is a possible `/` in the beginning, which refers to the root directory. Such an example can be seen in figure 1.1.

In a Unix filesystem there is complete freedom in the choice of the location of the files[6]. Fortunately, there are some universally accepted conventions respected by almost everyone. One expects to find *home directories* in the directory `/home`, configuration files in the directory `/etc`, application executables in directories with names such as `/bin, /usr/bin, /usr/local/bin`, software libraries in directories with names such as `/lib, /usr/lib` etc.

There are some important conventions in the naming of the paths. A single dot "`.`" refers to the current directory and a double dot "`..`" to the parent directory. Similarly, a tilde "`~`" refers to the home directory of the user. Assume, e.g., that we are the user `george` running a process with a current directory `/home/george/Music/Rock` (see figure 1.1). Then, the following paths refer to the same file `/home/george/Doc/lyrics.doc`:

```
../../Doc/lyrics.doc
~/Doc/lyrics.doc
~george/Doc/lyrics.doc
./../../Doc/lyrics.doc
```

Notice that `~` and `~george` refer to the home directory of the user `george` (ourselves), whereas `~mary` refer to the home directory of another user, `mary`.

---

[5]Of course, the capacity of the filesystem is finite, issue the command "`df -i .`" in order to see the number of inodes available in your filesystem. Every file corresponds to one and only one inode of the filesystem. Every path is mapped to a unique inode, but an inode maybe pointed to by more than one paths.

[6]This gives a great sense of freedom, but historically this was a important factor that led the Unix operating systems, although superior in quality, not to win a fair share of the market! The Linux family tries to keep things simple and universal to a large extent, but one should be aware that because of this freedom files in different version of Linuxes or Unices can be in different places.

We are now going to introduce the basic commands for filesystem navigation and manipulation[7]. The command `cd` (=change directory) changes the current directory, whereas the command `pwd` (=print working directory) prints the current directory:

```
> cd /usr/bin
> pwd
/usr/bin
> cd /usr/local/lib
> pwd
/usr/local/lib
> cd
> pwd
/home/george
> cd -
> pwd
/usr/local/lib
> cd ../../
> pwd
/usr
```

The argument of the command `cd` is an absolute or a relative path. If the path is correct and we have the necessary permissions, the command changes the current directory to this path. If no path is given, then the current directory changes to the home directory of the user. If the character `-` is given instead of a path, then the command changes the current directory to the previous current directory.

The command `mkdir` creates new directories, whereas the command `rmdir` removes empty directories. Try:

```
> mkdir new
> mkdir new/01
> mkdir new/01/02/03
mkdir: cannot create directory ‘new/01/02/03’: No such file or
       directory
> mkdir -p new/01/02/03
> rmdir new
rmdir: ‘new’: Directory not empty
> rmdir new/01/02/03
```

[7]Remember that lines that begin with the > character are commands. All other lines refer to the output of the commands.

```
> rmdir new/01/02
> rmdir new/01
> rmdir new
```

Note that the command `mkdir` cannot create directories more than one level down the filesystem, whereas the command `mkdir -p` can. The "switch" `-p` makes the behavior of the command different than the default one.

In order to list the contents of a directory, we use the command `ls` (=list):

```
> ls
BE.eps   Byz.eps   Programs       srBE_xyz.eps   srB_xyz.eps
B.eps    Bzy.eps   srBd_xyz.eps   srB_xy.eps
> ls Programs
Backup              rk3_Byz.cpp   rk3.cpp
plot-commands       rk3_Bz.cpp    rk3_g.cpp
```

The first command is given without an argument and it lists the contents of the current directory. The second one, lists the contents of the subdirectory of the current directory `Programs`. If the argument is a list of paths pointing to regular files, then the command prints the names of the paths. Another way of giving the command is

```
total 252
-rw-r--r--  1 george users 24284 May  1 12:08 BE.eps
-rw-r--r--  1 george users 22024 May  1 11:53 B.eps
-rw-r--r--  1 george users 29935 May  1 13:02 Byz.eps
-rw-r--r--  1 george users 48708 May  1 12:41 Bzy.eps
drwxr-xr-x  4 george users  4096 May  1 23:38 Programs
-rw-r--r--  1 george users 41224 May  1 22:56 srBd_xyz.eps
-rw-r--r--  1 george users 23187 May  1 21:13 srBE_xyz.eps
-rw-r--r--  1 george users 24610 May  1 20:29 srB_xy.eps
-rw-r--r--  1 george users 23763 May  1 20:29 srB_xyz.eps
```

The switch `-l` makes `ls` to list the contents of the current directory together with useful information on the files in 9 columns. The first column lists the *permissions* of the files (see below). The second one lists the number of links of the files[8]. The third one lists the user who is the *owner* of

[8]For a directory it means the number of its subdirectories plus 2 (the parent directory

each file. The fourth one lists the group that is assigned to the files. The fifth one lists the size of the file in bytes (=8 bits). The next three ones list the modification time of the file and the last one the paths of the files.

File permissions[9] are separated in three classes: owner permissions, group permissions and other permissions. Each class is given three specific permissions, r=**r**ead, w=**w**rite and x=e**x**ecute. For regular files, read permission effectively means access to the file for reading/copying, write permission means permission to modify the contents of the file and execute permission means permission to execute the file as a command[10]. For directories, read permission means that one is able to read the names of the files in the directory (but not make it as current directory with the cd command), write permission means to be able to modify its contents (i.e. create, delete, and rename files) and execute permission grants permission to access/modify the contents of the files (but not list the names of the files, this is granted by the read permission).

The command ls -l lists permissions in three groups. The owner (positions 2-4), the group (positions 5-7) and the rest of the world (others - positions 8-10). For example

```
-rw-r--r--
-rwxr-----
drwx--x--x
```

In the first case, the owner has read and write but not execute permissions and the group+others have only read permissions. In the second case, the user has read, write and execute permissions, the group has read permissions and others have no permissions at all. In the last case, the user has read, write and execute permissions, whereas the group and the world have only execute permissions. The first character d indicates a special file, which in this case is a **d**irectory. All special files have this position set to a character, while regular files have it set to -.

File permissions can be modified by using the command chmod:

---

and itself). For a regular file, it shows how many paths in the filesystem point to this file.

[9]See the "File system permissions" entry in en.wikipedia.org.

[10]Of course it is the user's responsibility to make sure the file with execute permission is actually a program that is possible to execute. An error results if this is not the case.

```
> chmod u+x  file
> chmod og-w file1 file2
> chmod a+r  file
```

Using the first command, the owner (u≡ user) obtains (+) permission to execute (x) the file named file. Using the second one, the rest of the world (o≡ others) and the group (g≡group) loose (-) the write (w) permission to the files named file1 and file2. Using the third one, everyone (a≡all) obtain read (r) permission on the file named file.

We will close this section by discussing some commands which are used for administering files in the filesystem. The command cp (copy) copies the contents of files into other files:

```
> cp file1.cpp file2.cpp
> cp file1.cpp file2.cpp file3.cpp Programs
```

If the file file2.cpp does not exist, the first command copies the contents of file1.cpp to a new file file2.cpp. If it already exists, it replaces its contents by the contents of the file file2.cpp. In order for the second command to be executed, Programs needs to be a directory. Then, the contents of the files file1.cpp, file2.cpp, file3.cpp are copied to indentical files in the directory Programs. Of course, we assume that the user has the appropriate privileges for the command to be executed successfully.

The command mv "moves", or renames, files:

```
> mv file1.cpp file2.cpp
> mv file1.cpp file2.cpp file3.cpp Programs
```

The first command renames the file file1.cpp to file2.cpp. The second one moves files file1.cpp, file2.cpp, file3.cpp into the directory Programs.

The command rm (remove) deletes files[11]. Beware, the command is unforgiving: after deletion, a file cannot be restored into the filesystem[12].

[11]Actually it removes "links" from files. A file may have more than one links in the same partition of a filesystem. A file is deleted when its last link is removed.

[12]This does not mean that its contents have been deleted from the disk. Deletion means marking for overwriting. Until the data is overwritten it can be recovered by the

Therefore, after executing successfully the following commands

```
> ls
file1.cpp  file2.cpp  file3.cpp  file4.csh
> rm file1.cpp file2.cpp file3.cpp
> ls
file4.csh
```

the files `file1.cpp`, `file2.cpp`, `file3.cpp` do not exist in the filesystem anymore. A more prudent use of the command demands the flag `-i`. Then, before deletion we are asked for confirmation:

```
> rm -i *
rm: remove regular file 'file1.cpp'? y
rm: remove regular file 'file2.cpp'? y
rm: remove regular file 'file3.cpp'? y
rm: remove regular file 'file4.csh'? n
> ls
file4.csh
```

When we type `y`, the file is deleted, when we type `n`, the file is not deleted.

We cannot remove directories the same way. It is possible to use the command `rmdir` in order to remove *empty* directories. In order to delete directories together with their contents (including subdirectories and their contents) use the command[13] `rm -r`. For example, assume that the contents of the directories `dir1` and `dir1/dir2` are the files:

```
./dir1
./dir1/file2.cpp
./dir1/file1.cpp
./dir1/dir2
./dir1/dir2/file3.cpp
```

Then the results of the following commands are:

```
> rm dir1
rm: cannot remove 'dir1': Is a directory
> rm dir1/dir2
```

use of special tools. Shredding sensitive data can be tricky business...

[13]A small mistake, like `rm -rf *` and your data is ... history!

```
rm: cannot remove 'dir1/dir2': Is a directory
> rmdir dir1
rmdir: dir1: Directory not empty
> rmdir dir1/dir2
rmdir: dir1/dir2: Directory not empty
> rm -r dir1
```

The last command removes all files (assuming that we have write permissions for all directories and subdirectories). Alternatively, we can empty the contents of all directories first, and then remove them with the command `rmdir`:

```
> cd dir1/dir2; rm file3.cpp
> cd .. ; rmdir dir2
> rm file1.cpp file2.cpp
> cd .. ; rmdir dir1
```

Note that by using a semicolon, we can execute two or more commands on the same line.

1.1.2 Commands

Commands in a Unix operating system are files with execute permission. When we write a sentence on the command line, like

```
> ls -l test.cpp test.dat
```

the shell reads its and interprets it. The shell is a program that creates a interface between a user and the operating system. The first word (`ls`) of the sentence is interpreted as a command. The rest of the words are the *arguments* of the command and the program can use them (or not) at the discretion of its programmer. There is a special convention for arguments that begin with a - (e.g. `-l`, `--help`, `--version`, `-O3`). They are called *options* or *switches*, and they act as virtual switches that make the program act in a particular way. We have already seen that the program `ls` gives a different output with the switch `-l`.

In order for a command to be executed, the shell looks for a file that has the same name as the command (here a file named `ls`). In order to understand where the shell looks for such a file, we should digress

a little bit and explain the use of *shell variables* and *environment variables*. These have a name, which is a string of permissible characters, and their values are obtained by preceding their name with the $ character. For example the variable `PATH` has value `$PATH`. The values of the environment variables can be set with the command[14] `setenv` and of the shell variables with the command `set`:

```
> setenv MYVAR test-env
> set myvar = test-shell
> echo $MYVAR $myvar
test-env test-shell
```

Two special variables are the variables `PATH` and `path`:

```
>echo $PATH
/usr/local/bin:/usr/bin:/bin:/usr/X11/bin
>echo $path
/usr/local/bin /usr/bin /bin /usr/X11/bin
```

The first one is an environment variable and the second one is a shell variable. Their values are set by the shell, and we don't need to worry about them, unless we want to change them. Their value is a string of characters whose components should be valid paths to directories. In the first case, the components are separated by a `:`, while in the second case, by one or more spaces. In the example shown above, the shell searches each component of the `path` or `PATH` variables (in this order) until it finds a file `ls` in their contents. If it succeeds and the file has execute permissions, then the program in this file is executed. If it fails, then it prints an error message. Try the commands:

```
> which ls
/bin/ls
> ls -l /bin/ls
-rwxr-xr-x 1 root root 93560 Sep 28  2006 /bin/ls
```

We see that the program that the `ls` command executes the program in the file `/bin/ls`.

[14]The command `setenv` is special to the `tcsh` shell. For example the `bash` shell uses the syntax `MYVAR=test-env` in order to set the value of an environment variable.

The arguments of a command are passed on to the program that the command executes for possible interpretation. For example:

```
> ls -l test.cpp test.dat
```

The argument `-l` is the switch that results in a long listing of the files. The arguments `test.cpp` and `test.dat` are interpreted by the program `ls` as paths that it will look up for file information.

You can use the `*` (wildcard) character as a shorthand notation for a group of files. For example, in the command shown below

```
> ls -l *.cpp *.dat
```

the shell will expand `*.cpp` and `*.dat` to a list of all files whose names end with `.cpp` or `.dat`. Therefore, if the current directory contains the files `test.cpp`, `test1.cpp`, `myprog.cpp`, `test.dat`, `hello.dat`, the arguments that will be passed on to the command `ls` are

```
> ls -l myprog.cpp test1.cpp test.cpp hello.dat test.dat
```

For each command there are three special files associated with it. The first one is the *standard input* (`stdin`), the second one is the *standard output* (`stdout`) and the third one the *standard error* (`stderr`). These are files where the program can print or read data from. By default, these files are the terminal that the user uses to execute the command. In this case, when the program reads data from the `stdin`, then it reads the data that we type to the terminal using the keyboard. When the program writes data to the `stdout` or to the `stderr`, then the data is written to the terminal.

The advantage of using these special files in order to read/write data is that the user can *redirect* the input/output to these files to any file she wants. Using the character > at the end of a command redirects the `stdout` to the file whose name is written after >. For example:

```
> ls
file1.cpp  file2.cpp  file3.cpp  file4.csh
> ls > results
> ls
```

```
file1.cpp  file2.cpp  file3.cpp  file4.csh  results
```

The first of the above commands, prints the contents of the current working directory to the terminal. The second command redirects data written to the `stdout` to the file `results`. After executing the command, the file `results` is created and its contents are the names of the files `file1.cpp file2.cpp file3.cpp file4.csh`. If the file `results` does not exist (as in the above example), the file is created. If it already exists, it is *truncated* and its contents replaced by the data written to the `stdout` of the command. If we want to *append* data without erasing the existing contents, then we should use the string of characters `>>`. Therefore, if we give the command

```
> ls >> results
```

after executing the previous commands, then the contents of the file `results` will be

```
file1.cpp  file2.cpp  file3.cpp  file4.csh
file1.cpp  file2.cpp  file3.cpp  file4.csh  results
```

The redirection of the `stdin` is accomplished by the use of the character `<` while that of the `stderr` by the use of the string of characters[15] `>&`. We will see more examples in section 1.2.

It is possible to redirect the `stdout` of a command to be the `stdin` of another command. This is very useful for creating *filters*. A filter is a command that creates a flow of data between two or more programs. This process is called *piping*. Pipes are creating by using the character `|`

```
> cmd1 | cmd2 | cmd3 | ... | cmdN
```

Using the syntax shown above, the `stdout` of the command `cmd1` is redirected to the `stdin` of the command `cmd2`, the `stdout` of the command `cmd2` is redirected to the `stdin` of the command `cmd3` etc. More examples will be presented in section 1.2.

[15]This syntax is particular to the `tcsh` shell. For other shells (`bash, sh, ...`) read their documentation.

1.1.3 Looking for Help

Unix got itself a reputation for not being user friendly. This is far from the truth. Although there is a steep learning curve, detailed documentation for almost everything is available online.

The key for a comfortable ride is to learn how to use the help system available on your computer and on the internet. Most of the commands are self documented. A simple test, like the one shown below, will help you with the basic usage of most of the commands:

```
> cmd --help
> cmd -h
> cmd -help
> cmd -\?
```

For example, try the command `ls --help`. For a window application, start from the menu "Help". You should not be afraid and/or lazy and you should proceed with careful searching and reading.

For example, let's assume that you have heard about a command that sounds like `printf`, or something like that. The first level of online help is the `man` (=manual) command that searches the "man pages". Read the output of the command

```
> man printf
```

The command `info` usually provides more detailed and user friendly documentation. It has basic browsing capabilities like the browsers you use to read pages from the internet. Try the command

```
> info printf
```

Furthermore, the commands

```
> man -k printf
> whatis printf
```

will inform you that there are other, possibly related, commands with names like `fprintf, fwprintf, wprintf, sprintf...`:

```
> whatis printf
printf               (1)   - format and print data
printf               (1p)  - write formatted output
printf               (3)   - formatted output conversion
printf               (3p)  - print formatted output
printf [builtins]    (1)   - bash built-in commands, see bash↩
   (1)
```

The second column printed by the `whatis` command is the "section" of the man pages. In order to gain access to the information in a particular section, you have to give it as an argument to the `man` command:

```
> man 1  printf
> man 1p printf
> man 3  printf
> man 3p printf
> man bash
```

Section 1 of the man pages contains information of ordinary command line commands, section 3 contains information on functions in libraries of the C language. Section 2 contains information on commands used for system administration. You may browse the directory `/usr/share/man`, or read the man page of the `man` command (use the command `man man` for that!).

By using the command

```
> printf --help
```

we obtain plenty of memory refreshing information. The command

```
> locate printf
```

shows us many files related to the command `printf`. The commands

```
> which printf
> where printf
```

give information on the location of the executable(s) of the command `printf`.

Another useful feature of the shell is the *command* or it filename completion. This means that we can write only the first characters of the name of a command or filename and then press simultaneously the keys `[Ctrl-d]`[16] (i.e. press the key `Ctrl` and the key of the letter `d` at the same time). Then the shell will complete the name of the command up to the point that is is unique with the given string of characters[17]:

```
> pri[Ctrl-d]
printafm     printf  printenv   printnodetest
```

Try to type an `x` on the command line and then type `[Ctrl-d]`. You will learn all the commands that are available and whose name begins with an `x`: `xterm, xeyes, xclock, xcalc, ...`

Finally, the internet contains a wealth of information. Google your blues... and you will be rewarded!

1.2 Text Processing Tools – Filters

For doing data analysis, we will need powerful tools for manipulating data in text files. These are files that consist solely of printable characters. Some tools that can be used in order to construct complicated and powerful filters are the programs `cat, less, head, tail, grep, sort` and `awk`.

Suppose that we have data in a file named `data`[18] which contains information on the contents of a food warehouse and their prices:

```
bananas  100 pieces 1.45
apples   325 boxes  1.18
pears     34 kilos  2.46
bread     62 kilos  0.60
ham       85 kilos  3.56
```

[16]If you use the `bash` shell press `[Tab]` once or twice.

[17]Use the same procedure to auto-complete the names of files in the arguments of commands.

[18]The particular file, as well as most of the files in this section, can be found in the accompanying software of the chapter. It is highly recommended that you try all the commands in this section by using all the provided files.

The command

```
> cat data
```

prints the contents of the file `data` to the `stdout`. In general, this command prints the contents of all files given in its arguments or the `stdin` if none is given. Since the `stdin` and the `stdout` can be redirected, the command

```
> cat <  data > data1
```

takes the contents of the file `data` from the `stdin` and prints them to the `stdout`, which in this case is the file `data1`. This command has the same result as the command:

```
> cp data data1
```

The command

```
> cat data data1 > data2
```

prints the contents of the file `data` and then the contents of the file `data1` to the `stdout`. Since the `stdout` is redirected to the file `data2`, `data2` contains the data of both files.

By giving the command

```
> less gfortran.txt
```

you can browse the data contained in the file `gfortran.txt` one page at a time. Press `[space]` in order to "turn" a page, `[b]` to turn back a page. Press the up and down arrows to move one line backwards/forward. Press `[g]` in order to jump to the beginning of the file and press `[G]` in order to jump to the end. Press `[h]` in order to get a help message and press `[q]` in order to quit.

The commands

```
> head -n 1 data
```

```
bananas  100 pieces 1.45
> tail -n 2 data
bread     62 kilos  0.60
ham       85 kilos  3.56
> tail -n 2 data | head -n 1
bread     62 kilos  0.60
```

print the first line, the last two lines and the second to the last line of the file `data` to the `stdout` respectively. Note that, by piping the `stdout` of the command `tail` to the `stdin` of the command `head`, we are able to construct the filter "print the line before the last one".

The command `sort` sorts the contents of a file by comparing each line of its text with all others. The sorting is alphabetical, unless otherwise set by using options. For example

```
> sort data
apples   325 boxes  1.18
bananas  100 pieces 1.45
bread     62 kilos  0.60
ham       85 kilos  3.56
pears     34 kilos  2.46
```

For reverse sorting, try `sort -r data`. We can also sort by comparing specific *fields* of each line. By default, fields are words separated by one or more spaces. For example, in order to sort w.r.t. the second column of the file `data`, we can use the switch `-k 2` (=second field). Furthermore, we can use the switch `-n` for numerical sorting:

```
> sort -k 2 -n data
pears     34 kilos  2.46
bread     62 kilos  0.60
ham       85 kilos  3.56
bananas  100 pieces 1.45
apples   325 boxes  1.18
```

If we omit the switch `-n`, the comparison of the lines is performed based on character sorting of the second field and the result is

```
> sort -k 2 data
bananas  100 pieces 1.45
apples   325 boxes  1.18
```

```
pears      34 kilos   2.46
bread      62 kilos   0.60
ham        85 kilos   3.56
```

The last column contains floating point numbers (not integers). In order to sort by the values of such numbers we should use the switch `-g`:

```
> sort -k 4 -g data
bread      62 kilos   0.60
apples    325 boxes   1.18
bananas   100 pieces  1.45
pears      34 kilos   2.46
ham        85 kilos   3.56
```

The command `grep` processes a text file line by line, searching for a given string of characters. When this string is found anywhere in a line, this line is printed to the `stdout`. The command

```
> grep kilos data
pears      34 kilos   2.46
bread      62 kilos   0.60
ham        85 kilos   3.56
```

prints each line containing the string "kilos". If we want to search for all line *not* containing the string "kilos", then we add the switch `-v`:

```
> grep -v kilos data
bananas   100 pieces  1.45
apples    325 boxes   1.18
```

We can use a *regular expression* for searching a whole family of strings of characters. These monsters need a full book for discussing them in detail! But it is not hard to learn how to use some simple forms of regular expressions. Here are some examples:

```
> grep ^b data
bananas   100 pieces  1.45
bread      62 kilos   0.60
> grep '0$' data
bread      62 kilos   0.60
> grep '3[24]' data
```

```
apples   325 boxes  1.18
pears     34 kilos  2.46
```

The first one, prints each line whose *first* character is a `b`. The second one, prints each line that *ends* with a `0`. The third one, prints each line contaning the strings `32` *or* `34`.

By far, the strongest tool in our toolbox is the `awk` program. By default, `awk` analyzes a text file line by line. Each word (or *field* in the awk jargon) of these lines is stored in a set of variables with names `$1, $2, ...`. The variable `$0` contains the full line currently processed, whereas the variable `NF` counts the number of fields in the current line. The variable `NR` counts the number of lines of the file processed so far by `awk`.

An `awk` program can be written in the command line. A set of commands within `{ ... }` is executed for each line of input. The constructs `BEGIN{ ... }` and `END{ ... }` contain commands executed, only once, before and after the processing of the file respectively. For example, the command

```
> awk '{print $1,"total value= ",$2*$4}' data
bananas total value=  145
apples  total value=  383.5
pears   total value=  83.64
bread   total value=  37.2
ham     total value=  302.6
```

prints the name of the product (1st column = `$1`) and the total value stored in the warehouse (2nd column = `$2`) × (4th column = `$4`). More examples are given below:

```
> awk '{value += $2*$4}END{print "Total= ",value}' data
Total=  951.94
> awk '{av += $4}END{print "Average Price= ",av/NR}' data
Average Price=  1.85
> awk '{print $2^2 * sin($4) + exp($4)}' data
```

The first one calculates the total value of all products: The processing of each line results in the increment (+=) of the variable `value` by the product of the second and fourth fields. In the end (`END{ ... }`), the string `Total=` is printed, together with the final value of the variable

`value`. This is an easy way for computing the sum of the values calculated for each line. The second command, calculates and prints an average. The sum is calculated in each line and stored in the variable `av`. In the end, we print the quotient of the sum of all values by the number of lines that have been processed (`NR`). The last command shows a (crazy) mathematical expression based on numerical values found in each line of the file `data`: It computes the square of the second field times the sine of the fourth field plus the exponential of the fourth field.

There is much more potential in the commands presented above. Reading the documentation and getting experience by using them will provide you with very strong tools in order to accomplish complicated tasks.

1.3 Programming with Emacs

For a programmer that spends many hours programming every day, the environment and the tools available for editing the commands of a large and complicated program determine, to a large extent, the quality of her life! An editor edits the contents of a text file, that consists solely of printable characters. Such editors, available in most Linux environments, are the programs `gedit, vim, pico, nano, zile...` They provide basic functionality such as adding, removing or changing text within a file as well as more complicated functions, such as copying, pasting, searching and replacing text etc. There are many functions that are particularly useful to a programmer, such as detecting and formatting keywords of a particular programming language, pretty printing, closing scopes etc, which can be very useful for comfortable programming and for spotting errors. A very powerful and "knowledgeable" editor, offering many such functions for several programming languages, is the GNU Emacs editor[19]. Emacs is open source software, it is available for free and can be used in most available operating systems. It is programmable[20] and the user

[19] `http://www.gnu.org/software/emacs/ (main site), http://www.emacswiki.org/ (expert tips), http://en.wikipedia.org/wiki/Emacs (general info)`

[20] Emacs is written in a dialect of the programming language Lisp, called Elisp. There is no need of an in-depth knowledge of the language in order to program simple functions, just see how others are doing it...

can automate most of her everyday repeated tasks and configure it to her liking. There is a full interaction with the operating system, in fact Emacs has been built with the ambition of becoming an operating system. For example, a programmer can edit a C++ file, compile it, debug it and run it, everything done with Emacs commands.

1.3.1 Calling Emacs

In the command line type

```
> emacs &
```

Note the character `&` at the end of the line. This makes the particular command to run in the *background*. Without it, the shell waits until a command exits in order to return the prompt.

In a desktop environment, Emacs starts in its own window. For a quick and dirty editing session, or in the case that a windows environment is not available[21], we can run Emacs in a terminal mode. Then, we omit the `&` at the end of the line and we run the command

```
> emacs -nw
```

The switch `-nw` forces Emacs to run in terminal mode.

1.3.2 Interacting with Emacs

We can interact with Emacs in various ways. Newbies will prefer buttons and menus that offer a simple and intuitive interface. For advanced usage, however, we recommend that you make an effort to learn the keyboard shortcuts. There are also thousands of functions available to be used interactively. They are called from a "command line", called *the minibuffer* in the Emacs jargon.

Keyboard shortcuts are usually combinations of keystrokes that consist of the simultaneous pressing of the `Ctrl` or `Alt` keys together with other keys. Our convention is that a key sequence starting with a `C-` means that the characters that follow are keys simultaneously pressed

[21] Quite handy when we edit files in a remote computer.

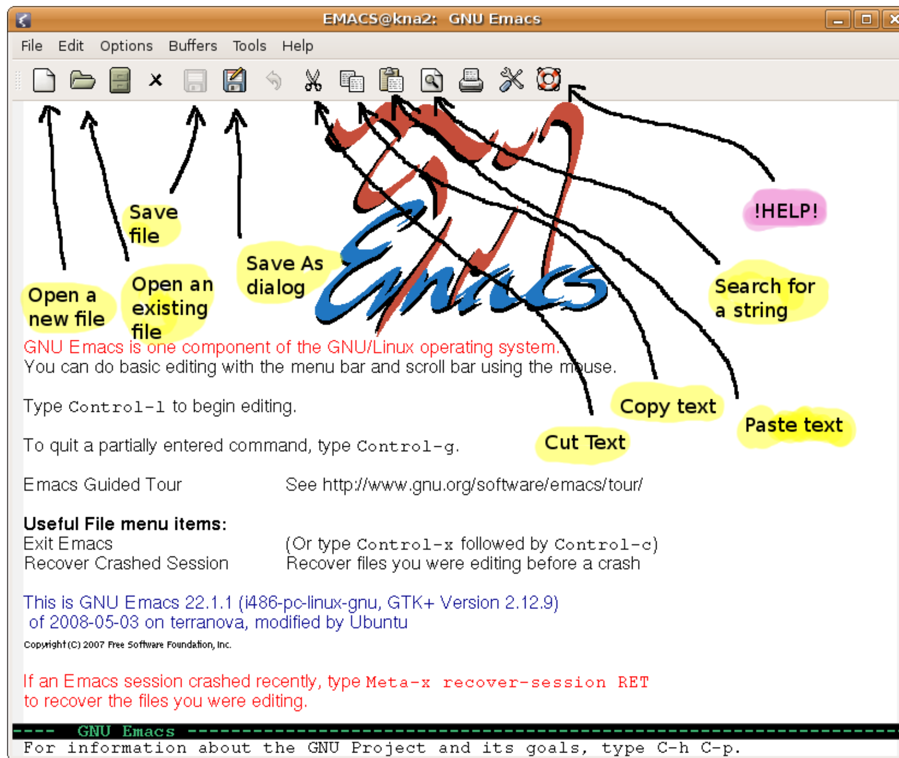

Figure 1.2: The Emacs window in a windows environment. The buttons of very basic functions found on its toolbar are shown and explained.

with the `Ctrl` key. A key sequance starting with a `M-` means that the characters that follow are keys simultaneously pressed with the `Alt` key[22]. Some commands have shortcuts consisting of two or more composite keystrokes. For example by `C-x C-c` we mean that we have to press simultaneously the `Ctrl` key together with `x` and *then* press simultaneously the `Ctrl` key together with `c`. This sequence is a shortcut to the command that exits Emacs. Another example is `C-x 2` which means to press the `Ctrl` key together with `x` and then press only the key `2`. This is a shortcut to the command that splits a window horizontally to two equal parts.

The most useful shortcuts are `M-x` (press the `Alt` key siumutaneously

[22]Actually, `M-` is the so called `Meta` key, usually bound to the `Alt` key. It is also bound to the `Esc` and `C-[` keys. The latter can be our *only* choices available in dumb terminals.

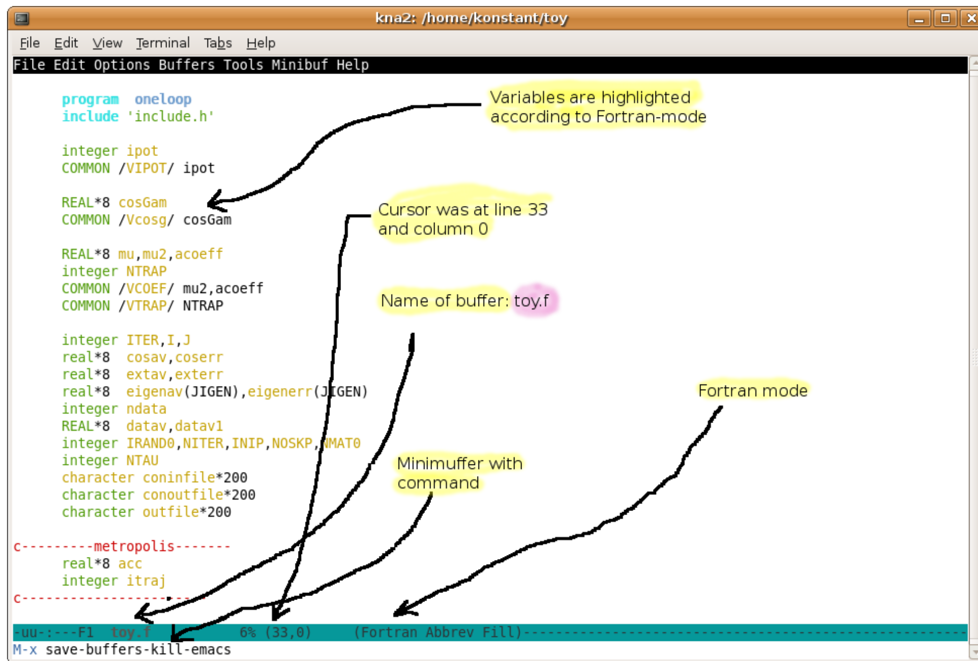

Figure 1.3: Emacs in a non-window mode running on the console. In this figure, we have typed the command `save-buffers-kill-emacs` in the minibuffer, a command that exits Emacs after saving edited data from all buffers. The same command can be given using the keyboard shortcut `C-x C-c`. We can see the *mode line* and the name of the buffer `toy.f` written on it, the percentage of the buffer (6%) shown in the window, the line and columns (33,0) where the point lies and the editing *mode* which is active on the buffer (Fortran mode (Fortran), Abbreviation mode (Abbrev), Auto Fill mode (Fill)).

with the `x` key) and `C-g`. The first command takes us to the minibuffer where we can give a command by typing its name. For example, type `M-x` and then type `save-buffers-kill-emacs` in the minibuffer (this will terminate Emacs). The second one is an "SOS button" that interrupts anything Emacs does and returns control to the working buffer. This can be pretty handy when a command hangs or destroys our work and we need to interrupt it.

The conventions for the mouse events are as follows: With `Mouse-1`, `Mouse-2` and `Mouse-3` we denote a simple click with the left, middle and right buttons of the mouse respectively. With `Drag-Mouse-1` we mean to press the left button of the mouse and at the same time drag the mouse.

Figure 1.4: The basic menus found in Emacs when run in a desktop environment. We can see the basic commands and the keyboard shortcut reminders in the parentheses. E.g. the command `File → Visit New File` can be given by typing `C-x C-f`. Note the commands `File → Visit New File` (open a file), `File→Save` (write contents of a buffer to a file), `File→Exit Emacs`, `File → Split Window` (split window in two), `File→New Frame` (open a new Emacs desktop window) and of course the well known commands `Cut, Copy, Paste, Undo` from the `Edit` menu. We can choose different buffers from the menu `Buffers`, which contain the contents of other files that we have opened for editing. We recommend trying the `Emacs Tutorial` and `Read Emacs Manual` in the `Help` menu.

We summarize the possible ways of giving a command in Emacs with the following examples that have the same effect: Open a file and put its contents in a buffer for editing.

- By pressing the toolbar button that looks like a white sheet of paper (see figure 1.2).
- By choosing the `File→Visit New File` menu entry.
- By typing the keyboard shortcut `C-x C-f`.
- By typing the name of the command in the minibuffer: `M-x find-file`

The number of available commands increases from the top to the bottom of the above list.

1.3.3 Basic Editing

In order to edit a file, Emacs places the *contents* of a file in a *buffer*. Such a buffer is a chunk of computer memory where the contents of the file are copied and it is not the file itself. When we make changes to the contents of a buffer, the file remains intact. For our changes to take effect and be written to the file, we have to save the buffer. Then, the contents of the buffer are written back to the file. It is important to understand the following cycle of events:

- Read a file's contents to a buffer.
- Edit buffer contents.
- Write (save) buffer's contents back into the file.

Emacs may have more than one buffers open for editing simultaneously. By default, the name of the buffer is the same as the name of the file that is edited, although this is not necessary[23]. The name of a buffer is written in the modeline of the window of the buffer, as can be seen in figure 1.3.

If Emacs crashes or exits before we save our edits, it is possible to recover (part of) them. There is a command `M-x recover-file` that will guide us through the necessary recovery steps, or we can look for a file that has the same name as the buffer we were editing surrounded by two `#`. For example, if we were editing the file `file.cpp`, the automatically saved changes can be found in the file `#file.cpp#`. Auto saving is done periodically by Emacs and its frequency can be controlled by the user.

The point where we insert text while editing is called "the point". This is right before the blinking cursor[24]. Each buffer has another position marked by "the mark". A point and the mark define a "region"

[23]The user can change the name of the buffer without affecting the name of the file it edits. Also, if we open more than one files with the same name, emacs gives each buffer a unique name. E.g. if we edit more than one files named `index.html` then the corresponding buffers are named `index.html`, `index.html<2>`, `index.html<3>`, ...

[24]Strictly speaking, the point lies between two characters and not on top of a character. The cursor lies on the character immediately to the right of the point. A point is assigned

in the buffer. This is a part of the text in the buffer where the functions of Emacs can act (e.g. copy, cut, change case, spelling etc.). We can set the region by setting a point and then press `C-SPC`[25] or give the command `M-x set-mark-command`. This defines the current point to be the mark. Then we can move the cursor to another point which will define a region together with the mark that we set. Alternatively we can use `Drag-Mouse-1` (hold the left mouse button and drag the mouse) and mark a region. The mark can be set with `Mouse-3`, i.e. with a simple click of the right button of the mouse. Therefore by `Mouse-1` at a point and then `Mouse-3` at a different point will set the region between the two points.

We can open a file in a buffer with the command `C-x C-f`, and then by typing its path. If the file already exists, its contents are copied to a buffer, otherwise a new buffer is created. Then:

- We can browse the buffer's contents with the Up/Down/Left/Right arrows. Alternatively, by using the commands `C-n, C-p, C-f` and `C-b`.

- If the buffer is large, we can browse its contents one page at a time by using the `Page Up`/`Page Dn` keys. Alternatively, by using the commands `C-v`, `M-v`.

- Enter text at the points simply by typing it.

- Delete characters before the point by using the `Backspace` key and after the point by using the `Delete` key. The command `C-d` deletes a forward character.

- Erase all the characters in a line that lie ahead of the point by using the command `C-k`.

- Open a new line by using `Enter` or `C-o`.

- Go to the first character of a line by using `Home` and the last one by using `End`. Alternatively, by using the commands `C-a` and `C-e`, respectively.

to every *window*, therefore a buffer can have multiple points, one for each window that displays its contents.

[25]Press the `Ctrl` and spacebar keys simultanesouly.

- Go to the first character of the buffer with the key C-Home and the last one with the key C-End. Alternatively, with M-x beginning -of -buffer and M-x end-of-buffer.
- Jump to any line we want: Type M-x goto-line and then the line number.
- Search for text after the point: Press C-s and then the text you are looking for. This is an incremental search and the point jumps immediately to the first string that matches the search. The same search can be repeated by pressing C-s repeatedely.

When we finish editing (or frequently enough so that we don't loose our work due to an unfortunate event), we save the changes in the buffer, either by pressing the save icon on the toolbar, or by pressing the keys C-s, or by giving the command M-x save-buffer.

1.3.4 Cut and Paste

Use the instructions below for slightly more advanced editing:

- Undo! Some of the changes described below can be catastrophic. Emacs has a great Undo function that keeps in its memory many of the changes inflicted by our editing commands. By repeatedely pressing C-/, we undo the changes we made. Alternatively, we can use C-x u or the menu entry Edit→Undo. Remember that C-g interrupts any Emacs process currently running in the buffer.
- Cut text by using the mouse: Click with Mouse-1 at the point before the beginning of the text and then Mouse-3 at the point after the end. A second Mouse-3 and the region is ... gone (in fact it is written in the "kill ring" and it is available for pasting)!
- Cut text by using a keyboard shortcut: Set the mark by C-SPC at the point before the beginning of the text that you want to cut. Then move the cursor after the last character of the text that you want to cut and type C-w.
- Copy text by using the mouse: Drag the mouse Drag-Mouse-1 and mark the region that you want to copy. Alternatively, Mouse-1 at

the point before the beginning of the text and then `Mouse-3` at the point after the end.

- Copy text by using a keyboard shortcut: Set the mark at the beginning of the text with `C-SPC` and then move the cursor after the last character of the text. Then type `M-w`.
- Pasting text with the mouse: We click the middle button[26] `Mouse-2` at the point that we want to insert the text from the kill ring (the copied text).
- Pasting text with a keyboard shortcut: We move the point to the desired insertion point and type `C-y`.
- Pasting text from previous copying: A fast choice is the menu entry `Edit→Paste from kill manu` and then select from the copied texts. The keyboard shortcut is to first type `C-y` and then `M-y` repeatedly, until the text that we want is yanked.
- Insert the contents of a file: Move the point to the desired place and type `C-x i` and the path of the file. Alternatively, give the command `M-x insert-file`.
- Insert the contents of a buffer: We can insert the contents of a whole buffer at a point by giving the command `M-x insert-buffer`.
- Replace text: We can replace text interactively with the command `M-x query-replace`, then type the string we want to replace, and then the replacement string. Then, we will be asked whether we want the change to be made and we can answer by typing `y` (yes), `n` (no), `q` (quit the replacements). A `,` (comma) makes only one replacement and quits (useful if we know that this is the last change that we want to make). If we are confident, we can change all string in a buffer, no questions asked, by giving the command `M-x replace-string`.
- Change case: We can change the case in the words of a region with the commands `M-x upcase-region`, `M-x capitalize-region` and `M-x downcase-region`. Try it.

[26]If it is a two button mouse, try clicking the left and right buttons simultaneously.

We note that cutting and pasting can be made between different windows of the same or different buffers.

1.3.5 Windows

Sometimes it is very convenient to edit one or more different buffers in two or more windows. The term “windows” in Emacs refers to regions of the same Emacs desktop window. In fact, a desktop window running an Emacs session is referred to as a *frame* in the Emacs jargon. Emacs can split a frame in two or more windows, horizontally or/and vertically. Study figure 1.5 on page 63 for details. We can also open a new frame and edit several buffers simultaneously[27]. We can manipulate windows and frames as follows:

- Position the point at the center of the window and clear the screen from garbage: `C-l` (careful: `l` not `1`).
- Split a window in two, horizontally: `C-x 2`.
- Split a window in two, vertically: `C-x 3`.
- Delete all other windows (remain only with the current one): `C-x 1`.
- Delete the current windows (the others remain): `C-x 0`.
- Move the cursor to the other window: `Mouse-1` or `C-x o`.
- Change the size of window: Use `Drag-Mouse-1` on the line separating two windows (the mode line). Use `C-^`, `C-}` for making a change of the horizontal/vertical size of a window respectively.
- Create a new frame: `C-x 5 2`.
- Delete a frame: `C-x 5 0`.
- Move the cursor to a different frame: With `Mouse-1` or with `C-x 5 o`.

[27]Be careful not to start a new Emacs session each time that all you need is a new frame. A new Emacs process takes time to start, binds computer resources and does not communicate with a different Emacs process.

You can have many windows in a dumb terminal. This is a blessing when a dekstop environment is not available. Of course, in that case you cannot have many frames.

1.3.6 Files and Buffers

- Open a file: `C-x C-f` or `M-x find-file`.
- Save a buffer: `C-x C-s` or `M-x save buffer`. With `C-x C-c` or `M-x save-buffers-kill-emacs` we can also exit Emacs. From the menu: `File`→`Save`. From the toolbar: click on the save icon.
- Save buffer contents to a different file: `C-x C-w` or `M-x write-file`. From the menu: `File`→`Save As`. From the toolbar: click on the "save as" icon.
- Save all buffers: `C-x s` or `M-x save-some-buffers`.
- Connect a buffer to a different file: `M-x set-visited-filename`.
- Kill a buffer: `C-x k`.
- Change the buffer of the current window: `C-x b`. Also, use the menu `Buffers`, then choose the name of the buffer.
- Show the list of all buffers: `C-x C-b`. From the menu: `Buffers` → `List All Buffers`. By typing `Enter` next to the name of the buffer, we make it appear in the window. There are several buffer administration commands. Learn about them by typing `C-h m` when the cursor is in the Bufer List window.
- Recover data from an edited buffer: If Emacs crashed, do not despair. Start a new Emacs and type `M-x recover-file` and follow the instructions. The command `M-x recover-session` recovers all unsaved buffers.
- Backup files: When you save a buffer, the previous contents of the file become a backup file. This is a file whose path is the same as the original's file with a `~` appended in the end. For example a file `test.cpp` will have as a backup the file `test.cpp~`. Emacs has

version control, and you can configure it to keep as many versions of your edits as you want.

- Directory browsing and directory administration commands: `C-x d` or `M-x dired`. You can act on the files of a directory (open, delete, rename, copy etc) by giving appropriate commands. When the cursor is in the `dired` window, type `C-h m` to read the relevant documentation.

1.3.7 Modes

Each buffer can be in different *modes*. Each mode may activate different commands or editing environment. For example each mode can color keywords relevant to the mode and/or bind keys to different commands. There exist *major modes*, and a buffer can be in only one of them. There are also *minor modes*, and a buffer can be in one or more of them. Emacs activates major and minor modes by default for each file. This usually depends on the filename but there are also other ways to control this. The user can change both major and minor modes at will, using appropriate commands.

Active modes are shown in a parenthesis on the mode line (see figures 1.3 and 1.5.

- `M-x c++-mode`: This mode is of special interest in this book since we will edit a lot of C++ code. We need it activated in buffers that contain a C++ program and its most useful characteristics are automatic code alignment by pressing the key `TAB`, the coloring of C++ statements, variables and other structural constructs (classes, if statements, for loops, variable declarations, comments etc). Another interesting function is the one that comments out a whole region of code, as well as the inverse function.

- `M-x c-mode`: For files containing programs written in the C language. Related modes are the `java-mode, perl-mode, awk-mode, python-mode, makefile-mode, octave-mode, gnuplot-mode` and others.

- `latex-mode`: For files containing LaTeX text formatting commands.

- `text-mode`: For editing simple text files (`.txt`).

- fundamental-mode: The basic mode, when one that fits better doesn't exist...

Some interesting minor modes are:

- M-x auto-fill-mode: When a line becomes too long, it is wrapped automatically. A related command to do that for the whole region is M-x fill-region, and for a paragraph M-x fill-paragraph.
- M-x overwite-mode: Instead of inserting characters at the point, overwrite the existing ones. By giving the command several times, we toggle between activating and deactivating the mode.
- M-x read-only mode: When visiting a file with valuable data that we don't want to change by mistake, we can activate this mode so that changes will not be allowed by Emacs. When we open a file with the command C-x C-r or M-x find-file-read-only this mode is activated. We can toggle this mode on and off with the command C-x C-q (M-x toggle-read-only). See the mode line of the buffer jack.c in figure 1.5 which contains a string %%. By clicking on the %% we can toggle the read-only mode on and off.
- flyspell-mode: Spell checking as we type.
- font-lock-mode: Colors the structural elements of the buffer which are defined by the major mode (e.g. the commands of a C++ program).

In a desktop environment, we can choose modes from the menu of the mode line. By clicking with Mouse-3 on the name of a mode we are offered options for (de)activating minor modes. With a Mouse-1 we can (de)activate the read-only mode with a click on :%% or :-- respectively. See figure 1.5.

1.3.8 Emacs Help

Emacs' documentation is impressive. For newbies, we recommend to follow the mini course offered by the Emacs tutorial. You can start the tutorial by typing C-h t or select Help → Emacs Tutorial from the menu. Enjoy... The Emacs man page (give the man emacs command in

the command line) will give you a summary of the basic options when calling Emacs from the command line.

A quite detailed manual can be found in the Emacs `info` pages[28]. Using `info` needs some training, but using the Emacs interface is quite intuitive and similar to using a web browser. Type the command `C-h r` (or choose `Help`→`Emacs Tutorial` from the menu) and you will open the front page of the emacs manual in a new window. By using the keys `SPC` and `Backspace` we can read the documentation page by page. When you find a link (similar to web page hyperlinks), you can click on it in order to open to read the topic it refers to. Using the navigation icons on the toolbar, you can go to the previous or to the next pages, go up one level etc. There are commands that can be given by typing single characters. For example, type `d` in order to jump to the main `info` directory. There you can find all the available manuals in the `info` system installed on your computer. Type `g (emacs)` and go to the top page of the Emacs manual. Type `g (info)` and read the `info` manual.

Emacs is structured in an intuitive and user friendly way. You will learn a lot from the names of the commands: Almost all names of Emacs commands consist of whole words, separated by a hyphen "-", which almost form a full sentence. These make them quite long sometimes, but by using auto completion of their names this does not pose a grave problem.

- `auto completion`: The names of the commands are auto completed by typing a `TAB` one or more times. E.g., type `M-x` in order to go to the minibuffer. Type `capi[TAB]` and the command autocompletes to `capitalize-`. By typing `[TAB]` for a second time, a new window opens and offers the options for completing to two possible commands: `capitalize-region` and `capitalize-word`. Type an extra `r[TAB]` and the command auto completes to the only possible choice `capitalize-region`. You can see all the commands that start with an `s` by typing `M-x s[TAB][TAB]`. Sure, there are many... Click on the `*Completions*` buffer and browse the possibilities. A lot will become clear just by reading the names of the commands. By typing `M-x [TAB][TAB]`, all available commands will appear in your buffer!

[28]If you prefer books in the form of `PDF` visit the page `www.gnu.org/software/emacs` and click on `Documentation`. You will find a 600 page book that has almost everything!

- keyboard shortcuts: If you don't remember what happens when you type C-s, no problem: Type C-h k and then the ... forgotten key sequence C-s. Conversely, have you forgotten what is the keyboard shortcut of the command save-buffer? Type C-h w and then the command.
- functions: Are you looking for a command, e.g. save-something-I-forgot? Type C-h f and then save-[TAB] in order to browse over different choices. Use Mouse-2 in order to select the command you are interested in, or type and complete the rest of its name (you may use [TAB] again). Read about the function in the *Help* buffer that opens.
- variables: Do the same after typing C-h v in order to see a variable's value and documentation.
- command apropos: Have you forgotten the exact name of a command? No problem... Type C-h a and a keyword. All commands related to the keyword you typed will appear in a buffer. Use C-h d for even more information.
- modes: When in a buffer, type C-h m and read information about the active modes of the buffer.
- info: Type C-h i
- Have you forgotten everything mentioned above? Just type C-h ?

1.3.9 Emacs Customization

You can customize everything in Emacs. From key bindings to programming your own functions in the Elisp language. The most common way for a user to customize her Emacs sessions, is to put all her customization commands in the file ~/.emacs in her home directory. Emacs reads and executes all these commands just before starting a session. Such a .emacs file is given below:

```
; Define F1 key to save the buffer
(global-set-key [f1]    'save-buffer)
```

```
; Define Control-c s to save the buffer
(global-set-key "\C-cs" 'save-some-buffers)
; Define Meta-s (Alt-s) to interactively search forward
(global-set-key "\M-s"  'isearch-forward)
; Define M-x is          to interactively search forward
(defalias       'is     'isearch-forward)
; Define M-x cm          to set c++-mode for the buffer
(defun cm()    (interactive) (c++-mode))
; Define M-x sign        to sign my name
(defun sign()  (interactive) (insert "K. N. Anagnostopoulos"))
```

Everything after a ; is a comment. Functions/commands are enclosed in parentheses. The first three ones bind the keys `F1`, `C-c s` and `M-s` to the commands `save-buffer`, `save-some-buffers` and `isearch-forward` respectively. The next one defines an *alias* of a command. This means that, when we give the command `M-x is` in the minibuffer, then the command `isearch-forward` will be executed. The last two commands are the definitions of the functions `(fm)` and `(sign)`, which can be called interactively from the minibuffer.

For more complicated examples google "emacs .emacs file" and you will see other users' `.emacs` files. You may also customize Emacs from the menu commands `Options`→`Customize Emacs`. For learning the Elisp language, you can read the manual "Emacs Lisp Reference Manual" found at the address
`www.gnu.org/software/emacs/manual/elisp.html`

1.4 The C++ Programming Language

In this section, we give a very basic introduction to the C++ programming language. This is not a systematic exposition and you are expected to learn what is needed in this book *by example*. So, please, if you have not already done it, get in front of a computer and *do* what you read. You can find many good tutorials and books introducing C++ in a more complete way in the bibliography.

1.4.1 The Foundation

The first program that one writes when learning a new programming language is the "Hello World!" program. This is the program that prints

"Hello World!" on your screen:

```
#include <iostream>
using namespace std;

int main(){

  // This is a comment.
  cout << "Hello World!\n";

}
```

Commands, or *statements*, in C++ are strings of characters separated by blanks ("words") and end with a semicolon (`;`). We can put more than one command on each line by separating them with a semicolon.

Everything after two slashes (`//`) is a comment. Proliferation of comments is necessary for documenting our code. Good documentation of our code is an integral part of programming. If we plan to have our code read by others (or by us) at a later time, we have to make sure to explain in detail what each line is supposed to do. You and your collaborators will save a lot of time in the process of debugging, improving and extending your code.

The first line of the code shown above is a *preprocessor directive*. These lines start with a `#` and are interpreted by a separate program, the preprocessor. The `#include` directive, inserts the contents of a file replacing the line where the directive is. This acts like an editor! Actually, the code that will be compiled is not the one shown above, but the result of adding the contents of a file whose name is `iostream`[29]. `iostream` is an example of a *header* file that has many definitions of functions and symbols used by the program. The particular header has the necessary definitions in order to perform standard input and standard output operations.

The execution of a C++ program starts by calling a *function* whose name is `main()`. Therefore, the line `int main(){` shows how to actually define a function in C++. Its name is the word before the parentheses `()` and the keyword `int` specifies that the function returns a value of integer

[29]The path to the file is determined by the compiler. If you are curious to see the file, search for it with the command `locate iostream`. In order to see the result of adding the contents of the file (and, actually several other files added by preprocessor directives in `iostream`), call the preprocessor with the command `cpp hello.cpp`

type[30]. Within the parentheses placed after the name of the function, we put the *arguments* that we pass to the function. In our case the parentheses contain nothing, showing how to define a function without arguments.

The curly brackets `{ ... }` define the *scope* or the *body* of the function and contain the statements to be executed when the function is called.

The line

```
  cout << "Hello World!\n";
```

is the only line that contains an executable statement that actually *does* something. Notice that it ends with a semicolon. This statement performs an output operation printing a *string* to the standard output. The sentence `Hello World!\n` is a constant string and contains a sequence of printable characters enclosed in double quotes. The last character `\n` is a `newline character`, that prints a new line to the `stdout`.

`cout` identifies the standard **c**haracter **out**put device, which gives access to the `stdout`. The characters << indicate that we write *to* `cout` the expression to the right. In order to make `cout` accessible to our program, we need both the inclusion of the header file `iostream` and the statement `using namespace std`[31].

Statements in C++ end with a semicolon. Splitting them over a number of lines is only a matter of making the code legible. Therefore, the following parts of the code have equivalent effect as the one written above:

```
int main()
{
  cout <<
   "Hello World!\n";
}
```

[30]The value returned by `main()` is useless within our program, but it is used by the operating system in order to inform us about the successful (or not) termination of the program. Of course, functions returning values is a very useful feature in general!

[31]Omitting `using namespace std` does not make `cout` inaccessible. One can use its "full name" `std::cout` instead. Remove the statement and try it. `cout` is part of the C++ *Standard Library*. All elements of the library belong to the `std` *namespace* and their names can be prefixed by `std::` Using the `using namespace std` statement, the prefix may be omitted.

```
int main(){cout <<"Hello World!\n";}
```

Finally notice that, for C++, uppercase and lowercase characters are different. Therefore `main()`, `Main()` and `MAIN()` are names of *different* functions.

In order to execute the commands in a program, it is necessary to *compile* it. This is a job done by a program called the *compiler* that translates the human language programming statements into binary commands that can be loaded to the computer memory for execution. There are many C++ compilers available, and you should learn which compilers are available for use in your computing environment. Typical names for C++ compilers are `g++, c++, icc, ....` You should find out which compiler is best suited for your program and spend time reading its documentation carefully. It is important to learn how to use a new compiler so that you can finely tune it to optimize the performance of *your* program.

We are going to use the open source and freely available compiler g++, which can be installed on most popular operating systems[32]. The compilation command is:

```
> g++ hello.cpp -o hello
```

The extension `.cpp` to the file name `hello.cpp` is important and instructs the compiler that the file contains source code in C++. Use your editor and edit a file with the name `hello.cpp` with the program shown above before executing the above command.

The switch `-o` defines the name of the executable file, which in our case is `hello`. If the compilation is successful, the program runs with the command:

```
> ./hello
Hello world!
```

The `./` is not a special symbol for running programs. The dot is the current working directory and `./hello` is the full path to the file `hello`.

[32]g++ is a *front end* to the GNU collection of compilers `gcc`. By installing `gcc`, you obtain a collection of compilers for several languages, like C, C++, Fortran, Java and others. See `http://gcc.gnu.org/`

Now, we will try a simple calculation. Given the radius of a circle we will compute its length and area. The program can be found in the file `area_01.cpp`:

```
#include <iostream>
using namespace std;

int main(){
  double PI = 3.14159265358979 32;
  double R  = 4.0;

  cout << "Perimeter= " << 2.0*PI*R << "\n";
  cout << "Area=      " << PI*R*R   << "\n";

}
```

The first two statements in `main()` declare the values of the *variables* `PI` and `R`. These variables are of type `double`, which are *floating point* numbers[33].

The following two commands have two effects: Computing the length $2\pi R$ and the area πR^2 of the circle and printing the results. The expressions `2.0*PI*R` and `PI*R*R` are evaluated before being printed to the `stdout`. Note the explicit decimal points at the constants `2.0` and `4.0`. If we write 2 or 4 instead, then these are going to be constants of the `int` type and by using them the wrong way we may obtain surprising results[34]. We compile and run the program with the commands:

```
> g++ area_01.cpp -o area
> ./area
Perimeter= 25.1327
Area=      50.2655
```

Now we will try a process that repeats itself for many times. We will calculate the length and area of 10 circles of different radii $R_i = 1.28 + i$,

[33]Don't confuse `double` variables with the real numbers. `double` variables take values that are finite approximations of real numbers and take values that are a subset of the rational numbers. This approximation becomes better by increasing the amount of memory allocated to store them. In most computing environments, `doubles` are allocated 8 bytes of memory, in which case they approximate real numbers with, more or less, 17 significant digits.

[34]Try adding the command `cout << 2/4 << 2.0/4.0;` and check the results.

$i = 1, 2, \ldots, 10$. We will store the values of the radii in an *array* `R[10]` of the `double` type. The code can be found in the file `area_02.cpp`:

```
#include <iostream>
using namespace std;

int main(){
  double PI = 3.1415926535897932;
  double R[10];
  double area, perimeter;
  int i;

  R[0] = 2.18;
  for(i=1;i<10;i++){
    R[i] = R[i-1] + 1.0;
  }

  for(i=0;i<10;i++){
    perimeter = 2.0*PI*R[i];
    area      = PI*R[i]*R[i];
    cout << (i+1) << ") R= " << R[i] << " perimeter= "
         << perimeter << '\n';
    cout << (i+1) << ") R= " << R[i] << " area      = "
         << area      << '\n';
  }

}
```

The declaration `double R[10]` defines an array of length 10. This way, the elements of the array are referred to by an index that takes values from 0 to 9. For example, `R[0]` is the first, `R[3]` is the fourth and `R[9]` is the last element of the array.

Between the lines

```
  for(i=1;i<10;i++){
    ...
  }
```

we can write commands that are repeatedly executed while the `int` variable `i` takes values from 1 to 9 with increasing step equal to 1. The way it works is the following: In the round brackets after the keyword `for`, there exist three statements separated by semicolons. The first, `i=1`, is

the statement executed once before the loop starts. The second, `i<10`, is a statement that is evaluated each time before the loop repeats itself. If it is `true`, then the statements in the loop are executed. If it is false, the control of the program is transferred after the end of the loop. The last statement, `i++`, is evaluated each time after the last statement in the loop has been executed. The operator `++` is the *increment* operator, and its effect is equivalent to the statement:

```
i = i + 1;
```

The value of `i` is increased by one. The command:

```
R[i] = R[i-1] + 1.0;
```

defines the `i`-th radius from the value `R[i-1]`. For the loop to work correctly, we must define the initial value of `R[0]`, otherwise[35] `R[0]=0.0`. The second loop uses the defined `R`-values in order to do the computation and print of the results.

Now we will write an interactive version of the program. Instead of *hard coding* the values of the radii, we will interact with the user asking her to give her own values. The program will read the 10 values of the radii from the standard input (`stdin`). We will also see how to write the results directly to a file instead of the standard output (`stdout`). The program can be found in the file `area_03.cpp`:

```
#include <iostream>
#include <fstream>
using namespace std;

int main(){
  const int    N  = 10;
  const double PI = 3.14159265358979 32;
  double R[N];
  double area,perimeter;
  int    i;

  for(i=0;i<N;i++){
    cout << "Enter radius of circle: ";
```

[35]Arrays in C++ are zero-initialized.

```
    cin  >> R[i];
    cout << "i= " << (i+1) << " R(i)= " << R[i] << '\n';
  }

  ofstream myfile ("AREA.DAT");
  for(i=0;i<N;i++){
    perimeter = 2.0*PI*R[i];
    area      = PI*R[i]*R[i];
    myfile << (i+1) << ") R= " << R[i]
           << " perimeter= "   << perimeter << '\n';
    myfile << (i+1) << ") R= " << R[i]
           << " area      = "  << area      << '\n';
  }

  myfile.close();

}
```

In the above program, the size of the array `R` is defined by a `const int`. A `const` declares a variable to be a parameter whose value does not change during the execution of the program and, if it is of `int` type, it can be used to declare the size of an array.

The array elements `R[i]` are read using the command:

```
    cin  >> R[i];
```

`cin` is the standard input stream, the same way that `cout` is the standard output stream[36]. We read input using the `>>` operator, which indicates that input is written *to* the variable on the right.

In order to interact with ordinary files, we need to include the header

```
#include <fstream>
```

In this header, the C++ class `ofstream` is defined and it can be used in order to write to files (output stream). An object in this class, like `myfile`, is defined ("instantiated") by the statement:

```
  ofstream myfile ("AREA.DAT");
```

[36]And `cerr` is the standard error stream.

This object's constructor is called by placing the parentheses ("AREA.DAT"), and then the output stream `myfile` is directed to the file AREA.DAT. Then we can write output to the file the same way we have already done with `cout`:

```
  myfile << (i+1) << ") R= " << R[i]
         << " perimeter= "   << perimeter << '\n';
```

When we are done writing to the file, we can close the stream with the statement:

```
  myfile.close();
```

Reading from files is done in a similar way by using the class `ifstream` instead of `ofstream`.

The next step will be to learn how to define and use functions. The program below shows how to define a function `area_of_circle()`, which computes the length and area of a circle of given radius. The following program can be found in the file area_04.cpp:

```
#include <iostream>
#include <fstream>
using namespace std;

const double PI = 3.1415926535897932;

void area_of_circle(const double& R, double& L, double& A);

int main(){
  const int    N  = 10;
  double R[N];
  double area,perimeter;
  int    i;

  for(i=0;i<N;i++){
    cout << "Enter radius of circle: ";
    cin  >> R[i];
    cout << "i= " << (i+1) << " R(i)= " << R[i] << '\n';
  }

  ofstream myfile ("AREA.DAT");
```

```
  for(i=0;i<N;i++){
    area_of_circle(R[i],perimeter,area);
    myfile << (i+1) << ") R= " << R[i] << " perimeter= "
           << perimeter << '\n';
    myfile << (i+1) << ") R= " << R[i] << " area      = "
           << area      << '\n';
  }

  myfile.close();

}
//-----------------------------------------------------------
void area_of_circle(const double& R, double& L, double& A){
  L = 2.0*PI*R;
  A =     PI*R*R;
}
```

The calculation of the length and the area of the circle is performed by the function

```
    area_of_circle(R[i],perimeter,area);
```

Calling a function, transfers the control of the program to the statements within the *body* of the function. The above function has *arguments* (R[i], perimeter, area). The argument R[i] is intended to be only an input variable whose value is not going to change during the calculation. The arguments perimeter and area are intended for output. Upon return of the function to the main program, they store the result of the computation. The user of a function must learn how to use its arguments in order to be able to call it in her program. These must be documented carefully by the programmer of the function.

In order to use a function, we need to declare it the same way we do with variables or, as we say, to provide its *prototype*. The prototype of a function can be declared without providing the function's definition. We may provide just enough details that determine the types of its arguments and the value returned. In our program this is done on the line:

```
void area_of_circle(const double& R, double& L, double& A);
```

This is the same syntax used later in the definition of the function, but replacing the body of the function with a semicolon. The argument list

does not need to include the argument names, only their types. We could have also used the following line in order to declare the function's prototype:

```
void area_of_circle(const double&  , double&  , double& );
```

We could also have used different names for the arguments, if we wished so. Including the names is a matter of style that improves legibility of the code.

The argument `R` is intended to be left unchanged during the function execution. This is why we used the keyword `const` in its declaration. The arguments `L` and `R`, however, will return a different value to the calling program. This is why `const` is not used for them.

The actual program executed by the function is between the lines:

```
void area_of_circle(const double& R, double& L, double& A){
  L = 2.0*PI*R;
  A =     PI*R*R;
}
```

The type of the value returned by a function is declared by the keyword before its name. In our case, this is `void` which declares that the function does not return a value.

The arguments `(R,L,A)` must be declared in the function and need not have the same names as the ones that we use when we call it. All arguments are declared to be of type `double`. The character `&` indicates that they are passed to the function *by reference*. This makes possible to change their values from within the function.

If `&` is omitted, then the arguments will be passed *by value* and a statement like `L = 2.0*PI*R` *will not* change the value of the variable passed by the calling program. This happens because, in this case, only the *value* of the variable `L` of the calling program is *copied* to a local variable which is used only within the function. This is important to understand and you are encouraged to run the program with and without the `&` and check the difference in the computed results.

The names of variables in a function are only valid within the *scope* of the function, i.e. between the curly brackets that contain the body of the function. Therefore the variable `const int N` is valid only within the

scope of `main()`. You may use any name you like, even if it is already used outside the scope of the function. The names of arguments need not be the same as the ones used in the calling program. Only their types have to match.

Variables in the *global scope* are accessible by all functions in the same file[37]. An example of such a variable is `PI`, which is accessible by `main()`, as well as by `area_of_circle()`.

We summarize all of the above in a program `trionymo.cpp`, which computes the roots of a second degree polynomial:

```
// ==========================================================
// Program to compute roots of a 2nd order polynomial
// Tasks: Input from user ,logical statements ,
//        use of functions ,exit
//
// Tests: a,b,c= 1  2  3 D=   -8
//        a,b,c= 1 -8 16 D=    0   x1=   4
//        a,b,c= 1 -1 -2 D=    9.  x1=   2.  x2=  -1.
//        a,b,c= 2.3 -2.99 -16.422 x1=   3.4 x2=  -2.1
// ==========================================================
#include <iostream>
#include <cstdlib>
#include <cmath>
using namespace std;

double Discriminant(double a, double b, double c);
void   roots(double a, double b, double c, double& x1,
             double& x2);

int main(){
  double a,b,c,D;
  double x1,x2;

  cout << "Enter a,b,c: ";
  cin  >> a >> b >> c;
  cout << a << " " << b << " " << c << " " << '\n';

  // Test if we have a well defined polynomial of 2nd degree:
  if( a == 0.0 ){
```

[37] If the code is spread over multiple files, then all files must use the keyword `external` in order to make the variable accessible to the functions that they contain. In *one* of the files, the variable must be defined without the word `external`. More on that later...

```
    cerr << "Trionymo: a=0\n";
    exit(1);
  }

  // Compute the discriminant
  D = Discriminant(a,b,c);
  cout << "Discriminant: D=  " << D << '\n';

  // Compute the roots in each case: D>0, D=0, D<0 (no roots)
  if     (D >  0.0) {
    roots(a,b,c,x1,x2);
    cout << "Roots:  x1= " << x1 << " x2= "<< x2 << '\n';
  }
  else if(D == 0.0) {
    roots(a,b,c,x1,x2);
    cout << "Double Root:  x1= " << x1 << '\n';
  }
  else{
    cout << "No real roots\n";
    exit(1);
  }

}
// ==========================================================
// This is the function that computes the discriminant
// A function returns a value. This value is returned using
// the return statement
// ==========================================================
double Discriminant(double a,double b,double c){
  return b * b - 4.0 * a * c;
}
// ==========================================================
// The function that computes the roots.
// a,b,c are passed by value: Their values cannot change
//                            within the function
// x1,x2 are passed by reference: Their values DO change
//                            within the function
// ==========================================================
void  roots(double a,double b,double c, double& x1,
            double& x2){
  double D;

  D = Discriminant(a,b,c);
  if(D >= 0.0){
    D = sqrt(D);
```

```
  }else{
    cerr << "roots: Sorry, cannot compute roots, D<0="
         << D << '\n';
  }

  x1 = (-b + D)/(2.0*a);
  x2 = (-b - D)/(2.0*a);
}
```

The program reads the coefficients of the polynomial $ax^2 + bx + c$. After a check whether $a \neq 0$, it computes the discriminant $D = b^2 - 4ac$ by calling the `Discriminant(a,b,c)`.

The type of the value returned must be declared at the function's prototype

```
double Discriminant(double a, double b, double c);
```

and at the function's definition

```
double Discriminant(double a,double b,double c){
  return b * b - 4.0 * a * c;
}
```

The value returned to the calling program is the value of the expression given as an argument to the `return` statement. `return` has also the effect of transferring the control of the program back to the calling statement.

1.5 Gnuplot

Plotting data is an indispensable tool for their qualitative, but also quantitative, analysis. Gnuplot is a high quality, open source, plotting program that can be used for generating publication quality plots, as well as for heavy duty analysis of a large amount of scientific data. Its great advantage is the possibility to use it from the command line, as well as from shell scripts and other programs. Gnuplot is programmable and it is possible to call external programs in order manipulate data and create complicated plots. There are many mathematical functions built in gnuplot and a `fit` command for non linear fitting of data. There exist

interactive *terminals* where the user can transform a plot by using the mouse and keyboard commands.

This section is brief and only the features, necessary for the following chapters, are discussed. For more information visit the official page of gnuplot `http://gnuplot.info`. Try the rich demo gallery at `http://gnuplot.info/screenshots/`, where you can find the type of graph that you want to create and obtain an easy to use recipe for it. The book [16] is an excellent place to look for many of gnuplot's secrets[38].

You can start a gnuplot session with the `gnuplot` command:

```
> gnuplot

  G N U P L O T
  Version X.XX
  ....
  The gnuplot FAQ is available from www.gnuplot.info/faq/
  ....
gnuplot>
```

There is a welcome message and then a prompt `gnuplot>` is issued waiting for your command. Type a command an press `[Enter]`. Type `quit` in order to quit the program. In the following, when we show a prompt `gnuplot>`, it is assumed that the command after the prompt is executed from within gnuplot.

Plotting a function is extremely easy. Use the command `plot` and `x` as the independent variable of the function[39]. The command

```
gnuplot> plot x
```

plots the function $y = f(x) = x$ which is a straight line with slope 1. In order to plot many functions simultaneously, you can write all of them in one line:

```
gnuplot> plot [-5:5][-2:4] x, x**2, sin(x),besj0(x)
```

[38] A the time of the writing of this book, there was a very nice site `www.gnuplotting.org` which shows how to create many beautiful and complicated plots.

[39] You can change the symbol of the independent variable. For example, the command `set dummy t` sets the independent variable to be `t`.

The above command plots the functions x, x^2, $\sin x$ and $J_0(x)$. Within the square brackets `[:]`, we set the limits of the x and y axes, respectively. The bracket `[-5:5]` sets $-5 \leq x \leq 5$ and the bracket `[-2:4]` sets $-2 \leq y \leq 4$. You may leave the job of setting such limits to gnuplot, by omitting some, or all of them, from the respective positions in the brackets. For example, typing `[1:][:5]` changes the lower and upper limits of x and y and leaves the upper and lower limits unchanged[40].

In order to plot data points (x_i, y_i), we can read their values from files. Assume that a file `data` has the following numbers recorded in it:

```
# x  y1   y2
0.5 1.0 0.779
1.0 2.0 0.607
1.5 3.0 0.472
2.0 4.0 0.368
2.5 5.0 0.287
3.0 6.0 0.223
```

The first line is taken by gnuplot as a *comment line*, since it begins with a `#`. In fact, gnuplot ignores everything after a `#`. In order to plot the second column as a function of the first, type the command:

```
gnuplot> plot "data" using 1:2 with points
```

The name of the file is within double quotes. After the keyword `using`, we instruct gnuplot which columns to use as the x and y coordinates, respectively. The keywords `with points` instructs gnuplot to add each pair (x_i, y_i) to the plot with points.

The command

```
gnuplot> plot "data" using 1:3 with lines
```

[40]By default, the x and y ranges are determined automatically. In order to *force* them to be automatic, you can insert a `*` in the brackets at the corresponding position(s). For example `plot [1:*][*:5]` sets the upper and lower limits of x and y to be determined automatically.

plots the third column as a function of the first, and the keywords `with lines` instruct gnuplot to connect each pair (x_i, y_i) with a straight line segment.

We can combine several plots together in one plot:

```
gnuplot> plot   "data" using 1:3 with points, exp(-0.5*x)
gnuplot> replot "data" using 1:2
gnuplot> replot 2*x
```

The first line plots the 1st and 3rd columns in the file `data` together with the function $e^{-x/2}$. The second line *adds* the plot of the 1st and 2nd columns in the file `data` and the third line adds the plot of the function $2x$.

There are many powerful ways to use the keyword `using`. Instead of column numbers, we can put mathematical expressions enclosed inside brackets, like `using (...):(...)`. Gnuplot evaluates each expression within the brackets and plots the result. In these expressions, the values of each column in the file `data` are represented as in the `awk` language. `$i` are variables that expand to the number read from columns `i=1,2,3,...`. Here are some examples:

```
gnuplot> plot "data" using 1:($2*sin($1)*$3) with points
gnuplot> replot 2*x*sin(x)*exp(-x/2)
```

The first line plots the 1st column of the file `data` together with the value $y_i sin(x_i) z_i$, where y_i, x_i and z_i are the numbers in the 2nd, 1st and 3rd columns respectively. The second line adds the plot of the function $2x\sin(x)e^{-x/2}$.

```
gnuplot> plot "data" using (log($1)):(log($2**2))
gnuplot> replot 2*x+log(4)
```

The first line plots the logarithm of the 1st column together with the logarithm of the square of the 2nd column.

We can plot the data written to the standard output of *any* command. Assume that there is a program called `area` that prints the perimeter and area of a circle to the `stdout` in the form shown below:

```
> ./area
R=    3.280000      area=    33.79851
R=    6.280000      area=    123.8994
R=    5.280000      area=    87.58257
R=    4.280000      area=    57.54895
```

The interesting data is at the second and fourth columns. These can be plotted directly with the gnuplot command:

```
gnuplot> plot "< ./area" using 2:4
```

All we have to do is to type the full command after the < within the double quotes. We can create complicated filters using pipes as in the following example:

```
gnuplot> plot \
 "< ./area|sort -g -k 2|awk '{print log($2),log($4)}'" \
 using 1:2
```

The filter produces data to the `stdout`, by combining the action of the commands `area`, `sort` and `awk`. The data printed by the last program is in two columns and we plot the results `using 1:2`.

In order to save plots in files, we have to change the *terminal* that gnuplot outputs the plots. Gnuplot can produce plots in several languages (e.g. PDF, postscript, SVG, LaTeX, jpeg, png, gif, etc), which can be interpreted and rendered by external programs. By redirecting the output to a file, we can save the plot to the hard disk. For example:

```
gnuplot> plot "data" using 1:3
gnuplot> set terminal jpeg
gnuplot> set output "data.jpg"
gnuplot> replot
gnuplot> set output
gnuplot> set terminal qt
```

The first line makes the plot as usual. The second one sets the output to be in the JPEG format and the third one sets the name of the file to which the plot will be saved. The fourth lines repeats all the previous plotting commands and the fifth one closes the file `data.jpg`. The last line chooses the interactive terminal `qt` to be the output of the next plot. High

quality images are usually saved in the PDF, encapsulated postscript or SVG format. Use `set terminal pdf,postscript eps` or `svg`, respectively.

And now a few words for 3-dimensional (3d) plotting. The next example uses the command `splot` in order to make a 3d plot of the function $f(x,y) = \mathrm{e}^{-x^2-y^2}$. After you make the plot, you can use the mouse in order to rotate it and view it from a different perspective:

```
gnuplot> set pm3d
gnuplot> set hidden3d
gnuplot> set size ratio 1
gnuplot> set isosamples 50
gnuplot> splot [-2:2][-2:2] exp(-x**2-y**2)
```

If you have data in the form (x_i, y_i, z_i) and you want to create a plot of $z_i = f(x_i, y_i)$, write the data in a file, like in the following example:

```
-1 -1 2.000
-1  0 1.000
-1  1 2.000

 0 -1 1.000
 0  0 0.000
 0  1 1.000

 1 -1 2.000
 1  0 1.000
 1  1 2.000
```

Note the empty line that follows the change of the value of the first column. If the name of the file is `data3`, then you can plot the data with the commands:

```
gnuplot> set pm3d
gnuplot> set hidden3d
gnuplot> set size ratio 1
gnuplot> splot "data3" with lines
```

We close this section with a few words on parametric plots. A parametric plot on the plane (2-dimensions) is a curve $(x(t), y(t))$, where t is a parameter. A parametric plot in space (3-dimensions) is a surface $(x(u,v)\ , y(u,v), z(u,v))$, where (u,v) are parameters. The following com-

mands plot the circle $(\sin t, \cos t)$ and the sphere $(\cos u \cos v,\ \cos u \sin v,\ \sin u)$:

```
gnuplot> set parametric
gnuplot> plot sin(t),cos(t)
gnuplot> splot cos(u)*cos(v),cos(u)*sin(v),sin(u)
```

1.6 Shell Scripting

A typical GNU/Linux environment offers very powerful tools for complicated system administration tasks. They are much more simple to use than to incorporate them into your program. This way, the programmer can concentrate on the high performance and scientific computing part of the project and leave the administration and trivial data analysis tasks to other, external, programs.

One can avoid repeating the same sequence of commands by coding them in a file. An example can be found in the file `script01.csh`:

```
#!/bin/tcsh -f
g++ area_01.cpp -o area
./area
g++ area_02.cpp -o area
./area
g++ area_03.cpp -o area
./area
g++ area_04.cpp -o area
./area
```

This is a very simple shell script. The first line instructs the operating system that the lines that follow are to be *interpreted* by the program `/bin/tcsh`[41]. This can be any program in the system, which in our case is the `tcsh` shell. The following lines are valid commands for the shell, one in each line. They compile the C++ programs found in the files that we created in section 1.4 with `g++`, and then they run the executable `./area`. In order to execute the commands in the file, we have to make

[41]Use `#!/bin/bash` if you prefer the `bash` shell.

sure that the file has the appropriate execute permissions. If not, we have to give the command:

```
> chmod u+x script01.csh
```

Then we simply type the path to the file `script01.csh`

```
> ./script01.csh
```

and the above commands are run the one after the other. Some of the versions of the programs that we wrote are asking for input from the `stdin`, which, normally, you have to type on the terminal. Instead of interacting directly with the program, we can write the input data to a file `Input`, and run the command

```
./area < Input
```

A more convenient solution is to use the, so called, "Here Document". A "Here Document" is a section of the script that is treated as if it were a separate file. As such, it can be used as input to programs by sending its "contents" to the `stdin` of the command that runs the program[42]. The "Here Document" does not appear in the filesystem and we don't need to administer it as a regular file. An example of using a "Here Document" can be found in the file `script02.csh`:

```
#!/bin/tcsh -f
g++ area_04.cpp -o area
./area <<EOF
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
```

[42] Their great advantage is that we can use variable and command substitution in them, therefore sending this information to the program that we want to run.

```
10.0
EOF
```

The `stdin` of the command `./area` is redirected to the contents between the lines

```
./area <<EOF
...
EOF
```

The string `EOF` marks the beginning and the end of the "Here Document", and can be any string you like. The last `EOF` has to be placed exactly in the beginning of the line.

The power of shell scripting lies in its programming capabilities: Variables, arrays, loops and conditionals can be used in order to create a complicated program. Shell variables can be used as discussed in section 1.1.2: The value of a variable `name` is `$name` and it can be set with the command `set name = value`. An array is defined, for example, by the command

```
set R = (1.0 2.0 3.0 4.0 5.0 6.0 7.0 8.0 9.0 10.0)
```

and its data can be accessed using the syntax `$R[1] ... $R[10]`.

Lets take a look at the following script:

```
#!/bin/tcsh -f

set files = (area_01.cpp area_02.cpp area_03.cpp area_04.cpp)
set R     = (1.0 2.0 3.0 4.0 5.0 6.0 7.0 8.0 9.0 10.0)

echo "Hello $USER Today is " `date`
foreach file ($files)
 echo "# ----------- Working on file $file "
 g++ $file -o area
 ./area <<EOF
$R[1]
$R[2]
$R[3]
$R[4]
$R[5]
$R[6]
```

```
$R[7]
$R[8]
$R[9]
$R[10]
EOF
 echo "# ———————— Done "
 if( -f AREA.DAT ) cat AREA.DAT
end
```

The first two lines of the script define the values of the arrays `files` (4 values) and `R` (10 values). The command `echo` echoes its argument to the `stdin`. `$USER` is the name of the user running the script. `` `date` `` is an example of *command substitution*: When a command is enclosed between backquotes and is part of a string, then the command is executed and its `stdout` is pasted back to the string. In the example shown above, `` `date` `` is replaced by the current date and time in the format produced by the `date` command.

The `foreach` loop

```
foreach file ($files)
 ...
end
```

is executed once for each of the 4 values of the array `files`. Each time the value of the variable `file` is set equal to one of the values `area_01.cpp`, `area_02.cpp`, `area_03.cpp`, `area_04.cpp`. These values can be used by the commands in the loop. Therefore, the command `g++ $file -o area` compiles a different file each time that it is executed by the loop.

The last line in the loop

```
 if( -f AREA.DAT ) cat AREA.DAT
```

is a conditional. It executes the command `cat AREA.DAT` if the condition `-f AREA.DAT` is true. In this case, `-f` constructs a logical expression which is true when the file `AREA.DAT` exists.

We close this section by presenting a more complicated and advanced script. It only serves as a demonstration of the shell scripting capabilities. For more information, the reader is referred to the bibliography [18, 19, 20, 21, 22]. Read carefully the commands, as well as the comments which

follow the # mark. Then, write the commands to a file `script04.csh`[43], make it an executable file with the command `chmod u+x script04.csh` and give the command

```
> ./script04.csh This is my first serious tcsh script
```

The script will run with the words "This is my first serious tcsh script" as its arguments. Notice how these arguments are manipulated by the script. Then, the script asks for the values of the radii of ten or more circles interactively, so that it will compute their perimeter and area. Type them on the terminal and then observe the script's output, so that you understand the function of each command. You will not regret the time investment!

```
#!/bin/tcsh -f
# Run this script as:
# ./script04.csh Hello this is a tcsh script
#-------------------------------------------------------------------
# `command` is command substitution: it is replaced by stdout of command
set now = `date` ; set mypc = `uname -a`
# Print information: variables are expanded within double quotes
echo "I am user $user working on the computer $HOST" #HOST is predefined
echo "Today the date is      :  $now"                #now  is defined above
echo "My home directory is   :  $home"               #home is predefined
echo "My current directory is:  $cwd"                #cwd changes with cd
echo "My computer runs       :  $mypc"               #mypc is defined above
echo "My process id is       :  $$    "              #$$   is predefined
# Manipulate the command line: ($#argv is number of elements in array argv)
echo "The command line has $#argv arguments"
echo "The name of the command I am running is: $0"
echo "Arguments 3rd to last of the command   : $argv[3-]"    #third to last
echo "The last argument is                   : $argv[$#argv]" #last element
echo "All arguments                          : $argv"

# Ask user for input: enter radii of circles
echo -n "Enter radii of circles: " # variable $< stores one line of input
set  Rs = ($<)  #Rs is now an array with all words entered by user
if($#Rs < 10 )then #make a test, need at least 10 of them
 echo "Need more than 10 radii. Exiting...."
 exit(1)
endif
echo "You entered $#Rs radii, the first is $Rs[1] and the last $Rs[$#Rs]"
echo "Rs= $Rs"
# Now, compute the perimeter of each circle:
foreach R ($Rs)
 # -v rad=$R set the awk variable rad equal to $R. pi=atan2(0,-1)=3.14...
 set l = `awk -v rad=$R 'BEGIN{print 2*atan2(0,-1)*rad}'`
```

[43]You will find it also in the accompanying software

```
 echo "Circle with R= $R has perimeter $l"
end
# alias defines a command to do what you want: use awk as a calculator
alias acalc  'awk "BEGIN{ print \!* }"' # \!* substitutes args of acalc
echo "Using acalc to compute       2+3=" `acalc 2+3`
echo "Using acalc to compute cos(2*pi)=" `acalc cos(2*atan2(0,-1))`
# Now do the same loop over radii as above in a different way
# while( expression ) is executed as long as "expression" is true
while($#Rs > 0) #executed as long as $Rs contains radii
 set R = $Rs[1] #take first element of $Rs
 shift Rs       #now $Rs has one less element:old $Rs[1] has vanished
 set a = `acalc atan2(0,-1)*${R}*${R}` # =pi*R*R calculated by acalc
 # construct a filename to save the result from the value of R:
 set file = area${R}.dat
 echo "Circle with R= $R has area $a" > $file #save result in a file
end            #end while
# Now look for our files: save their names in an array files:
set files = (`ls -1 area*.dat`)
if( $#files == 0) echo "Sorry, no area files found"
echo "----------------------------------------------"
echo "files: $files"
ls -l $files
echo "----------------------------------------------"
echo "And the results for the area are:"
foreach f ($files)
 echo -n "file ${f}: "
 cat $f
end
# now play a little bit with file names:
echo "----------------------------------------------"
set f = $files[1] # test permissions on first file
# -f, -r, -w, -x, -d test existence of file, rwxd permissions
# the ! negates the expression (true -> false, false -> true)
echo "testing permissions on files:"
if(  -f $f      ) echo "$file exists"
if(  -r $f      ) echo "$file is readable by me"
if(  -w $f      ) echo "$file is writable by be"
if(! -w /bin/ls) echo "/bin/ls is NOT writable by me"
if(! -x $f      ) echo "$file is NOT an executable"
if(  -x /bin/ls) echo "/bin/ls is executable by me"
if(! -d $f      ) echo "$file is NOT a directory"
if(  -d /bin    ) echo "/bin is a directory"
echo "----------------------------------------------"
# transform the name of a file
set f = $cwd/$f       # add the full path in $f
set filename  = $f:r  # removes extension .dat
set extension = $f:e  # gets    extension .dat
set fdir      = $f:h  # gets    directory of $f
set base      = `basename $f` # removes directory name
echo "file      is: $f"
echo "filename  is: $filename"
echo "extension is: $extension"
echo "directory is: $fdir"
echo "basename  is: $base"
# now transform the name to one with different extension:
set newfile = ${filename}.jpg
echo "jpeg name is: $newfile"
```

```
echo "jpeg base is:" `basename $newfile`
if($newfile:e == jpg)echo `basename $newfile` " is a picture"
echo "--------------------------------------------"
# Now save all data in a file using a "here document"
# A here document starts with <<EOF and ends with a line
# starting exactly with EOF (EOF can be any string as below)
# In a "here document" we can use variables and command
# substitution:
cat <<AREAS >> areas.dat
# This file contains the areas of circle of given radii
# Computation done by ${user} on ${HOST}. Today is `date`
`cat $files`
AREAS
# now see what we got:
if( -f areas.dat) cat areas.dat
# You can use a "here document" as standard input to any command:
# use gnuplot to save a plot: gnuplot does the job and exits...
gnuplot <<GNU
set terminal jpeg
set output   "areas.jpg"
plot "areas.dat" using 4:7 title "areas.dat",\
     pi*x*x                title "pi*R^2"
set output
GNU
# check our results: display the jpeg file using eog
if( -f areas.jpg) eog areas.jpg &
```

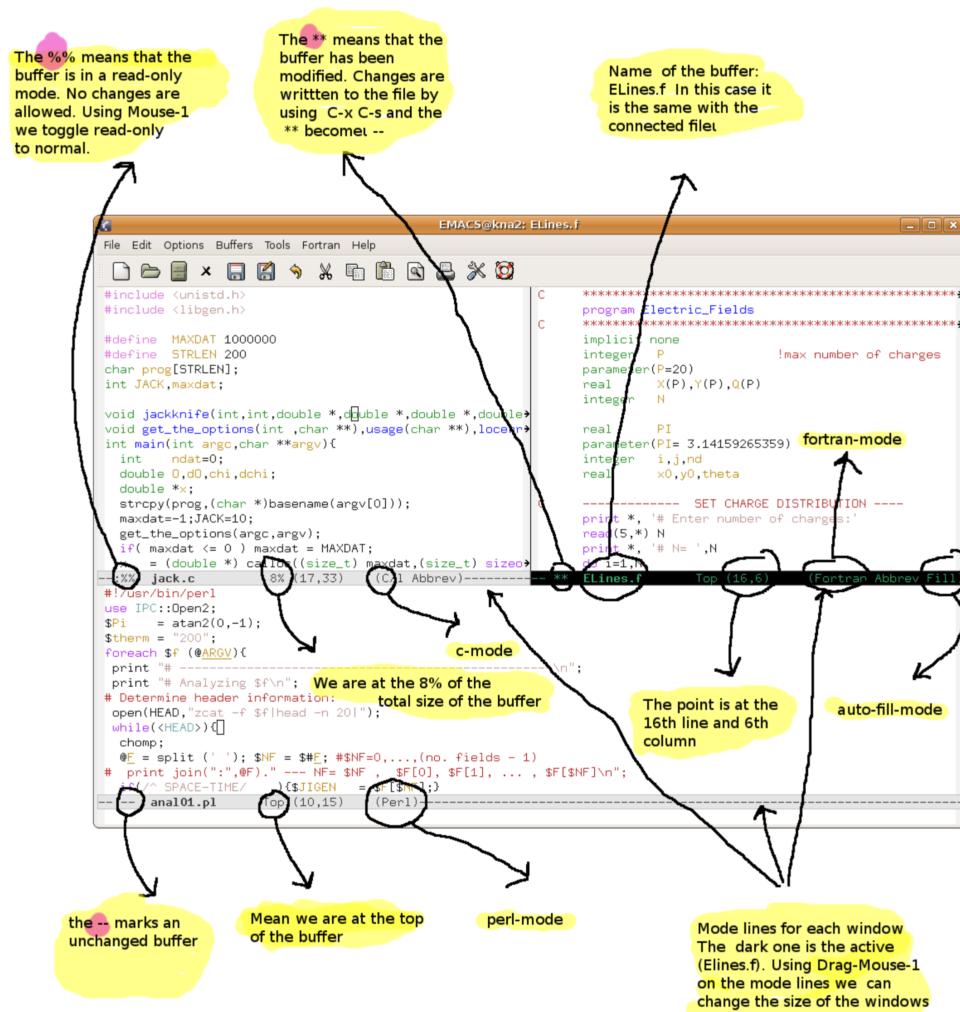

Figure 1.5: In this figure, the Emacs window has been split in three *windows*. The splitting was done horizontally first (`C-x 2`), and then vertically (`C-x 3`). By dragging the mouse (`Drag-Mouse-1`) on the horizontal *mode lines* and vertical lines that separate the windows, we can change window sizes. Notice the useful information diplayed on the mode lines. Each window has one *point* and the *cursor* is on the active window (in this case the window of the buffer named `ELines.f`). A buffer with no active changes in its contents is marked by a `--`, an edited buffer is marked by `**` and a buffer in *read only* mode with (%%). With a mouse click on a %%, we can change them to `--` (so that we can edit) and vice versa. With `Mouse-3` on the name of a *mode* we can activate a choice of *minor modes*. With `Mouse-1` on the name of a mode we ca have access to commands relevant to the mode. The numbers (17,31), (16,6) and (10,15) on the mode lines show the (line,column) of the point location on the respective windows.

awk	search for and process patterns in a file,
cat	display, or join, files
cd	change working directory
chmod	change the access mode of a file
cp	copy files
date	display current time and date
df	display the amount of available disk space
diff	display the differences between two files
du	display information on disk usage
echo	echo a text string to output
find	find files
grep	search for a pattern in files
gzip	compress files in the gzip (.gz) format (gunzip to uncompress)
head	display the first few lines of a file
kill	send a signal (like KILL) to a process
locate	search for files stored on the system (faster than find)
less	display a file one screen at a time
ln	create a link to a file
lpr	print files
ls	list information about files
man	search information about command in man pages
mkdir	create a directory
mv	move and/or rename a file
ps	report information on the processes run on the system
pwd	print the working directory
rm	remove (delete) files
rmdir	remove (delete) a directory
sort	sort and/or merge files
tail	display the last few lines of a file
tar	store or retrieve files from an archive file
top	dynamic real-time view of processes
wc	counts lines, words and characters in a file
whatis	list man page entries for a command
where	show where a command is located in the path (alternatively: whereis)
which	locate an executable program using ”path”
zip	create compressed archive in the zip format (.zip)
unzip	get/list contents of zip archive

Table 1.1: Basic Unix commands.

Table 1.2: Basic Emacs commands.

Leaving Emacs	
suspend Emacs (or iconify it under X)	C-z
exit Emacs permanently	C-x C-c

Files	
read a file into Emacs	C-x C-f
save a file back to disk	C-x C-s
save **all** files	C-x s
insert contents of another file into this buffer	C-x i
toggle read-only status of buffer	C-x C-q

Getting Help

The help system is simple. Type C-h (or F1) and follow the directions. If you are a first-time user, type C-h t for a **tutorial**.

remove help window	C-x 1
apropos: show commands matching a string	C-h a
describe the function a key runs	C-h k
describe a function	C-h f
get mode-specific information	C-h m

Error Recovery	
abort partially typed or executing command	C-g
recover files lost by a system crash	M-x recover-session
undo an unwanted change	C-x u, C-_ or C-/
restore a buffer to its original contents	M-x revert-buffer
redraw garbaged screen	C-l

Incremental Search	
search forward	C-s
search backward	C-r
regular expression search	C-M-s
abort current search	C-g

Use C-s or C-r again to repeat the search in either direction. If Emacs is still searching, C-g cancels only the part not matched.

Motion

entity to move over	**backward**	**forward**
character	C-b	C-f
word	M-b	M-f
line	C-p	C-n

Continued...

Table 1.2: Continued...

go to line beginning (or end)	`C-a`	`C-e`
go to buffer beginning (or end)	`M-<`	`M->`
scroll to next screen	`C-v`	
scroll to previous screen	`M-v`	
scroll left	`C-x <`	
scroll right	`C-x >`	
scroll current line to center of screen	`C-u C-l`	
Killing and Deleting		
entity to kill	**backward**	**forward**
character (delete, not kill)	`DEL`	`C-d`
word	`M-DEL`	`M-d`
line (to end of)	`M-0 C-k`	`C-k`
kill **region**	`C-w`	
copy region to kill ring	`M-w`	
yank back last thing killed	`C-y`	
replace last yank with previous kill	`M-y`	
Marking		
set mark here	`C-@` or `C-SPC`	
exchange point and mark	`C-x C-x`	
mark **paragraph**	`M-h`	
mark entire **buffer**	`C-x h`	
Query Replace		
interactively replace a text string	`M-%` *or* `M-x query-replace`	
using regular expressions	`M-x query-replace-regexp`	
Buffers		
select another buffer	`C-x b`	
list all buffers	`C-x C-b`	
kill a buffer	`C-x k`	
Multiple Windows		
When two commands are shown, the second is a similar command for a frame instead of a window.		
delete all other windows	`C-x 1`	`C-x 5 1`
split window, above and below	`C-x 2`	`C-x 5 2`
delete this window	`C-x 0`	`C-x 5 0`

Continued...

Table 1.2: Continued...

split window, side by side	C-x 3	
switch cursor to another window	C-x o	C-x 5 o
grow window taller	C-x ^	
shrink window narrower	C-x {	
grow window wider	C-x }	
Formatting		
indent current **line** (indent code etc)	TAB	
insert newline after point	C-o	
fill paragraph	M-q	
Case Change		
uppercase word	M-u	
lowercase word	M-l	
capitalize word	M-c	
uppercase region	C-x C-u	
lowercase region	C-x C-l	
The Minibuffer		
The following keys are defined in the minibuffer.		
complete as much as possible	TAB	
complete up to one word	SPC	
complete and execute	RET	
abort command	C-g	
Type C-x ESC ESC to edit and repeat the last command that used the minibuffer. Type F10 to activate menu bar items on text terminals.		
Spelling Check		
check spelling of current word	M-$	
check spelling of all words in region	M-x ispell-region	
check spelling of entire buffer	M-x ispell-buffer	
On the fly spell checking	M-x flyspell-mode	
Info – Getting Help Within Emacs		
enter the Info documentation reader	C-h i	
scroll forward	SPC	
scroll reverse	DEL	
next node	n	

Continued...

Table 1.2: Continued...

previous node	`p`
move **up**	`u`
select menu item by name	`m`
return to last node you saw	`l`
return to directory node	`d`
go to top node of Info file	`t`
go to any node by name	`g`
quit Info	`q`

Chapter 2

Kinematics

In this chapter we show how to program simple kinematic equations of motion of a particle and how to do basic analysis of numerical results. We use simple methods for plotting and animating trajectories on the two dimensional plane and three dimensional space. In section 2.3 we study numerical errors in the calculation of trajectories of freely moving particles bouncing off hard walls and obstacles. This will be a prelude to the study of the integration of the *dynamical* equations of motion that we will introduce in the following chapters.

2.1 Motion on the Plane

When a particle moves on the plane, its position can be given in Cartesian coordinates $(x(t), y(t))$. These, as a function of time, describe the particle's trajectory. The position vector is $\vec{r}(t) = x(t)\,\hat{x} + y(y)\,\hat{y}$, where $\hat{x}$ and $\hat{y}$ are the unit vectors on the x and y axes respectively. The velocity vector is $\vec{v}(t) = v_x(t)\,\hat{x} + v_y(t)\,\hat{y}$ where

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$$

$$v_x(t) = \frac{dx(t)}{dt} \qquad v_y(t) = \frac{dy(t)}{dt}\,, \tag{2.1}$$

The acceleration $\vec{a}(t) = a_x(t)\,\hat{x} + a_y(t)\,\hat{y}$ is given by

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} \quad = \quad \frac{d^2\vec{r}(t)}{dt^2}$$
$$a_x(t) = \frac{dv_x(t)}{dt} = \frac{d^2x(t)}{dt^2} \qquad a_y(t) = \frac{dv_y(t)}{dt} = \frac{d^2y(t)}{dt^2}\,. \tag{2.2}$$

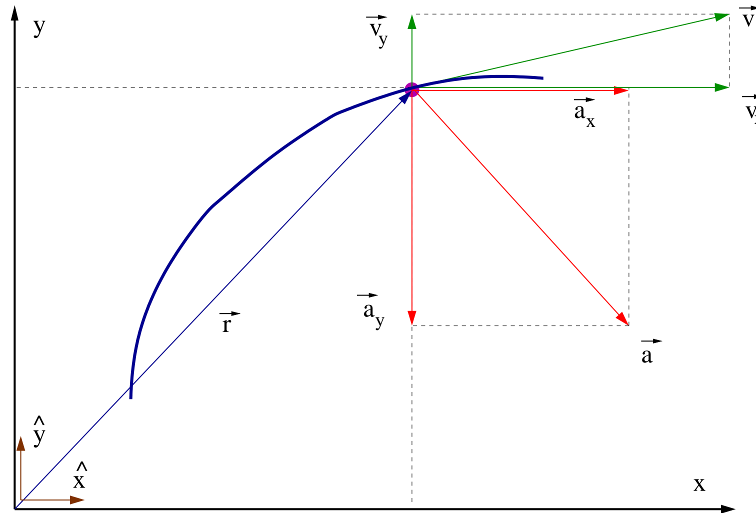

Figure 2.1: The trajectory of a particle moving in the plane. The figure shows its position vector $\vec{r}$, velocity $\vec{v}$ and acceleration $\vec{a}$ and their Cartesian components in the chosen coordinate system at a point of the trajectory.

In this section we study the kinematics of a particle trajectory, therefore we assume that the functions $(x(t), y(t))$ are known. By taking their derivatives, we can compute the velocity and the acceleration of the particle in motion. We will write simple programs that compute the values of these functions in a time interval $[t_0, t_f]$, where t_0 is the initial and t_f is the final time. The continuous functions $x(t), y(t), v_x(t), v_y(t)$ are approximated by a discrete sequence of their values at the times $t_0, t_0 + \delta t, t_0 + 2\delta t, t_0 + 3\delta t, \ldots$ such that $t_0 + n\delta t \leq t_f$.

We will start the design of our program by forming a generic template to be used in all of the problems of interest. Then we can study each problem of particle motion by programming only the equations of motion without worrying about the less important tasks, like input/output,

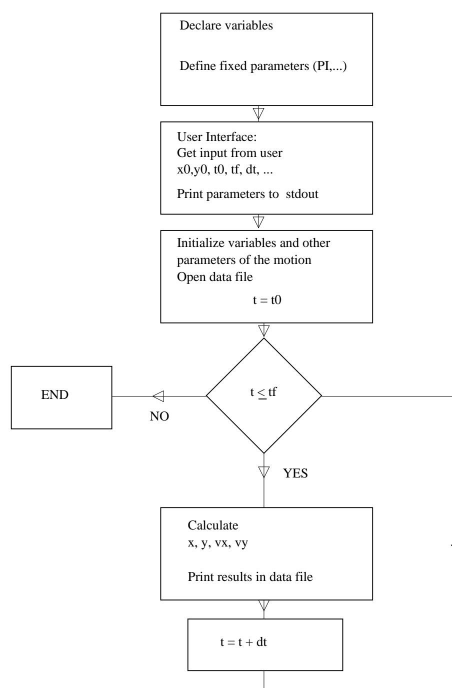

Figure 2.2: The flowchart of a typical program computing the trajectory of a particle from its (kinematic) equations of motion.

user interface etc. Figure 2.2 shows a *flowchart* of the basic steps in the algorithm. The first part of the program declares variables and defines the values of the fixed parameters (like $\pi = 3.1459\ldots$, $g = 9.81$, etc). The program starts by interacting with the user ("user interface") and asks for the values of the variables x_0, y_0, t_0, t_f, $\delta t\ldots$. The program prints these values to the `stdout` so that the user can check them for correctness and store them in her data.

The main calculation is performed in a *loop* executed while $t \leq t_f$. The values of the positions and the velocities $x(t), y(t), v_x(t), v_y(t)$ are calculated and printed in a file together with the time t. At this point we fix the format of the program output, something that is very important to do it in a consistent and convenient way for easing data analysis. We choose to print the values t, x, y, vx, vy in five columns in each line of the output file.

The specific problem that we are going to solve is the computation of

the trajectory of the circular motion of a particle on a circle with center (x_0, y_0) and radius R with constant angular velocity ω. The position on the circle can be defined by the angle θ, as can be seen in figure 2.3. We define the initial position of the particle at time t_0 to be $\theta(t_0) = 0$.

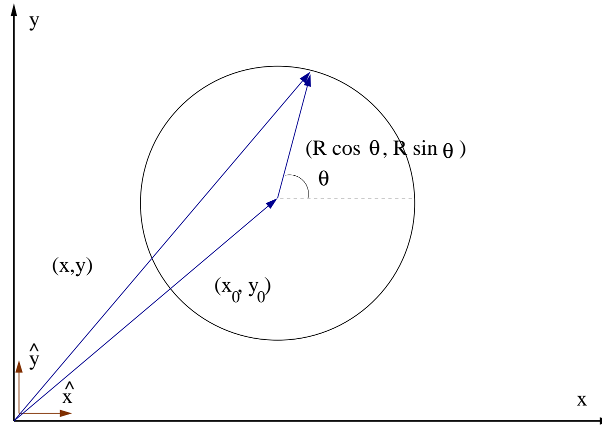

Figure 2.3: The trajectory of a particle moving on a circle with constant angular velocity calculated by the program `Circle.cpp`.

The equations giving the position of the particle at time t are

$$\begin{aligned} x(t) &= x_0 + R\cos\left(\omega(t - t_0)\right) \\ y(t) &= y_0 + R\sin\left(\omega(t - t_0)\right) . \end{aligned} \tag{2.3}$$

Taking the derivative w.r.t. t we obtain the velocity

$$\begin{aligned} v_x(t) &= -\omega R\sin\left(\omega(t - t_0)\right) \\ v_y(t) &= \omega R\cos\left(\omega(t - t_0)\right) , \end{aligned} \tag{2.4}$$

and the acceleration

$$\begin{aligned} a_x(t) &= -\omega^2 R\cos\left(\omega(t - t_0)\right) = -\omega^2(x(t) - x_0) \\ a_y(t) &= -\omega^2 R\sin\left(\omega(t - t_0)\right) = -\omega^2(y(t) - y_0) . \end{aligned} \tag{2.5}$$

We note that the above equations imply that $\vec{R}\cdot\vec{v} = 0$ ($\vec{R} \equiv \vec{r} - \vec{r}_0$, $\vec{v} \perp \vec{R}$, $\vec{v}$ tangent to the trajectory) and $\vec{a} = -\omega^2\vec{R}$ ($\vec{R}$ and $\vec{a}$ anti-parallel, $\vec{a} \perp \vec{v}$).

The data structure is quite simple. The constant angular velocity ω is stored in the `double` variable `omega`. The center of the circle (x_0, y_0), the

radius R of the circle and the angle θ are stored in the double variables x0, y0, R, theta. The times at which we calculate the particle's position and velocity are defined by the parameters $t_0, t_f, \delta t$ and are stored in the double variables t0, tf, dt. The current position $(x(t), y(t))$ is calculated and stored in the double variables x, y and the velocity $(v_x(t), v_y(t))$ in the double variables vx, vy. The declarations of the variables are put in the beginning of the program:

```
double x0,y0,R,x,y,vx,vy,t,t0,tf,dt;
double theta,omega;
```

The user interface of the program is the interaction of the program with the user and, in our case, it is the part of the program where the user enters the parameters omega, x0, y0, R, t0, tf, dt. The program issues a prompt with the names the variables expected to be read. The variables are read from the stdin by reading from the stream cin and the values entered by the user are printed to the stdout using the stream cout[1]:

```
cout << "# Enter omega:\n";
cin  >> omega;              getline(cin,buf);
cout << "# Enter center of circle (x0,y0) and radius R:\n";
cin  >> x0 >> y0 >> R;     getline(cin,buf);
cout << "# Enter t0,tf,dt:\n";
cin  >> t0 >> tf >> dt;    getline(cin,buf);
cout <<"# omega= " << omega << endl;
cout <<"# x0= "    << x0    << " y0= " << y0
     << " R=  "    << R     << endl;
cout <<"# t0= "    << t0    << " tf= " << tf
     << " dt= "    << dt    << endl;
```

There are a couple of things to explain. Notice that after reading each variable from the standard input stream cin, we call the function getline. By calling getline(cin,buf), a whole line is read from the input stream cin into the *string* buf[2]. Then the statement

[1]This is done so that the used can check for typos and see the actual value read by the program. By redirecting the stdout of a file on the hard disk, the parameters can be saved for future reference and used in data analysis.

[2]In fact it is possible to call getline(cin,buf,char) and read a line until the character char is encountered.

```
cin >> x0 >> y0 >> R;   getline(cin,buf);
```

has the effect of reading three doubles from the `stdin` and put the rest of the line in the string `buf`. Since we never use `buf`, this is a mechanism to discard the rest of the line of input. The reason for doing so will become clear later.

Objects of type `string` in C++ store character sequences. In order to use them you have to include the header

```
#include <string>
```

and, e.g., declare them like

```
string buf,buf1,buf2;
```

Then you can store data in the obvious way, like `buf="Hello World!"`, manipulate `string` data using operators like `buf=buf1` (assign `buf1` to `buf`), `buf=buf1+buf2` (concatenate `buf1` and `buf2` and store the result in `buf`), `buf1==buf2` (compare strings) etc.

Finally, `endl` is used to end all the `cout` statements. This has the effect of adding a newline to the output stream and flush the output[3].

Next, the program initializes the state of the computation. This includes checking the validity of the parameters entered by the user, so that the computation will be possible. For example, the program computes the expression `2.0*PI/omega`, where it is assumed that `omega` has a non zero value. We will also demand that $R > 0$ and $\omega > 0$. An `if` statement will make those checks and if the parameters have illegal values, the `exit` statement[4] will stop the program execution and print an

[3]When *buffered* output is used, it is not written out immediately but stored in a temporary memory location (a *buffer*). When the buffer fills, it is automatically flushed to the output stream. If we want to force flushing before the buffer is full, then we have to *flush* the buffer. There are several methods to flush an output stream `os` (like `os.flush()`).

[4]The `exit(1)` statement returns 1 as exit code for the program. This is the `int` that `main()` returns. `exit(0)` is conventionally used for a normal exit and a non zero value is used when an error occurs. In order to use `exit()` you must include the header `cstdlib`.

informative message to the standard error stream `cerr`[5]. The program opens the file `Circle.dat` for writing the calculated values of the position and the velocity of the particle.

```
if(R     <=0.0){cerr <<"Illegal value of R     \n";exit(1);}
if(omega<=0.0){cerr <<"Illegal value of omega\n";exit(1);}
cout     << "# T= "  << 2.0*PI/omega              << endl;
ofstream myfile("Circle.dat");
myfile.precision(17);
```

The line `myfile.precision(17)` sets the precision of the floating point numbers (like `double`) printed to `myfile` to 17 significant digits accuracy. The default is 6 which is a pity, because `doubles` have up to 17 significant digits accuracy.

If $R \leq 0$ or $\omega \leq 0$ the corresponding `exit` statements are executed which end the program execution. The optional error messages are included after the `stop` statements which are printed to the `stderr`. The value of the period $T = 2\pi/\omega$ is also calculated and printed for reference.

The main calculation is performed within the loop

```
t = t0;
while( t <= tf ){
  .........
  t   =   t + dt;
}
```

The first statement sets the initial value of the time. The statements between within the scope of the `while`(*condition*) are executed as long as *condition* has a `true` value. The statement `t=t+dt` increments the time and this is necessary in order not to enter into an infinite loop. The statements put in place of the dots .... calculate the position and the velocity and print them to the file `Circle.dat`:

```
#include <cmath>
................
    theta = omega * (t-t0);
    x   =   x0+R*cos(theta);
```

[5]Note that there are more assumptions that need to be checked by the program. We leave this as an exercise for the reader.

```
    y  =  y0+R*sin(theta);
    vx =  -omega*R*sin(theta);
    vy =   omega*R*cos(theta);
    myfile << t << " "
       << x  << " " << y << " "
       << vx << " " << vy << endl;
```

Notice the use of the functions `sin` and `cos` that calculate the sine and cosine of an angle expressed in radians. The header `cmath` is necessary to be included.

The program is stored in the file `Circle.cpp` and can be found in the accompanied software. The *extension* `.cpp` is used to inform the compiler that the file contains source code written in the C++ language. Compilation and running can be done using the commands:

```
> g++ Circle.cpp -o cl
> ./cl
```

The switch `-o cl` forces the compiler `g++` to write the binary commands executed by the program to the file[6] `cl`. The command `./cl` loads the program instructions to the computer memory for execution. When the programs starts execution, it first asks for the parameter data and then performs the calculation. A typical session looks like:

```
> g++ Circle.cpp -o cl
> ./cl
# Enter omega:
1.0
# Enter center of circle (x0,y0) and radius R:
1.0 1.0 0.5
# Enter t0,tf,dt:
0.0 20.0 0.01
# omega= 1
# x0= 1 y0= 1 R=  0.5
# t0= 0 tf= 20 dt= 0.01
# T= 6.28319
```

The lines shown above that start with a # character are printed by the program and the lines without # are the values of the parameters entered

[6]If omitted, the executable file has the default name `a.out`.

interactively by the user. The user types in the parameters and then presses the `Enter` key in order for the program to read them. Here we have used $\omega = 1.0$, $x_0 = y_0 = 1.0$, $R = 0.5$, $t_0 = 0.0$, $t_f = 20.0$ and $\delta t = 0.01$.

You can execute the above program many times for different values of the parameters by writing the parameter values in a file using an editor. For example, in the file `Circle.in` type the following data:

```
1.0              omega
1.0   1.0  0.5   (x0, y0) , R
0.0  20.0  0.01  t0 tf dt
```

Each line has the parameters that we want to pass to the program with each call to `cout`. The rest of the line consists of comments that explain to the user what each number is there for. We want to discard these characters during input and this is the reason for using `getline` to complete reading the rest of the line. The program can read the above values of the parameters with the command:

```
> ./cl < Circle.in > Circle.out
```

The command `./cl` runs the commands found in the executable file `./cl`. The `< Circle.in` redirects the contents of the file `Circle.in` to the standard input (`stdin`) of the command `./cl`. This way the program reads in the values of the parameters from the contents of the file `Circle.in`. The `> Circle.out` redirects the standard output (`stdout`) of the command `./cl` to the file `Circle.out`. Its contents can be inspected after the execution of the program with the command `cat`:

```
> cat Circle.out
# Enter omega:
# Enter center of circle (x0,y0) and radius R:
# Enter t0,tf,dt:
# omega= 1
# x0= 1 y0= 1 R=  0.5
# t0= 0 tf= 20 dt= 0.01
# T= 6.28319
```

We list the full program in `Circle.cpp` below:

```cpp
//============================================================
//File Circle.cpp
//Constant angular velocity circular motion
//Set (x0,y0) center of circle, its radius R and omega.
//At t=t0, the particle is at theta=0
//------------------------------------------------------------
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;

#define PI 3.1415926535897932

int main(){
//------------------------------------------------------------
//Declaration of variables
  double x0,y0,R,x,y,vx,vy,t,t0,tf,dt;
  double theta,omega;
  string buf;
//------------------------------------------------------------
//Ask user for input:
  cout << "# Enter omega:\n";
  cin  >> omega;           getline(cin,buf);
  cout << "# Enter center of circle (x0,y0) and radius R:\n";
  cin  >> x0 >> y0 >> R;   getline(cin,buf);
  cout << "# Enter t0,tf,dt:\n";
  cin  >> t0 >> tf >> dt;  getline(cin,buf);
  cout <<"# omega= " << omega << endl;
  cout <<"# x0= "    << x0    << " y0= " << y0
       << " R=  "    << R     << endl;
  cout <<"# t0= "    << t0    << " tf= " << tf
       << " dt= "    << dt    << endl;
//------------------------------------------------------------
//Initialize
  if(R    <=0.0){cerr <<"Illegal value of R    \n";exit(1);}
  if(omega<=0.0){cerr <<"Illegal value of omega\n";exit(1);}
  cout    << "# T= "  << 2.0*PI/omega              << endl;
  ofstream myfile("Circle.dat");
  // Set precision for numeric output to myfile to 17 digits
  myfile.precision(17);
//------------------------------------------------------------
//Compute:
  t = t0;
```

```
  while( t <= tf ){
    theta = omega * (t-t0);
    x   =  x0+R*cos(theta);
    y   =  y0+R*sin(theta);
    vx  =  -omega*R*sin(theta);
    vy  =   omega*R*cos(theta);
    myfile <<  t  << " "
       <<  x  << " " <<  y << " "
       << vx  << " " << vy << endl;
    t   =  t + dt;
  }
} //main()
```

2.1.1 Plotting Data

We use `gnuplot` for plotting the data produced by our programs. The file `Circle.dat` has the time `t` and the components `x`, `y`, `vx`, `vy` in five columns. Therefore we can plot the functions $x(t)$ and $y(t)$ by using the `gnuplot` commands:

```
gnuplot> plot   "Circle.dat" using 1:2 with lines title "x(t)"
gnuplot> replot "Circle.dat" using 1:3 with lines title "y(t)"
```

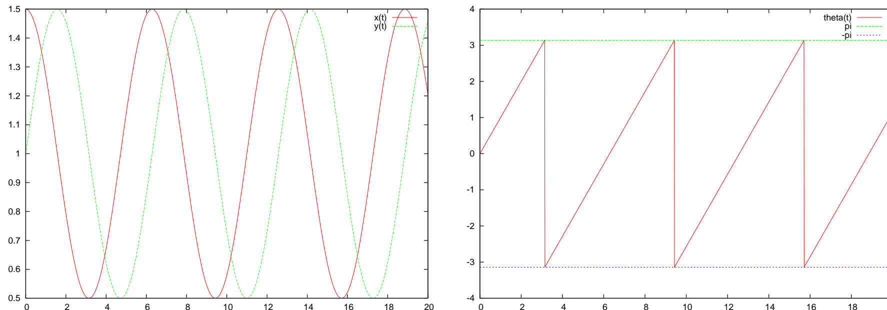

Figure 2.4: The plots $(x(t), y(t))$ (left) and $\theta(t)$ (right) from the data in `Circle.dat` for $\omega = 1.0$, $x_0 = y_0 = 1.0$, $R = 0.5$, $t_0 = 0.0$, $t_f = 20.0$ and $\delta t = 0.01$.

The second line puts the second plot together with the first one. The results can be seen in figure 2.4.

Let's see now how we can make the plot of the function $\theta(t)$. We can do that using the raw data from the file `Circle.dat` within `gnuplot`, without having to write a new program. Note that $\theta(t) = \tan^{-1}((y-y_0)/(x-x_0))$. The function `atan2` is available in `gnuplot`[7] as well as in C++. Use the online help system in `gnuplot` in order to see its usage:

```
gnuplot> help atan2
 The `atan2(y,x)` function returns the arc tangent (inverse
 tangent) of the ratio of the real parts of its arguments.
 `atan2` returns its argument in radians or degrees, as
 selected by `set angles`, in the correct quadrant.
```

Therefore, the right way to call the function is `atan2(y-y0,x-x0)`. In our case `x0=y0=1` and `x, y` are in the 2nd and 3rd columns of the file `Circle.dat`. We can construct an *expression* after the `using` command as in page 53, where `$2` is the value of the second and `$3` the value of the third column:

```
gnuplot> x0 = 1 ; y0 = 1
gnuplot> plot "Circle.dat" using 1:(atan2($3-y0,$2-x0)) \
                           with lines title "theta(t)",pi,-pi
```

The second command is broken in two lines by using the character `\` so that it fits conveniently in the text[8]. Note how we defined the values of the variables `x0, y0` and how we used them in the expression `atan2($3-x0,$2-y0)`. We also plot the lines which graph the constant functions $f_1(t) = \pi$ and $f_2(t) = -\pi$ which mark the limit values of $\theta(t)$. The `gnuplot` variable[9] `pi` is predefined and can be used in formed expressions. The result can be seen in the left plot of figure 2.4.

The velocity components $(v_x(t), v_y(t))$ as function of time as well as the trajectory $\vec{r}(t)$ can be plotted with the commands:

```
gnuplot> plot   "Circle.dat" using 1:4 title "v_x(t)" \
                             with lines
gnuplot> replot "Circle.dat" using 1:5 title "v_y(t)" \
```

[7]The command `help functions` will show you all the available functions in `gnuplot`.
[8]This can be done on the `gnuplot` command line as well.
[9]Use the command `show variables` in order to see the current/default values of `gnuplot` variables.

```
                                   with lines
gnuplot> plot   "Circle.dat" using 2:3 title "x-y"
                                   with lines
```

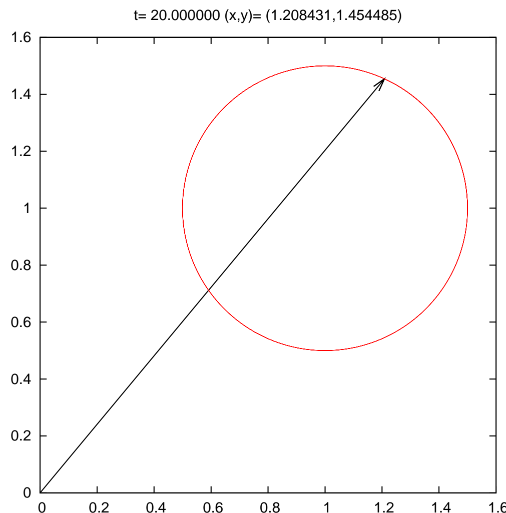

Figure 2.5: The particle trajectory plotted by the `gnuplot` program in the file `animate2D.gnu` of the accompanied software. The position vector is shown at a given time `t`, which is marked on the title of the plot together with the coordinates `(x,y)`. The data is produced by the program `Circle.cpp` described in the text.

We close this section by showing how to do a simple animation of the particle trajectory using `gnuplot`. There is a file `animate2D.gnu` in the accompanied software which you can copy in the directory where you have the data file `Circle.dat`. We are not going to explain how it works[10] but how to use it in order to make your own animations. The final result is shown in figure 2.5. All that you need to do is to define the data file[11], the initial time `t0`, the final time `tf` and the time step `dt`. These times can be different from the ones we used to create the data in `Circle.dat`. A full animation session can be launched using the commands:

[10]You are most welcome to study the commands in the script and guess how it works of course!

[11]It can be *any* file that has (t, x, y) in the 1st, 2nd and 3rd columns respectively.

```
gnuplot> file = "Circle.dat"
gnuplot> set xrange [0:1.6]; set yrange [0:1.6]
gnuplot> t0   = 0; tf = 20 ; dt = 0.1
gnuplot> load "animate2D.gnu"
```

The first line defines the data file that `animate2D.gnu` reads data from. The second line sets the range of the plots and the third line defines the time parameters used in the animation. The final line launches the animation. If you want to rerun the animation, you can repeat the last two commands as many times as you want using the same or different parameters. E.g. if you wish to run the animation at "half the speed" you should simply redefine `dt=0.05` and set the initial time to `t0=0`:

```
gnuplot> t0   = 0; dt = 0.05
gnuplot> load "animate2D.gnu"
```

2.1.2 More Examples

We are now going to apply the steps described in the previous section to other examples of motion on the plane. The first problem that we are going to discuss is that of the small oscillations of a simple pendulum. Figure 2.6 shows the single oscillating degree of freedom $\theta(t)$, which is the small angle that the pendulum forms with the vertical direction. The motion is periodic with angular frequency $\omega = \sqrt{g/l}$ and period

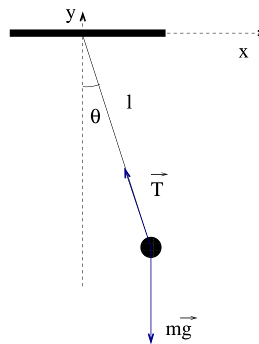

Figure 2.6: The simple pendulum whose motion for $\theta \ll 1$ is described by the program `SimplePendulum.cpp`.

$T = 2\pi/\omega$. The angular velocity is computed from $\dot{\theta} \equiv d\theta/dt$ which gives

$$\begin{aligned} \theta(t) &= \theta_0 \cos(\omega(t - t_0)) \\ \dot{\theta}(t) &= -\omega\theta_0 \sin(\omega(t - t_0)) \end{aligned} \tag{2.6}$$

We have chosen the initial conditions $\theta(t_0) = \theta_0$ and $\dot{\theta}(t_0) = 0$. In order to write the equations of motion in the Cartesian coordinate system shown in figure 2.6 we use the relations

$$\begin{aligned} x(t) &= l \sin(\theta(t)) \\ y(t) &= -l \cos(\theta(t)) \\ v_x(t) &= \frac{dx(t)}{dt} = l\dot{\theta}(t) \cos(\theta(t)) \\ v_y(t) &= \frac{dy(t)}{dt} = l\dot{\theta}(t) \sin(\theta(t)) \ . \end{aligned} \tag{2.7}$$

These are similar to the equations (2.3) and (2.4) that we used in the case of the circular motion of the previous section. Therefore the structure of the program is quite similar. Its final form, which can be found in the file `SimplePendulum.cpp`, is:

```
//==========================================================
//File SimplePendulum.cpp
//Set pendulum original position at theta0
//with no initial speed
//----------------------------------------------------------
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;

#define PI 3.1415926535897932
#define g  9.81

int main(){
//----------------------------------------------------------
//Declaration of variables
  double l,x,y,vx,vy,t,t0,tf,dt;
  double theta,theta0,dtheta_dt,omega;
```

```
  string buf;
//--------------------------------------------------------------
//Ask user for input:
  cout << "# Enter l:\n";
  cin  >> l;            getline(cin,buf);
  cout << "# Enter theta0:\n";
  cin  >> theta0;   getline(cin,buf);
  cout << "# Enter t0,tf,dt:\n";
  cin  >> t0 >> tf >> dt;  getline(cin,buf);
  cout <<"# l= "   << l  << " theta0= " << theta0 << endl;
  cout <<"# t0= "  << t0 << " tf= " << tf
       << " dt= "  << dt << endl;
//--------------------------------------------------------------
//Initialize
  omega = sqrt(g/l);
  cout << "# omega= " << omega
       << " T= "      << 2.0*PI/omega << endl;
  ofstream myfile("SimplePendulum.dat");
  myfile.precision(17);
//--------------------------------------------------------------
//Compute:
  t    =  t0;
  while( t <= tf ){
    theta     =        theta0*cos(omega*(t-t0));
    dtheta_dt = -omega*theta0*sin(omega*(t-t0));
    x  =  l*sin(theta);
    y  = -l*cos(theta);
    vx =  l*dtheta_dt*cos(theta);
    vy =  l*dtheta_dt*sin(theta);
    myfile <<  t    << " "
       <<  x    << " " <<  y << " "
       << vx    << " " << vy << " "
       << theta << dtheta_dt
       << endl;
    t  =  t + dt;
  }
} //main()
```

We note that the acceleration of gravity g is hard coded in the program and that the user can only set the length l of the pendulum. The data file `SimplePendulum.dat` produced by the program, contains two extra columns with the current values of $\theta(t)$ and the angular velocity $\dot{\theta}(t)$.

A simple session for the study of the above problem is shown below[12]:

[12]Notice that we replaced the command "`using 1:2 with lines title`" with "`u`

```
> g++ SimplePendulum.cpp -o sp
> ./sp
# Enter l:
1.0
# Enter theta0:
0.314
# Enter t0,tf,dt:
0 20 0.01
# l= 1 theta0= 0.314
# t0= 0 tf= 20 dt= 0.01
# omega= 3.13209 T= 2.00607
> gnuplot
gnuplot> plot   "SimplePendulum.dat" u 1:2 w l t "x(t)"
gnuplot> plot   "SimplePendulum.dat" u 1:3 w l t "y(t)"
gnuplot> plot   "SimplePendulum.dat" u 1:4 w l t "v_x(t)"
gnuplot> replot "SimplePendulum.dat" u 1:5 w l t "v_y(t)"
gnuplot> plot   "SimplePendulum.dat" u 1:6 w l t "theta(t)"
gnuplot> replot "SimplePendulum.dat" u 1:7 w l t "theta'(t)"
gnuplot> plot   [-0.6:0.6][-1.1:0.1] "SimplePendulum.dat" \
                                     u 2:3 w l t "x-y"
gnuplot> file = "SimplePendulum.dat"
gnuplot> t0=0;tf=20.0;dt=0.1
gnuplot> set xrange  [-0.6:0.6];set yrange [-1.1:0.1]
gnuplot> load "animate2D.gnu"
```

The next example is the study of the trajectory of a particle shot near the earth's surface[13] when we consider the effect of air resistance to be negligible. Then, the equations describing the trajectory of the particle and its velocity are given by the parametric equations

$$\begin{aligned} x(t) &= v_{0x}t \\ y(t) &= v_{0y}t - \frac{1}{2}gt^2 \\ v_x(t) &= v_{0x} \\ v_y(t) &= v_{0y} - gt\,, \end{aligned} \tag{2.8}$$

where t is the parameter. The initial conditions are $x(0) = y(0) = 0$, $v_x(0) = v_{0x} = v_0 \cos\theta$ and $v_y(0) = v_{0y} = v_0 \sin\theta$, as shown in figure 2.7.

`1:2 w lines t`". These abbreviations can be done with every `gnuplot` command if an abbreviation uniquely determines a command.

[13] I.e. $\vec{g} =$ const. and the Coriolis force can be ignored.

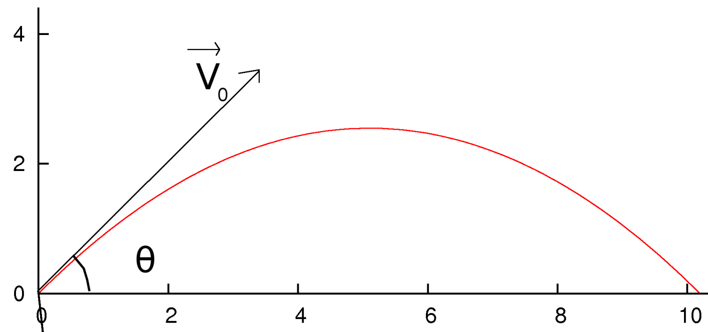

Figure 2.7: The trajectory of a particle moving under the influence of a constant gravitational field. The initial conditions are set to $x(0) = y(0) = 0$, $v_x(0) = v_{0x} = v_0 \cos\theta$ and $v_y(0) = v_{0y} = v_0 \sin\theta$.

The structure of the program is similar to the previous ones. The user enters the magnitude of the particle's initial velocity and the shooting angle θ in *degrees*. The initial time is taken to be $t_0 = 0$. The program calculates v_{0x} and v_{0y} and prints them to the `stdout`. The data is written to the file `Projectile.dat`. The full program is listed below and it can be found in the file `Projectile.cpp` in the accompanied software:

```
//==========================================================
//File Projectile.cpp
//Shooting a progectile near the earth surface.
//No air resistance.
//Starts at (0,0), set (v0,theta).
//----------------------------------------------------------
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;

#define PI 3.1415926535897932
#define  g 9.81

int main(){
//----------------------------------------------------------
//Declaration of variables
```

```
  double x0,y0,R,x,y,vx,vy,t,tf,dt;
  double theta,v0x,v0y,v0;
  string buf;
//----------------------------------------------------------
//Ask user for input:
  cout << "# Enter v0,theta (in degrees):\n";
  cin  >> v0 >> theta;      getline(cin,buf);
  cout << "# Enter tf,dt:\n";
  cin  >> tf >> dt;         getline(cin,buf);
  cout <<"# v0= "    << v0
       << " theta=  "<< theta << "o (degrees)" << endl;
  cout <<"# t0= "    << 0.0    << " tf= "       << tf
       << " dt= "    << dt     << endl;
//----------------------------------------------------------
//Initialize
  if(v0   <= 0.0)
    {cerr <<"Illegal value of v0   <= 0\n";exit(1);}
  if(theta<= 0.0)
    {cerr <<"Illegal value of theta<= 0\n";exit(1);}
  if(theta>=90.0)
    {cerr <<"Illegal value of theta>=90\n";exit(1);}
  theta    = (PI/180.0)*theta; //convert to radians
  v0x      = v0*cos(theta);
  v0y      = v0*sin(theta);
  cout     << "# v0x= "  << v0x
     << "  v0y= "  << v0y  << endl;
  ofstream myfile("Projectile.dat");
  myfile.precision(17);
//----------------------------------------------------------
//Compute:
  t = 0.0;
  while( t <= tf ){
    x  =  v0x * t;
    y  =  v0y * t - 0.5*g*t*t;
    vx =  v0x;
    vy =  v0y      -      g*t;
    myfile <<  t  << " "
       <<  x  << " " <<  y << " "
       << vx  << " " << vy << endl;
    t  =  t + dt;
  }
} //main()
```

A typical session for the study of this problem is shown below:

```
> g++ Projectile.cpp -o pj
> ./pj
# Enter v0,theta (in degrees):
10 45
# Enter tf,dt:
1.4416 0.001
# v0= 10 theta= 45o (degrees)
# t0= 0 tf= 1.4416 dt= 0.001
# v0x= 7.07107 v0y= 7.07107
> gnuplot
gnuplot> plot   "Projectile.dat" using 1:2 w l t "x(t)"
gnuplot> replot "Projectile.dat" using 1:3 w l t "y(t)"
gnuplot> plot   "Projectile.dat" using 1:4 w l t "v_x(t)"
gnuplot> replot "Projectile.dat" using 1:5 w l t "v_y(t)"
gnuplot> plot   "Projectile.dat" using 2:3 w l t "x-y"
gnuplot> file = "Projectile.dat"
gnuplot> set xrange [0:10.3];set yrange [0:10.3]
gnuplot> t0=0;tf=1.4416;dt=0.05
gnuplot> load "animate2D.gnu"
```

Next, we will study the effect of air resistance of the form $\vec{F} = -mk\vec{v}$. The solutions to the equations of motion

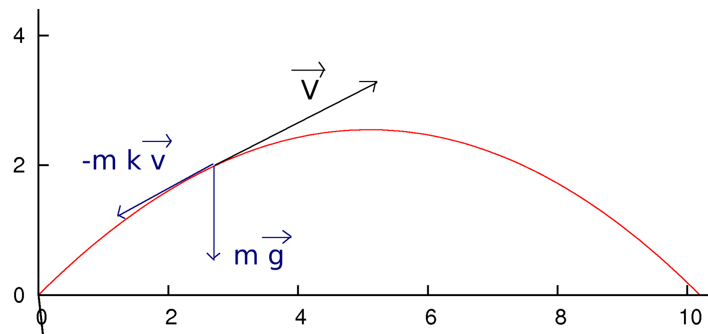

Figure 2.8: The forces that act on the particle of figure 2.7 when we assume air resistance of the form $\vec{F} = -mk\vec{v}$.

$$
\begin{aligned}
a_x &= \frac{dv_x}{dt} = -kv_x \\
a_y &= \frac{dv_y}{dt} = -kv_y - g
\end{aligned} \tag{2.9}
$$

with initial conditions $x(0) = y(0) = 0$, $v_x(0) = v_{0x} = v_0 \cos\theta$ and $v_y(0) = v_{0y} = v_0 \sin\theta$ are[14]

$$\begin{aligned}
v_x(t) &= v_{0x}\mathrm{e}^{-kt} \\
v_y(t) &= \left(v_{0y} + \frac{g}{k}\right)\mathrm{e}^{-kt} - \frac{g}{k} \\
x(t) &= \frac{v_{0x}}{k}\left(1 - \mathrm{e}^{-kt}\right) \\
y(t) &= \frac{1}{k}\left(v_{0y} + \frac{g}{k}\right)\left(1 - \mathrm{e}^{-kt}\right) - \frac{g}{k}t
\end{aligned} \tag{2.10}$$

Programming the above equations is as easy as before, the only difference being that the user needs to provide the value of the constant k. The full program can be found in the file `ProjectileAirResistance.cpp` and it is listed below:

```
//============================================================
//File ProjectileAirResistance.cpp
//Shooting a progectile near the earth surface.
//No air resistance.
//Starts at (0,0), set k, (v0,theta).
//------------------------------------------------------------
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;

#define PI 3.1415926535897932
#define  g 9.81

int main(){
//------------------------------------------------------------
//Declaration of variables
  double x0,y0,R,x,y,vx,vy,t,tf,dt,k;
  double theta,v0x,v0y,v0;
  string buf;
//------------------------------------------------------------
//Ask user for input:
  cout << "# Enter k,v0,theta (in degrees):\n";
```

[14] The proof of equations (2.10) is left as an exercise for the reader.

```
  cin  >> k >> v0 >> theta; getline(cin,buf);
  cout << "# Enter tf,dt:\n";
  cin  >> tf >> dt;          getline(cin,buf);
  cout <<"# k = "     << k      << endl;
  cout <<"# v0= "     << v0
       << " theta=  "<< theta << "o (degrees)" << endl;
  cout <<"# t0= "     << 0.0    << " tf= "        << tf
       << " dt= "     << dt     << endl;
//------------------------------------------------------------
//Initialize
  if(v0   <= 0.0)
    {cerr <<"Illegal value of v0   <= 0\n";exit(1);}
  if(k    <= 0.0)
    {cerr <<"Illegal value of k    <= 0\n";exit(1);}
  if(theta<= 0.0)
    {cerr <<"Illegal value of theta<= 0\n";exit(1);}
  if(theta>=90.0)
    {cerr <<"Illegal value of theta>=90\n";exit(1);}
  theta    = (PI/180.0)*theta; //convert to radians
  v0x      = v0*cos(theta);
  v0y      = v0*sin(theta);
  cout    << "# v0x= " << v0x
      << " v0y= " << v0y  << endl;
  ofstream myfile("ProjectileAirResistance.dat");
  myfile.precision(17);
//------------------------------------------------------------
//Compute:
  t = 0.0;
  while( t <= tf ){
    x  =  (v0x/k)*(1.0-exp(-k*t));
    y  =  (1.0/k)*(v0y+(g/k))*(1.0-exp(-k*t))-(g/k)*t;
    vx =  v0x*exp(-k*t);
    vy =  (v0y+(g/k))*exp(-k*t)-(g/k);
    myfile <<  t  << " "
       <<  x  << " " <<  y << " "
       << vx  << " " << vy << endl;
    t  =  t + dt;
  }
} //main()
```

We also list the commands of a typical session of the study of the problem:

```
> g++ ProjectileAirResistance.cpp -o pja
```

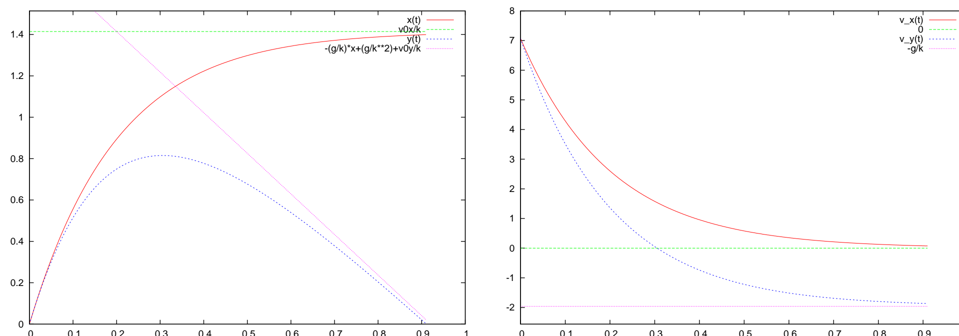

Figure 2.9: The plots of $x(t)$,$y(t)$ (left) and $v_x(t)$,$v_y(t)$ (right) from the data produced by the program `ProjectileAirResistance.cpp` for $k = 5.0$, $v_0 = 10.0$, $\theta = \pi/4$, $t_f = 0.91$ and $\delta t = 0.001$. We also plot the asymptotes of these functions as $t \to \infty$.

```
# Enter k,v0,theta (in degrees):
5.0 10.0 45
# Enter tf,dt:
0.91 0.001
# k = 5
# v0= 10 theta=  45o (degrees)
# t0= 0 tf= 0.91 dt= 0.001
# v0x= 7.07107  v0y= 7.07107
> gnuplot
gnuplot> v0x = 10*cos(pi/4) ; v0y = 10*sin(pi/4)
gnuplot> g = 9.81 ; k = 5
gnuplot> plot [:][:v0x/k+0.1]  "ProjectileAirResistance.dat" \
         using 1:2 with lines title "x(t)",v0x/k
gnuplot> replot                "ProjectileAirResistance.dat" \
         using 1:3 with lines title "y(t)",\
         -(g/k)*x+(g/k**2)+v0y/k
gnuplot> plot [:][-g/k-0.6:]   "ProjectileAirResistance.dat" \
         using 1:4 with lines title "v_x(t)",0
gnuplot> replot                "ProjectileAirResistance.dat" \
         using 1:5 with lines title "v_y(t)",-g/k
gnuplot> plot                  "ProjectileAirResistance.dat" \
         using 2:3 with lines title "With air resistance k=5.0"
gnuplot> replot                "Projectile.dat"              \
         using 2:3 with lines title "No air resistance k=0.0"
gnuplot> file = "ProjectileAirResistance.dat"
gnuplot> set xrange [0:1.4];set yrange [0:1.4]
gnuplot> t0=0;tf=0.91;dt=0.01
gnuplot> load "animate2D.gnu"
```

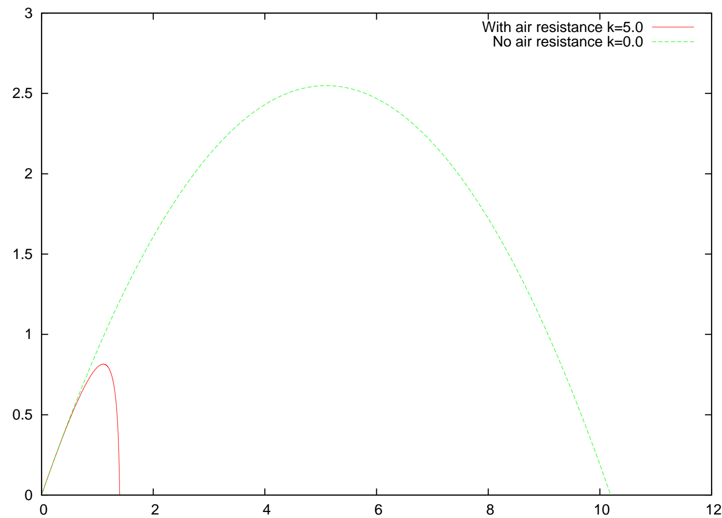

Figure 2.10: Trajectories of the particles shot with $v_0 = 10.0$, $\theta = \pi/4$ in the absence of air resistance and when the air resistance is present in the form $\vec{F} = -mk\vec{v}$ with $k = 5.0$.

Long commands have been continued to the next line as before. We defined the `gnuplot` variables `v0x`, `v0y`, `g` and `k` to have the values that we used when running the program. We can use them in order to construct the asymptotes of the plotted functions of time. The results are shown in figures 2.9 and 2.10.

The last example of this section will be that of the anisotropic harmonic oscillator. The force on the particle is

$$F_x = -m\omega_1^2 x \qquad F_y = -m\omega_2^2 y \tag{2.11}$$

where the “spring constants” $k_1 = m\omega_1^2$ and $k_2 = m\omega_2^2$ are different in the directions of the axes x and y. The solutions of the dynamical equations of motion for $x(0) = A$, $y(0) = 0$, $v_x(0) = 0$ and $v_y(0) = \omega_2 A$ are

$$\begin{aligned} x(t) &= A\cos(\omega_1 t) \qquad y(t) = A\sin(\omega_2 t) \\ v_x(t) &= -\omega_1 A\sin(\omega_1 t) \qquad v_y(t) = \omega_2 A\cos(\omega_2 t)\,. \end{aligned} \tag{2.12}$$

If the angular frequencies ω_1 and ω_2 satisfy certain relations, the trajectories of the particle are closed and self intersect at a given number of

points. The proof of these relations, as well as their numerical confirmation, is left as an exercise for the reader. The program listed below is in the file `Lissajoux.cpp`:

```
//============================================================
//File Lissajous.cpp
//Lissajous curves (special case)
//x(t)= cos(o1 t), y(t)= sin(o2 t)
//------------------------------------------------------------
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;

#define PI 3.1415926535897932

int main(){
//------------------------------------------------------------
//Declaration of variables
  double x0,y0,R,x,y,vx,vy,t,t0,tf,dt;
  double o1,o2,T1,T2;
  string buf;
//------------------------------------------------------------
//Ask user for input:
  cout << "# Enter omega1 and omega2:\n";
  cin  >> o1      >> o2;getline(cin,buf);
  cout << "# Enter tf,dt:\n";
  cin  >> tf      >> dt;getline(cin,buf);
  cout <<"# o1= " << o1  << " o2= " << o2 << endl;
  cout <<"# t0= " << 0.0 << " tf= " << tf
       << " dt= " << dt  <<  endl;
//------------------------------------------------------------
//Initialize
  if(o1 <=0.0){cerr <<"Illegal value of o1\n";exit(1);}
  if(o2 <=0.0){cerr <<"Illegal value of o2\n";exit(1);}
  T1     = 2.0*PI/o1;
  T2     = 2.0*PI/o2;
  cout   << "# T1= "  << T1 << " T2= " << T2 << endl;
  ofstream myfile("Lissajous.dat");
  myfile.precision(17);
//------------------------------------------------------------
//Compute:
```

```
  t = t0;
  while( t <= tf ){
    x  =  cos(o1*t);
    y  =  sin(o2*t);
    vx = -o1*sin(o1*t);
    vy =  o2*cos(o2*t);
    myfile <<  t  << " "
       <<  x  << " " <<  y << " "
       << vx  << " " << vy << endl;
    t  =  t + dt;
  }
} //main()
```

We have set $A = 1$ in the program above. The user must enter the two angular frequencies ω_1 and ω_2 and the corresponding times. A typical session for the study of the problem is shown below:

```
> g++  Lissajous.cpp -o lsj
> ./lsj
# Enter omega1 and omega2:
3 5
# Enter tf,dt:
10.0 0.01
# o1= 3 o2= 5
# t0= 0 tf= 10 dt= 0.01
# T1= 2.0944 T2= 1.25664
>gnuplot
gnuplot> plot   "Lissajous.dat" using 1:2 w l t "x(t)"
gnuplot> replot "Lissajous.dat" using 1:3 w l t "y(t)"
gnuplot> plot   "Lissajous.dat" using 1:4 w l t "v_x(t)"
gnuplot> replot "Lissajous.dat" using 1:5 w l t "v_y(t)"
gnuplot> plot   "Lissajous.dat" using 2:3 w l t "x-y for 3:5"
gnuplot> file = "Lissajous.dat"
gnuplot> set xrange [-1.1:1.1];set yrange [-1.1:1.1]
gnuplot> t0=0;tf=10;dt=0.1
gnuplot> load "animate2D.gnu"
```

The results for $\omega_1 = 3$ and $\omega_2 = 5$ are shown in figure 2.11.

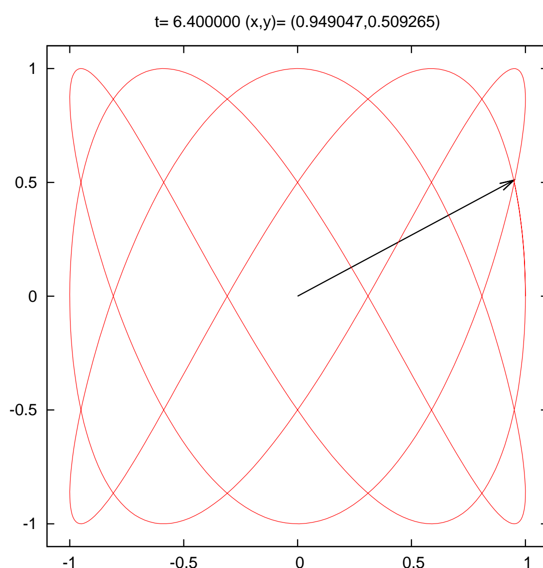

Figure 2.11: The trajectory of the anisotropic oscillator with $\omega_1 = 3$ and $\omega_2 = 5$.

2.2 Motion in Space

By slightly generalizing the methods described in the previous section, we will study the motion of a particle in three dimensional space. All we have to do is to add an extra equation for the coordinate $z(t)$ and the component of the velocity $v_z(t)$. The structure of the programs will be exactly the same as before.

The first example is the conical pendulum, which can be seen in figure 2.12. The particle moves on the xy plane with constant angular velocity ω. The equations of motion are derived from the relations

$$T_z = T\cos\theta = mg \qquad T_{xy} = T\sin\theta = m\omega^2 r\,, \tag{2.13}$$

where $r = l\sin\theta$. Their solution[15] is

$$\begin{aligned} x(t) &= r\cos\omega t \\ y(t) &= r\sin\omega t \\ z(t) &= -l\cos\theta\,, \end{aligned} \tag{2.14}$$

[15]One has to choose appropriate initial conditions. Exercise: find them!

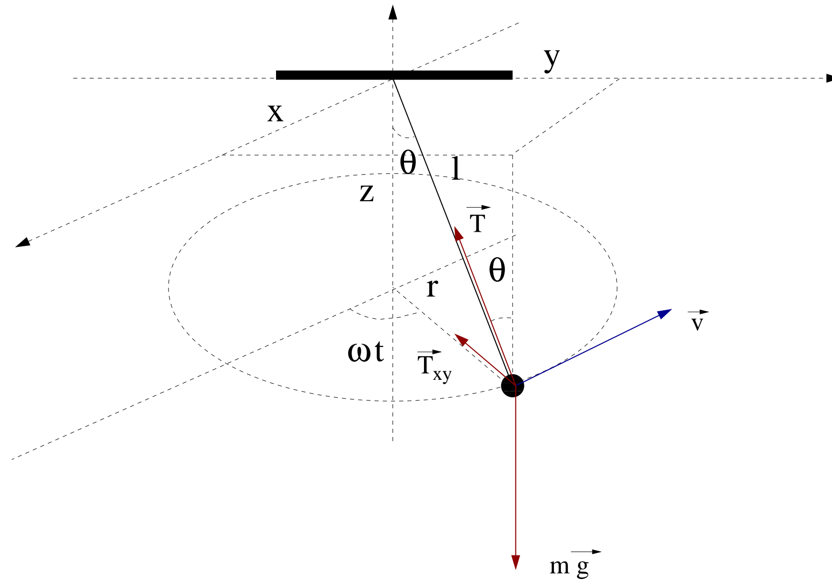

Figure 2.12: The conical pendulum of the program `ConicalPendulum.cpp`.

where we have to substitute the values

$$\begin{aligned} \cos\theta &= \frac{g}{\omega^2 l} \\ \sin\theta &= \sqrt{1-\cos^2\theta} \\ r &= \frac{g}{\omega^2}\frac{\sin\theta}{\cos\theta}\,. \end{aligned} \tag{2.15}$$

For the velocity components we obtain

$$\begin{aligned} v_x &= -r\omega\sin\omega t \\ v_y &= r\omega\cos\omega t \\ v_z &= 0\,. \end{aligned} \tag{2.16}$$

Therefore we must have

$$\omega \geq \omega_{\text{min}} = \sqrt{\frac{g}{l}}\,, \tag{2.17}$$

and when $\omega\to\infty$, $\theta\to\pi/2$.

In the program that we will write, the user must enter the parameters l, ω, the final time t_f and the time step δt. We take $t_0 = 0$. The convention that we follow for the output of the results is that they should be written in a file where the first 7 columns are the values of t, x, y, z, v_x, v_y and v_z. The full program is listed below:

```
//===========================================================
//File ConicalPendulum.cpp
//Set pendulum angular velocity omega and display motion in 3D
//-----------------------------------------------------------
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;

#define PI 3.1415926535897932
#define g  9.81

int main(){
//-----------------------------------------------------------
//Declaration of variables
  double l,r,x,y,z,vx,vy,vz,t,tf,dt;
  double theta,cos_theta,sin_theta,omega;
  string buf;
//-----------------------------------------------------------
//Ask user for input:
  cout << "# Enter l,omega:\n";
  cin  >> l  >> omega;          getline(cin,buf);
  cout << "# Enter tf,dt:\n";
  cin  >> tf >> dt;             getline(cin,buf);
  cout << "# l= "  << l   << " omega= " << omega << endl;
  cout << "# T=  " << 2.0*PI/omega
       << " omega_min= "  << sqrt(g/l)  << endl;
  cout <<"# t0= "  << 0.0 << " tf= "    << tf
       << " dt= "  << dt  << endl;
//-----------------------------------------------------------
//Initialize
  cos_theta = g/(omega*omega*l);
  if( cos_theta >= 1.0){
    cerr << "cos(theta)>= 1\n";
    exit(1);
```

```
  }
  sin_theta = sqrt(1.0-cos_theta*cos_theta);
  z = -g/(omega*omega); //they remain constant throught;
  vz= 0.0;               //the  motion
  r =  g/(omega*omega)*sin_theta/cos_theta;
  ofstream myfile("ConicalPendulum.dat");
  myfile.precision(17);
//------------------------------------------------------------
//Compute:
  t    =  0.0;
  while( t <= tf ){
    x  =  r*cos(omega*t);
    y  =  r*sin(omega*t);
    vx = -r*sin(omega*t)*omega;
    vy =  r*cos(omega*t)*omega;
    myfile <<  t    << " "
       <<  x    << " " <<  y << " " <<  z << " "
       << vx    << " " << vy << " " << vz << " "
       << endl;
    t  =  t + dt;
  }
} //main()
```

In order to compile and run the program we can use the commands shown below:

```
> g++  ConicalPendulum.cpp -o cpd
> ./cpd
# Enter l,omega:
1.0 6.28
# Enter tf,dt:
10.0 0.01
# l= 1 omega= 6.28
# T=  1.00051 omega_min= 3.13209
# t0= 0 tf= 10 dt= 0.01
```

The results are recorded in the file `ConicalPendulum.dat`. In order to plot the functions $x(t)$, $y(t)$, $z(t)$, $v_x(t)$, $v_y(t)$, $v_z(t)$ we give the following `gnuplot` commands:

```
> gnuplot
gnuplot> plot   "ConicalPendulum.dat" u 1:2 w l t "x(t)"
gnuplot> replot "ConicalPendulum.dat" u 1:3 w l t "y(t)"
```

```
gnuplot> replot "ConicalPendulum.dat" u 1:4 w l t "z(t)"
gnuplot> plot   "ConicalPendulum.dat" u 1:5 w l t "v_x(t)"
gnuplot> replot "ConicalPendulum.dat" u 1:6 w l t "v_y(t)"
gnuplot> replot "ConicalPendulum.dat" u 1:7 w l t "v_z(t)"
```

The results are shown in figure 2.13. In order to make a three dimen-

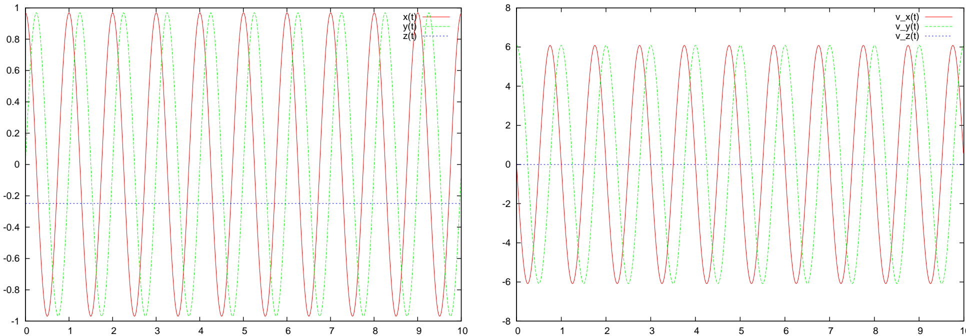

Figure 2.13: The plots of the functions $x(t), y(t), z(t), v_x(t), v_y(t), v_z(t)$ of the program ConicalPendulum.cpp for $\omega = 6.28$, $l = 1.0$.

sional plot of the trajectory, we should use the gnuplot command splot:

```
gnuplot> splot "ConicalPendulum.dat" u 2:3:4 w l t "r(t)"
```

The result is shown in figure 2.14. We can click on the trajectory and rotate it and view it from a different angle. We can change the plot limits with the command:

```
gnuplot> splot [-1.1:1.1][-1.1:1.1][-0.3:0.0] \
 "ConicalPendulum.dat" using 2:3:4 w l t "r(t)"
```

We can animate the trajectory of the particle by using the file animate3D.gnu from the accompanying software. The commands are similar to the ones we had to give in the two dimensional case for the planar trajectories when we used the file animate2D.gnu:

```
gnuplot> file = "ConicalPendulum.dat"
gnuplot> set xrange [-1.1:1.1];set yrange [-1.1:1.1]
gnuplot> set zrange [-0.3:0]
```

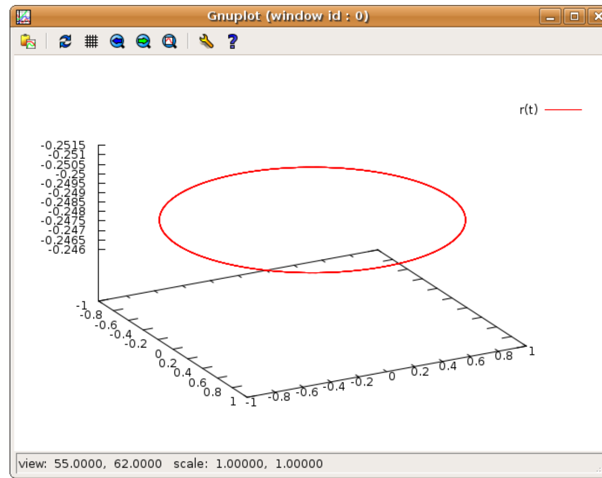

Figure 2.14: The plot of the particle trajectory $\vec{r}(t)$ of the program `ConicalPendulum.cpp` for $\omega = 6.28$, $l = 1.0$. We can click and drag with the mouse on the window and rotate the curve and see it from a different angle. At the bottom left of the window, we see the viewing direction, given by the angles $\theta = 55.0$ degrees (angle with the z axis) and $\phi = 62$ degrees (angle with the x axis).

```
gnuplot> t0=0;tf=10;dt=0.1
gnuplot> load "animate3D.gnu"
```

The result can be seen in figure 2.15. The program `animate3D.gnu` can be used on the data file of any program that prints `t x y z` as the first words on each of its lines. All we have to do is to change the value of the `file` variable in gnuplot.

Next, we will study the trajectory of a charged particle in a homogeneous magnetic field $\vec{B} = B\hat{z}$. At time t_0, the particle is at $\vec{r}_0 = x_0\hat{x}$ and its velocity is $\vec{v}_0 = v_{0y}\hat{y} + v_{0z}\hat{z}$, see figure 2.16. The magnetic force on the particle is $\vec{F} = q(\vec{v} \times \vec{B}) = qBv_y\hat{x} - qBv_x\hat{y}$ and the equations of motion are

$$\begin{aligned} a_x &= \frac{dv_x}{dt} = \omega v_y \qquad \omega \equiv \frac{qB}{m} \\ a_y &= \frac{dv_y}{dt} = -\omega v_x \\ a_z &= 0\,. \end{aligned} \tag{2.18}$$

By integrating the above equations with the given initial conditions we

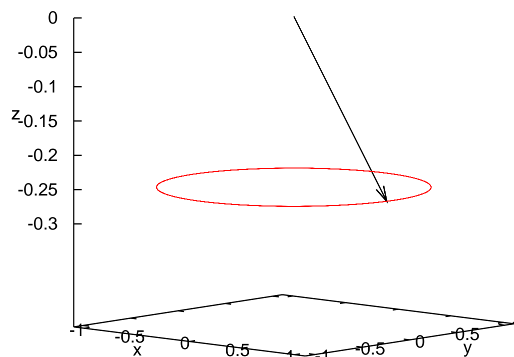

Figure 2.15: The particle trajectory $\vec{r}(t)$ computed by the program `ConicalPendulum.cpp` for $\omega = 6.28$, $l = 1.0$ and plotted by the `gnuplot` script `animate3D.gnu`. The title of the plot shows the current time and the particles coordinates.

obtain

$$
\begin{aligned}
v_x(t) &= v_{0y}\sin\omega t\\
v_y(t) &= v_{0y}\cos\omega t\\
v_z(t) &= v_{0z}\,.
\end{aligned}
\tag{2.19}
$$

Integrating once more, we obtain the position of the particle as a function of time

$$
\begin{aligned}
x(t) &= \left(x_0+\frac{v_{0y}}{\omega}\right)-\frac{v_{0y}}{\omega}\cos\omega t = x_0\cos\omega t\\
y(t) &= \frac{v_{0y}}{\omega}\sin\omega t = -x_0\sin\omega t \quad \text{με} \quad x_0 = -\frac{v_{0y}}{\omega}\\
z(t) &= v_{0z}t\,,
\end{aligned}
\tag{2.20}
$$

where we have chosen $x_0 = -v_{0y}/\omega$. This choice places the center of the circle, which is the projection of the trajectory on the xy plane, to be at

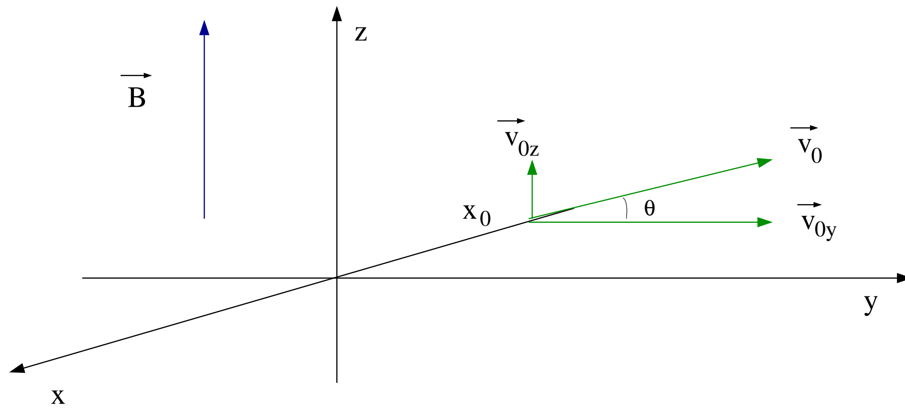

Figure 2.16: A particle at time $t_0 = 0$ is at the position $\vec{r}_0 = x_0\hat{x}$ with velocity $\vec{v}_0 = v_{0y}\hat{y} + v_{0z}\hat{z}$ in a homogeneous magnetic field $\vec{B} = B\hat{z}$.

the origin of the coordinate system. The trajectory is a helix with radius $R = -x_0$ and pitch $v_{0z}T = 2\pi v_{0z}/\omega$.

We are now ready to write a program that calculates the trajectory given by (2.20). The user enters the parameters v_0 and θ, shown in figure 2.16, as well as the angular frequency ω (Larmor frequency). The components of the initial velocity are $v_{0y} = v_0 \cos\theta$ and $v_{0z} = v_0 \sin\theta$. The initial position is calculated from the equation $x_0 = -v_{0y}/\omega$. The program can be found in the file `ChargeInB.cpp`:

```
//=========================================================
//File ChargeInB.cpp
//A charged particle of mass m and charge q enters a magnetic
//field B in +z direction. It enters with velocity
//v0x=0,v0y=v0 cos(theta),v0z=v0 sin(theta), 0<=theta<pi/2
//at the position x0=-v0y/omega, omega=q B/m
//
//Enter v0 and theta and see trajectory from
//t0=0 to tf at step dt
//---------------------------------------------------------
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
```

```
using namespace std;

#define PI 3.1415926535897932

int main(){
//--------------------------------------------------------------------
//Declaration of variables
  double x,y,z,vx,vy,vz,t,tf,dt;
  double x0,y0,z0,v0x,v0y,v0z,v0;
  double theta,omega;
  string buf;
//--------------------------------------------------------------------
//Ask user for input:
  cout << "# Enter omega:\n";
  cin  >> omega;            getline(cin,buf);
  cout << "# Enter v0, theta (degrees):\n";
  cin  >> v0 >> theta;      getline(cin,buf);
  cout << "# Enter tf,dt:\n";
  cin  >> tf >> dt;               getline(cin,buf);
  cout << "# omega= " << omega
       << " T= "      << 2.0*PI/omega << endl;
  cout << "# v0=    " << v0
       << " theta=  " << theta
       << "o(degrees)"<< endl;
  cout <<"# t0= "     << 0.0 << " tf= "    << tf
       << " dt= "     << dt  << endl;
//--------------------------------------------------------------------
//Initialize
  if(theta<0.0 || theta>=90.0) exit(1);
  theta = (PI/180.0)*theta; //convert to radians
  v0y   = v0*cos(theta);
  v0z   = v0*sin(theta);
  cout << "# v0x= " << 0.0
       << "  v0y= " << v0y
       << "  v0z= " << v0z  << endl;
  x0    = - v0y/omega;
  cout << "# x0= " << x0
       << "  y0= " << y0
       << "  z0= " << z0    << endl;
  cout << "# xy plane: Circle with center (0,0) and R= "
       << abs(x0)           << endl;
  cout << "# step of helix: s=v0z*T= "
       <<  v0z*2.0*PI/omega << endl;
  ofstream myfile("ChargeInB.dat");
  myfile.precision(17);
```

```
//--------------------------------------------------------------------
//Compute:
  t    =  0.0;
  vz   = v0z;
  while( t <= tf ){
    x  =  x0*cos(omega*t);
    y  = -x0*sin(omega*t);
    z  =  v0z*t;
    vx =  v0y*sin(omega*t);
    vy =  v0y*cos(omega*t);
    myfile <<  t    << " "
       <<  x    << " " <<  y << " " <<  z << " "
       << vx    << " " << vy << " " << vz << " "
       << endl;
    t  =  t + dt;
  }
} //main()
```

A typical session in which we calculate the trajectories shown in figures 2.17 and 2.18 is shown below:

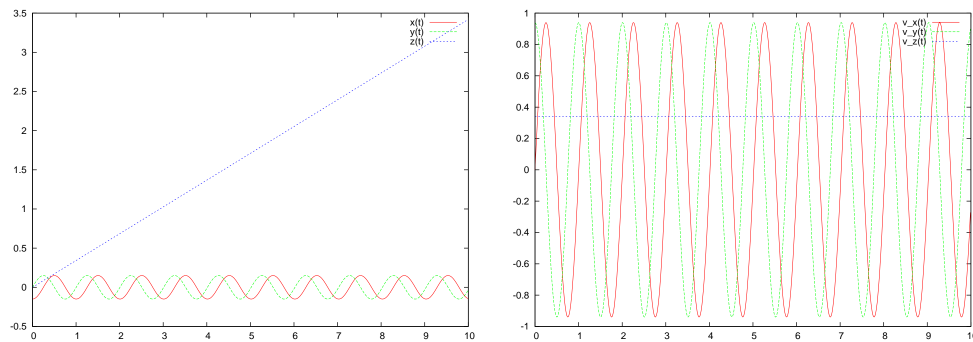

Figure 2.17: The plots of the $x(t), y(t), z(t), v_x(t), v_y(t), v_z(t)$ functions calculated by the program in `ChargeInB.cpp` for $\omega = 6.28$, $x_0 = 1.0$, $\theta = 20$ degrees.

```
> g++  ChargeInB.cpp -o chg
> ./chg
# Enter omega:
6.28
# Enter v0, theta (degrees):
1.0 20
```

```
# Enter tf,dt:
10 0.01
# omega= 6.28 T= 1.00051
# v0=     1 theta=  20o(degrees)
# t0= 0 tf= 10 dt= 0.01
# v0x= 0  v0y= 0.939693  v0z= 0.34202
# x0= -0.149633  y0= 0  z0= 3.11248e-317
# xy plane: Circle with center (0,0) and R= 0.149633
# step of helix: s=v0z*T= 0.342194
> gnuplot
gnuplot> plot   "ChargeInB.dat" u 1:2   w l title "x(t)"
gnuplot> replot "ChargeInB.dat" u 1:3   w l title "y(t)"
gnuplot> replot "ChargeInB.dat" u 1:4   w l title "z(t)"
gnuplot> plot   "ChargeInB.dat" u 1:5   w l title "v_x(t)"
gnuplot> replot "ChargeInB.dat" u 1:6   w l title "v_y(t)"
gnuplot> replot "ChargeInB.dat" u 1:7   w l title "v_z(t)"
gnuplot> splot  "ChargeInB.dat" u 2:3:4 w l title "r(t)"
gnuplot> file = "ChargeInB.dat"
gnuplot> set xrange [-0.65:0.65];set yrange [-0.65:0.65]
gnuplot> set zrange [0:1.3]
gnuplot> t0=0;tf=3.5;dt=0.1
gnuplot> load "animate3D.gnu"
```

2.3 Trapped in a Box

In this section we will study the motion of a particle that is free, except when bouncing elastically on a wall or on certain obstacles. This motion is calculated by *approximate* algorithms that introduce systematic errors. These types of errors[16] are also encountered in the study of more complicated dynamics, but the simplicity of the problem will allow us to control them in a systematic and easy to understand way.

2.3.1 The One Dimensional Box

The simplest example of such a motion is that of a particle in a "one dimensional box". The particle moves freely on the x axis for $0 < x < L$,

[16] In the previous sections, our calculations had a small systematic error due to the approximate nature of numerical floating point operations which approximate exact real number calculations. But the algorithms used were not introducing systematic errors like in the cases discussed in this section.

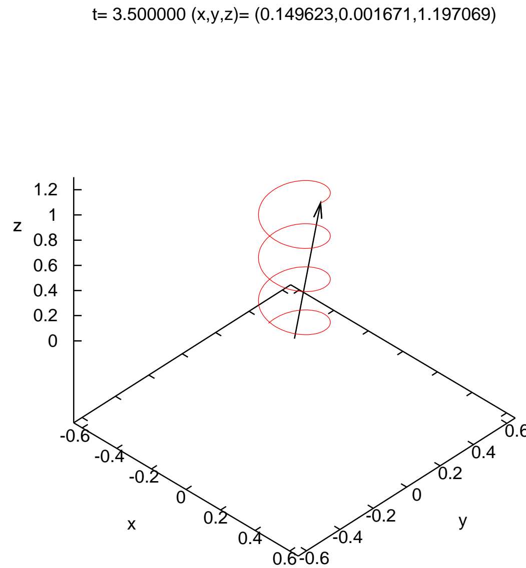

Figure 2.18: The trajectory $\vec{r}(t)$ calculated by the program in `ChargeInB.cpp` for $\omega = 6.28$, $v_0 = 1.0$, $\theta = 20$ degrees as shown by the program `animate3D.gnu`. The current time and the coordinates of the particle are printed on the title of the plot.

as can be seen in figure 2.19. When it reaches the boundaries $x = 0$ and $x = L$ it bounces and its velocity instantly reversed. Its potential energy is

$$V(x) = \begin{cases} 0 & 0 < x < L \\ +\infty & \text{elsewhere} \end{cases} , \tag{2.21}$$

which has the shape of an infinitely deep well. The force $F = -dV(x)/dx = 0$ within the box and $F = \pm\infty$ at the position of the walls.

Initially we have to know the position of the particle x_0 as well as its velocity v_0 (the sign of v_0 depends on the direction of the particle's motion) at time t_0. As long as the particle moves within the box, its motion is free and

$$\begin{aligned} x(t) &= x_0 + v_0(t - t_0) \\ v(t) &= v_0 \,. \end{aligned} \tag{2.22}$$

For a small enough change in time δt, so that there is no bouncing on

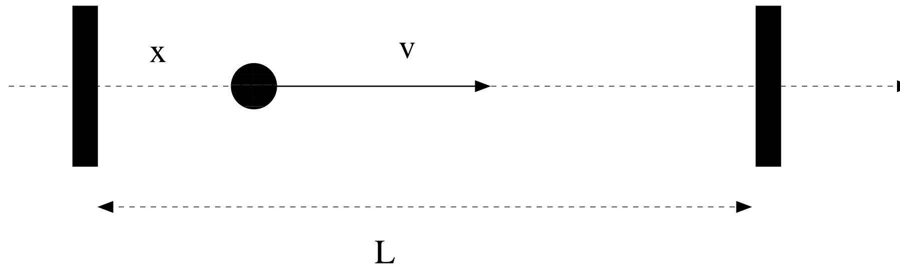

Figure 2.19: A particle in a one dimensional box with its walls located at $x = 0$ and $x = L$.

the wall in the time interval $(t, t + \delta t)$, we have that

$$\begin{aligned} x(t+\delta t) &= x(t) + v(t)\delta t \\ v(t+\delta t) &= v(t)\,. \end{aligned} \tag{2.23}$$

Therefore we could use the above relations in our program and when the particle bounces off a wall we could simple reverse its velocity $v(t) \to -v(t)$. The devil is hiding in the word "when". Since the time interval δt is finite in our program, there is no way to know the instant of the collision with accuracy better than $\sim \delta t$. However, our algorithm will change the direction of the velocity at time $t + \delta t$, when the particle will have already crossed the wall. This will introduce a systematic error, which is expected to decrease with decreasing δt. One way to implement the above idea is by constructing the loop

```
while(t <= tf){
  x += v*dt;
  t +=   dt;
  if(x < 0.0 || x > L) v = -v;
}
```

where the last line gives the testing condition for the wall collision and the subsequent change of the velocity.

The full program that realizes the proposed algorithm is listed below and can be found in the file `box1D_1.cpp`. The user can set the size of the box `L`, the initial conditions `x0` and `v0` at time `t0`, the final time `tf` and the time step `dt`:

```
//==========================================================
// File box1D_1.cpp
// Motion of a free particle in a box  0<x<L
// Use integration with time step dt: x = x + v*dt
//----------------------------------------------------------
#include <iostream>
#include <iomanip>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;

int main(){
//----------------------------------------------------------
// Declaration of variables
  float L,x0,v0,t0,tf,dt,t,x,v;
  string buf;
//----------------------------------------------------------
// Ask user for input:
  cout << "# Enter L:\n";
  cin  >> L;             getline(cin,buf);
  cout << "# L = " << L  << endl;
  cout << "# Enter x0,v0:\n";
  cin  >> x0 >> v0;     getline(cin,buf);
  cout << "# x0= " << x0 << " v0= "  << v0 << endl;
  cout << "# Enter t0,tf,dt:\n";
  cin  >> t0 >> tf >> dt;  getline(cin,buf);
  cout << "# t0= " << t0 << " tf= " << tf
       << " dt= "  << dt << endl;
  if( L <= 0.0f){cerr << "L <=0\n"; exit(1);}
  if( x0<  0.0f){cerr << "x0<=0\n"; exit(1);}
  if( x0>  L   ){cerr << "x0> L\n"; exit(1);}
  if( v0== 0.0f){cerr << "v0 =0\n"; exit(1);}
//----------------------------------------------------------
// Initialize
  t = t0;
  x = x0;
  v = v0;
  ofstream myfile("box1D_1.dat");
  myfile.precision(9); // float precision (and a bit more...)
//----------------------------------------------------------
//Compute:
  while(t <= tf){
    myfile << setw(17) << t << " "    // set width of field
```

```
        << setw(17) << x << " "    // to 17 characters
        << setw(17) << v << '\n';  // using setw(17)
    x += v*dt;
    t +=   dt;
    if(x < 0.0f || x > L) v = -v;
  }
  myfile.close();
} //main()
```

In this section we will study the effects of roundoff errors in numerical computations. Computers store numbers in memory, which is finite. Therefore, real numbers are represented in some approximation that depends on the amount of memory that is used for their storage. This approximation corresponds to what is termed as floating point numbers. C++ is supposed to provide at least three basic types of floating point numbers, `float`, `double` and `long double`. In most implementations[17], `float` uses 4 bytes of memory and `double` 8. In this case, `float` has an accuracy to, approximately, 7 significant digits and `double` 17. See Chapter 1 of [8] and [14] for details. Moreover, `float` represent numbers with magnitude in the, approximate, range $(10^{-38}, 10^{38})$ while `double` in $(10^{-308}, 10^{308})$. Note that variables of the integer type (`int`, `long`, ...) are exact representations of integers, whereas floating point numbers are approximations to reals.

In the program shown above, we used numbers of the `float` type instead of `double` in order to exaggerate roundoff errors. This way we can study the dependence of this type of errors on the accuracy of the floating point numbers used in a program[18]. In order to do that, we declared the floating point variables as `float`:

[17]Notice that the C++ standard states that the value representation of floating-point types is implementation-defined. The C standard requires that the type double provides at least as much precision as float, and the type long double provides at least as much precision as double. The `gcc` 5.4 version that we are using in this book represents `float` using 4 bytes and `double` with 8, but you should check whether this is true with the compiler that you are using.

[18]The use of `float` can be the preferred choice of a programmer for some applications. First, in order to save memory, because `float` occupies half the memory of a `double`. Second, it is not always true that increasing the accuracy of floating point numbers will increase the accuracy of a computation, although in most of the cases it will. The wisdom of the field is to always to use as much accuracy as you need and no more!

```
float L,x0,v0,t0,tf,dt,t,x,v;
```

We also used *numerical constants* of type `float`. This is indicated by the letter `f` at the end of their names: `2.0` is a constant of type `double` (the C++ default), whereas `2.0f` is a constant of type `float`. Determining the accuracy of floating point constants is a thorny issue that can be the cause on introducing subtle bugs in a program and the programmer should be very careful about doing it carefully.

Finally we changed the form of the output. Since a `float` represents a real number with at most 7 significant digits, there is no point of printing more. That is why we used the statements

```
myfile.precision(9);
myfile.setw(17);
```

For purposes of studying the numerical accuracy of our results, we used 9 digits of output, which is, of course, slightly redundant. `setw(17)` prints the numbers of the next output of the stream `myfile` using *at least* 17 character spaces. This improves the legibility of the results when inspecting the output files. The use of `setw` requires the header `iomanip`.

The computed data is recorded in the file `box1D_1.dat` in three columns. Compiling, running and plotting the trajectory using `gnuplot` can be done as follows:

```
> g++ box1D_1.cpp -o box1
> ./box1
# Enter L:
10
# L = 10
# Enter x0,v0:
0 1.0
# x0= 0 v0= 1
# Enter t0,tf,dt:
0 100 0.01
# t0= 0 tf= 100 dt= 0.01
> gnuplot
gnuplot> plot "box1D_1.dat" using  1:2 w l title "x(t)",\
                                   0 notitle,10 notitle
gnuplot> plot [:][-1.2:1.2] "box1D_1.dat" \
                            using  1:3 w l title "v(t)"
```

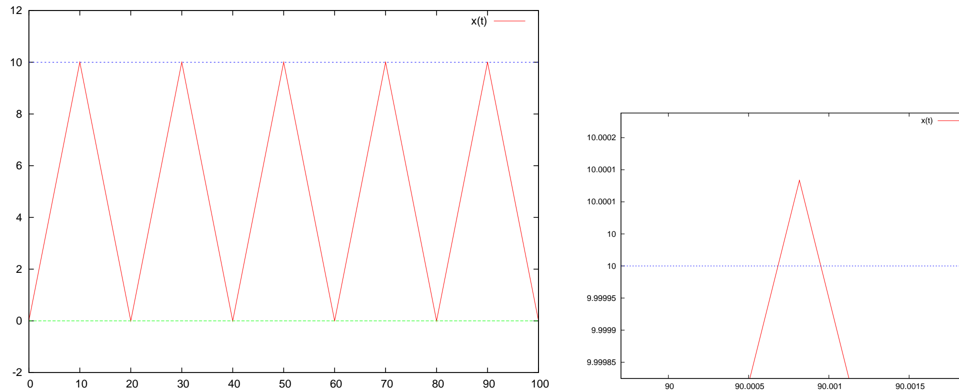

Figure 2.20: The trajectory $x(t)$ of a particle in a box with $L = 10$, $x_0 = 0.0$, $v_0 = 1.0$, $t_0 = 0$, $\delta t = 0.01$. The plot to the right magnifies a detail when $t \approx 90$ which exposes the systematic errors in determining the exact moment of the collision of the particle with the wall at $t_k = 90$ and the corresponding maximum value of $x(t)$, $x_m = L = 10.0$.

The trajectory $x(t)$ is shown in figure 2.20. The effects of the systematic errors can be easily seen by noting that the expected collisions occur every $T/2 = L/v = 10$ units of time. Therefore, on the plot to the right of figure 2.20, the reversal of the particle's motion should have occurred at $t = 90$, $x = L = 10$.

The reader should have already realized that the above mentioned error can be made to vanish by taking arbitrarily small δt. Therefore, we naively expect that as long as we have the necessary computer power to take δt as small as possible and the corresponding time intervals as many as possible, we can achieve any precision that we want. Well, that is true only up to a point. The problem is that the next position is determined by the addition operation `x+v*dt` and the next moment in time by `t+dt`. Floating point numbers of the `float` type have a maximum accuracy of approximately 7 significant decimal digits. Therefore, if the operands `x` and `v*dt` are real numbers differing by more than 7 orders of magnitude (`v*dt`$\lesssim 10^{-7}$ `x`), the effect of the addition `x+v*dt=x`, which is null! The reason is that the floating point unit of the processor has to convert both numbers `x` and `v*dt` into a representation having the same exponent and in doing so, the corresponding significant digits of the smaller number `v*dt` are lost. The result is less catastrophic when `v*dt`$\lesssim 10^{-a}$ `x` with $0 < a < 7$, but some degree of accuracy is also lost at

each addition operation. And since we have accumulation of such errors over many intervals `t→t+dt`, the error can become significant and destroy our calculation for large enough times. A similar error accumulates in the determination of the next instant of time `t+dt`, but we will discuss below how to make this contribution to the total error negligible. The above mentioned errors can become less detrimental by using floating point numbers of greater accuracy than the `float` type. For example `double` numbers have approximately 17 significant decimal digits. But again, the precision is finite and the same type of errors are there only to be revealed by a more demanding and complicated calculation.

The remedy to such a problem can only be a change in the algorithm. This is not always possible, but in the case at hand this is easy to do. For example, consider the equation that gives the position of a particle in free motion

$$x(t) = x_0 + v_0(t - t_0)\,. \tag{2.24}$$

Let's use the above relation for the parts of the motion between two collisions. Then, all we have to do is to reverse the direction of the motion and reset the initial position and time to be the position and time of the collision. This can be done by using the loop:

```
while(t <= tf){
  x = x0 + v0*(t-t0);
  if(x < 0.0f || x > L) {
    x0=  x;
    t0=  t;
    v0= -v0;
  }
  t  += dt;
}
```

In the above algorithm, the error in the time of the collision is not vanishing but we don't have the "instability" problem of the `dt`$\to 0$ limit[19]. Therefore we can isolate and study the effect of each type of error. The full program that implements the above algorithm is given below and can be found in the file `box1D_2.cpp`:

[19] We still have this problem in the `t=t+dt` operation. See discussion in the next section.

```
//=========================================================
// File box1D_2.cpp
// Motion of a free particle in a box  0<x<L
// Use constant velocity equation: x = x0 + v0*(t-t0)
// Reverse velocity and redefine x0,t0 on boundaries
//---------------------------------------------------------
#include <iostream>
#include <iomanip>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;

int main(){
//---------------------------------------------------------
// Declaration of variables
  float L,x0,v0,t0,tf,dt,t,x,v;
  string buf;
//---------------------------------------------------------
// Ask user for input:
  cout << "# Enter L:\n";
  cin  >> L;             getline(cin,buf);
  cout << "# L = " << L  << endl;
  cout << "# Enter x0,v0:\n";
  cin  >> x0 >> v0;      getline(cin,buf);
  cout << "# x0= " << x0 << " v0= "  << v0 << endl;
  cout << "# Enter t0,tf,dt:\n";
  cin  >> t0 >> tf >> dt;  getline(cin,buf);
  cout << "# t0= " << t0 << " tf= " << tf
       << " dt= "  << dt << endl;
  if( L <= 0.0f){cerr << "L <=0\n"; exit(1);}
  if( x0<  0.0f){cerr << "x0<=0\n"; exit(1);}
  if( x0>  L   ){cerr << "x0> L\n"; exit(1);}
  if( v0== 0.0f){cerr << "v0 =0\n"; exit(1);}
//---------------------------------------------------------
// Initialize
  t = t0;
  ofstream myfile("box1D_2.dat");
  myfile.precision(9); // float precision (and a bit more...)
//---------------------------------------------------------
//Compute:
  while(t <= tf){
    x = x0 + v0*(t-t0);
    myfile << setw(17) << t  << " "
```

```
      << setw(17) << x  << " "
      << setw(17) << v0 << '\n';
    if(x < 0.0f || x > L) {
      x0=  x;
      t0=  t;
      v0= -v0;
    }
    t  += dt;
  }
  myfile.close();
} //main()
```

Compiling and running the above program is done as before and the results are stored in the file `box1D_2.dat`.

2.3.2 Errors

In this section we will study the effect of the systematic errors that we encountered in the previous section in more detail. We considered two types of errors: First, the systematic error of determining the instant of the collision of the particle with the wall. This error is *reduced* by taking a smaller time step δt. Then, the systematic error that accumulates with each addition of two numbers with increasing difference in their orders of magnitude. This error is *increased* with decreasing δt. The competition of the two effects makes the optimal choice of δt the result of a careful analysis. Such a situation is found in many interesting problems, therefore it is quite instructive to study it in more detail.

When the exact solution of the problem is not known, the systematic errors are controlled by studying the behavior of the solution as a function of δt. If the solutions are converging in a region of values of δt, one gains confidence that the true solution has been determined up to the accuracy of the convergence.

In the previous sections, we studied two different algorithms, programmed in the files `box1D_1.cpp` and `box1D_2.cpp`. We will refer to them as "method 1" and "method 2" respectively. We will study the convergence of the results as $\delta t \to 0$ by fixing all the parameters except δt and then study the dependence of the results on δt. We will take $L = 10$, $v_0 = 1.0$, $x_0 = 0.0$, $t_0 = 0.0$, $t_f = 95.0$, so that the particle will collide with the wall every 10 units of time. We will measure the position of

the particle $x(t \approx 95)$[20] as a function of δt and study its convergence to a limit[21] as $\delta t \to 0$.

The analysis requires a lot of repetitive work: Compiling, setting the parameter values, running the program and calculating the value of $x(t \approx 95)$ for many values of δt. We write the values of the parameters read by the program in a file `box1D_anal.in`:

```
10          L
0  1.0      x0  v0
0  95  0.05 t0  tf  dt
```

Then we compile the program

```
> g++ box1D_1.cpp -o box
```

and run it with the command:

```
> cat box1D_anal.in | ./box
```

By using the pipe `|`, we send the contents of `box1D_anal.in` to the `stdin` of the command `./box` by using the command `cat`. The result $x(t \approx 95)$ can be found in the last line of the file `box1D_1.dat`:

```
> tail -n 1  box1D_1.dat
  94.9511948  5.45000267  -1.
```

The third number in the above line is the value of the velocity. In a file `box1D_anal.dat` we write δt and the first two numbers coming out from the command `tail`. Then we decrease the value $\delta t \to \delta t/2$ in the file `box1D_anal.in` and run again. We repeat for 12 more times until δt reaches the value[22] 0.000012. We do the same[23] using method 2 and we place the results for $x(t \approx 95)$ in two new columns in the file `box1D_anal.dat`. The result is

---

[20] Note the $\approx$!

[21] Of course we know the answer: $x(95) = 5$.

[22] Try the command `sed 's/0.05/0.025/' box1D_anal.in | ./box` by changing `0.025` with the desired value of δt.

[23] See the shell script `box1D_anal.csh` as a suggestion on how to automate this boring process.

```
# ------------------------------------------------------------
#   dt       t1_95      x1(95)     x2(95)
# ------------------------------------------------------------
0.050000 94.95119 5.450003 5.550126
0.025000 94.97849 5.275011 5.174837
0.012500 94.99519 5.124993 5.099736
0.006250 94.99850 4.987460 5.063134
0.003125 94.99734 5.021894 5.035365
0.001563 94.99923 5.034538 5.017764
0.000781 94.99939 4.919035 5.011735
0.000391 94.99979 4.695203 5.005493
0.000195 95.00000 5.434725 5.001935
0.000098 94.99991 5.528124 5.000745
0.000049 94.99998 3.358000 5.000330
0.000024 94.99998 2.724212 5.000232
0.000012 94.99999 9.240705 5.000158
```

Convergence is studied in figure 2.21. The 1st method maximizes its accuracy for $\delta t \approx 0.01$, whereas for $\delta t < 0.0001$ the error becomes $> 10\%$ and the method becomes useless. The 2nd method has much better behavior that the 1st one.

We observe that as δt decreases, the final value of t approaches the expected $t_f = 95$. Why don't we obtain $t = 95$, especially when $t/\delta t$ is an integer? How many steps does it really take to reach $t \approx 95$, when the expected number of those is $\approx 95/\delta t$? Each time you take a measurement, issue the command

```
> wc -l box1D_1.dat
```

which measures the number of lines in the file `box1D_1.dat` and compare this number with the expected one. The result is interesting:

```
# --------------------------------
#    dt     N          N0
# --------------------------------
0.050000 1900      1900
0.025000 3800      3800
0.012500 7601      7600
0.006250 15203     15200
0.003125 30394     30400
0.001563 60760     60780
```

```
0.000781 121751  121638
0.000391 243753  242966
0.000195 485144  487179
0.000098 962662  969387
0.000049 1972589 1938775
0.000024 4067548 3958333
0.000012 7540956 7916666
```

where the second column has the number of steps computed by the program and the third one has the expected number of steps. We observe that the accuracy decreases with decreasing δt and in the end the difference is about 5%! Notice that the last line should have given $t_f = 0.000012 \times 7540956 \approx 90.5$, an error comparable to the period of the particle's motion.

We conclude that one important source of accumulation of systematic errors is the calculation of time. This type of errors become more significant with decreasing δt. We can improve the accuracy of the calculation significantly if we use the multiplication `t=t0+i*dt` instead of the addition `t=t+dt`, where `i` is a step counter:

```
//t = t  + dt;    // Not accurate,    avoid
  t = t0 + i*dt;  // Better accuracy, prefer
```

The main loop in the program `box1D_1.cpp` becomes:

```
  t = t0;
  x = x0;
  v = v0;
  i =  0;
  while(t <= (tf+1.0e-5f)){
    i +=    1;
    x += v*dt;
    t  = t0 + i*dt;
    if(x < 0.0f || x > L) v = -v;
  }
```

The full program can be found in the file `box1D_4.cpp` of the accompanying software. We call this "method 3". We perform the same change in the file `box1D_2.cpp`, which we store in the file `box1D_5.cpp`. We call this "method 4". We repeat the same analysis using methods 3 and 4 and we find that the problem of calculating time accurately practically

vanishes. The result of the analysis can be found on the right plot of figure 2.21. Methods 2 and 4 have no significant difference in their results, whereas methods 1 and 3 do have a dramatic difference, with method 3 decreasing the error more than tenfold. The problem of the increase of systematic errors with decreasing δt does not vanish completely due to the operation `x=x+v*dt`. This type of error is harder to deal with and one has to invent more elaborate algorithms in order to reduce it significantly. This will be discussed further in chapter 4.

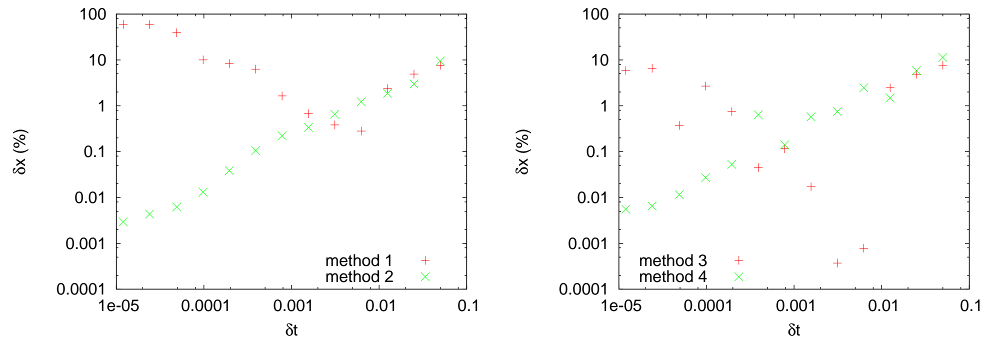

Figure 2.21: The error $\delta x = 2|x_i(95) - x(95)|/|x_i(95) + x(95)| \times 100$ where $x_i(95)$ is the value calculated by method $i = 1, 2, 3, 4$ and $x(95)$ the exact value according to the text.

2.3.3 The Two Dimensional Box

A particle is confined to move on the plane in the area $0 < x < L_x$ and $0 < y < L_y$. When it reaches the boundaries of this two dimensional box, it bounces elastically off its walls. The particle is found in an infinite depth orthogonal potential well. The particle starts moving at time t_0 from (x_0, y_0) and our program will calculate its trajectory until time t_f with time step δt. Such a trajectory can be seen in figure 2.23.

If the particle's position and velocity are known at time t, then at time

$t + \delta t$ they will be given by the relations

$$\begin{aligned} x(t+\delta t) &= x(t) + v_x(t)\delta t \\ y(t+\delta t) &= y(t) + v_y(t)\delta t \\ v_x(t+\delta t) &= v_x(t) \\ v_y(t+\delta t) &= v_y(t)\,. \end{aligned} \tag{2.25}$$

The collision of the particle off the walls is modeled by reflection of the normal component of the velocity when the respective coordinate of the particle crosses the wall. This is a source of the systematic errors that we discussed in the previous section. The central loop of the program is:

```
    i ++;
    t  = t0 + i*dt;
    x += vx*dt;
    y += vy*dt;
    if(x < 0.0 || x > Lx){
      vx = -vx;
      nx++;
    }
    if(y < 0.0 || y > Ly){
      vy = -vy;
      ny++;
    }
```

The full program can be found in the file `box2D_1.cpp`. Notice that we introduced two counters `nx` and `ny` of the particle's collisions with the walls:

```
//==========================================================
//File box2D_1.cpp
//Motion of a free particle in a box  0<x<Lx 0<y<Ly
//Use integration with time step dt: x = x + vx*dt y=y+vy*dt
//----------------------------------------------------------
#include <iostream>
#include <iomanip>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
```

```
int main(){
//------------------------------------------------------------------
//Declaration of variables
  double Lx,Ly,x0,y0,v0x,v0y,t0,tf,dt,t,x,y,vx,vy;
  int    i,nx,ny;
  string buf;
//------------------------------------------------------------------
//Ask user for input:
  cout << "# Enter Lx,Ly:\n";
  cin  >> Lx >> Ly;                 getline(cin,buf);
  cout << "# Lx = "<< Lx  << " Ly= "  << Ly  << endl;
  cout << "# Enter x0,y0,v0x,v0y:\n";
  cin  >> x0 >> y0 >> v0x >> v0y;   getline(cin,buf);
  cout << "# x0= " << x0  << " y0= "  << y0
       << " v0x= " << v0x << " v0y= " << v0y << endl;
  cout << "# Enter t0,tf,dt:\n";
  cin  >> t0 >> tf >> dt;           getline(cin,buf);
  cout << "# t0= " << t0  << " tf= "  << tf
       << " dt= "  << dt  << endl;
  if(Lx<= 0.0){cerr  << "Lx<=0 \n"; exit(1);}
  if(Ly<= 0.0){cerr  << "Ly<=0 \n"; exit(1);}
  if(x0<  0.0){cerr  << "x0<=0 \n"; exit(1);}
  if(x0>  Lx ){cerr  << "x0> Lx\n"; exit(1);}
  if(y0<  0.0){cerr  << "x0<=0 \n"; exit(1);}
  if(y0>  Ly ){cerr  << "y0> Ly\n"; exit(1);}
  if(v0x*v0x+v0y*v0y == 0.0 ){cerr << "v0 =0\n"; exit(1);}
//------------------------------------------------------------------
//Initialize
  i  =  0 ;
  nx =  0 ;  ny = 0  ;
  t  = t0 ;
  x  = x0 ;  y  = y0 ;
  vx = v0x;  vy = v0y;
  ofstream myfile("box2D_1.dat");
  myfile.precision(17);
//------------------------------------------------------------------
//Compute:
  while(t  <= tf){
    myfile << setw(28) << t  << " "
       << setw(28) << x  << " "
       << setw(28) << y  << " "
       << setw(28) << vx << " "
       << setw(28) << vy << '\n';
    i +=     1;
```

```
    t  = t0 + i*dt;
    x += vx*dt;
    y += vy*dt;
    if(x < 0.0 || x > Lx){
      vx = -vx;
      nx++;
    }
    if(y < 0.0 || y > Ly){
      vy = -vy;
      ny++;
    }
  }
  myfile.close();
  cout << "# Number of collisions:\n";
  cout << "# nx= " << nx << " ny= " << ny << endl;
} //main()
```

A typical session for the study of a particle's trajectory could be:

```
> g++ box2D_1.cpp -o box
> ./box
# Enter Lx,Ly:
10.0 5.0
# Lx = 10 Ly= 5
# Enter x0,y0,v0x,v0y:
5.0 0.0 1.27 1.33
# x0= 5 y0= 0 v0x= 1.27 v0y= 1.33
# Enter t0,tf,dt:
0 50 0.01
# t0= 0 tf= 50 dt= 0.01
# Number of collisions:
# nx= 6 ny= 13
> gnuplot
gnuplot> plot   "box2D_1.dat" using 1:2 w l title "x (t)"
gnuplot> replot "box2D_1.dat" using 1:3 w l title "y (t)"
gnuplot> plot   "box2D_1.dat" using 1:4 w l title "vx(t)"
gnuplot> replot "box2D_1.dat" using 1:5 w l title "vy(t)"
gnuplot> plot   "box2D_1.dat" using 2:3 w l title "x-y"
```

Notice the last line of output from the program: The particle bounces off the vertical walls 6 times (`nx=6`) and from the horizontal ones 13 (`ny=13`). The `gnuplot` commands construct the diagrams displayed in figures 2.22 and 2.23.

In order to animate the particle's trajectory, we can copy the file

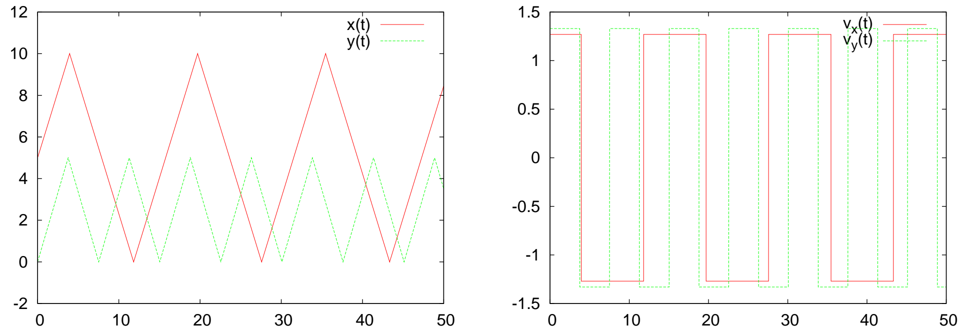

Figure 2.22: The results for the trajectory of a particle in a two dimensional box given by the program `box2D_1.cpp`. The parameters are $L_x = 10$, $L_y = 5$, $x_0 = 5$, $y_0 = 0$, $v_{0x} = 1.27$, $v_{0y} = 1.33$, $t_0 = 0$, $t_f = 50$, $\delta t = 0.01$.

`box2D_animate.gnu` of the accompanying software to the current directory and give the `gnuplot` commands:

```
gnuplot> file = "box2D_1.dat"
gnuplot> Lx = 10 ; Ly = 5
gnuplot> t0 = 0  ; tf = 50; dt = 1
gnuplot> load "box2D_animate.gnu"
gnuplot> t0 = 0  ; dt = 0.5; load "box2D_animate.gnu"
```

The last line repeats the same animation at half speed. You can also use the file `animate2D.gnu` discussed in section 2.1.1. We add new commands in the file `box2D_animate.gnu` so that the plot limits are calculated automatically and the box is drawn on the plot. The arrow drawn is not the position vector with respect to the origin of the coordinate axes, but the one connecting the initial with the current position of the particle.

The next step should be to test the accuracy of your results. This can be done by generalizing the discussion of the previous section and it is left as an exercise for the reader.

2.4 Applications

In this section we will study simple examples of motion in a box with different types of obstacles. We will start with a game of ... mini golf.

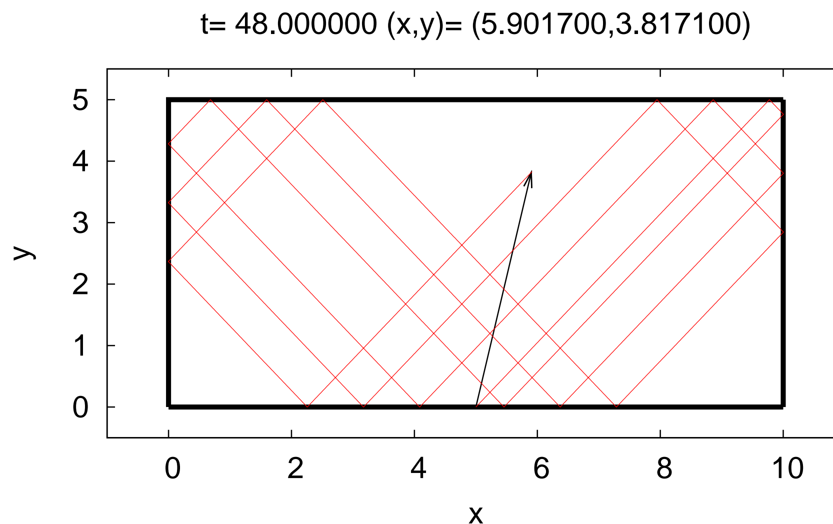

Figure 2.23: The trajectory of the particle of figure 2.22 until $t = 48$. The origin of the arrow is at the initial position of the particle and its end is at its current position. The bold lines mark the boundaries of the box.

The player shoots a (point) "ball" which moves in an orthogonal box of linear dimensions L_x and L_y and which is open on the $x = 0$ side. In the box there is a circular "hole" with center at (x_c, y_c) and radius R. If the "ball" falls in the "hole", the player wins. If the ball leaves out of the box through its open side, the player loses. In order to check if the ball is in the hole when it is at position (x, y), all we have to do is to check whether $(x - x_c)^2 + (y - y_c)^2 \le R^2$.

Initially we place the ball at the position $(0, L_y/2)$ at time $t_0 = 0$. The player hits the ball which leaves with initial velocity of magnitude v_0 at an angle θ degrees with the x axis. The program is found in the file `MiniGolf.cpp` and is listed below:

```
//========================================================
//File MiniGolf.cpp
//Motion of a free particle in a box  0<x<Lx 0<y<Ly
//The box is open at x=0 and has a hole at (xc,yc) of radius R
```

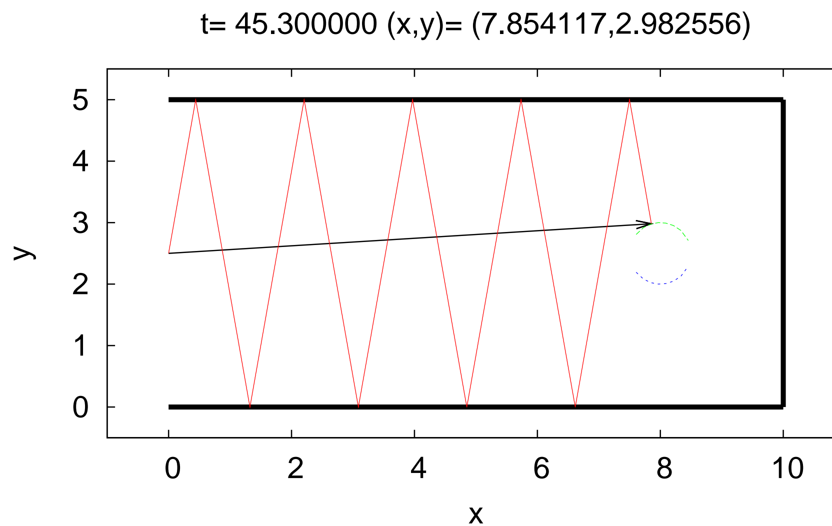

Figure 2.24: The trajectory of the particle calculated by the program `MiniGolf.cpp` using the parameters chosen in the text. The moment of ... success is shown. At time $t = 45.3$ the particle enters the hole's region which has its center at $(8, 2.5)$ and its radius is 0.5.

```
//Ball is shot at (0,Ly/2) with speed v0, angle theta (degrees)
//Use integration with time step dt: x = x + vx*dt y=y+vy*dt
//Ball stops in hole (success) or at x=0 (failure)
//----------------------------------------------------------------
#include <iostream>
#include <iomanip>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;

#define PI 3.14159265358979324

int main(){
//----------------------------------------------------------------
//Declaration of variables
  double Lx,Ly,x0,y0,v0x,v0y,t0,tf,dt,t,x,y,vx,vy;
```

```
  double v0,theta,xc,yc,R,R2;
  int    i,nx,ny;
  string result;
  string buf;
//--------------------------------------------------------------
//Ask user for input:
  cout << "# Enter Lx,Ly:\n";
  cin  >> Lx >> Ly;          getline(cin,buf);
  cout << "# Lx = " << Lx  << " Ly= "  << Ly  << endl;
  cout << "# Enter hole position and radius: (xc,yc), R:\n";
  cin  >> xc >> yc  >> R;   getline(cin,buf);
  cout << "# (xc,yc)= ( " << xc << " , "  << yc << ") "
       << " R= "    << R  << endl;
  cout << "# Enter v0, theta(degrees):\n";
  cin  >> v0 >> theta;      getline(cin,buf);
  cout << "# v0= " << v0  << " theta= "  << theta
       << " degrees "     << endl;
  cout << "# Enter dt:\n";
  cin  >> dt;               getline(cin,buf);
  if(Lx<=          0.0){cerr << "Lx<=0     \n"; exit(1);}
  if(Ly<=          0.0){cerr << "Ly<=0     \n"; exit(1);}
  if(v0<=          0.0){cerr << "v0<=0     \n"; exit(1);}
  if(abs(theta) > 90.0){cerr << "theta > 90\n"; exit(1);}
//--------------------------------------------------------------
//Initialize
  t0    = 0.0;
  x0    = 0.00001; // small but non-zero
  y0    = Ly/2.0;
  R2    = R*R;
  theta = (PI/180.0)*theta;
  v0x   = v0*cos(theta);
  v0y   = v0*sin(theta);
  cout << "# x0= " << x0  << "  y0= " << y0
       << " v0x= " << v0x << " v0y= " << v0y << endl;
  i  =  0 ;
  nx =  0 ;  ny = 0  ;
  t  = t0 ;
  x  = x0 ;  y  = y0 ;
  vx = v0x;  vy = v0y;
  ofstream myfile("MiniGolf.dat");
  myfile.precision(17);
//--------------------------------------------------------------
//Compute:
  while(true){
    myfile << setw(28) << t  << " "
```

```
        << setw(28) << x  << " "
        << setw(28) << y  << " "
        << setw(28) << vx << " "
        << setw(28) << vy << '\n';
     i ++;
     t  = t0 + i*dt;
     x += vx*dt;
     y += vy*dt;
     if(x > Lx ){vx = -vx; nx++;}
     if(y < 0.0){vy = -vy; ny++;}
     if(y > Ly ){vy = -vy; ny++;}
     if(x <=0.0)
       {result="Failure";break;} // exit loop
     if(((x-xc)*(x-xc)+(y-yc)*(y-yc)) <=  R2)
       {result="Success";break;} // exit loop
   }
   myfile.close();
   cout << "# Number of collisions:\n";
   cout << "# Result= "  << result
        << " nx= " << nx << " ny= " << ny << endl;
} //main()
```

In order to run it, we can use the commands:

```
> g++ MiniGolf.cpp -o mg
> ./mg
# Enter Lx,Ly:
10 5
# Lx = 10 Ly= 5
# Enter hole position and radius: (xc,yc), R:
8 2.5 0.5
# (xc,yc)= ( 8  ,  2.5)  R= 0.5
# Enter v0, theta(degrees):
1 80
# v0= 1 theta= 80 degrees
# Enter dt:
0.01
# x0= 1e-05  y0= 2.5 v0x= 0.173648 v0y= 0.984808
# Number of collisions:
# Result= Success nx= 0 ny= 9
```

You should construct the plots of the position and the velocity of the particle. You can also use the animation program found in the file `MiniGolf_animate.gnu` for fun. Copy it from the accompanying software

to the current directory and give the `gnuplot` commands:

```
gnuplot> file = "MiniGolf.dat"
gnuplot> Lx = 10;Ly = 5
gnuplot> xc = 8; yc = 2.5 ; R = 0.5
gnuplot> t0 = 0; dt = 0.1
gnuplot> load "MiniGolf_animate.gnu"
```

The results are shown in figure 2.24.

The next example with be three dimensional. We will study the motion of a particle confined within a cylinder of radius R and height L. The collisions of the particle with the cylinder are elastic. We take the axis of the cylinder to be the z axis and the two bases of the cylinder to be located at $z=0$ and $z=L$. This is shown in figure 2.26.

The collisions of the particle with the bases of the cylinder are easy to program: we follow the same steps as in the case of the simple box. For the collision with the cylinder's side, we consider the projection of the motion on the $x-y$ plane. The projection of the particle moves within a circle of radius R and center at the intersection of the z axis with the plane. This is shown in figure 2.25. At the collision, the r component of the velocity is reflected $v_r \to -v_r$, whereas v_θ remains the same. The velocity of the particle before the collision is

$$\begin{aligned} \vec{v} &= v_x\hat{x} + v_y\hat{y} \\ &= v_r\hat{r} + v_\theta\hat{\theta} \end{aligned} \tag{2.26}$$

and after the collision is

$$\begin{aligned} \vec{v}' &= v_x'\hat{x} + v_y'\hat{y} \\ &= -v_r\hat{r} + v_\theta\hat{\theta} \end{aligned} \tag{2.27}$$

From the relations

$$\begin{aligned} \hat{r} &= \cos\theta\hat{x} + \sin\theta\hat{y} \\ \hat{\theta} &= -\sin\theta\hat{x} + \cos\theta\hat{y}\,, \end{aligned} \tag{2.28}$$

and $v_r = \vec{v}\cdot\hat{r}$, $v_\theta = \vec{v}\cdot\hat{\theta}$, we have that

$$\begin{aligned} v_r &= v_x\cos\theta + v_y\sin\theta \\ v_\theta &= -v_x\sin\theta + v_y\cos\theta\,. \end{aligned} \tag{2.29}$$

The inverse relations are

$$\begin{aligned} v_x &= v_r \cos\theta - v_\theta \sin\theta \\ v_y &= v_r \sin\theta + v_\theta \cos\theta \,. \end{aligned} \tag{2.30}$$

With the transformation $v_r \to -v_r$, the new velocity in Cartesian coordinates will be

$$\begin{aligned} v'_x &= -v_r \cos\theta - v_\theta \sin\theta \\ v'_y &= -v_r \sin\theta + v_\theta \cos\theta \,. \end{aligned} \tag{2.31}$$

The transformation $v_x \to v'_x$, $v_y \to v'_y$ will be performed in the function

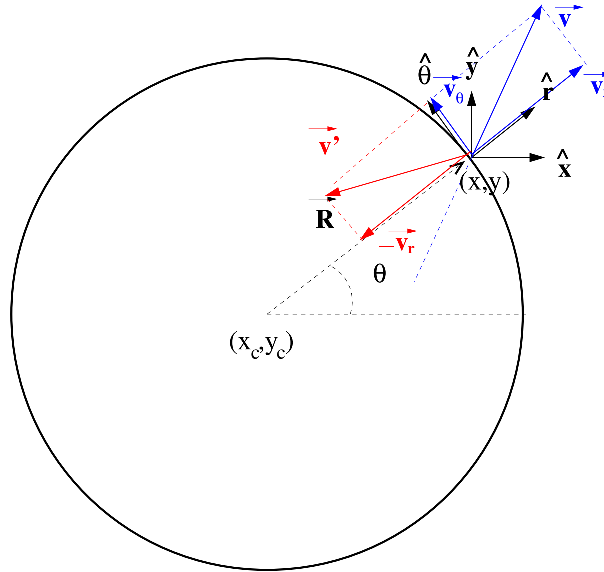

Figure 2.25: The elastic collision of the particle moving within the circle of radius $R = |\vec{R}|$ and center $\vec{r}_c = x_c\hat{x} + y_c\hat{y}$ at the point $\vec{r} = x\hat{x} + y\hat{y}$. We have that $\vec{R} = (x - x_c)\hat{x} + (y - y_c)\hat{y}$. The initial velocity is $\vec{v} = v_r\hat{r} + v_\theta\hat{\theta}$ where $\hat{r} \equiv \vec{R}/R$. After reflecting $v_r \to -v_r$ the new velocity of the particle is $\vec{v}' = -v_r\hat{r} + v_\theta\hat{\theta}$.

`reflectVonCircle(vx,vy,x,y,xc,yc,R)`. Upon entry to the function, we provide the initial velocity `(vx,vy)`, the collision point `(x,y)`, the center of the circle `(xc,yc)` and the radius of the circle[24] `R`. Upon exit from the

[24]Of course one expects $R^2 = (x - x_c)^2 + (y - y_c)^2$, but because of systematic errors, we require R to be given.

function, (vx,vy) have been replaced with the new values[25] (v'_x, v'_y).

The program can be found in the file Cylinder3D.cpp and is listed below:

```
//==========================================================
//File Cylinder3D.cpp
//Motion of a free particle in a cylinder with axis the z-axis,
//radius R and 0<z<L
//Use integration with time step dt: x = x + vx*dt
//                                   y = y + vy*dt
//                                   z = z + vz*dt
//Use function reflectVonCircle for colisions at r=R
//----------------------------------------------------------
#include <iostream>
#include <iomanip>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;

void reflectVonCircle(double& vx,double& vy,
              double& x ,double& y ,
              const       double& xc,
              const       double& yc,
              const       double& R );

int main(){
//----------------------------------------------------------
//Declaration of variables
  double x0,y0,z0,v0x,v0y,v0z,t0,tf,dt,t,x,y,z,vx,vy,vz;
  double L,R,R2,vxy,rxy,r2xy,xc,yc;
  int    i,nr,nz;
  string buf;
//----------------------------------------------------------
//Ask user for input:
  cout << "# Enter R,L:\n";
  cin  >> R  >> L;                     getline(cin,buf);
  cout << "# R= " << R  << " L= "  << L  << endl;
  cout << "# Enter x0,y0,z0,v0x,v0y,v0z:\n";
  cin  >> x0>>y0>>z0>>v0x>>v0y>>v0z;   getline(cin,buf);
  rxy  = sqrt(x0*x0+y0*y0);
```

[25] Notice that upon exit, the particle is also placed exactly on the circle.

```
  cout << "# x0 = " << x0
       << "  y0 = " << y0
       << "  z0 = " << z0
       << "  rxy= " << rxy << endl;
  cout << "# v0x= " << v0x
       << "  v0y= " << v0y
       << "  v0z= " << v0z << endl;
  cout << "# Enter t0,tf,dt:\n";
  cin  >> t0 >> tf  >> dt;               getline(cin,buf);
  cout << "# t0= "  << t0  << " tf= " << tf
       << "  dt= "  << dt  << endl;
  if(R   <= 0.0){cerr << "R<=0   \n"; exit(1);}
  if(L   <= 0.0){cerr << "L<=0   \n"; exit(1);}
  if(z0  <  0.0){cerr << "z0<0   \n"; exit(1);}
  if(z0  >    L){cerr << "z0>L   \n"; exit(1);}
  if(rxy >    R){cerr << "rxy>R  \n"; exit(1);}
  if(v0x*v0x+v0y*v0y+v0z*v0z == 0.0)
                {cerr << "v0=0   \n"; exit(1);}
//------------------------------------------------------------
//Initialize
  i  =  0 ;
  nr =  0 ;  nz = 0  ;
  t  = t0 ;
  x  = x0 ;  y  = y0 ;  z  = z0 ;
  vx = v0x;  vy = v0y;  vz = v0z;
  R2 = R*R;
  xc = 0.0; //center of circle which is the projection
  yc = 0.0; //of the cylinder on the xy plane
  ofstream myfile("Cylinder3D.dat");
  myfile.precision(17);
//------------------------------------------------------------
//Compute:
  while(t <= tf){
    myfile << setw(28) << t  << " "
       << setw(28) << x  << " "
       << setw(28) << y  << " "
       << setw(28) << z  << " "
       << setw(28) << vx << " "
       << setw(28) << vy << " "
       << setw(28) << vz << '\n';
    i ++;
    t  = t0 + i*dt;
    x += vx*dt;
    y += vy*dt;
    z += vz*dt;
```

```
    if( z <= 0.0 || z > L){vz = -vz; nz++;}
    r2xy = x*x+y*y;
    if( r2xy > R2){
      reflectVonCircle(vx,vy,x,y,xc,yc,R);
      nr++;
    }
  }
  myfile.close();
  cout << "# Number of collisions:\n";
  cout << "# nr= " << nr << " nz= " << nz << endl;
} //main()
//--------------------------------------------------------------
//==============================================================
//--------------------------------------------------------------
void reflectVonCircle(double& vx,double& vy,
                      double& x ,double& y ,
                      const      double& xc,
                      const      double& yc,
                      const      double& R ){
  double theta,cth,sth,vr,vth;

  theta = atan2(y-yc,x-xc);
  cth   = cos(theta);
  sth   = sin(theta);

  vr    =  vx*cth + vy *sth;
  vth   = -vx*sth + vy *cth;

  vx    = -vr*cth - vth*sth; //reflect vr -> -vr
  vy    = -vr*sth + vth*cth;

  x     =  xc     + R*cth;   //put x,y on the circle
  y     =  yc     + R*sth;
} //reflectVonCircle()
```

Note that the function `atan2` is used for computing the angle `theta`. This function, when called with two arguments `atan2(y,x)`, returns the angle $\theta = \tan^{-1}(y/x)$ in radians. The correct quadrant of the circle where (x, y) lies is chosen. The angle that we want to compute is given by `atan2(y-yc,x-xc)`. Then we apply equations (2.29) and (2.31) and in the last two lines we enforce the particle to be at the point $(x_c + R\cos\theta, y_c + R\sin\theta)$, exactly on the circle.

A typical session is shown below:

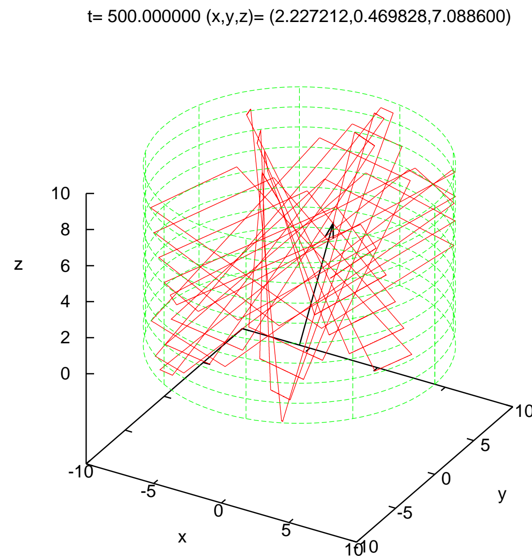

Figure 2.26: The trajectory of a particle moving inside a cylinder with $R = 10$, $L = 10$, computed by the program `Cylinder3D.cpp`. We have chosen $\vec{r}_0 = 1.0\hat{x} + 2.2\hat{y} + 3.1\hat{z}$, $\vec{v}_0 = 0.93\hat{x} - 0.89\hat{y} + 0.74\hat{z}$, $t_0 = 0$, $t_f = 500.0$, $\delta t = 0.01$.

```
> g++ Cylinder3D.cpp -o cl
> ./cl
# Enter R,L:
10.0 10.0
# R= 10 L= 10
# Enter x0,y0,z0,v0x,v0y,v0z:
1.0 2.2 3.1   0.93 -0.89 0.74
# x0 = 1  y0 = 2.2  z0 = 3.1  rxy= 2.41661
# v0x= 0.93  v0y= -0.89  v0z= 0.74
# Enter t0,tf,dt:
0.0 500.0 0.01
# t0= 0 tf= 500  dt= 0.01
# Number of collisions:
# nr= 33 nz= 37
```

In order to plot the position and the velocity as a function of time, we use the following `gnuplot` commands:

```
gnuplot> file="Cylinder3D.dat"
gnuplot> plot file using 1:2 with lines title "  x(t)",\
              file using 1:3 with lines title "  y(t)",\
              file using 1:4 with lines title "  z(t)"
gnuplot> plot file using 1:5 with lines title "v_x(t)",\
              file using 1:6 with lines title "v_y(t)",\
              file using 1:7 with lines title "v_z(t)"
```

We can also compute the distance of the particle from the cylinder's axis $r(t) = \sqrt{x(t)^2 + y(t)^2}$ as a function of time using the command:

```
gnuplot> plot file using 1:(sqrt($2**2+$3**2)) w l t "r(t)"
```

In order to plot the trajectory, together with the cylinder, we give the commands:

```
gnuplot> file="Cylinder3D.dat"
gnuplot> L = 10 ; R = 10
gnuplot> set urange [0:2.0*pi]
gnuplot> set vrange [0:L]
gnuplot> set parametric
gnuplot> splot file using 2:3:4 with lines notitle,\
                   R*cos(u),R*sin(u),v notitle
```

The command `set parametric` is necessary if one wants to make a parametric plot of a surface $\vec{r}(u,v) = x(u,v)\,\hat{x} + y(u,v)\,\hat{y} + z(u,v)\,\hat{z}$. The cylinder (without the bases) is given by the parametric equations $\vec{r}(u,v) = R\cos u\,\hat{x} + R\sin u\,\hat{y} + v\,\hat{z}$ with $u \in [0, 2\pi)$, $v \in [0, L]$.

We can also animate the trajectory with the help of the `gnuplot` script file `Cylinder3D_animate.gnu`. Copy the file from the accompanying software to the current directory and give the `gnuplot` commands:

```
gnuplot> file="Cylinder3D.dat"
gnuplot> R=10;L=10;t0=0;tf=500;dt=10
gnuplot> load "Cylinder3D_animate.gnu"
```

The result is shown in figure 2.26.

The last example will be that of a simple model of a spacetime wormhole. This is a simple spacetime geometry which, in the framework of the theory of general relativity, describes the connection of two distant

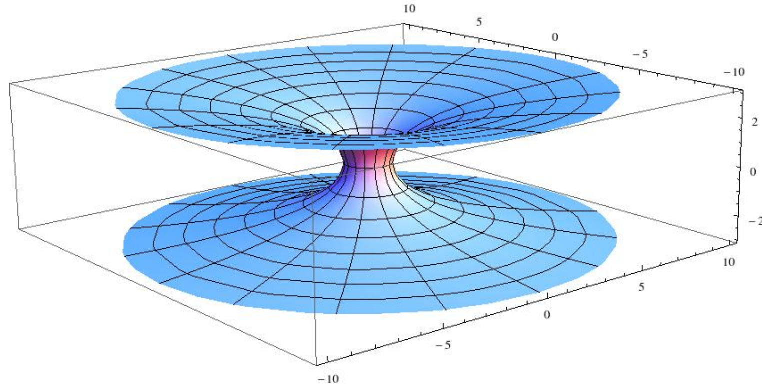

Figure 2.27: A typical geometry of space near a wormhole. Two asymptotically flat regions of space are connected through a "neck" which can be arranged to be of small length compared to the distance of the wormhole mouths when traveled from the outside space.

areas in space which are asymptotically flat. This means, that far enough from the wormhole's mouths, space is almost flat - free of gravity. Such a geometry is depicted in figure 2.27. The distance traveled by someone through the mouths could be much smaller than the distance traveled outside the wormhole and, at least theoretically, traversable wormholes could be used for interstellar/intergalactic traveling and/or communications between otherwise distant areas in the universe. Of course we should note that such macroscopic and stable wormholes are not known to be possible to exist in the framework of general relativity. One needs an exotic type of matter with negative energy density which has never been observed. Such exotic geometries may realize microscopically as quantum fluctuations of spacetime and make the small scale structure of the geometry[26] a "spacetime foam".

We will study a very simple model of the above geometry on the plane

[26] See K.S. Thorne "Black Holes and Time Wraps: Einstein's Outrageous Legacy", W.W. Norton, New York for a popular review of these concepts.

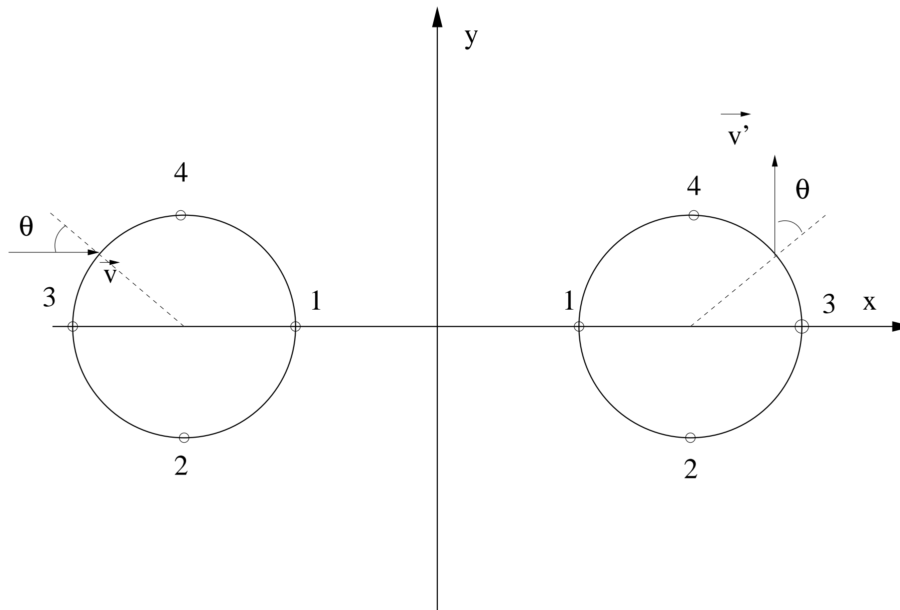

Figure 2.28: A simple model of the spacetime geometry of figure 2.27. The particle moves on the whole plane except withing the two disks that have been removed. The neck of the wormhole is modeled by the two circles $x(\theta) = \pm d/2 \pm R\cos\theta$, $y(\theta) = R\sin\theta$, $-\pi < \theta \leq \pi$ and has zero length since their points have been identified. There is a given direction in this identification, so that points with the same θ are the same (you can imagine how this happens by folding the plane across the y axis and then glue the two circles together). The entrance of the particle through one mouth and exit through the other is done as shown for the velocity vector $\vec{v} \to \vec{v}'$.

with a particle moving freely in it[27]. We take the two dimensional plane and cut two equal disks of radius R with centers at distance d like in figure 2.28. We identify the points on the two circles such that the point 1 of the left circle is the same as the point 1 on the right circle, the point 2 on the left with the point 2 on the right etc. The two circles are given by the parametric equations $x(\theta) = d/2 + R\cos\theta$, $y(\theta) = R\sin\theta$, $-\pi < \theta \leq \pi$ for the right circle and $x(\theta) = -d/2 - R\cos\theta$, $y(\theta) = R\sin\theta$, $-\pi < \theta \leq \pi$ for the left. Points on the two circles with the same θ are identified. A particle entering the wormhole from the left circle with velocity v is immediately exiting from the right with velocity v' as shown in figure

[27]This idea can be found as an exercise in the excellent introductory general relativity textbook J. B. Hartle, "Gravity: An Introduction to Einstein's General Relativity", Addison Wesley 2003, Ch. 7, Ex. 25.

2.28.

Then we will do the following:

1. Write a program that computes the trajectory of a particle moving in the geometry of figure 2.28. We set the limits of motion to be $-L/2 \leq x \leq L/2$ and $-L/2 \leq y \leq L/2$. We will use periodic boundary conditions in order to define what happens when the particle attempts to move outside these limits. This means that we identify the $x = -L/2$ line with the $x = +L/2$ line as well as the $y = -L/2$ line with the $y = +L/2$ line. The user enters the parameters R, d and L as well as the initial conditions (x_0, y_0), (v_0, ϕ) where $\vec{v}_0 = v_0(\cos\phi\hat{x} + \sin\phi\hat{y})$. The user will also provide the time parameters t_f and dt for motion in the time interval $t \in [t_0 = 0, t_f]$ with step dt.

2. Plot the particle's trajectory with $(x_0, y_0) = (0, -1)$, $(v_0, \phi) = (1, 10^o)$ με $t_f = 40$, $dt = 0.05$ in the geometry with $L = 20, d = 5, R = 1$.

3. Find a closed trajectory which does not cross the boundaries $|x| = L/2$, $|y| = L/2$ and determine whether it is stable under small perturbations of the initial conditions.

4. Find other closed trajectories that go through the mouths of the wormhole and study their stability under small perturbations of the initial conditions.

5. Add to the program the option to calculate the distance traveled by the particle. If the particle starts from $(-x_0, 0)$ and moves in the $+x$ direction to the $(x_0, 0)$, $x_0 > R + d/2$ position, draw the trajectory and calculate the distance traveled on paper. Then confirm your calculation from the numerical result coming from your program.

6. Change the boundary conditions, so that the particle bounces off elastically at $|x| = L/2$, $|y| = L/2$ and replot all the trajectories mentioned above.

Define the right circle c_1 by the parametric equations

$$x(\theta) = \frac{d}{2} + R\cos\theta\,, \qquad y(\theta) = R\sin\theta\,, \qquad -\pi < \theta \leq \pi\,, \tag{2.32}$$

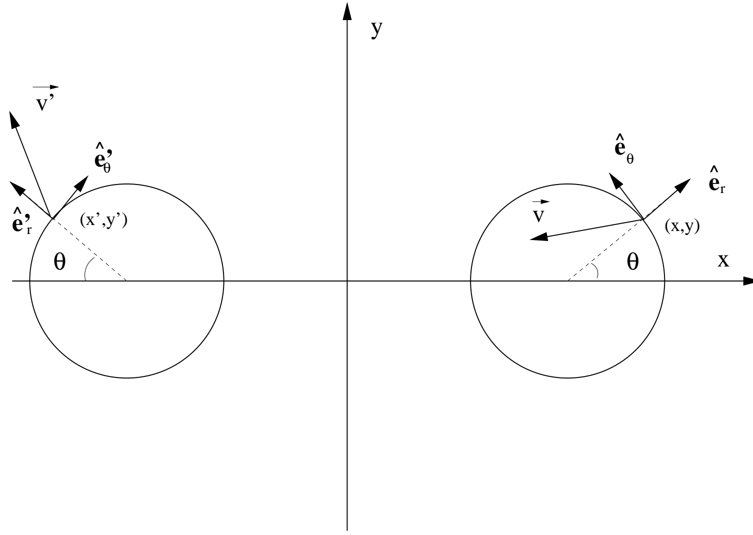

Figure 2.29: The particle crossing the wormhole through the right circle c_1 with velocity $\vec{v}$. It emerges from c_2 with velocity $\vec{v}'$. The unit vectors $(\hat{e}_r, \hat{e}_\theta)$, $(\hat{e}'_r, \hat{e}'_\theta)$ are computed from the parametric equations of the two circles c_1 and c_2.

and the left circle c_2 by the parametric equations

$$x(\theta) = -\frac{d}{2} - R\cos\theta\,, \qquad y(\theta) = R\sin\theta\,, \qquad -\pi < \theta \leq \pi\,. \tag{2.33}$$

The particle's position changes at time dt by

$$\begin{aligned} t_i &= idt \\ x_i &= x_{i-1} + v_x dt \\ y_i &= y_{i-1} + v_y dt \end{aligned} \tag{2.34}$$

for $i = 1, 2, \ldots$ for given (x_0, y_0), $t_0 = 0$ and as long as $t_i \leq t_f$. If the point (x_i, y_i) is outside the boundaries $|x| = L/2$, $|y| = L/2$, we redefine $x_i \to x_i \pm L$, $y_i \to y_i \pm L$ in each case respectively. Points defined by the same value of θ are identified, i.e. they represent the *same* points of space. If the point (x_i, y_i) crosses either one of the circles c_1 or c_2, then we take the particle out from the other circle.

Crossing the circle c_1 is determined by the relation

$$\left(x_i - \frac{d}{2}\right)^2 + y_i^2 \leq R^2\,. \tag{2.35}$$

The angle θ is calculated from the equation

$$\theta = \tan^{-1}\left(\frac{y_i}{x_i - \frac{d}{2}}\right) , \tag{2.36}$$

and the point (x_i, y_i) is mapped to the point (x_i', y_i') where

$$x_i' = -\frac{d}{2} - R\cos\theta \,, \qquad y_i' = y_i \,, \tag{2.37}$$

as can be seen in figure 2.29. For mapping $\vec{v} \to \vec{v}'$, we first calculate the vectors

$$\left.\begin{array}{rrcr} \hat{e}_r = & \cos\theta\,\hat{x} & + & \sin\theta\,\hat{y} \\ \hat{e}_\theta = & -\sin\theta\,\hat{x} & + & \cos\theta\,\hat{y} \end{array}\right\} \to \left\{\begin{array}{rrcr} \hat{e}_r' = & -\cos\theta\,\hat{x} & + & \sin\theta\,\hat{y} \\ \hat{e}_\theta' = & \sin\theta\,\hat{x} & + & \cos\theta\,\hat{y} \end{array}\right. , \tag{2.38}$$

so that the velocity

$$\vec{v} = v_r\,\hat{e}_r + v_\theta\,\hat{e}_\theta \quad \to \quad \vec{v}' = -v_r\,\hat{e}_r' + v_\theta\,\hat{e}_\theta' \,, \tag{2.39}$$

where the radial components are $v_r = \vec{v}\cdot\hat{e}_r$ and $v_\theta = \vec{v}\cdot\hat{e}_\theta$. Therefore, the relations that give the "emerging" velocity $\vec{v}'$ are:

$$\begin{array}{rrcr} v_r = & v_x\cos\theta & + & v_y\sin\theta \\ v_\theta = & -v_x\sin\theta & + & v_y\cos\theta \\ v_x' = & v_r\cos\theta & + & v_\theta\sin\theta \\ v_y' = & -v_r\sin\theta & + & v_\theta\cos\theta \end{array} . \tag{2.40}$$

Similarly we calculate the case of entering from c_2 and emerging from c_1. The condition now is:

$$\left(x_i + \frac{d}{2}\right)^2 + y_i^2 \le R^2 \,. \tag{2.41}$$

The angle θ is given by

$$\theta = \pi - \tan^{-1}\left(\frac{y_i}{x_i + \frac{d}{2}}\right) , \tag{2.42}$$

and the point (x_i, y_i) is mapped to the point (x_i', y_i') where

$$x_i' = \frac{d}{2} + R\cos\theta \,, \qquad y_i' = y_i \,. \tag{2.43}$$

For mapping $\vec{v} \to \vec{v}'$, we calculate the vectors

$$\left.\begin{array}{rrcr} \hat{e}_r = & -\cos\theta\,\hat{x} & + & \sin\theta\,\hat{y} \\ \hat{e}_\theta = & \sin\theta\,\hat{x} & + & \cos\theta\,\hat{y} \end{array}\right\} \to \left\{\begin{array}{rrcr} \hat{e}'_r = & \cos\theta\,\hat{x} & + & \sin\theta\,\hat{y} \\ \hat{e}'_\theta = & -\sin\theta\,\hat{x} & + & \cos\theta\,\hat{y} \end{array}\right. , \tag{2.44}$$

so that the velocity

$$\vec{v} = v_r\,\hat{e}_r + v_\theta\,\hat{e}_\theta \quad \to \quad \vec{v}' = -v_r\,\hat{e}'_r + v_\theta\,\hat{e}'_\theta\,. \tag{2.45}$$

The emerging velocity $\vec{v}'$ is:

$$\begin{array}{rrcr} v_r = & -v_x\cos\theta & + & v_y\sin\theta \\ v_\theta = & v_x\sin\theta & + & v_y\cos\theta \\ v'_x = & -v_r\cos\theta & - & v_\theta\sin\theta \\ v'_y = & -v_r\sin\theta & + & v_\theta\cos\theta \end{array} . \tag{2.46}$$

Systematic errors are now coming from crossing the two mouths of the wormhole. There are no systematic errors from crossing the boundaries $|x| = L/2$, $|y| = L/2$ (why?). Try to think of ways to control those errors and study them.

The closed trajectories that we are looking for come from the initial conditions

$$(x_0, y_0, v_0, \phi) = (0, 0, 1, 0) \tag{2.47}$$

and they connect points 1 of figure 2.28. They are unstable, as can be seen by taking $\phi \to \phi + \epsilon$.

The closed trajectories that cross the wormhole and "wind" through space can come from the initial conditions

$$\begin{array}{rcl} (x_0, y_0, v_0, \phi) & = & (-9, 0, 1, 0) \\ (x_0, y_0, v_0, \phi) & = & (2.5, -3, 1, 90^o) \end{array}$$

and cross the points $3 \to 3$ and $2 \to 2 \to 4 \to 4$ respectively. They are also unstable, as can be easily verified by using the program that you will write. The full program is listed below:

```
//==========================================================
//File Wormhole.cpp
//----------------------------------------------------------
#include <iostream>
```

```
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;

#define PI 3.1415926535897932

void  crossC1(       double&  x,        double&  y,
                     double& vx,        double& vy,
               const double& dt,  const double&  R,
               const double& d);
void  crossC2(       double&  x,        double&  y,
                     double& vx,        double& vy,
               const double& dt,  const double&  R,
               const double& d);

int main(){
//------------------------------------------------------------
//Declaration of variables
  double Lx,Ly,L,R,d;
  double x0,y0,v0,theta;
  double t0,tf,dt;
  double t,x,y,vx,vy;
  double xc1,yc1,xc2,yc2,r1,r2;
  int    i;
  string buf;
//------------------------------------------------------------
//Ask user for input:
  cout << "# Enter L,d,R:\n";
  cin  >> L  >> d  >> R;                getline(cin,buf);
  cout << "# Enter (x0,y0), v0, theta(degrees):\n";
  cin  >> x0 >> y0 >> v0 >> theta;      getline(cin,buf);
  cout << "# Enter tf,dt:\n";
  cin  >> tf >> dt;  getline(cin,buf);
  cout << "# L=  " << L  << " d=     " << d
       << "  R=  " << R  << endl;
  cout << "# x0= " << x0 << " y0=    " << y0    << endl;
  cout << "# v0= " << v0 << " theta= "
       << theta    << " degrees"               << endl;
  cout << "# tf= " << tf << " dt=    " << dt    << endl;
  if(L <=  d+2.0*R){cerr <<"L <= d+2*R \n";exit(1);}
  if(d <=    2.0*R){cerr <<"d <=   2*R \n";exit(1);}
  if(v0<=    0.0  ){cerr <<"v0<=     0 \n";exit(1);}
//------------------------------------------------------------
```

```
//Initialize
  theta = (PI/180.0)*theta;
  i     =  0;
  t     =  0.0;
  x     =  x0           ; y     =  y0;
  vx    =  v0*cos(theta); vy    =  v0*sin(theta);
  cout << "# x0= " << x0 << "  y0= " << y0
       << " v0x= " << vx << " v0y= " << vy << endl;
//Wormhole's centers:
  xc1   =  0.5*d; yc1   =  0.0;
  xc2   = -0.5*d; yc2   =  0.0;
//Box limits coordinates:
  Lx    =  0.5*L; Ly    =  0.5*L;
//Test if already inside cut region:
  r1    = sqrt((x-xc1)*(x-xc1)+(y-yc1)*(y-yc1));
  r2    = sqrt((x-xc2)*(x-xc2)+(y-yc2)*(y-yc2));
  if(r1<=        R){cerr <<"r1 <=    R \n";exit(1);}
  if(r1<=        R){cerr <<"r2 <=    R \n";exit(1);}
//Test if outside box limits:
  if(abs(x) >=  Lx){cerr <<"|x|>=   Lx \n";exit(1);}
  if(abs(y) >=  Ly){cerr <<"|y|>=   Ly \n";exit(1);}
  ofstream myfile("Wormhole.dat");
  myfile.precision(17);
//------------------------------------------------------------
//Compute:
  while( t <  tf ){
    myfile <<  t  << " "
           <<  x  << " " <<  y << " "
           << vx  << " " << vy << endl;
    i++;
    t  =  i*dt;
    x += vx*dt; y += vy*dt;
// Toroidal boundary conditions:
    if( x >  Lx) x  = x - L;
    if( x < -Lx) x  = x + L;
    if( y >  Ly) y  = y - L;
    if( y < -Ly) y  = y + L;
    r1    = sqrt((x-xc1)*(x-xc1)+(y-yc1)*(y-yc1));
    r2    = sqrt((x-xc2)*(x-xc2)+(y-yc2)*(y-yc2));
// Notice: we pass r1 as radius of circle, not R
    if      (r1 < R)
      crossC1(x,y,vx,vy,dt,r1,d);
    else if(r2 < R)
      crossC2(x,y,vx,vy,dt,r2,d);
// small chance here that still in C1 or C2, but OK since
```

```
// another dt-advance given at the beginning of for-loop
  }// while( t <= tf )
} // main ()
//----------------------------------------------------------------
void  crossC1(       double&  x,       double&  y,
                     double& vx,       double& vy,
               const double& dt, const double&  R,
               const double& d){

  double vr,v0,theta,xc,yc;
  cout << "# Inside C1: (x,y,vx,vy,R)= "
       << x  << " " << y  << " "
       << vx << " " << vy << " " <<R << endl;
  xc    =  0.5*d;               //center of C1
  yc    =  0.0;
  theta =  atan2(y-yc,x-xc);
  x     = -xc - R*cos(theta); //new x-value, y invariant
//Velocity transformation:
  vr    =  vx*cos(theta)+vy*sin(theta);
  v0    = -vx*sin(theta)+vy*cos(theta);
  vx    =  vr*cos(theta)+v0*sin(theta);
  vy    = -vr*sin(theta)+v0*cos(theta);
//advance x,y, hopefully outside C2:
  x     = x + vx*dt;
  y     = y + vy*dt;
  cout << "# Exit   C2: (x,y,vx,vy   )= "
       << x  << " " << y  << " "
       << vx << " " << vy << endl;
}//void  crossC1( )
//----------------------------------------------------------------
void  crossC2(       double&  x,       double&  y,
                     double& vx,       double& vy,
               const double& dt, const double&  R,
               const double& d){

  double vr,v0,theta,xc,yc;
  cout << "# Inside C2: (x,y,vx,vy,R)= "
       << x  << " " << y  << " "
       << vx << " " << vy << " " <<R << endl;
  xc    = -0.5*d;               //center of C2
  yc    =  0.0;
  theta =  PI-atan2(y-yc,x-xc);
  x     = -xc + R*cos(theta); //new x-value, y invariant
//Velocity transformation:
  vr    = -vx*cos(theta)+vy*sin(theta);
```

```
  v0     =  vx*sin(theta)+vy*cos(theta);
  vx     = -vr*cos(theta)-v0*sin(theta);
  vy     = -vr*sin(theta)+v0*cos(theta);
//advance x,y, hopefully outside C1:
  x      = x + vx*dt;
  y      = y + vy*dt;
  cout << "# Exit   C1: (x,y,vx,vy   )= "
       << x  << " " << y  << " "
       << vx << " " << vy << endl;
}//void  crossC2( )
```

It is easy to compile and run the program. See also the files `Wormhole.csh` and `Wormhole_animate.gnu` of the accompanying software and run the `gnuplot` commands:

```
gnuplot> file  = "Wormhole.dat"
gnuplot> R=1;d=5;L=20;
gnuplot> ! ./Wormhole.csh
gnuplot> t0=0;dt=0.2;load "Wormhole_animate.gnu"
```

You are now ready to answer the rest of the questions that we asked in our list.

2.5 Problems

2.1 Change the program `Circle.cpp` so that it prints the number of full circles traversed by the particle.

2.2 Add all the necessary tests on the parameters entered by the user in the program `Circle.cpp`, so that the program is certain to run without problems. Do the same for the rest of the programs given in the same section.

2.3 A particle moves with constant angular velocity ω on a circle that has the origin of the coordinate system at its center. At time $t_0 = 0$, the particle is at (x_0, y_0). Write the program `CircularMotion.cpp` that will calculate the particle's trajectory. The user should enter the parameters $\omega, x_0, y_0, t_0, t_f, \delta t$. The program should print the results like the program `Circle.cpp` does.

2.4 Change the program `SimplePendulum.cpp` so that the user could enter a non zero initial velocity.

2.5 Study the $k \to 0$ limit in the projectile motion given by equations (2.10). Expand $\mathrm{e}^{-kt} = 1 - kt + \frac{1}{2!}(kt)^2 + \ldots$ and keep the non vanishing terms as $k \to 0$. Then keep the next order leading terms which have a smaller power of k. Program these relations in a file `ProjectileSmallAirResistance.cpp`. Consider the initial conditions $\vec{v}_0 = \hat{x} + \hat{y}$ and calculate the range of the trajectory numerically by using the two programs `ProjectileSmallAirResistance.cpp`, `ProjectileAirResistance.cpp`. Determine the range of values of k for which the two results agree within 5% accuracy.

2.6 Write a program for a projectile which moves through a fluid with fluid resistance proportional to the square of the velocity. Compare the range of the trajectory with the one calculated by the program `ProjectileAirResistance.cpp` for the parameters shown in figure 2.10.

2.7 Change the program `Lissajous.cpp` so that the user can enter a different amplitude and initial phase in each direction. Study the case where the amplitudes are the same and the phase difference

in the two directions are $\pi/4, \pi/2, \pi, -\pi$. Repeat by taking the amplitude in the y direction to be twice as much the amplitude in the x direction.

2.8 Change the program `ProjectileAirResistance.cpp`, so that it can calculate also the $k = 0$ case.

2.9 Change the program `ProjectileAirResistance.cpp` so that it can calculate the trajectory of the particle in three dimensional space. Plot the position coordinates and the velocity components as a function of time. Plot the three dimensional trajectory using `splot` in `gnuplot` and animate the trajectory using the `gnuplot` script `animate3D.gnu`.

2.10 Change the program `ChargeInB.cpp` so that it can calculate the number of full revolutions that the projected particle's position on the $x - y$ plane makes during its motion.

2.11 Change the program `box1D_1.cpp` so that it prints the number of the particle's collisions on the left wall, on the right wall and the total number of collisions to the `stdout`.

2.12 Do the same for the program `box1D_2.cpp`. Fill the table on page 115 the number of calculated collisions and comment on the results.

2.13 Run the program `box1D_1.cpp` and choose `L= 10, v0=1`. Decrease the step `dt` up to the point that the particle stops to move. For which value of `dt` this happens? Increase `v0=10,100`. Until which value of `dt` the particle moves now? Why?

2.14 Change the `float` declarations to `double` in the program `box1D_1.cpp`. Make sure that all the constants that you use become double precision (e.g. `1.0f` changes to `1.0`). Compare your results to those obtained in section 2.3.2. Repeat problem 2.13. What do you observe?

2.15 Change the program `box1D_1.cpp` so that you can study non elastic collisions $v' = -ev$, $0 < e \leq 1$ with the walls.

2.16 Change the program `box2D_1.cpp` so that you can study inelastic collisions with the walls, such that $v'_x = -ev_x$, $v'_y = -ev_y$, $0 < e \leq 1$.

2.17 Use the method of calculating time in the programs `box1D_4.cpp` and `box1D_5.cpp` in order to produce the results in figure 2.21.

2.18 Particle falls freely moving in the vertical direction. It starts with zero velocity at height h. Upon reaching the ground, it bounces inelastically such that $v'_y = -ev_y$ with $0 < e \leq 1$ a parameter. Write the necessary program in order to study numerically the particle's motion and study the cases $e = 0.1, 0.5, 0.9, 1.0$.

2.19 Generalize the program of the previous problem so that you can study the case $\vec{v}_0 = v_{0x}\,\hat{x}$. Animate the calculated trajectories.

2.20 Study the motion of a particle moving inside the box of figure 2.30. Count the number of collisions of the particle with the walls before it leaves the box.

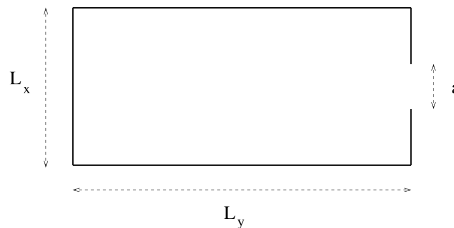

Figure 2.30: Problem 2.20.

2.21 Study the motion of the point particle on the "billiard table" of figure 2.31. Count the number of collisions with the walls before the particle enters into a hole. The program should print from which hole the particle left the table.

2.22 Write a program in order to study the motion of a particle in the box of figure 2.32. At the center of the box there is a disk on which the particle bounces off elastically (Hint: use the routine `reflectVonCircle` of the program `Cylinder3D.cpp`).

2.23 In the box of the previous problem, put four disks on which the particle bounces of elastically like in figure 2.33.

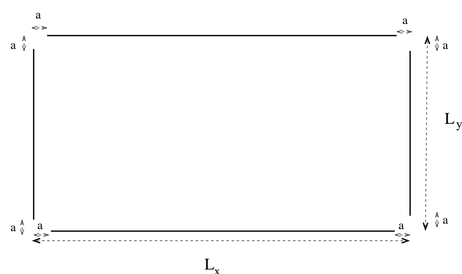

Figure 2.31: Problem 2.21.

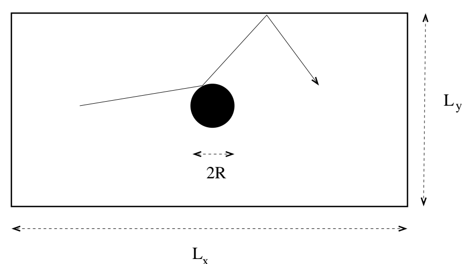

Figure 2.32: Problem 2.22.

2.24 Consider the arrangement of figure 2.34. Each time the particle bounces elastically off a circle, the circle disappears. The game is over successfully if all the circles vanish. Each time the particle bounces off on the wall to the left, you lose a point. Try to find trajectories that minimize the number of lost points.

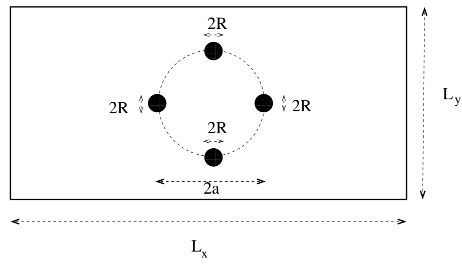

Figure 2.33: Problem 2.23.

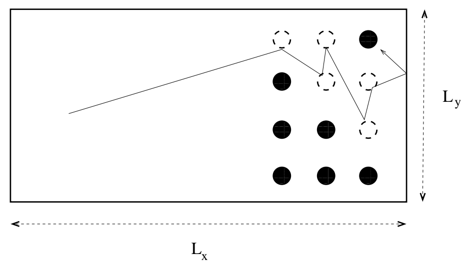

Figure 2.34: Problem 2.24.

Chapter 3

Logistic Map

Nonlinear differential equations model interesting dynamical systems in physics, biology and other branches of science. In this chapter we perform a numerical study of the discrete logistic map as a "simple mathematical model with complex dynamical properties" [23] similar to the ones encountered in more complicated and interesting dynamical systems. For certain values of the parameter of the map, one finds chaotic behavior giving us an opportunity to touch on this very interesting topic with important consequences in physical phenomena. Chaotic evolution restricts out ability for *useful* predictions in an otherwise fully deterministic dynamical system: measurements using slightly different initial conditions result in a distribution which is indistinguishable from the distribution coming from sampling a random process. This scientific field is huge and active and we refer the reader to the bibliography for a more complete introduction [23, 24, 25, 26, 27, 28, 29, 40].

3.1 Introduction

The most celebrated application of the logistic map comes from the study of population growth in biology. One considers populations which reproduce at fixed time intervals and whose generations do not overlap.

The simplest (and most naive) model is the one that makes the reasonable assumption that the rate of population growth $dP(t)/dt$ of a

population $P(t)$ is proportional to the current population:

$$\frac{dP(t)}{dt} = kP(t)\,. \tag{3.1}$$

The general solution of the above equation is $P(t) = P(0)\mathrm{e}^{kt}$ showing an *exponential* population growth for $k > 0$ an decline for $k < 0$. It is obvious that this model is reasonable as long as the population is small enough so that the interaction with its environment (adequate food, diseases, predators etc) can be neglected. The simplest model that takes into account some of the factors of the interaction with the environment (e.g. starvation) is obtained by the introduction of a simple non linear term in the equation so that

$$\frac{dP(t)}{dt} = kP(t)(1 - bP(t))\,. \tag{3.2}$$

The parameter k gives the maximum growth rate of the population and b controls the ability of the species to maintain a certain population level. The equation (3.2) can be discretized in time by assuming that each generation reproduces every δt and that the n-th generation has population $P_n = P(t_n)$ where $t_n = t_0 + (n-1)\delta t$. Then $P(t_{n+1}) \approx P(t_n) + \delta t P'(t_n)$ and equation (3.1) becomes

$$P_{n+1} = rP_n\,, \tag{3.3}$$

where $r = 1 + k\delta t$. The solutions of the above equation are well approximated by $P_n \sim P_0\mathrm{e}^{kt_n} \propto \mathrm{e}^{(r-1)n}$ so that we have population growth when $r > 1$ and decline when $r < 1$. Equation (3.2) can be discretized as follows:

$$P_{n+1} = P_n(r - bP_n)\,. \tag{3.4}$$

Defining $x_n = (b/r)P_n$ we obtain the logistic map

$$x_{n+1} = rx_n(1 - x_n)\,. \tag{3.5}$$

We define the functions

$$f(x) = rx(1-x), \qquad F(x,r) = rx(1-x) \tag{3.6}$$

(their only difference is that, in the first one, r is considered as a given parameter), so that

$$x_{n+1} = f(x_n) = f^{(2)}(x_{n-1}) = \ldots = f^{(n)}(x_1) = f^{(n+1)}(x_0)\,, \tag{3.7}$$

where we use the notation $f^{(1)}(x) = f(x)$, $f^{(2)}(x) = f(f(x))$, $f^{(3)}(x) = f(f(f(x)))$, ... for function composition. In what follows, the derivative of f will be useful:

$$f'(x) = \frac{\partial F(x,r)}{\partial x} = r(1-2x)\,. \tag{3.8}$$

Since we interpret x_n to be the fraction of the population with respect to its maximum value, we should have $0 \leq x_n \leq 1$ for each[1] n. The function $f(x)$ has one global maximum for $x = 1/2$ which is equal to $f(1/2) = r/4$. Therefore, if $r > 4$, then $f(1/2) > 1$, which for an appropriate choice of x_0 will lead to $x_{n+1} = f(x_n) > 1$ for some value of n. Therefore, the interval of values of r which is of interest for our model is

$$0 < r \leq 4\,. \tag{3.9}$$

The logistic map (3.5) may be viewed as a finite difference equation and it is a one step inductive relation. Given an initial value x_0, a sequence of values $\{x_0,\, x_1,\, \ldots,\, x_n,\, \ldots\}$ is produced. This will be referred[2] to as the *trajectory* of x_0. In the following sections we will study the properties of these trajectories as a function of the parameter r.

The solutions of the logistic map are not known except in special cases. For $r = 2$ we have

$$x_n = \frac{1}{2}\left(1 - (1-x_0)^{2n}\right)\,, \tag{3.10}$$

and for[3] $r = 4$

$$x_n = \sin^2(2^n \pi\theta)\,, \qquad \theta = \frac{1}{\pi}\sin^{-1}\sqrt{x_0}\,. \tag{3.11}$$

For $r = 2$, $\lim_{n\to\infty} x_n = 1/2$ whereas for $r = 4$ we have periodic trajectories resulting in rational θ and non periodic resulting in irrational θ. For other values of r we have to resort to a numerical computation of the trajectories of the logistic map.

---

[1]Note that if $x_n > 1$ then $x_{n+1} < 0$, so that if we want $x_n \geq 0$ for each n, then we should have $x_n \leq 1$ for each n.

[2]In the bibliography, the term "splinter of x_0" is frequently used.

[3]E. Schröder, "Über iterierte Funktionen", Math. Ann. **3** (1870) 296; E. Lorenz, "The problem of deducing the climate from the governing equations", Tellus **16** (1964) 1

3.2 Fixed Points and 2^n Cycles

It is obvious that if the point x^* is a solution of the equation $x = f(x)$, then $x_n = x^* \Rightarrow x_{n+k} = x^*$ for every $k \geq 0$. For the function $f(x) = rx(1-x)$ we have two solutions

$$x_1^* = 0 \qquad \text{and} \qquad x_2^* = 1 - 1/r\,. \tag{3.12}$$

We will see that for appropriate values of r, these solutions are *attractors* of most of the trajectories. This means that for a range of values for the initial point $0 \leq x_0 \leq 1$, the sequence $\{x_n\}$ approaches asymptotically one of these points as $n \to \infty$. Obviously the (measure zero) sets of initial values $\{x_0\} = \{x_1^*\}$ and $\{x_0\} = \{x_2^*\}$ result in trajectories attracted by x_1^* and x_2^* respectively. In order to determine which one of the two values is preferred, we need to study the *stability* of the fixed points x_1^* and x_2^*. For this, assume that for some value of n, x_n is infinitesimally close to the fixed point x^* so that

$$\begin{aligned} x_n &= x^* + \epsilon_n \\ x_{n+1} &= x^* + \epsilon_{n+1}\,. \end{aligned} \tag{3.13}$$

Since

$$x_{n+1} = f(x_n) = f(x^* + \epsilon_n) \approx f(x^*) + \epsilon_n f'(x^*) = x^* + \epsilon_n f'(x^*)\,, \tag{3.14}$$

where we used the Taylor expansion of the analytic function $f(x^* + \epsilon_n)$ about x^* and the relation $x^* = f(x^*)$, we have that $\epsilon_{n+1} = \epsilon_n f'(x^*)$. Then we obtain

$$\left|\frac{\epsilon_{n+1}}{\epsilon_n}\right| = |f'(x^*)|\ . \tag{3.15}$$

Therefore, if $|f'(x^*)| < 1$ we obtain $\lim_{n\to\infty} \epsilon_n = 0$ and the fixed point x^* is *stable*: the sequence $\{x_{n+k}\}$ approaches x^* asymptotically. If $|f'(x^*)| > 1$ then the sequence $\{x_{n+k}\}$ deviates away from x^* and the fixed point is *unstable*. The limiting case $|f'(x^*)| = 1$ should be studied separately and it indicates a change in the stability properties of the fixed point. In the following discussion, these points will be shown to be bifurcation points.

For the function $f(x) = rx(1-x)$ with $f'(x) = r(1-2x)$ we have that $f'(0) = r$ and $f'(1-1/r) = 2 - r$. Therefore, if $r < 1$ the point $x_1^* = 0$ is an attractor, whereas the point $x_2^* = 1 - 1/r < 0$ is irrelevant. When

$r > 1$, the point $x_1^* = 0$ results in $|f'(x_1^*)| = r > 1$, therefore x_1^* is unstable. Any initial value x_0 near x_1^* deviates from it. Since for $1 < r < 3$ we have that $0 \leq |f'(x_2^*)| = |2 - r| < 1$, the point x_2^* is an attractor. Any initial value $x_0 \in (0,1)$ approaches $x_2^* = 1 - 1/r$. When $r = r_c^{(1)} = 1$ we have the limiting case $x_1^* = x_2^* = 0$ and we say that at the critical value $r_c^{(1)} = 1$ the fixed point x_1^* *bifurcates* to the two fixed points x_1^* and x_2^*.

As r increases, the fixed points continue to bifurcate. Indeed, when $r = r_c^{(2)} = 3$ we have that $f'(x_2^*) = 2 - r = -1$ and for $r > r_c^{(2)}$ the point x_2^* becomes unstable. Consider the solution of the equation $x = f^{(2)}(x)$. If $0 < x^* < 1$ is one of its solutions and for some n we have that $x_n = x^*$, then $x_{n+2} = x_{n+4} = \ldots = x_{n+2k} = \ldots = x^*$ and $x_{n+1} = x_{n+3} = \ldots = x_{n+2k+1} = \ldots = f(x^*)$ (therefore $f(x^*)$ is also a solution). If $0 < x_3^* < x_4^* < 1$ are two such *different* solutions with $x_3^* = f(x_4^*)$, $x_4^* = f(x_3^*)$, then the trajectory is periodic with period 2. The points x_3^*, x_4^* are such that they are *real* solutions of the equation

$$f^{(2)}(x) = r^2 x(1-x)(1 - rx(1-x)) = x\,, \tag{3.16}$$

and at the same time they are *not* the solutions $x_1^* = 0$ $x_2^* = 1 - 1/r$ of the equation[4] $x = f^{(2)}(x)$, the polynomial above can be written in the form (see [24] for more details)

$$x\left(x - \left(1 - \frac{1}{r}\right)\right)(Ax^2 + Bx + C) = 0\,. \tag{3.17}$$

By expanding the polynomials (3.16), (3.17) and comparing their coefficients we conclude that $A = -r^3$, $B = r^2(r+1)$ and $C = -r(r+1)$. The roots of the trinomial in (3.17) are determined by the discriminant $\Delta = r^2(r+1)(r-3)$. For the values of r of interest ($1 < r \leq 4$), the discriminant becomes positive when $r > r_c^{(2)} = 3$ and we have two different solutions

$$x_\alpha^* = ((r+1) \mp \sqrt{r^2 - 2r - 3})/(2r) \qquad \alpha = 3, 4\,. \tag{3.18}$$

When $r = r_c^{(2)}$ we have one double root, therefore a unique fixed point.

The study of the stability of the solutions of $x = f^{(2)}(x)$ requires the same steps that led to the equation (3.15) and we determine if the

[4] Because, if $x^* = f(x^*) \Rightarrow f^{(2)}(x^*) = f(f(x^*)) = f(x^*) = x^*$ etc, the point x^* is also a solution of $x^* = f^{(n)}(x^*)$.

absolute value of $f^{(2)\prime}(x)$ is greater, less or equal to one. By noting that[5] $f^{(2)\prime}(x_3) = f^{(2)\prime}(x_4) = f'(x_3)f'(x_4) = -r^2 + 2r + 4$, we see that for $r = r_c^{(2)} = 3$, $f^{(2)\prime}(x_3^*) = f^{(2)\prime}(x_4^*) = 1$ and for $r = r_c^{(3)} = 1 + \sqrt{6} \approx 3.4495$, $f^{(2)\prime}(x_3) = f^{(2)\prime}(x_4) = -1$. For the intermediate values $3 < r < 1 + \sqrt{6}$ the derivatives $|f^{(2)\prime}(x_\alpha^*)| < 1$ for $\alpha = 3, 4$. Therefore, these points are stable solutions of $x = f^{(2)}(x)$ and the points x_1^*, x_2^* bifurcate to x_α^*, $\alpha = 1, 2, 3, 4$ for $r = r_c^{(2)} = 3$. Almost all trajectories with initial points in the interval $[0, 1]$ are attracted by the periodic trajectory with period 2, the "2-cycle" $\{x_3^*, x_4^*\}$.

Using similar arguments we find that the fixed points x_α^*, $\alpha = 1, 2, 3, 4$ bifurcate to the eight fixed points x_α^*, $\alpha = 1, \ldots, 8$ when $r = r_c^{(3)} = 1 + \sqrt{6}$. These are real solutions of the equation that gives the 4-cycle $x = f^{(4)}(x)$. For $r_c^{(3)} < r < r_c^{(4)} \approx 3.5441$, the points x_α^*, $\alpha = 5, \ldots, 8$ are a stable 4-cycle which is an attractor of almost all trajectories of the logistic map[6]. Similarly, for $r_c^{(4)} < r < r_c^{(5)}$ the 16 fixed points of the equation $x = f^{(8)}(x)$ give a stable 8-cycle, for $r_c^{(5)} < r < r_c^{(6)}$ a stable 16-cycle etc[7]. This is the phenomenon which is called *period doubling* which continues ad infinitum. The points $r_c^{(n)}$ are getting closer to each other as n increases so that $\lim_{n\to\infty} r_c^{(n)} = r_c \approx 3.56994567$. As we will see, r_c marks the onset of the non-periodic, chaotic behavior of the trajectories of the logistic map.

Computing the bifurcation points becomes quickly intractable and we have to resort to a numerical computation of their values. Initially we will write a program that computes trajectories of the logistic map for chosen values of r and x_0. The program can be found in the file `logistic.cpp` and is listed below:

```
#include <iostream>
#include <fstream>
```

[5]The chain rule $dh(g(x))/dx = h'(g(x))g'(x)$ gives that $f^{(2)\prime}(x_3^*) = df(f(x_3^*))/dx = f'(f(x_3^*))f'(x_3^*) = f'(x_4^*)f'(x_3^*)$ and similarly for $f^{(2)\prime}(x_4^*)$. We can prove by induction that for the n solutions $x_{n+1}^*, x_{n+2}^*, \ldots, x_{2n}^*$ that belong to the n-cycle of the equation $x = f^{(n)}(x)$ we have that $f^{(n)\prime}(x_{n+i}) = f'(x_{n+1})\ f'(x_{n+2}) \ldots f'(x_{2n})$ for every $i = 1, \ldots, n$.

[6]The points x_α^*, $\alpha = 1, \ldots, 4$ are unstable fixed points and 2-cycle.

[7]Generally, for $r_c^{(n)} < r < r_c^{(n+1)} < r_c \approx 3.56994567$ we have 2^n fixed points of the equation $x = f^{(2^{n-1})}(x)$ and stable 2^{n-1}-cycles, which are attractors of almost all trajectories.

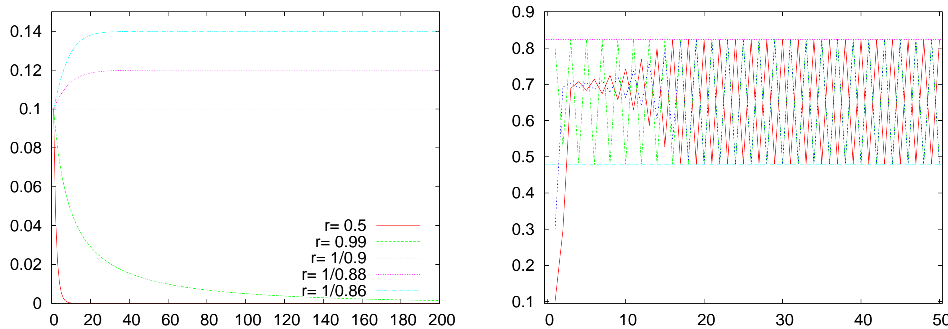

Figure 3.1: (Left) Some trajectories of the logistic map with $x_0 = 0.1$ and various values of r. We can see the first bifurcation for $r_c^{(1)} = 1$ from $x_1^* = 0$ to $x_2^* = 1 - 1/r$. (Right) Trajectories of the logistic map for $r_c^{(2)} < r = 3.5 < r_c^{(3)}$. The three curves start from three different initial points. After a transient period, depending on the initial point, one obtains a periodic trajectory which is a 2-cycle. The horizontal lines are the expected values $x_{3,4}^* = ((r+1) \mp \sqrt{r^2 - 2r - 3})/(2r)$ (see text).

```
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;

int main(){
  int    NSTEPS,i;
  double r,x0,x1;
  string buf;
  // ------ Input:
  cout << "# Enter NSTEPS, r, x0:\n";
  cin  >> NSTEPS   >> r >> x0;    getline(cin,buf);
  cout << "# NSTEPS = " << NSTEPS << endl;
  cout << "# r      = " << r      << endl;
  cout << "# x0     = " << x0     << endl;
  // ------ Initialize:
  ofstream myfile("log.dat");
  myfile.precision(17);
  // ------ Calculate:
  myfile << 0 << x0;
  for(i=1;i<=NSTEPS;i++){
    x1 = r * x0 * (1.0-x0);
    myfile << i << " " << x1 << "\n";
    x0 = x1;
```

```
  }
  myfile.close();
}//main()
```

The program is compiled and run using the commands:

```
> g++ logistic.cpp -o l
> echo "100 0.5 0.1" | ./l
```

The command `echo` prints to the `stdout` the values of the parameters `NSTEPS=100, r=0.5` and `x0=0.1`. Its `stdout` is redirected to the `stdin` of the command `./l` by using a pipe via the symbol `|`, from which the program reads their value and uses them in the calculation. The results can be found in two columns in the file `log.dat` and can be plotted using `gnuplot`. The plots are put in figure 3.1 and we can see the first two bifurcations when r goes past the values $r_c^{(1)}$ and $r_c^{(2)}$. Similarly, we can study trajectories which are 2^n-cycles when r crosses the values $r_c^{(n-1)}$.

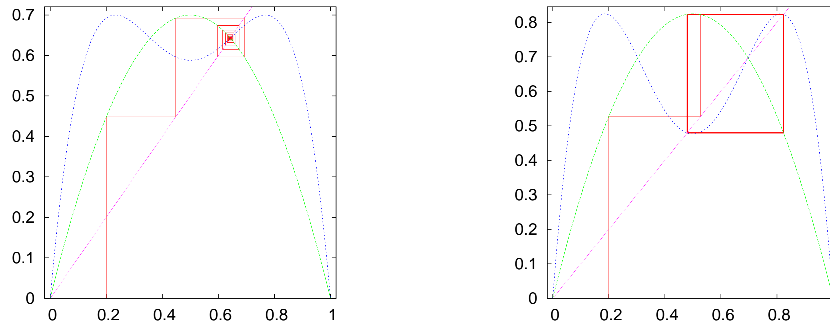

Figure 3.2: Cobweb plots of the logistic map for $r = 2.8$ and 3.3. (Left) The left plot is an example of a fixed point $x^* = f(x^*)$. The green line is $y = f(x)$ and the blue line is $y = f^{(2)}(x)$. The trajectory ends at the unique non zero intersection of the diagonal and $y = f(x)$ which is $x_2^* = 1 - 1/r$. The trajectory intersects the curve $y = f^{(2)}(x)$ at the same point. $y = f^{(2)}(x)$ does not intersect the diagonal anywhere else. (Right) The right plot shows an example of a 2-cycle. $y = f^{(2)}(x)$ intersects the diagonal at two additional points determined by x_3^* and x_4^*. The trajectory ends up on the orthogonal (x_3^*, x_3^*), (x_4^*, x_3^*), (x_4^*, x_4^*), (x_3^*, x_4^*).

Another way to depict the 2-cycles is by constructing the cobweb plots: We start from the point $(x_0, 0)$ and we calculate the point (x_0, x_1), where

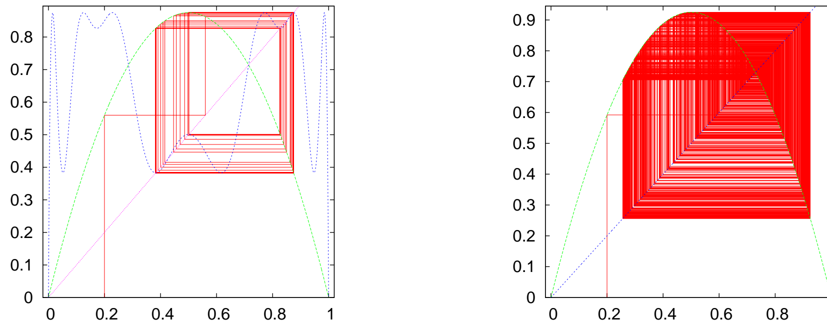

Figure 3.3: (Left) A 4-cycle for $r = 3.5$. The blue curve is $y = f^{(4)}(x)$ which intersects the diagonal at four points determined by x_α, $\alpha = 5, 6, 7, 8$. The four cycle passes through these points. (Right) a non periodic orbit for $r = 3.7$ when the system exhibits chaotic behavior.

$x_1 = f(x_0)$. This point belongs on the curve $y = f(x)$. The point (x_0, x_1) is then projected on the diagonal $y = x$ and we obtain the point (x_1, x_1). We repeat n times obtaining the points (x_n, x_{n+1}) and (x_{n+1}, x_{n+1}) on $y = f(x)$ and $y = x$ respectively. The fixed points $x^* = f(x^*)$ are at the intersections of these curves and, if they are attractors, the trajectories will converge on them. If we have a 2^n-cycle, we will observe a periodic trajectory going through points which are solutions to the equation $x = f^{(2^n)}(x)$. This exercise can be done by using the following program, which can be found in the file `logistic1.cpp`:

```
//============================================================
// Discrete Logistic Map: Cobweb diagram
// Map the trajectory in 2d space (plane)
//============================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;

int main(){
  int    NSTEPS,i;
  double r,x0,x1;
```

```
  string buf;
  // ------ Input:
  cout << "# Enter NSTEPS, r, x0:\n";
  cin  >> NSTEPS   >> r  >> x0;        getline(cin,buf);
  cout << "# NSTEPS = "  << NSTEPS     << endl;
  cout << "# r      = "  << r          << endl;
  cout << "# x0     = "  << x0         << endl;
  // ------ Initialize:
  ofstream myfile("trj.dat");
  myfile.precision(17);
  // ------ Calculate:
  myfile   << 0     << " " << x0 << " " << 0  << '\n';
  for(i=1;i<=NSTEPS;i++){
    x1 = r * x0 * (1.0-x0);
    myfile << 2*i-3 << " " << x0 << " " << x1 << '\n';
    myfile << 2*i-2 << " " << x1 << " " << x1 << '\n';
    x0 = x1;
  }
  myfile.close();
}//main()
```

Compiling and running this program is done exactly as in the case of the program in `logistic.cpp`. We can plot the results using `gnuplot`. The plot in figure 3.2 can be constructed using the commands:

```
gnuplot> set size square
gnuplot> f(x) = r*x*(1.0-x)
gnuplot> r = 3.3
gnuplot> plot "<echo 50 3.3 0.2|./l;cat trj.dat" using 2:3 w l
gnuplot> replot f(x) ,f(f(x)),x
```

The `plot` command shown above, runs the program exactly as it is done on the command line. This is accomplished by using the symbol `<`, which reads the plot from the `stdout` of the command `"echo 50 3.3 0.2|./l;cat trj.dat"`. Only the second command `"echo trj.dat"` writes to the `stdout`, therefore the plot is constructed from the contents of the file `trj.dat`. The following line adds the plots of the functions $f(x)$, $f^{(2)}(x) = f(f(x))$ and of the diagonal $y = x$. Figures 3.2 and 3.3 show examples of attractors which are fixed points, 2-cycles and 4-cycles. An example of a non periodic trajectory is also shown, which exhibits chaotic behavior which can happen when $r > r_c \approx 3.56994567$.

3.3 Bifurcation Diagrams

The bifurcations of the fixed points of the logistic map discussed in the previous section can be conveniently shown on the "bifurcation diagram". We remind to the reader that the first bifurcations happen at the critical values of r

$$r_c^{(1)} < r_c^{(2)} < r_c^{(3)} < \ldots < r_c^{(n)} < \ldots < r_c\,, \tag{3.19}$$

where $r_c^{(1)} = 1$, $r_c^{(2)} = 3$, $r_c^{(3)} = 1 + \sqrt{6}$ and $r_c = \lim_{n\to\infty} r_c^{(n)} \approx 3.56994567$. For $r_c^{(n)} < r < r_c^{(n+1)}$ we have 2^n fixed points x^*_α, $\alpha = 1, 2, ..., 2^n$ of $x = f^{(2^n)}(x)$. By plotting these points $x^*_\alpha(r)$ as a function of r we construct the bifurcation diagram. These can be calculated numerically by using the program `bifurcate.cpp`. In this program, the user selects the values of r that she needs to study and for each one of them the program records the point of the 2^{n-1}-cycles[8] $x^*_\alpha(r)$, $\alpha = 2^{n-1}+1, 2^{n-1}+2, \ldots, 2^n$. This is easily done by computing the logistic map several times until we are sure that the trajectories reach the stable state. The parameter `NTRANS` in the program determines the number of points that we throw away, which should contain all the *transient* behavior. After `NTRANS` steps, the program records `NSTEPS` points, where `NSTEPS` should be large enough to cover all the points of the 2^{n-1}-cycles or depict a dense enough set of values of the non periodic orbits. The program is listed below:

```
//============================================================
// Bifurcation Diagram of the Logistic Map
//============================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;

int main(){
  const double rmin   = 2.5;
  const double rmax   = 4.0;
  const double NTRANS = 500;    //Number of discarted steps
```

[8]If we want to be more precise, the bifurcation diagram contains also the unstable points. What we really construct is the orbit diagram which contains only the stable points.

```
  const double NSTEPS = 100;   //Number of recorded  steps
  const double RSTEPS = 2000;  //Number of values of r
  int i;
  double r,dr,x0,x1;

  // ------ Initialize:
  dr      = (rmax-rmin)/RSTEPS; //Increment in r
  ofstream myfile("bif.dat");
  myfile.precision(17);
  // ------ Calculate:
  r       = rmin;
  while( r <= rmax){
    x0    = 0.5;
    // ---- Transient steps: skip
    for(i=1;i<=NTRANS;i++){
      x1 = r * x0 * (1.0-x0);
      x0 = x1;
    }
    for(i=1;i<=NSTEPS;i++){
      x1 = r * x0 * (1.0-x0);
      myfile << r << " " << x1 << '\n';
      x0 = x1;
    }
    r += dr;
  }//while( r <= rmax)
  myfile.close();
}//main()
```

The program can be compiled and run using the commands:

```
> g++ bifurcate.cpp -o b
> ./b;
```

The left plot of figure 3.4 can be constructed by the `gnuplot` commands:

```
gnuplot> plot "bif.dat" with dots
```

We observe the fixed points and the 2^n-cycles for $r < r_c$. When r goes past r_c, the trajectories become non-periodic and exhibit chaotic behavior. Chaotic behavior will be discussed more extensively in the next section. For the time being, we note that if we measure the distance between the points $\Delta r^{(n)} = r_c^{(n+1)} - r_c^{(n)}$, we find that it decreases constantly with n so

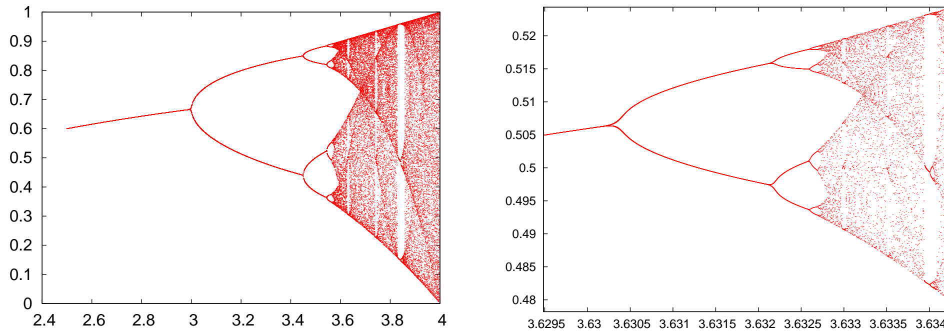

Figure 3.4: (Left) The bifurcation diagram computed by the program `bifurcate.cpp` for $2.5 < r < 4$. Notice the first bifurcation points followed by intervals of chaotic, non-periodic orbits interrupted by intermissions of stable periodic trajectories. The chaotic trajectories take values in subsets of the interval $(0, 1)$. For $r = 4$ they take values within the whole $(0, 1)$. One can see that for $r = 1 + \sqrt{8} \approx 3.8284$ we obtain a 3-cycle which subsequently bifurcates to $3 \cdot 2^n$-cycles. (Right) The diagram on the left is magnified in a range of r showing the self-similarity of the diagram at all scales.

that

$$\lim_{n\to\infty} \frac{\Delta r^{(n)}}{\Delta r^{(n+1)}} = \delta \approx 4.669\,201\,609\,, \tag{3.20}$$

where δ is the Feigenbaum constant. An additional constant α, defined by the quotient of the separation of adjacent elements Δw_n of period doubled attractors from one double to the next Δw_{n+1}, is

$$\lim_{n\to\infty} \frac{\Delta w_n}{\Delta w_{n+1}} = \alpha \approx 2.502\,907\,875\,. \tag{3.21}$$

It is also interesting to note the appearance of a 3-cycle right after $r = 1 + \sqrt{8} \approx 3.8284 > r_c$! By using the theorem of Sharkovskii, Li and Yorke[9] showed that any one dimensional system has 3-cycles, therefore it will have cycles of any length and chaotic trajectories. The stability of the 3-cycle can be studied from the solutions of $x = f^{(3)}(x)$ in exactly the same way that we did in equations (3.16) and (3.17) (see [24] for details). Figure 3.5 magnifies a branch of the 3-cycle. By magnifying different regions in the bifurcation plot, as shown in the right plot of figure 3.4, we find similar shapes to the branching of the 3-cycle. Figure 3.4 shows that

[9] T.Y. Li, J.A. Yorke, "Period Three Implies Chaos", American Mathematical Monthly 82 (1975) 985.

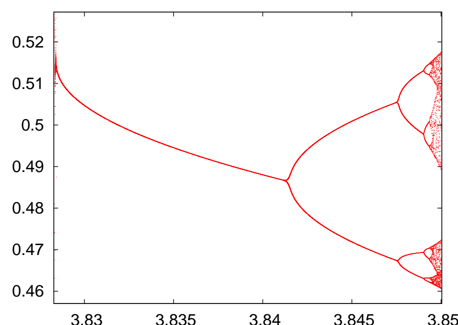

Figure 3.5: Magnification of one of the three branches of the 3-cycle for $r > 1 + \sqrt{8}$. To the left, we observe the temporary halt of the chaotic behavior of the trajectory, which comes back as shown in the plot to the right after an intermission of stable periodic trajectories.

between intervals of chaotic behavior we obtain "windows" of periodic trajectories. These are infinite but countable. It is also quite interesting to note that if we magnify a branch withing these windows, we obtain a diagram that is similar to the whole diagram! We say that the bifurcation diagram exhibits self similarity. There are more interesting properties of the bifurcation diagram and we refer the reader to the bibliography for a more complete exposition.

We close this section by mentioning that the qualitative properties of the bifurcation diagram are the same for a whole class of functions. Feigenbaum discovered that if one takes any function that is concave and has a unique global maximum, its bifurcation diagram behaves qualitatively the same way as that of the logistic map. Examples of such functions[10] studied in the literature are $g(x) = x\mathrm{e}^{r(1-x)}$, $u(x) = r\sin(\pi x)$ and $w(x) = b - x^2$. The constants δ and α of equations (3.20) and (3.21) are the same of all these mappings. The functions that result in chaotic behavior are studied extensively in the literature and you can find a list of those in [30].

[10] The function $x\exp(r(1-x))$ has been used as a model for populations whose large density is restricted by epidemics. The populations are always positive independently of the (positive) initial conditions and the value of r.

3.4 The Newton-Raphson Method

In order to determine the bifurcation points, one has to solve the nonlinear, polynomial, algebraic equations $x = f^{(n)}(x)$ and $f^{(n)\prime}(x) = -1$. For this reason, one has to use an approximate numerical calculation of the roots, and the simple Newton-Raphson method will prove to be a good choice.

Newton-Raphson's method uses an initial guess x_0 for the solution of the equation $g(x) = 0$ and computes a sequence of points $x_1, x_2, \ldots, x_n, x_{n+1}, \ldots$ that presumably converges to one of the roots of the equation. The computation stops at a finite n, when we decide that the desired level of accuracy has been achieved. In order to understand how it works, we assume that $g(x)$ is an analytic function for all the values of x used in the computation. Then, by Taylor expanding around x_n we obtain

$$g(x_{n+1}) = g(x_n) + (x_{n+1} - x_n)g'(x) + \ldots . \tag{3.22}$$

If we wish to have $g(x_{n+1}) \approx 0$, we choose

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} . \tag{3.23}$$

The equation above gives the Newton-Raphson method for one equation $g(x) = 0$ of one variable x. Different choices for x_0 will possibly lead to different roots. When $g'(x)$, $g''(x)$ are non zero at the root and $g'''(x)$ is bounded, the convergence of the method is quadratic with the number of iterations. This means that there is a neighborhood of the root α such that the distance $\Delta x_{n+1} = x_{n+1} - \alpha$ is $\Delta x_{n+1} \propto (\Delta x_n)^2$. If the root α has multiplicity larger than 1, convergence is slower. The proofs of these statements are simple and can be found in [31].

The Newton-Raphson method is simple to program and, most of the times, sufficient for the solution of many problems. In the general case it works well only close enough to a root. We should also keep in mind that there are simple reasons for the method to fail. For example, when $g'(x_n) = 0$ for some n, the method stops. For functions that tend to 0 as $x \to \pm\infty$, it is easy to make a bad choice for x_0 that does not lead to convergence to a root. Sometimes it is a good idea to combine the Newton-Raphson method with the *bisection* method. When the derivative $g'(x)$ diverges at the root we might get into trouble. For example, the

equation $|x|^\nu = 0$ with $0 < \nu < 1/2$, does not lead to a convergent sequence. In some cases, we might enter into non-convergent cycles [8]. For some functions the basin of attraction of a root (the values of x_0 that will converge to the root) can be tiny. See problem 13.

As a test case of our program, consider the equation

$$\epsilon \tan \epsilon = \sqrt{\rho^2 - \epsilon^2} \tag{3.24}$$

which results from the solution of Schrödinger's equation for the energy spectrum of a quantum mechanical particle of mass m in a one dimensional potential well of depth V_0 and width L. The parameters $\epsilon = \sqrt{mL^2E/(2\hbar)}$ and $\rho = \sqrt{mL^2V_0/(2\hbar)}$. Given ρ, we solve for ϵ which gives the energy E. The function $g(x)$ and its derivative $g'(x)$ are

$$\begin{aligned} g(x) &= x\tan x - \sqrt{\rho^2 - x^2} \\ g'(x) &= \frac{x}{\sqrt{\rho^2 - x^2}} + \frac{x}{\cos^2 x} + \tan x\,. \end{aligned} \tag{3.25}$$

The program of the Newton-Raphson method for solving the equation $g(x) = 0$ can be found in the file `nr.cpp`:

```
//===========================================================
//Newton Raphson of function of one variable
//===========================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;

int main(){
  const double rho  = 15.0;
  const double eps  = 1.0e-6;
  const int    NMAX = 1000;
  double x0, x1, err, g, gp;
  int    i;
  string buf;
  // ------ Input:
  cout << "# Enter x0:\n";
  cin  >> x0;    getline(cin,buf);
  err = 1.0;
```

```
  cout << "iter             x                        error     \n";
  cout << "-----------------------------------------------------\n";
  cout << 0   << " "   << x0   << " "   <<         err  << '\n';

  cout.precision(17);
  for(i=1;i<=NMAX;i++){
    //value of function g(x):
    g   = x0*tan(x0)-sqrt(rho*rho-x0*x0);
    //value of the derivative g'(x):
    gp  = x0/sqrt(rho*rho-x0*x0)+x0/(cos(x0)*cos(x0))+tan(x0);
    x1  = x0 - g/gp;
    err = abs(x1-x0);
    cout  << i << " " << x1 << " " << err << '\n';
    if(err < eps) break;
    x0  = x1;
  }
}//main()
```

In the program listed above, the user is asked to set the initial point x_0. We fix $\rho =$ `rho` $= 15$. It is instructive to make the plot of the left and right hand sides of (3.24) and make a graphical determination of the roots from their intersections. Then we can make appropriate choices of the initial point x_0. Using `gnuplot`, the plots are made with the commands:

```
gnuplot> g1(x) = x*tan(x)
gnuplot> g2(x) = sqrt(rho*rho-x*x)
gnuplot> plot [0:20][0:20]  g1(x), g2(x)
```

The compilation and running of the program can be done as follows:

```
> g++ nr.cpp -o n
> echo "1.4"|./n
# Enter x0:
iter            x                         error
-------------------------------------------------

0 1.4                  1
1 1.5254292024457967 0.12542920244579681
2 1.5009739120496131 0.024455290396183666
3 1.48072070172022   0.020253210329393099
4 1.4731630533073483 0.0075576484128716537
5 1.4724779331237687 0.00068512018357957949
6 1.4724731072313519 4.8258924167932093e-06
7 1.4724731069952235 2.3612845012621619e-10
```

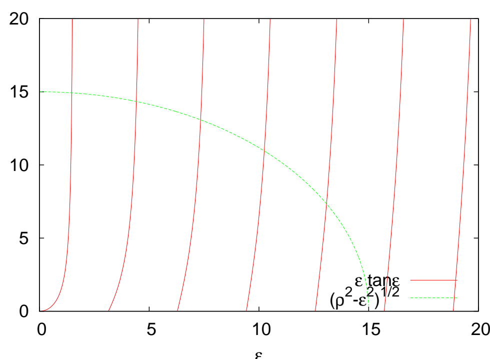

Figure 3.6: Plots of the right and left hand sides of equation (3.24). The intersections of the curves determine the solutions of the equation and their approximate graphical estimation can serve as initial points x_0 for the Newton-Raphson method.

We conclude that one of the roots of the equation is $\epsilon \approx 1.472473107$. The reader can compute more of these roots by following these steps by herself.

The method discussed above can be easily generalized to the case of two equations. Suppose that we need to solve simultaneously two algebraic equations $g_1(x_1, x_2) = 0$ and $g_2(x_1, x_2) = 0$. In order to compute a sequence (x_{10}, x_{20}), (x_{11}, x_{21}), ..., (x_{1n}, x_{2n}), $(x_{1(n+1)}, x_{2(n+1)})$, ... that may converge to a root of the above system of equations, we Taylor expand the two functions around (x_{1n}, x_{2n})

$$
\begin{aligned}
g_1(x_{1(n+1)}, x_{2(n+1)}) &= g_1(x_{1n}, x_{2n}) + (x_{1(n+1)} - x_{1n})\frac{\partial g_1(x_{1n}, x_{2n})}{\partial x_1} \\
&+ \quad (x_{2(n+1)} - x_{2n})\frac{\partial g_1(x_{1n}, x_{2n})}{\partial x_2} + \ldots \\
g_2(x_{1(n+1)}, x_{2(n+1)}) &= g_2(x_{1n}, x_{2n}) + (x_{1(n+1)} - x_{1n})\frac{\partial g_2(x_{1n}, x_{2n})}{\partial x_1} \\
&+ \quad (x_{2(n+1)} - x_{2n})\frac{\partial g_2(x_{1n}, x_{2n})}{\partial x_2} + \ldots . \qquad (3.26)
\end{aligned}
$$

Defining $\delta x_1 = (x_{1(n+1)} - x_{1n})$ and $\delta x_2 = (x_{2(n+1)} - x_{2n})$ and setting

$g_1(x_{1(n+1)}, x_{2(n+1)}) \approx 0$, $g_2(x_{1(n+1)}, x_{2(n+1)}) \approx 0$, we obtain

$$\begin{aligned} \delta x_1 \frac{\partial g_1}{\partial x_1} + \delta x_2 \frac{\partial g_1}{\partial x_2} &= -g_1 \\ \delta x_1 \frac{\partial g_2}{\partial x_1} + \delta x_2 \frac{\partial g_2}{\partial x_2} &= -g_2 \,. \end{aligned} \tag{3.27}$$

This is a linear 2×2 system of equations

$$\begin{aligned} A_{11}\delta x_1 + A_{12}\delta x_2 &= b_1 \\ A_{21}\delta x_1 + A_{22}\delta x_2 &= b_2 \,, \end{aligned} \tag{3.28}$$

where $A_{ij} = \partial g_i / \partial x_j$ and $b_i = -g_i$, with $i, j = 1, 2$. Solving for δx_i we obtain

$$\begin{aligned} x_{1(n+1)} &= x_{1n} + \delta x_1 \\ x_{2(n+1)} &= x_{2n} + \delta x_2 \,. \end{aligned} \tag{3.29}$$

The iterations stop when δx_i become small enough.

As an example, consider the equations with $g_1(x) = 2x^2 - 3xy + y - 2$, $g_2(x) = 3x + xy + y - 1$. We have $A_{11} = 4x - 3y$, $A_{12} = 1 - 3x$, $A_{21} = 3 + y$, $A_{22} = 1 + x$. The program can be found in the file `nr2.cpp`:

```
//============================================================
//Newton Raphson of two functions of two variables
//============================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;

void solve2x2(double A[2][2],double b[2],double dx[2]);

int main(){
  const double eps  = 1.0e-6;
  const int    NMAX = 1000;
  double A[2][2],b[2],dx[2];
  double x, y, err;
  int    i;
  string buf;
```

```
  // ------ Input:
  cout << "# Enter x0,y0:\n";
  cin  >> x >> y;    getline(cin,buf);
  err = 1.0;
  cout << "iter         x              y                error     \n";
  cout << "-----------------------------------------------------------\n";
  cout << 0 << " " << x << " " << y << " " <<    err  << '\n';

  cout.precision(17);
  for(i=1;i<=NMAX;i++){
    b[0] =  -(2.0*x*x-3.0*x*y + y - 2.0); // -g1(x,y)
    b[1] =  -(3.0*x  +    x*y + y - 1.0); // -g2(x,y)
    // dg1/dx                        dg1/dy
    A[0][0] = 4.0*x-3.0*y; A[0][1] = 1.0-3.0*x;
    // dg2/dx                        dg2/dy
    A[1][0] = 3.0  +    y; A[1][1] = 1.0+    x;
    solve2x2(A,b,dx);
    x += dx[0];
    y += dx[1];
    err = 0.5*sqrt(dx[0]*dx[0]+dx[1]*dx[1]);
    cout << i << " " << x << " " << y << " " << err << endl;
    if(err < eps) break;
  }
}//main()
void solve2x2(double A[2][2],double b[2],double dx[2]){
  double num0,num1,det;

  num0   = A[1][1] * b[0]    - A[0][1] * b[1];
  num1   = A[0][0] * b[1]    - A[1][0] * b[0];
  det    = A[0][0] * A[1][1] - A[0][1] * A[1][0];
  if(det == 0.0){cerr << "solve2x2: det=0\n";exit(1);}
  dx[0]  = num0/det;
  dx[1]  = num1/det;

}//solve2x2()
```

In order to guess the region where the real roots of the systems lie, we make a 3-dimensional plot using `gnuplot`:

```
gnuplot> set isosamples 20
gnuplot> set hidden3d
gnuplot> splot 2*x**2-3*x*y+y-2,3*x+y*x+y-1,0
```

We plot the functions $g_i(x,y)$ together with the plane $x=0$. The in-

tersection of the three surfaces determine the roots we are looking for. Compiling and running the program can be done by using the commands:

```
> g++ nr2.cpp -o n
> echo 2.2 1.5 |./n
# Enter x0,y0:
iter     x            y       error
-------------------------------------------------
0   2.20000000   1.50000000 1.0000
1   0.76427104   0.26899383 0.9456
2   0.73939531  -0.68668275 0.4780
3   0.74744506  -0.71105605 1.2834e-2
4   0.74735933  -0.71083147 1.2019e-4
5   0.74735932  -0.71083145 1.2029e-8
> echo 0 1 |./n
.................
5 -0.10899022   1.48928857 4.3461e-12
> echo -5 0|./n
6 -6.13836909  -3.77845711 3.2165e-13
```

The computation above leads to the roots $(0.74735932, -0.71083145)$, $(-0.10899022, 1.48928857)$, $(-6.13836909, -3.77845711)$.

The Newton-Raphson method for many variables becomes hard quite soon: One needs to calculate the functions as well as their derivatives, which is prohibitively expensive for many problems. It is also hard to determine the roots, since the method converges satisfactorily only very close to the roots. We refer the reader to [8] for more information on how one can deal with these problems.

3.5 Calculation of the Bifurcation Points

In order to determine the bifurcation points for $r < r_c$ we will solve the algebraic equations $x = f^{(k)}(x)$ and $f^{(k)\prime}(x) = -1$. At these points, k-cycles become unstable and $2k$-cycles appear and are stable. This happens when $r = r_c^{(n)}$, where $k = 2^{n-2}$. We will look for solutions $(x^*_\alpha, r_c^{(n)})$ for $\alpha = k+1, k+2, \ldots, 2k$.

We define the functions $F(x,r) = f(x) = rx(1-x)$ and $F^{(k)}(x,r) =$

$f^{(k)}(x)$ as in equation (3.6). We will solve the algebraic equations:

$$\begin{aligned} g_1(x,r) &= x - F^{(k)}(x,r) = 0 \\ g_2(x,r) &= \frac{\partial F^{(k)}(x,r)}{\partial x} + 1 = 0\,. \end{aligned} \tag{3.30}$$

According to the discussion of the previous section, in order to calculate the roots of these equations we have to solve the linear system (3.28), where the coefficients are

$$\begin{aligned} b_1 &= -g_1(x,r) = -x + F^{(k)}(x,r) \\ b_2 &= -g_2(x,r) = -\frac{\partial F^{(k)}(x,r)}{\partial x} - 1 \\ A_{11} &= \frac{\partial g_1(x,r)}{\partial x} = 1 - \frac{\partial F^{(k)}(x,r)}{\partial x} \\ A_{12} &= \frac{\partial g_1(x,r)}{\partial r} = -\frac{\partial F^{(k)}(x,r)}{\partial r} \\ A_{21} &= \frac{\partial g_2(x,r)}{\partial x} = \frac{\partial^2 F^{(k)}(x,r)}{\partial x^2} \\ A_{22} &= \frac{\partial g_2(x,r)}{\partial r} = \frac{\partial^2 F^{(k)}(x,r)}{\partial x \partial r}\,. \end{aligned} \tag{3.31}$$

The derivatives will be calculated approximately using finite differences

$$\begin{aligned} \frac{\partial F^{(k)}(x,r)}{\partial x} &\approx \frac{F^{(k)}(x+\epsilon,r) - F^{(k)}(x-\epsilon,r)}{2\epsilon} \\ \frac{\partial F^{(k)}(x,r)}{\partial r} &\approx \frac{F^{(k)}(x,r+\epsilon) - F^{(k)}(x,r-\epsilon)}{2\epsilon}\,, \end{aligned} \tag{3.32}$$

and similarly for the second derivatives

$$
\begin{aligned}
\frac{\partial^2 F^{(k)}(x,r)}{\partial x^2} &\approx \frac{\frac{\partial F^{(k)}(x+\frac{\epsilon}{2},r)}{\partial x} - \frac{\partial F^{(k)}(x-\frac{\epsilon}{2},r)}{\partial x}}{2\frac{\epsilon}{2}} \\
&= \frac{1}{\epsilon}\left\{\frac{F^{(k)}(x+\epsilon,r)-F^{(k)}(x,r)}{\epsilon} - \frac{F^{(k)}(x,r)-F^{(k)}(x-\epsilon,r)}{\epsilon}\right\} \\
&= \frac{1}{\epsilon^2}\left\{F^{(k)}(x+\epsilon,r)-2F^{(k)}(x,r)+F^{(k)}(x-\epsilon,r)\right\} \\
\frac{\partial^2 F^{(k)}(x,r)}{\partial x \partial r} &\approx \frac{\frac{\partial F^{(k)}(x+\epsilon_x,r)}{\partial r} - \frac{\partial F^{(k)}(x-\epsilon_x,r)}{\partial r}}{2\epsilon_x} \\
&= \frac{1}{2\epsilon_x}\left\{\frac{F^{(k)}(x+\epsilon_x,r+\epsilon_r)-F^{(k)}(x+\epsilon_x,r-\epsilon_r)}{2\epsilon_r}\right. \\
&\qquad \left. - \frac{F^{(k)}(x-\epsilon_x,r+\epsilon_r)-F^{(k)}(x-\epsilon_x,r-\epsilon_r)}{2\epsilon_r}\right\} \\
&= \frac{1}{4\epsilon_x\epsilon_r}\left\{F^{(k)}(x+\epsilon_x,r+\epsilon_r)-F^{(k)}(x+\epsilon_x,r-\epsilon_r)\right. \\
&\qquad \left. -F^{(k)}(x-\epsilon_x,r+\epsilon_r)+F^{(k)}(x-\epsilon_x,r-\epsilon_r)\right\}
\end{aligned} \tag{3.33}
$$

We are now ready to write the program for the Newton-Raphson method like in the previous section. The only difference is the approximate calculation of the derivatives using the relations above and the calculation of the function $F^{(k)}(x,r)$ by a routine that will compose the function $f(x)$ k-times. The program can be found in the file `bifurcationPoints.cpp`:

```
//========================================================
//        bifurcationPoints.cpp
// Calculate bifurcation points of the discrete logistic map
// at period k by solving the condition
// g1(x,r) = x - F(k,x,r)   = 0
// g2(x,r) = dF(k,x,r)/dx+1 = 0
// determining when the Floquet multiplier bacomes 1
// F(k,x,r) iterates F(x,r) = r*x*(x-1) k times
// The equations are solved by using a Newton-Raphson method
//========================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
```

```
#include <cmath>
using namespace std;
double F      (const int& k,const double& x,const double& r);
double dFdx   (const int& k,const double& x,const double& r);
double dFdr   (const int& k,const double& x,const double& r);
double d2Fdx2 (const int& k,const double& x,const double& r);
double d2Fdrdx(const int& k,const double& x,const double& r);
void solve2x2 (double A[2][2],double b[2],double dx[2]);

int main(){
  const double tol  = 1.0e-10;
  double r0,x0;
  double A[2][2],B[2],dX[2];
  double error;
  int k,iter;
  string buf;

  // ------ Input:
  cout << "# Enter k,r0,x0:\n";
  cin  >> k >> r0 >> x0;          getline(cin,buf);
  cout << "# Period k= "              << k << endl;
  cout << "# r0= " << r0 << " x0= " << x0 << endl;
  // ------ Initialize
  error = 1.0; //initial large value of error>tol
  iter  =   0;
  cout.precision(17);
  while(error > tol){
    // ---- Calculate jacobian matrix
    A[0][0] = 1.0  -dFdx(k,x0,r0);
    A[0][1] = -dFdr     (k,x0,r0);
    A[1][0] = d2Fdx2    (k,x0,r0);
    A[1][1] = d2Fdrdx   (k,x0,r0);
    B[0]    = -x0 +    F(k,x0,r0);
    B[1]    = -dFdx     (k,x0,r0)-1.0;
    // ---- Solve a 2x2 linear system:
    solve2x2(A,B,dX);
    x0      = x0 + dX[0];
    r0      = r0 + dX[1];
    error   = 0.5*sqrt(dX[0]*dX[0]+dX[1]*dX[1]);
    iter++;
    cout  <<iter
          << " x0=  " << x0
          << " r0=  " << r0
          << " err= " << error << '\n';
  }//while(error > tol)
```

```
}//main()
//=======================================================
//Function F(k,x,r) and its derivatives
double F      (const int& k,const double& x,const double& r){
  double x0;
  int     i;
  x0  = x;
  for(i=1;i<=k;i++) x0 = r*x0*(1.0-x0);
  return x0;
}
// --------------------------------------
double dFdx   (const int& k,const double& x,const double& r){
  double eps;
  eps     = 1.0e-6*x;
  return (F(k,x+eps,r)-F(k,x-eps,r))/(2.0*eps);
}
// --------------------------------------
double dFdr   (const int& k,const double& x,const double& r){
  double eps;
  eps     = 1.0e-6*r;
  return (F(k,x,r+eps)-F(k,x,r-eps))/(2.0*eps);
}
// --------------------------------------
double d2Fdx2 (const int& k,const double& x,const double& r){
  double eps;
  eps     = 1.0e-6*x;
  return (F(k,x+eps,r)-2.0*F(k,x,r)+F(k,x-eps,r))/(eps*eps);
}
// --------------------------------------
double d2Fdrdx(const int& k,const double& x,const double& r){
  double epsx,epsr;
  epsx     = 1.0e-6*x;
  epsr     = 1.0e-6*r;
  return ( F(k,x+epsx,r+epsr)-F(k,x+epsx,r-epsr)
          -F(k,x-epsx,r+epsr)+F(k,x-epsx,r-epsr))
          /(4.0*epsx*epsr);
}
//=======================================================
void solve2x2(double A[2][2],double b[2],double dx[2]){
  double num0,num1,det;

  num0   = A[1][1] * b[0]    - A[0][1] * b[1];
  num1   = A[0][0] * b[1]    - A[1][0] * b[0];
  det    = A[0][0] * A[1][1] - A[0][1] * A[1][0];
  if(det == 0.0){cerr << "solve2x2: det=0\n";exit(1);}
```

```
  dx[0]   = num0/det;
  dx[1]   = num1/det;

}//solve2x2()
```

Compiling and running the program can be done as follows:

```
> g++ bifurcationPoints.cpp -o b
> echo 2 3.5 0.5 |./b
# Enter k,r0,x0:
# Period k=            2
# r0= 3.5000000000000 x0= 0.50000000000
1 x0= 0.4455758353187 r0= 3.38523275827 err= 6.35088e-2
2 x0= 0.4396562547624 r0= 3.45290970406 err= 3.39676e-2
3 x0= 0.4399593001407 r0= 3.44949859951 err= 1.71226e-3
4 x0= 0.4399601690333 r0= 3.44948974267 err= 4.44967e-6
5 x0= 0.4399601689937 r0= 3.44948974281 err= 7.22160e-11
> echo 2 3.5 0.85 | ./b
 .................
4 x0= 0.8499377795512 r0= 3.44948974275 err= 1.85082e-11
> echo 4 3.5 0.5 |./b
 .................
5 x0= 0.5235947861540 r0= 3.54409035953 err= 1.86318e-11
> echo 4 3.5 0.35 | ./b
 .................
5 x0= 0.3632903374118 r0= 3.54409035955 err= 5.91653e-13
```

The above listing shows the points of the 2-cycle and some of the points of the 4-cycle. It is also possible to compare the calculated value $r_c^{(3)} = 3.449490132$ with the expected one $r_c^{(3)} = 1+\sqrt{6} \approx 3.449489742$. Improving the accuracy of the calculation is left as an exercise for the reader who has to control the systematic errors of the calculations and achieve better accuracy in the computation of $r_c^{(4)}$.

3.6 Liapunov Exponents

We have seen that when $r > r_c \approx 3.56994567$, the trajectories of the logistic map become non periodic and exhibit chaotic behavior. Chaotic behavior mostly means sensitivity of the evolution of a dynamical system to the choice of initial conditions. More precisely, it means that two different trajectories constructed from infinitesimally close initial conditions,

diverge very fast from each other. This implies that there is a set of initial conditions that densely cover subintervals of $(0,1)$ whose trajectories do not approach arbitrarily close to any cycle of finite length.

Assume that two trajectories have x_0, $\tilde{x}_0$ as initial points and $\Delta x_0 = x_0 - \tilde{x}_0$. When the points x_n, $\tilde{x}_n$ have a distance $\Delta x_n = \tilde{x}_n - x_n$ that for small enough n increases exponentially with n (the "time"), i.e.

$$\Delta x_n \sim \Delta x_0 \mathrm{e}^{\lambda n}\,, \qquad \lambda > 0\,, \tag{3.34}$$

the system is most likely exhibiting chaotic behavior[11]. The exponent λ is called a *Liapunov exponent*. A useful equation for the calculation of λ is

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |f'(x_k)|\,. \tag{3.35}$$

This relation can be easily proved by considering infinitesimal $\epsilon \equiv |\Delta x_0|$ so that $\lambda = \lim_{n\to\infty}\lim_{\epsilon\to 0} \frac{1}{n}\ln|\Delta x_n|/\epsilon$. Then we obtain

$$\begin{aligned}
\tilde{x}_1 &= f(\tilde{x}_0) = f(x_0+\epsilon) \approx f(x_0) + \epsilon f'(x_0) \\
&= x_1 + \epsilon f'(x_0) \Rightarrow \\
\frac{\Delta x_1}{\epsilon} &= \frac{\tilde{x}_1 - x_1}{\epsilon} \approx f'(x_0) \\
\\
\tilde{x}_2 &= f(\tilde{x}_1) = f(x_1 + \epsilon f'(x_0)) \approx f(x_1) + (\epsilon f'(x_0)) f'(x_1) \\
&= x_2 + \epsilon f'(x_0) f'(x_1) \Rightarrow \\
\frac{\Delta x_2}{\epsilon} &= \frac{\tilde{x}_2 - x_2}{\epsilon} \approx f'(x_0) f'(x_1) \\
\\
\tilde{x}_3 &= f(\tilde{x}_2) = f(x_2 + \epsilon f'(x_0) f'(x_1)) \approx f(x_2) + (\epsilon f'(x_0) f'(x_1)) f'(x_2) \\
&= x_3 + \epsilon f'(x_0) f'(x_1) f'(x_2) \Rightarrow \\
\frac{\Delta x_3}{\epsilon} &= \frac{\tilde{x}_3 - x_3}{\epsilon} \approx f'(x_0) f'(x_1) f'(x_2)\,.
\end{aligned} \tag{3.36}$$

We can show by induction that $|\Delta x_n|/\epsilon \approx f'(x_0)f'(x_1)f'(x_2)\ldots f'(x_{n-1})$ and by taking the logarithm and the limits we can prove (3.35).

[11] Sensitivity to the initial condition alone does not necessarily imply chaos. It is necessary to have topological mixing and dense periodic orbits. Topological mixing means that every open set in phase space will evolve to a set that for large enough time will have non zero intersection with any open set. Dense periodic orbits means that every point in phase space lies infinitesimally close to a periodic orbit.

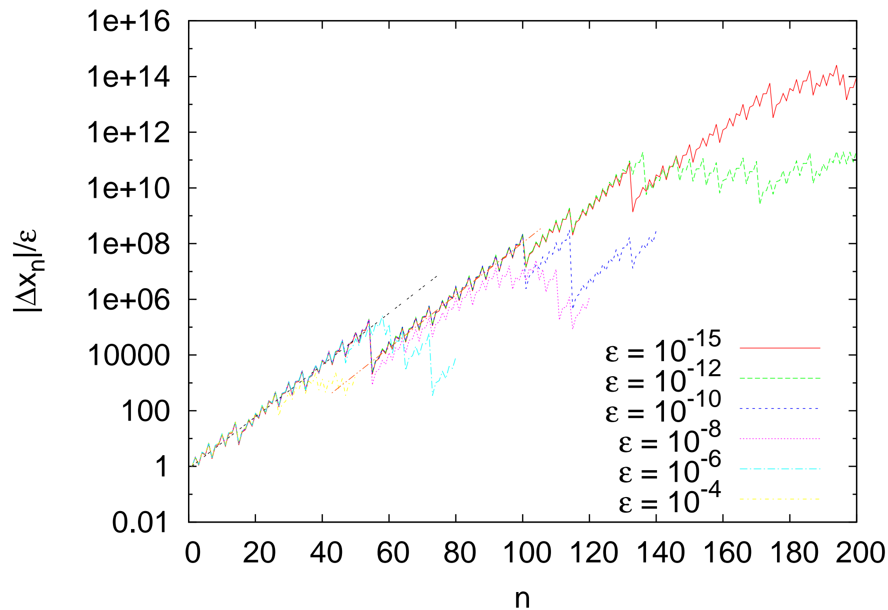

Figure 3.7: A plot of $|\Delta x_n|/\epsilon$ for the logistic map for $r = 3.6$, $x_0 = 0.2$. Note the convergence of the curves as $\epsilon \to 0$ and the approximate exponential behavior in this limit. The two lines are fits to the equation (3.34) and give $\lambda = 0.213(4)$ and $\lambda = 0.217(6)$ respectively.

A first attempt to calculate the Liapunov exponents could be made by using the definition (3.34). We modify the program `logistic.cpp` so that it calculates two trajectories whose initial distance is $\epsilon =$ `epsilon`:

```
//============================================================
//Discrete Logistic Map:
//Two trajectories with close initial conditions.
//============================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;

int main(){
```

```
  int     NSTEPS,i;
  double r,x0,x1,x0t,x1t,epsilon;
  string buf;
  // ------ Input:
  cout << "# Enter NSTEPS, r, x0, epsilon:\n";
  cin  >> NSTEPS    >> r >> x0 >> epsilon;getline(cin,buf);
  cout << "# NSTEPS  = " << NSTEPS  << endl;
  cout << "# r       = " << r       << endl;
  cout << "# x0      = " << x0      << endl;
  cout << "# epsilon = " << epsilon << endl;

  x0t = x0+epsilon;
  // ------ Initialize:
  ofstream myfile("lia.dat");
  myfile.precision(17);
  // ------ Calculate:
  myfile   << 1   << " "   << x0  << " "  << x0t << " "
           << abs(x0t-x0)/epsilon << "\n";
  for(i=2;i<=NSTEPS;i++){
    x1  = r * x0  * (1.0-x0 );
    x1t = r * x0t * (1.0-x0t);
    myfile << i   << " "   << x1  << " "  << x1t << " "
           << abs(x1t-x1)/epsilon << "\n";
    x0  = x1; x0t  = x1t;
  }
  myfile.close();
}//main()
```

After running the program, the quantity $|\Delta x_n|/\epsilon$ is found at the fourth column of the file `lia.dat`. The curves of figure 3.7 can be constructed by using the commands:

```
> g++ liapunov1.cpp -o l
> gnuplot
gnuplot> set logscale y
gnuplot> plot \
  "<echo 200 3.6 0.2 1e-15 |./l;cat lia.dat" u 1:4 w l
```

The last line plots the `stdout` of the command `"echo 200 3.6 0.2 1e-15 |./l;cat lia.dat"`, i.e. the contents of the file `lia.dat` produced after running our program using the parameters `NSTEPS` $= 200$, `r` $= 3.6$, `x0` $= 0.2$ and `epsilon` $= 10^{-15}$. The gnuplot command `set logscale y`, puts the y axis in a logarithmic scale. Therefore an exponential function

is shown as a straight line and this is what we see in figure 3.7: The points $|\Delta x_n|/\epsilon$ tend to lie on a straight line as ϵ decreases. The slopes of these lines are equal to the Liapunov exponent λ. Deviations from the straight line behavior indicates corrections and systematic errors, as we point out in figure 3.7. A different initial condition results in a slightly different value of λ, and the true value can be estimated as the average over several such choices. We estimate the error of our computation from the standard error of the mean. The reader should perform such a computation as an exercise.

One can perform a fit of the points $|\Delta x_n|/\epsilon$ to the exponential function in the following way: Since $|\Delta x_n|/\epsilon \sim C\exp(\lambda n) \Rightarrow \ln(|\Delta x_n|/\epsilon) = \lambda n + c$, we can make a fit to a straight line instead. Using `gnuplot`, the relevant commands are:

```
gnuplot> fit [5:53] a*x+b \
         "<echo 500 3.6 0.2 1e-15 |./l;cat lia.dat"\
         using 1:(log($4)) via a,b
gnuplot> replot exp(a*x+b)
```

The command shown above fits the data to the function `a*x+b` by taking the 1st column and the logarithm of the 4th column (`using 1:(log($4))`) of the `stdout` of the command that we used for creating the previous plot. We choose data for which $5 \leq n \leq 53$ (`[5:53]`) and the fitting parameters are `a,b` (`via a,b`). The second line, adds the fitted function to the plot.

Now we are going to use equation (3.35) for calculating λ. This equation is approximately correct when (a) we have already reached the steady state and (b) in the large n limit. For this reason we should study if we obtain a satisfactory convergence when we (a) "throw away" a number of `NTRANS` steps, (b) calculate the sum (3.35) for increasing `NSTEPS=` n (c) calculate the sum (3.35) for many values of the initial point x_0. This has to be carefully repeated for all values of r since each factor will contribute differently to the quality of the convergence: In regions that manifest chaotic behavior (large λ) convergence will be slower. The program can be found in the file `liapunov2.cpp`:

```
//========================================================
//Discrete Logistic Map:
//Liapunov exponent from sum_i ln|f'(x_i)|
```

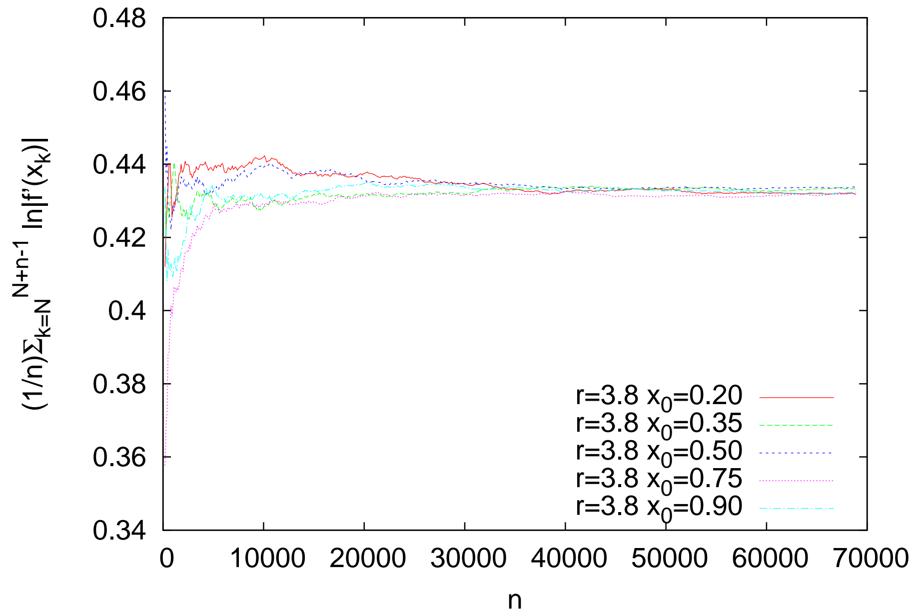

Figure 3.8: Plot of the sum $(1/n)\sum_{k=N}^{N+n-1}\ln|f'(x_k)|$ as a function of n for the logistic map with $r=3.8$, $N=2000$ for different initial conditions $x_0=0.20, 0.35, 0.50, 0.75, 0.90$. The different curves converge in the limit $n\to\infty$ to $\lambda=0.4325(10)$.

```
// NTRANS: number of discarted iterations in order to discart
//         transient behaviour
// NSTEPS: number of terms in the sum
//=============================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;

int main(){
  int    NTRANS,NSTEPS,i;
  double r,x0,x1,sum;
  string buf;
  // ------ Input:
  cout  << "# Enter NTRANS,NSTEPS, r, x0:\n";
  cin   >> NTRANS >> NSTEPS >> r   >>   x0; getline(cin,buf);
```

```
  cout  << "# NTRANS = " << NTRANS << endl;
  cout  << "# NSTEPS = " << NSTEPS << endl;
  cout  << "# r      = " << r      << endl;
  cout  << "# x0     = " << x0     << endl;

  for(i=1;i<=NTRANS;i++){
    x1   = r * x0  * (1.0 - x0);
    x0   = x1;
  }
  sum    = log(abs(r*(1.0 - 2.0*x0)));
  // ------ Initialize:
  ofstream myfile("lia.dat");
  myfile.precision(17);
  // ------ Calculate:
  myfile   << 1   << " " << x0  << " " << sum   << "\n";
  for(i=2;i<=NSTEPS;i++){
    x1   = r * x0  * (1.0-x0 );
    sum += log(abs(r*(1.0-2.0*x1)));
    myfile << i   << " " << x1  << " " << sum/i << "\n";
    x0   = x1;
  }
  myfile.close();
}//main()
```

After `NTRANS` steps, the program calculates `NSTEPS` times the sum of the terms $\ln|f'(x_k)| = \ln|r(1-2x_k)|$. At each step the sum divided by the number of steps `i` is printed to the file `lia.dat`. Figure 3.6 shows the results for $r = 3.8$. This is a point where the system exhibits strong chaotic behavior and convergence is achieved after we compute a large number of steps. Using `NTRANS` $= 2\,000$ and `NSTEPS` $\approx 70\,000$ the achieved accuracy is about 0.2% with $\lambda = 0.4325 \pm 0.0010 \equiv 0.4325(10)$. The main contribution to the error comes from the different paths followed by each initial point chosen. The plot can be constructed with the `gnuplot` commands:

```
> g++ liapunov2.cpp -o l
> gnuplot
gnuplot> plot \
 "<echo 2000 70000 3.8 0.20 |./l;cat lia.dat" u 1:3 w l,\
 "<echo 2000 70000 3.8 0.35 |./l;cat lia.dat" u 1:3 w l,\
 "<echo 2000 70000 3.8 0.50 |./l;cat lia.dat" u 1:3 w l,\
 "<echo 2000 70000 3.8 0.75 |./l;cat lia.dat" u 1:3 w l,\
 "<echo 2000 70000 3.8 0.90 |./l;cat lia.dat" u 1:3 w l
```

The `plot` command runs the program using the parameters `NTRANS` = 2 000, `NSTEPS` = 70 000, `r` = 3.8 and `x0` = 0.20, 0.35, 0.50, 0.75, 0.90 and plots the results from the contents of the file `lia.dat`.

In order to determine the regions of chaotic behavior we have to study the dependence of the Liapunov exponent λ on the value of r. Using our experience coming from the careful computation of λ before, we will run the program for several values of r using the parameters `NTRANS` = 2 000, `NSTEPS` = 60 000 from the initial point `x0` = 0.2. This calculation gives accuracy of the order of 1%. If we wish to measure λ carefully and estimate the error of the results, we have to follow the steps described in figures 3.7 and 3.8. The program can be found in the file `liapunov3.cpp` and it is a simple modification of the previous program so that it can perform the calculation for many values of r.

```
//============================================================
//Discrete Logistic Map:
//Liapunov exponent from sum_i ln|f'(x_i)|
//Calculation for r in [rmin,rmax] with RSTEPS steps
// RSTEPS: values or r studied: r=rmin+(rmax-rmin)/RSTEPS
// NTRANS: number of discarted iterations in order to discart
//         transient behaviour
// NSTEPS: number of terms in the sum
// xstart: value of initial x0 for every r
//============================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;

int main(){
  const double rmin   = 2.5;
  const double rmax   = 4.0;
  const double xstart = 0.2;
  const int    RSTEPS = 1000;
  const int    NSTEPS = 60000;
  const int    NTRANS = 2000;
  int    i,ir;
  double r,x0,x1,sum,dr;
```

```
  string buf;

  // ------ Initialize:
  ofstream myfile("lia.dat");
  myfile.precision(17);
  // ------ Calculate:
  dr        = (rmax-rmin)/(RSTEPS-1);
  for(ir    = 0; ir < RSTEPS;ir++){
    r       = rmin+ir*dr;
    x0      = xstart;
    for(i = 1; i <= NTRANS; i++){
      x1   = r * x0   * (1.0-x0 );
      x0   = x1;
    }
    sum    = log(abs(r*(1.0-2.0*x0)));
    //Calculate:
    for(i = 2; i <= NSTEPS; i++){
      x1   = r * x0   * (1.0-x0 );
      sum+= log(abs(r*(1.0-2.0*x1)));
      x0   = x1;
    }
    myfile << r << " " << sum/NSTEPS << '\n';
  }//for(ir=0;ir<RSTEPS;ir++)
  myfile.close();
}//main()
```

The program can be compiled and run using the commands:

```
> g++ liapunov3.cpp -o l
> ./l &
```

The character & makes the program `./l` to run in the background. This is recommended for programs that run for a long time, so that the shell returns the prompt to the user and the program continues to run even after the shell is terminated.

The data are saved in the file `lia.dat` and we can make the plot shown in figure 3.7 using gnuplot:

```
gnuplot> plot  "lia.dat"  with lines notitle ,0 notitle
```

Now we can compare figure 3.9 with the bifurcation diagram shown in figure 3.4. The intervals with $\lambda < 0$ correspond to stable k-cycles. The

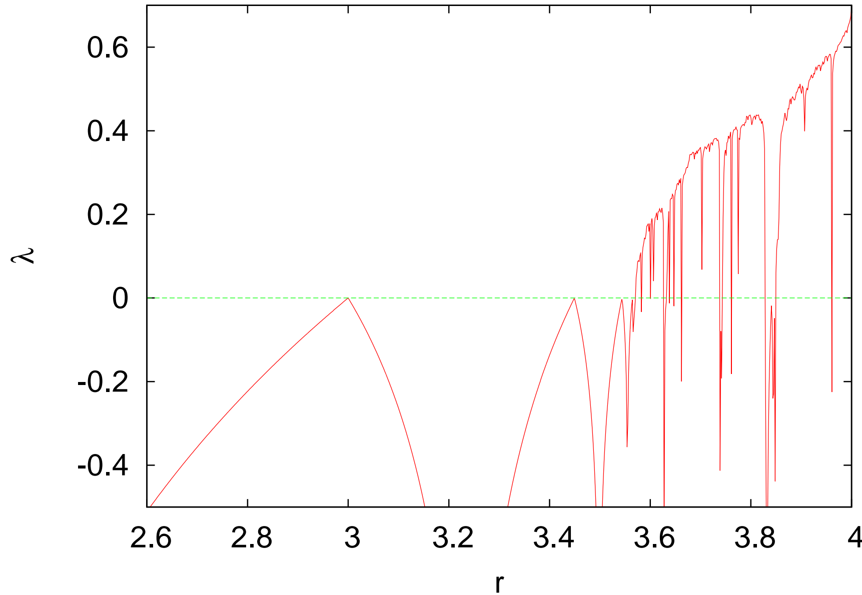

Figure 3.9: The Liapunov exponent λ of the logistic map calculated via equation (3.35). Note the chaotic behavior that manifests for the values of r where $\lambda > 0$ and the windows of stable k-cycles where $\lambda < 0$. Compare this plot with the bifurcation diagram of figure 3.4. At the points where $\lambda = 0$ we have onset of chaos (or "edge of chaos") with manifestation of weak chaos (i.e. $|\Delta x_n| \sim |\Delta x_0| n^\omega$). At these points we have transitions from stable k-cycles to strong chaos. We observe the onset of chaos for the first time when $r = r_c \approx 3.5699$, at which point $\lambda = 0$ (for smaller r the plot seems to touch the $\lambda = 0$ line, but in fact λ takes negative values with $|\lambda|$ very small).

intervals where $\lambda > 0$ correspond to manifestation of *strong chaos*. These intervals are separated by points with $\lambda = 0$ where the system exhibits *weak chaos*. This means that neighboring trajectories diverge from each other with a power law $|\Delta x_n| \sim |\Delta x_0| n^\omega$ instead of an exponential, where $\omega = 1/(1-q)$ is a positive exponent that needs to be determined. The parameter q is the one usually used in the literature. Strong chaos is obtained in the $q \to 1$ limit. For larger r, switching between chaotic and stable periodic trajectories is observed each time λ changes sign. The critical values of r can be computed with relatively high accuracy by restricting the calculation to a small enough neighborhood of the critical point. You can do this using the program listed above by setting the

parameters `rmin` and `rmax`.

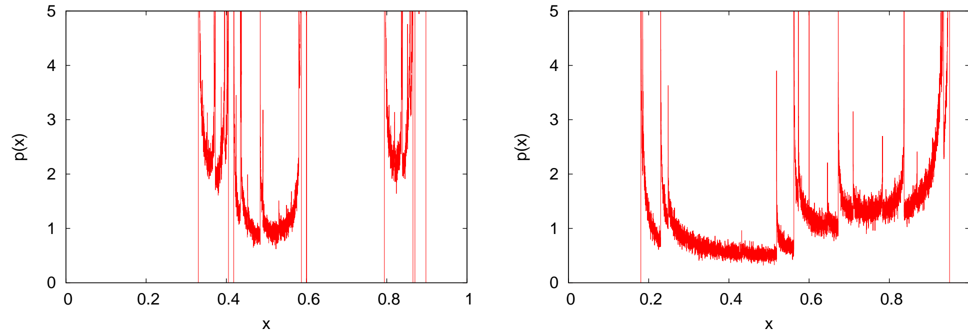

Figure 3.10: The distribution functions $p(x)$ of x of the logistic map for $r = 3.59$ (left) and 3.8 (right). The chaotic behavior appears to be weaker for $r = 3.59$, and this is reflected on the value of the entropy. One sees that there exist intervals of x with $p(x) = 0$ which become smaller and vanish as r gets close to 4. This distribution is very hard to be distinguished from a truly random distribution.

We can also study the chaotic properties of the trajectories of the logistic map by computing the distribution $p(x)$ of the values of x in the interval $(0, 1)$. After the transitional period, the distribution $p(x)$ for the k cycles will have support only at the points of the k cycles, whereas for the chaotic regimes it will have support on subintervals of $(0, 1)$. The distribution function $p(x)$ is independent for most of the initial points of the trajectories. If one obtains a large number of points from many trajectories of the logistic map, it will be practically impossible to understand that these are produced by a deterministic rule. For this reason, chaotic systems can be used for the production of *pseudorandom* numbers, as we will see in chapter 11. By measuring the *entropy*, which is a measure of disorder in a system, we can quantify the "randomness" of the distribution. As we will see in chapter 12, it is given by the equation

$$S = -\sum_k p_k \ln p_k \,, \tag{3.37}$$

where p_k is the probability of observing the state k. In our case, we can make an approximate calculation of S by dividing the interval $(0, 1)$ to N subintervals of width Δx. For given r we obtain a large number M of values x_n of the logistic map and we compute the histogram h_k of

their distribution in the intervals $(x_k, x_k + \Delta x)$. The probability density is obtained from the limit of $p_k = h_k/(M\Delta x)$ as M becomes large and Δx small (large N). Indeed, $\sum_{k=1}^{N} p_k \Delta x = 1$ converges to $\int_0^1 p(x)\, dx = 1$. We will define $S = -\sum_{k=1}^{N} p_k \ln p_k \Delta x$.

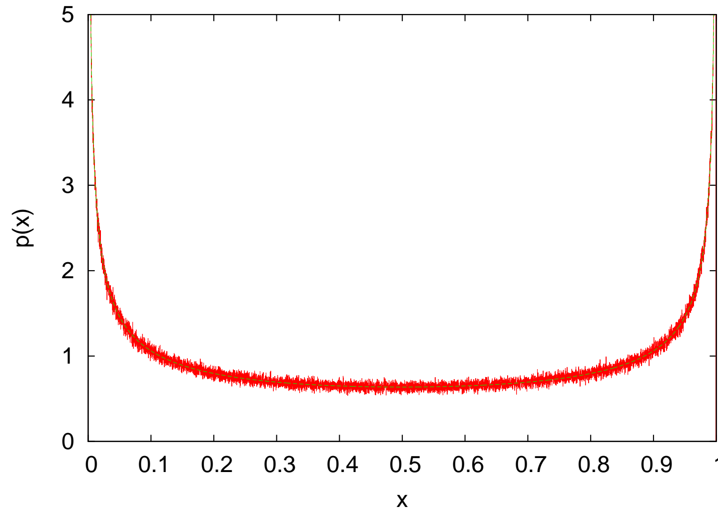

Figure 3.11: The distribution $p(x)$ of x for the logistic map for $r = 4$. We observe strong chaotic behavior, $p(x)$ has support over the whole interval $(0, 1)$ and the entropy is large. The solid line is the analytic form of the distribution $p(x) = \pi^{-1} x^{-1/2}(1-x)^{-1/2}$ which is known for $r = 4$ [32]. This is the beta distribution for $a = 1/2$, $b = 1/2$.

The program listed below calculates p_k for chosen values of r, and then the entropy S is calculated using (3.37). It is a simple modification of the program in `liapunov3.cpp` where we add the parameter `NHIST` counting the number of intervals N for the histograms. The probability density is calculated in the array `p[NHIST]`. The program can be found in the file `entropy.cpp`:

```
//=============================================================
//Discrete Logistic Map:
//Liapunov exponent from sum_i ln|f'(x_i)|
//Calculation for r in [rmin,rmax] with RSTEPS steps
// RSTEPS: values or r studied: r=rmin+(rmax-rmin)/RSTEPS
// NTRANS: number of discarted iterations in order to discart
//         transient behaviour
```

```
// NSTEPS: number of terms in the sum
// xstart: value of initial x0 for every r
//===============================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;

int main(){
  const double rmin   = 2.5;
  const double rmax   = 4.0;
  const double xstart = 0.2;
  const int    RSTEPS = 1000;
  const int    NHIST  = 10000;
  const int    NTRANS = 2000;
  const int    NSTEPS = 5000000;
  const double xmin=0.0,xmax=1.0;
  int     i,ir,isum,n;
  double r,x0,x1,sum,dr,dx;
  double p[NHIST],S;
  string buf;

  // ------ Initialize:
  ofstream myfile("entropy.dat");
  myfile.precision(17);
  // ------ Calculate:
  for(i=0;i<NHIST;i++) p[i] = 0.0;
  dr = (rmax-rmin)/(RSTEPS-1);
  dx = (xmax-xmin)/(NHIST -1);
  for(ir=0;ir<RSTEPS;ir++){
    r = rmin+ir*dr;
    x0= xstart;
    for(i=1;i<=NTRANS;i++){
      x1  = r * x0  * (1.0-x0 );
      x0  = x1;
    }
    //make histogram:
    n=int(x0/dx); p[n]+=1.0;
    for(i=2;i<=NSTEPS;i++){
      x1    = r * x0  * (1.0-x0 );
      n     = int(x1/dx);
      p[n] += 1.0;
      x0    = x1;
```

```
    }
    //p[k] is now histogram of x-values.
    //Normalize so that sum_k p[k]*dx=1
    //to get probability distribution:
    for(i=0;i < NHIST;i++) p[i] /= (NSTEPS*dx);
    //sum all non zero terms: p[n]*log(p[n])*dx
    S = 0.0;
    for(i=0;i < NHIST;i++)
      if(p[i] > 0.0)
        S -= p[i]*log(p[i])*dx;
    myfile << r << " " << S << '\n';
  }//for(ir=0;ir<RSTEPS;ir++)
  myfile.close();

  myfile.open("entropy_hist.dat");
  myfile.precision(17);
  for(n=0;n<NHIST;n++){
    x0 = xmin + n*dx + 0.5*dx;
    myfile << r << " " << x0 << " " << p[n] << '\n';
  }
  myfile.close();
}//main()
```

For the calculation of the distribution functions and the entropy we have to choose the parameters which control the systematic error. The parameter NTRANS should be large enough so that the transitional behavior will not contaminate our results. Our measurements must be checked for being independent of its value. The same should be done for the initial point xstart. The parameter NHIST controls the partitioning of the interval $(0,1)$ and the width Δx, so it should be large enough. The parameter NSTEPS is the number of "measurements" for each value of r and it should be large enough in order to reduce the "noise" in p_k. It is obvious that NSTEPS should be larger when Δx becomes smaller. Appropriate choices lead to the plots shown in figures 3.10 and 3.11 for $r = 3.59$, 3.58 and 4. We see that stronger chaotic behavior means a wider distribution of the values of x.

The entropy is shown in figure 3.12. The stable periodic trajectories lead to small entropy, whereas the chaotic ones lead to large entropy. There is a sudden increase in the value of the entropy at the beginning of chaos at $r = r_c$, which increases even further as the chaotic behavior becomes stronger. During the intermissions of the chaotic behavior there are sudden drops in the value of the entropy. It is quite instructive to

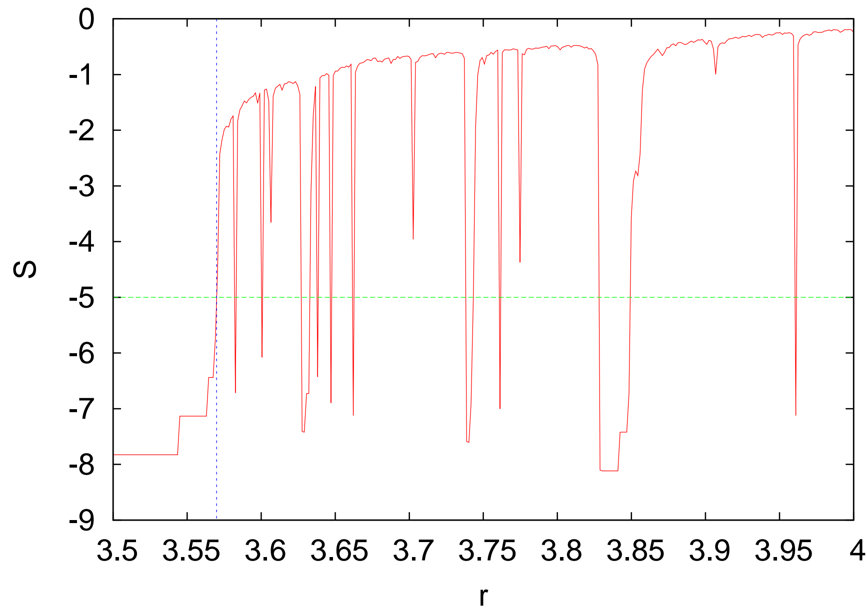

Figure 3.12: The entropy $S = -\sum_k p_k \ln p_k \, \Delta x$ for the logistic map as a function of r. The vertical line is $r_c \approx 3.56994567$ which marks the beginning of chaos and the horizontal is the corresponding entropy. The entropy is low for small values of r, where we have the stable 2^n cycles, and large in the chaotic regimes. S drops suddenly when we pass to a (temporary) periodic behavior interval. We clearly observe the 3-cycle for $r = 1+\sqrt{8} \approx 3.8284$ and the subsequent bifurcations that we observed in the bifurcation diagram (figure 3.4) and the Liapunov exponent diagram (figure 3.9).

compare the entropy diagrams with the corresponding bifurcation diagrams (see figure 3.4) and the Liapunov exponent diagrams (see figure 3.9). The entropy is increasing until r reaches its maximum value 4, but this is not done smoothly. By magnifying the corresponding areas in the plot, we can see an infinite number of sudden drops in the entropy in intervals of r that become more and more narrow.

3.7 Problems

Several of the programs that you need to write for solving the problems of this chapter can be found in the `Problems` directory of the accompanying software of this chapter.

3.1 Confirm that the trajectories of the logistic map for $r < 1$ are falling off exponentially for large enough n.

(a) Choose $r = 0.5$ and plot the trajectories for $x_0 = 0.1 - 0.9$ with step 0.1 for $n = 1, \ldots, 1000$. Put the y axis in a logarithmic scale. From the resulting curves discuss whether you obtain an exponential falloff.

(b) Fit the points x_n for $n > 20$ to the function $c\,\mathrm{e}^{-ax}$ and determine the fitting parameters a and c. How do these parameters depend on the initial point x_0? You can use the following `gnuplot` commands for your calculation:

```
gnuplot> !g++ logistic.cpp -o l
gnuplot> a=0.7;c=0.4;
gnuplot> fit [10:] c*exp(-a*x) \
 "<echo 1000 0.5  0.5|./l;cat log.dat" via a,c
gnuplot> plot c*exp(-a*x),\
 "<echo 1000 0.5  0.5|./l;cat log.dat" w l
```

As you can see, we set `NSTEPS = 1000, r = 0.5, x0 = 0.5`. By setting the limits `[10:]` to the `fit` command, the fit includes only the points $x_n \geq 10$, therefore avoiding the transitional period and the deviation from the exponential falloff for small n.

(c) Repeat for $r = 0.3 - 0.9$ with step 0.1 and for $r = 0.99, 0.999$. As you will be approaching $r = 1$, you will need to discard more points from near the origin. You might also need to increase `NSTEPS`. You should always check graphically whether the fitted exponential function is a good fit to the points x_n for large n. Construct a table for the values of a as a function of r.

The solutions of the equation (3.3) is $\mathrm{e}^{(r-1)x}$. How is this related to the values that you computed in your table?

3.2 Consider the logistic map for $r = 2$. Choose NSTEPS=100 and calculate the corresponding trajectories for x0=0.2, 0.3, 0.5, 0.7, 0.9. Plot them on the same graph. Calculate the fixed point x_2^* and compare your result to the known value $1 - 1/r$. Repeat for x0= $10^{-\alpha}$ for $\alpha = -1, -2, -5, -10, -20, -25$. What do you conclude about the point $x_1^* = 0$?

3.3 Consider the logistic map for $r = 2.9, 2.99, 2.999$. Calculate the stable point x_2^* and compare your result to the known value $1-1/r$. How large should NSTEPS be chosen each time? You may choose x0=0.3.

3.4 Consider the logistic map for $r = 3.2$. Take x0=0.3, 0.5, 0.9 and NSTEPS=300 and plot the resulting trajectories. Calculate the fixed points x_3^* and x_4^* by using the command tail log.dat. Increase NSTEPS and repeat so that you make sure that the trajectory has converged to the 2-cycle. Compare their values to the ones given by equation (3.18). Make the following plots:

```
gnuplot> plot   \
 "<echo 300 3.2 0.3|./l;awk 'NR%2==0' log.dat"  w l
gnuplot> replot \
 "<echo 300 3.2 0.3|./l;awk 'NR%2==1' log.dat"  w l
```

What do you observe?

3.5 Repeat the previous problem for $r = 3.4494$. How big should NSTEPS be chosen so that you obtain x_3^* and x_4^* with an accuracy of 6 significant digits?

3.6 Repeat the previous problem for $r = 3.5$ and 3.55. Choose NSTEPS = 1000, x0 = 0.5. Show that the trajectories approach a 4-cycle and an 8-cycle respectively. Calculate the fixed points x_5^*-x_8^* and x_9^*-x_{16}^*.

3.7 Plot the functions $f(x)$, $f^{(2)}(x)$, $f^{(4)}(x)$, x for given r on the same graph. Use the commands:

```
gnuplot> set samples 1000
gnuplot> f(x) = r*x*(1-x)
gnuplot> r=1;plot [0:1] x,f(x),f(f(x)),f(f(f(f(x))))
```

The command `r=1` sets the value of r. Take $r = 2.5, 3, 3.2, 1+\sqrt{6}, 3.5$. Determine the fixed points and the k-cycles from the intersections of the plots with the diagonal $y = x$.

3.8 Construct the cobweb plots of figures 3.2 and 3.4 for $r = 2.8, 3.3$ and 3.5. Repeat by dropping from the plot an increasing number of initial points, so that in the end only the k-cycles will remain. Do the same for $r = 3.55$.

3.9 Construct the bifurcation diagrams shown in figure 3.4.

3.10 Construct the bifurcation diagram of the logistic map for $3.840 < r < 3.851$ and for $0.458 < x < 0.523$. Compute the first four bifurcation points with an accuracy of 5 significant digits by magnifying the appropriate parts of the plots. Take `NTRANS=15000`.

3.11 Construct the bifurcation diagram of the logistic map for $2.9 < r < 3.57$. Compute graphically the bifurcation points $r_c^{(n)}$ for $n = 2, 3, 4, 5, 6, 7, 8$. Make sure that your results are stable against variations of the parameters `NTRANS, NSTEPS` as well as from the choice of branching point. From the known values of $r_c^{(n)}$ for $n = 2, 3$, and from the dependence of your results on the choices of `NTRANS`, `NSTEPS`, estimate the accuracy achieved by this graphical method. Compute the ratios $(r_c^{(n)} - r_c^{(n-1)})/(r_c^{(n+1)} - r_c^{(n)})$ and compare your results to equation (3.20).

3.12 Choose the values of ρ in equation (3.24) so that you obtain only one energy level. Compute the resulting value of the energy. When do we have three energy levels?

3.13 Consider the polynomial $g(x) = x^3 - 2x^2 - 11x + 12$. Find the roots obtained by the Newton-Raphson method when you choose $x_0 = 2.35287527, 2.35284172, 2.35283735, 2.352836327, 2.352836323$. What do you conclude concerning the basins of attraction of each root of the polynomial? Make a plot of the polynomial in a neighborhood of its roots and try other initial points that will converge to each one of the roots.

3.14 Use the Newton-Raphson method in order to compute the 4-cycle $x_5^*, \ldots, x_8^*$ of the logistic map. Use appropriate areas of the bifur-

n	$r_c^{(n)}$	n	$r_c^{(n)}$
2	3.0000000000	10	3.56994317604
3	3.4494897429	11	3.569945137342
4	3.544090360	12	3.5699455573912
5	3.564407266	13	3.569945647353
6	3.5687594195	14	3.5699456666199
7	3.5696916098	15	3.5699456707464
8	3.56989125938	16	3.56994567163008
9	3.56993401837	17	3.5699456718193
$r_c = 3.56994567\ldots$			

Table 3.1: The values of $r_c^{(n)}$ for the logistic map calculated for problem 17. $r_c^{(\infty)} \equiv r_c$ is taken from the bibliography.

cation diagram so that you can choose the initial points correctly. Check that your result for $r_c^{(4)}$ is the same for all x_α^*. Tune the parameters chosen in your calculation on order to improve the accuracy of your measurements.

3.15 Repeat the previous problem for the 8-cycle $x_9^*, \ldots, x_{16}^*$ and $r_c^{(5)}$.

3.16 Repeat the previous problem for the 16-cycle $x_{17}^*, \ldots, x_{32}^*$ and $r_c^{(6)}$.

3.17 Calculate the critical points $r_c^{(n)}$ for $n = 3, \ldots, 17$ of the logistic map using the Newton-Raphson method. In order to achieve that, you should determine the bifurcation points graphically in the bifurcation diagram first and then choose the initial points in the Newton-Raphson method appropriately. The program in `bifurcationPoints.cpp` should read the parameters `eps, epsx, epsr` from the `stdin` so that they can be tuned for increasing n. If these parameters are too small the convergence will be unstable and if they are too large you will have large systematic errors. Using this method, try to reproduce table 3.1

3.18 Calculate the ratios $\Delta r^{(n)}/\Delta r^{(n+1)}$ of equation (3.20) using the results of table 3.1. Calculate Feigenbaum's constant and comment on the accuracy achieved by your calculation.

3.19 Estimate Feigenbaum's constant δ and the critical value r_c by assuming that for large enough n, $r_c^{(n)} \approx r_c - C\delta^{-n}$. This behavior is a result of equation (3.20). Fit the results of table 3.1 to this function and calculate δ and r_c. This hypothesis is confirmed in figure 3.13 where we can observe the exponential convergence of $r_c^{(n)}$ to r_c. Construct the same plot using the parameters of your calculation.
Hint: You can use the following `gnuplot` commands:

```
gnuplot> nmin=2;nmax=17
gnuplot> r(x)= rc-c*d**(-x)
gnuplot> fit [nmin:nmax] r(x)  "rcrit" u 1:2 via rc,c,d
gnuplot> plot "rcrit", r(x)
gnuplot> print rc,d
```

The file `rcrit` contains the values of table 3.1. You should vary the parameters `nmin`, `nmax` and repeat until you obtain a stable fit.

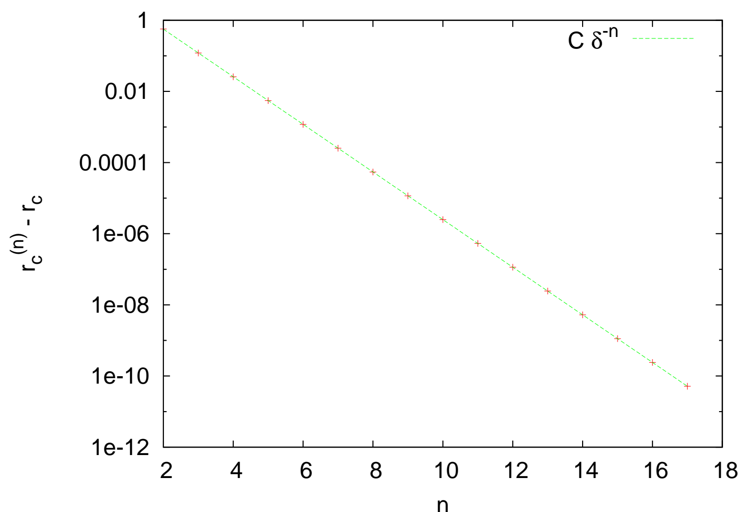

Figure 3.13: Test of the relation $r_c^{(n)} \approx r_c - C\delta^{-n}$ discussed in problem 17. The parameters used in the plot are approximately $r_c = 3.5699457$, $\delta = 4.669196$ and $C = 12.292$.

3.20 Use the Newton-Raphson method to calculate the first three bifurcation points after the appearance of the 3-cycle for $r = 1 + \sqrt{8}$. Choose one bifurcation point of the 3-cycle, one of the 6-cycle and one of the 12-cycle and magnify the bifurcation diagram in their neighborhood.

3.21 Consider the map describing the evolution of a population

$$x_{n+1} = p(x_n) = x_n \mathrm{e}^{r(1-x_n)} . \tag{3.38}$$

(a) Plot the functions x, $p(x)$, $p^{(2)}(x)$, $p^{(4)}(x)$ for $r = 1.8, 2, 2.6, 2.67$, 2.689 for $0 < x < 8$. For which values of r do you expect to obtain stable k-cycles?

(b) For the same values of r plot the trajectories with initial points $x_0 = 0.2, 0.5, 0.7$. For each r make a separate plot.

(c) Use the Newton-Raphson method in order to determine the points $r_c^{(n)}$ for $n = 3, 4, 5$ as well as the first two bifurcation points of the 3-cycle.

(d) Construct the bifurcation diagram for $1.8 < r < 4$. Determine the point marking the onset of chaos as well as the point where the 3-cycle starts. Magnify the diagram around a branch that you will choose.

(e) Estimate Feigenbaum's constant δ as in problem 17. Is your result compatible with the expectation of universality for the value of δ? Is the value of r_c the same as that of the logistic map?

3.22 Consider the sine map:

$$x_{n+1} = s(x_n) = r \sin(\pi x_n) . \tag{3.39}$$

(a) Plot the functions x, $s(x)$, $s^{(2)}(x)$, $s^{(4)}(x)$, $s^{(8)}(x)$ for $r = 0.65$, 0.75, 0.84, 0.86, 0.88. Which values of r are expected to lead to stable k-cycles?

(b) For the same values of r, plot the trajectories with initial points $x_0 = 0.2, 0.5, 0.7$. Make one plot for each r.

(c) Use the Newton-Raphson method in order to determine the points $r_c^{(n)}$ for $n = 3, 4, 5$ as well as the first two bifurcation points of the 3-cycle.

(d) Construct the bifurcation diagram for $0.6 < r < 1$. Within which limits do the values of x lie in? Repeat for $0.6 < r < 2$. What do you observe? Determine the point marking the onset of chaos as well as the point where the 3-cycle starts. Magnify the diagram around a branch that you will choose.

3.23 Consider the map:

$$x_{n+1} = 1 - r x_n^2 \,. \tag{3.40}$$

(a) Construct the bifurcation diagram for $0 < r < 2$. Within which limits do the values of x lie in? Determine the point marking the onset of chaos as well as the point where the 3-cycle starts. Magnify the diagram around a branch that you will choose.

(b) Use the Newton-Raphson method in order to determine the points $r_c^{(n)}$ for $n = 3, 4, 5$ as well as the first two bifurcation points of the 3-cycle.

3.24 Consider the tent map:

$$x_{n+1} = r \min\{x_n, 1 - x_n\} = \begin{cases} r x_n & 0 \le x_n \le \frac{1}{2} \\ r(1 - x_n) & \frac{1}{2} < x_n \le 1 \end{cases} \,. \tag{3.41}$$

Construct the bifurcation diagram for $0 < r < 2$. Within which limits do the values of x lie in? On the same graph, plot the functions $r/2$, $r - r^2/2$.

Magnify the diagram in the area $1.407 < r < 1.416$ and $0.580 < x < 0.588$. At which point do the two disconnected intervals within which x_n take their values merge into one? Magnify the areas $1.0 < r < 1.1$, $0.4998 < x < 0.5004$ and $1.00 < r < 1.03$, $0.4999998 < x < 0.5000003$ and determine the merging points of two disconnected intervals within which x_n take their values.

3.25 Consider the Gauss map (or mouse map):

$$x_{n+1} = \mathrm{e}^{-r x_n^2} + q \,. \tag{3.42}$$

Construct the bifurcation diagram for $-1 < q < 1$ and $r = 4.5$, 4.9, 7.5. Make your program to take as the initial point of the new trajectory to be the last one of the previous trajectory and choose

$x_0 = 0$ for $q = -1$. Repeat for $x_0 = 0.7, 0.5, -0.7$. What do you observe? Note that as q is increased, we obtain bifurcations and "anti-bifurcations".

3.26 Consider the circle map:

$$x_{n+1} = [x_n + r - q\sin(2\pi x_n)] \quad \text{mod } 1\,. \tag{3.43}$$

(Make sure that your program keeps the values of x_n so that $0 \leq x_n < 1$). Construct the bifurcation diagram for $0 < q < 2$ and $r = 1/3$.

3.27 Use the program in `liapunov.cpp` in order to compute the distance between two trajectories of the logistic map for $r = 3.6$ that originally are at a distance $\Delta x_0 = 10^{-15}$. Choose $x_0 = 0.1,\ 0.2, 0.3,\ 0.4, 0.5, 0.6,\ 0.7, 0.8,\ 0.9,\ 0.99, 0.999$ and calculate the Liapunov exponent by fitting to a straight line appropriately. Compute the mean value and the standard error of the mean.

3.28 Calculate the Liapunov exponent for $r = 3.58,\ 3.60,\ 3.65,\ 3.70,\ 3.80$ for the logistic map. Use both ways mentioned in the text. Choose at least 5 different initial points and calculate the mean and the standard error of the mean of your results. Compare the values of λ that you obtain with each method and comment.

3.29 Compute the critical value r_c numerically as the limit $\lim\limits_{n\to\infty} r_c^{(n)}$ for the logistic map with an accuracy of nine significant digits. Use the calculation of the Liapunov exponent λ given by equation (3.35).

3.30 Compute the values of r of the logistic map numerically for which we (a) enter a stable 3-cycle (b) reenter into the chaotic behavior. Do the calculation by computing the Liapunov exponent λ and compare your results with the ones obtained from the bifurcation diagram.

3.31 Calculate the Liapunov exponent using equation (3.35) for the fol-

lowing maps:

$$
\begin{aligned}
x_{n+1} &= x_n \mathrm{e}^{r(1-x_n)}\,, \qquad 1.8 < r < 4 \\
x_{n+1} &= r\sin(\pi x_n)\,, \qquad 0.6 < r < 1 \\
x_{n+1} &= 1 - r x_n^2\,, \qquad 0 < r < 2 \\
x_{n+1} &= \mathrm{e}^{-r x_n^2} + q\,, \qquad r = 7.5, -1 < q < 1 \\
x_{n+1} &= \left[x_n + \frac{1}{3} - q\sin(2\pi x_n)\right] \mod 1\,, \quad 0 < q < 2\,, \qquad (3.44)
\end{aligned}
$$

and construct the diagrams similar to the ones in figure 3.9. Compare your plots with the respective bifurcation diagrams (you may put both graphs on the same plot). Use two different initial points $x_0 = 0, 0.2$ for the Gauss map ($x_{n+1} = \mathrm{e}^{-r x_n^2} + q$) and observe the differences. For the circle map ($x_{n+1} = [x_n + 1/3 - q\sin(2\pi x_n)] \mod 1$) study carefully the values $0 < q < 0.15$.

3.32 Reproduce the plots in figures 3.10, 3.11 and 3.12. Compute the function $p(x)$ for $r = 3.68$, 3.80, 3.93 and 3.98. Determine the points where you have stronger chaos by observing $p(x)$ and the corresponding values of the entropy. Compute the entropy for $r \in (3.95, 4.00)$ by taking `RSTEPS=2000` and estimate the values of r where we enter to and exit from chaos. Compare your results with the computation of the Liapunov exponent.

3.33 Consider the Hénon map:

$$
\begin{aligned}
x_{n+1} &= y_n + 1 - a x_n^2 \\
y_{n+1} &= b x_n
\end{aligned} \qquad (3.45)
$$

(a) Construct the two bifurcation diagrams for x_n and y_n for $b = 0.3$, $1.0 < a < 1.5$. Check if the values $a = 1.01, 1.4$ that we will use below correspond to stable periodic trajectories or chaotic behavior.

(b) Write a program in a file `attractor.cpp` which will take `NINIT` = `NL` $\times$ `NL` initial conditions $(x_0(i), y_0(i))$ $i = 1, \ldots,$`NL` on a `NL`$\times$`NL` lattice of the square $x_m \le x_0 \le x_M$, $y_m \le y_0 \le y_M$. Each of the points $(x_0(i), y_0(i))$ will evolve according to equation (3.45) for $n =$ `NSTEPS` steps. The program will print the

points $(x_n(i), y_n(i))$ to the `stdout`. Choose $x_m = y_m = 0.6$, $x_M = y_M = 0.8$, `NL`$= 200$.

(c) Choose $a = 1.01$, $b = 0.3$ and plot the points $(x_n(i), y_n(i))$ for $n = 0,\ 1,\ 2,\ 3,\ 10,\ 20,\ 30,\ 40,\ 60,\ 1000$ on the same diagram.

(d) Choose $a = 1.4$, $b = 0.3$ and plot the points $(x_n(i), y_n(i))$ for $n = 0, \ldots, 7$ on the same diagram.

(e) Choose $a = 1.4$, $b = 0.3$ and plot the points $(x_n(i), y_n(i))$ for $n = 999$ on the same diagram. Observe the Hénon strange attractor and its fractal properties. It is characterized by a Hausdorff[12] dimension $d_H = 1.261 \pm 0.003$. Then magnify the regions

$$\begin{array}{lll}
\{(x,y)| & -1.290 < x < -1.270, & 0.378 < y < 0.384\}\,, \\
\{(x,y)| & 1.150 < x < -1.130, & 0.366 < y < 0.372\}\,, \\
\{(x,y)| & 0.108 < x < 0.114, & 0.238 < y < 0.241\}\,, \\
\{(x,y)| & 0.300 < x < 0.320, & 0.204 < y < 0.213\}\,, \\
\{(x,y)| & 1.076 < x < 1.084, & 0.090 < y < 0.096\}\,, \\
\{(x,y)| & 1.216 < x < 1.226, & 0.032 < y < 0.034\}\,.
\end{array}$$

3.34 Consider the Duffing map:

$$\begin{aligned}
x_{n+1} &= y_n \\
y_{n+1} &= -b x_n + a y_n - y_n^3\,.
\end{aligned} \tag{3.46}$$

(a) Construct the two bifurcation diagrams for x_n and y_n for $b = 0.3$, $0 < a < 2.78$. Choose four different initial conditions $(x_0, y_0) = (\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$. What do you observe?

(b) Use the program `attractor.cpp` from problem 33 in order to study the attractor of the map for $b = 0.3$, $a = 2.75$.

3.35 Consider the Tinkerbell map:

$$\begin{aligned}
x_{n+1} &= x_n^2 - y_n^2 + a x_n + b y_n \\
y_{n+1} &= 2 x_n y_n + c x_n + d y_n\,.
\end{aligned} \tag{3.47}$$

[12]D.A. Russel, J.D. Hanson, and E. Ott, "Dimension of strange attractors", Phys. Rev. Lett. **45** (1980) 1175. See "Hausdorff dimension" in Wikipedia.

(a) Choose $a = 0.9$, $b = -0.6013$, $c = 2.0$, $d = 0.50$. Plot a trajectory on the plane by plotting the points (x_n, y_n) for $n = 0, \ldots, 10\,000$ with $(x_0, y_0) = (-0.72, -0.64)$.

(b) Use the program `attractor.cpp` from problem 33 in order to study the attractor of the map for the values of the parameters a, b, c, d given above. Choose $x_m = -0.68$, $x_M = -0.76$, $y_m = -0.60$, $y_M = -0.68$, $n = 10\,000$.

(c) Repeat the previous question by taking $d = 0.27$.

Chapter 4

Motion of a Particle

In this chapter we will study the numerical solution of classical equations of motion of one dimensional mechanical systems, e.g. a point particle moving on the line, the simple pendulum etc. We will make an introduction to the numerical integration of ordinary differential equations with initial conditions and in particular to the Euler and Runge-Kutta methods. We study in detail the examples of the damped harmonic oscillator and of the damped pendulum under the influence of an external periodic force. The latter system is nonlinear and exhibits interesting chaotic behavior.

4.1 Numerical Integration of Newton's Equations

Consider the problem of the solution of the dynamical equations of motion of one particle under the influence of a dynamical field given by Newton's law. The equations can be written in the form

$$\frac{d^2\vec{x}}{dt^2} = \vec{a}(t,\vec{x},\vec{v})\,, \tag{4.1}$$

where

$$\vec{a}(t,\vec{x},\vec{v}) \equiv \frac{\vec{F}}{m} \qquad \vec{v} = \frac{d\vec{x}}{dt}\,. \tag{4.2}$$

From the numerical analysis point of view, the problems that we will discuss are initial value problems for ordinary differential equations where

the initial conditions

$$\vec{x}(t_0) = \vec{x}_0 \qquad \vec{v}(t_0) = \vec{v}_0\,, \tag{4.3}$$

determine a unique solution $\vec{x}(t)$. The equations (4.1) are of second order with respect to time and it is convenient to write them as a system of twice as many first order equations:

$$\frac{d\vec{x}}{dt} = \vec{v} \qquad \frac{d\vec{v}}{dt} = \vec{a}(t,\vec{x},\vec{v})\,. \tag{4.4}$$

In particular, we will be interested in the study of the motion of a particle moving on a line (1 dimension), therefore the above equations become

$$\begin{array}{ll} \dfrac{dx}{dt} = v & \dfrac{dv}{dt} = a(t,x,v) \quad \text{1-dimension} \\ x(t_0) = x_0 & v(t_0) = v_0\,. \end{array} \tag{4.5}$$

When the particle moves on the plane (2 dimensions) the equations of motion become

$$\begin{array}{ll} \dfrac{dx}{dt} = v_x & \dfrac{dv_x}{dt} = a_x(t,x,v_x,y,v_y) \quad \text{2-dimensions} \\ \dfrac{dy}{dt} = v_y & \dfrac{dv_y}{dt} = a_y(t,x,v_x,y,v_y) \\ x(t_0) = x_0 & v_x(t_0) = v_{0x} \\ y(t_0) = y_0 & v_y(t_0) = v_{0y}\,, \end{array} \tag{4.6}$$

4.2 Prelude: Euler Methods

As a first attempt to tackle the problem, we will study a simple pendulum of length l in a homogeneous gravitational field g (figure 4.1). The equations of motion are given by the differential equations

$$\begin{aligned} \frac{d^2\theta}{dt^2} &= -\frac{g}{l}\sin\theta \\ \frac{d\theta}{dt} &= \omega\,, \end{aligned} \tag{4.7}$$

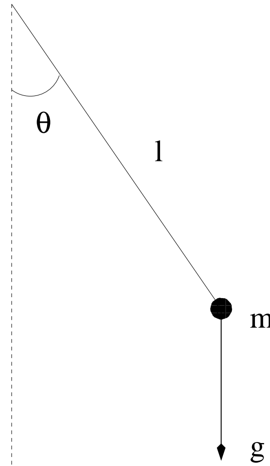

Figure 4.1: A simple pendulum of length l in a homogeneous gravitational field g.

which can be rewritten as a first order system of differential equations

$$\begin{aligned} \frac{d\theta}{dt} &= \omega \\ \frac{d\omega}{dt} &= -\frac{g}{l}\sin\theta \qquad , \end{aligned} \tag{4.8}$$

The equations above need to be written in a discrete form appropriate for a numerical solution with the aid of a computer. We split the interval of time of integration $[t_i, t_f]$ to $N-1$ equal intervals[1] of width $\Delta t \equiv h$, where $h = (t_f - t_i)/(N-1)$. The derivatives are approximated by the relations $(x_{n+1} - x_n)/\Delta t \approx x'_n$, so that

$$\begin{aligned} \omega_{n+1} &= \omega_n + \alpha_n \Delta t \\ \theta_{n+1} &= \theta_n + \omega_n \Delta t\,. \end{aligned} \tag{4.9}$$

where $\alpha = -(g/l)\sin\theta$ is the angular acceleration. This is the so-called *Euler* method. The error at each step is estimated to be of order $(\Delta t)^2$. This is most easily seen by Taylor expanding around the point t_n and neglecting all terms starting from the second derivative and beyond[2].

[1]We have N discrete time points $t_i \equiv t_1, \ldots, t_{N-1}, t_N \equiv t_f$

[2]See appendix 4.7 for retails.

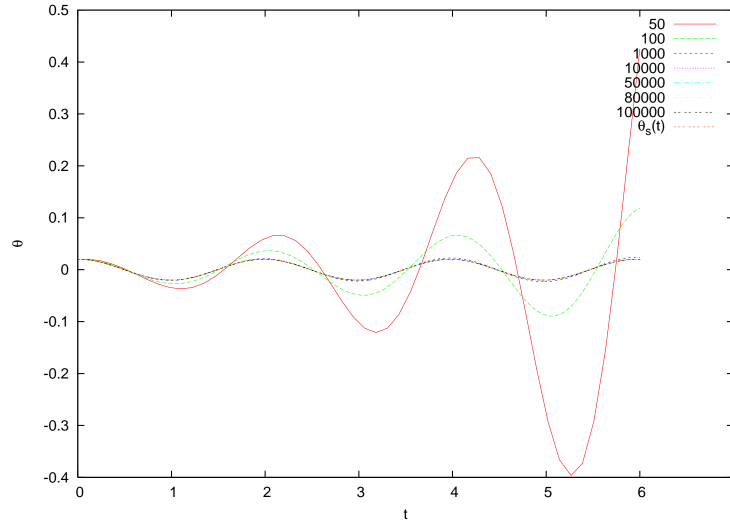

Figure 4.2: Convergence of Euler's method for a simple pendulum with period $T \approx 1.987 (\omega^2 = 10.0)$ for several values of the time step Δt which is determined by the number of integration steps `Nt=` $50 - 100,000$. The solution is given for $\theta_0 = 0.2$, $\omega_0 = 0.0$ and we compare it with the known solution for small angles with $\alpha(t) \approx -(g/l)\,\theta$.

What we are mostly interested in is in the *total* error of the estimate of the functions we integrate for at time t_f! We expect that errors accumulate in an additive way at each integration step, and since the number of steps is $N \propto 1/\Delta t$ the total error should be $\propto (\Delta t)^2 \times (1/\Delta t) = \Delta t$. This is indeed what happens, and we say that Euler's method is a first order method. Its range of applicability is limited and we only study it for academic reasons. Euler's method is *asymmetric* because it uses information only from the beginning of the integration interval $(t, t + \Delta t)$. It can be put in a more balanced form by using the velocity *at the end* of the interval $(t, t + \Delta t)$. This way we obtain the Euler-Cromer method with a slightly improved behavior, but which is still of first order

$$
\begin{aligned}
\omega_{n+1} &= \omega_n + \alpha_n \Delta t \\
\theta_{n+1} &= \theta_n + \omega_{n+1} \Delta t \,.
\end{aligned}
\tag{4.10}
$$

An improved algorithm is the Euler–Verlet method which is of second

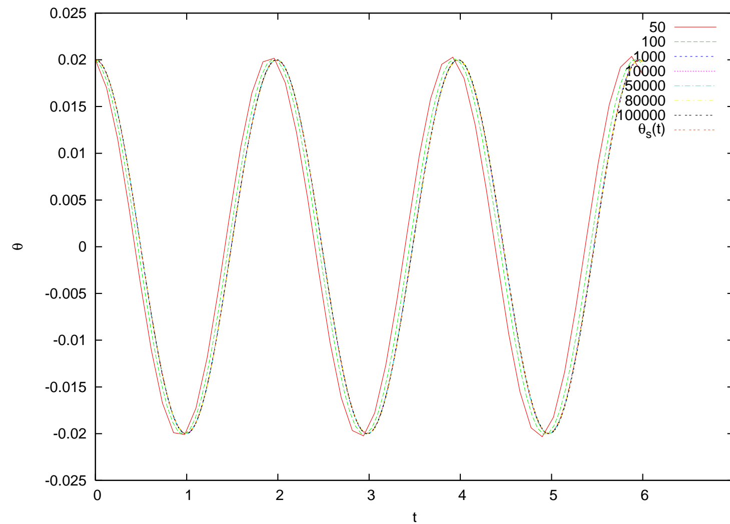

Figure 4.3: Convergence of the Euler-Cromer method, similarly to figure 4.2. We observe a faster convergence compared to Euler's method.

order and gives total error[3] $\sim (\Delta t)^2$. This is given by the equations

$$\begin{aligned} \theta_{n+1} &= 2\theta_n - \theta_{n-1} + \alpha_n (\Delta t)^2 \\ \omega_n &= \frac{\theta_{n+1} - \theta_{n-1}}{2\Delta t} \,. \end{aligned} \qquad (4.11)$$

The price that we have to pay is that we have to use a two step relation in order to advance the solution to the next step. This implies that we have to carefully determine the initial conditions of the problem which are given only at one given time t_i. We make one Euler time step *backwards* in order to define the value of θ_0. If the initial conditions are $\theta_1 = \theta(t_i)$, $\omega_1 = \omega(t_i)$, then we *define*

$$\theta_0 = \theta_1 - \omega_1 \Delta t + \frac{1}{2}\alpha_1 (\Delta t)^2 \,. \qquad (4.12)$$

It is important that at this step the error introduced is not larger than $\mathcal{O}(\Delta t^2)$, otherwise it will spoil and eventually dominate the $\mathcal{O}(\Delta t^2)$ total error of the method introduced by the intermediate steps. At the last

[3] See appendix 4.7 for details.

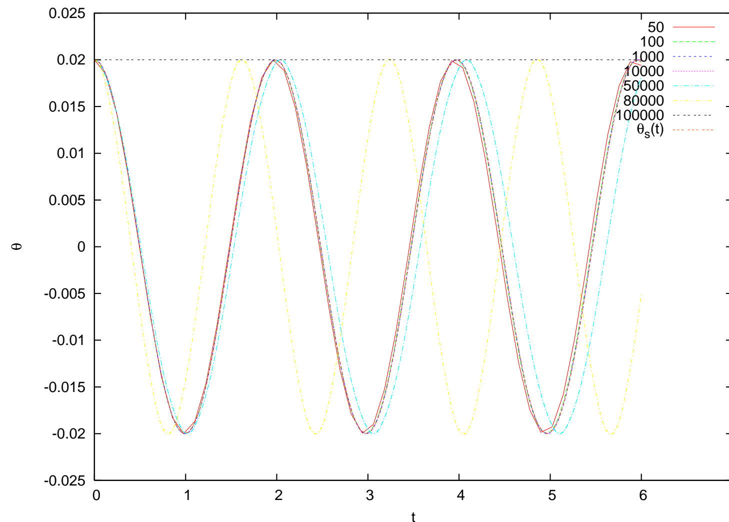

Figure 4.4: Convergence of the Euler-Verlet method, similarly to figure 4.2. We observe a faster convergence than Euler's method, but the roundoff errors make the results useless for Nt$\gtrsim 50,000$ (note what happens when Nt$= 100,000$. Why?).

step we also have to take

$$\omega_N = \frac{\theta_N - \theta_{N-1}}{\Delta t}\,. \tag{4.13}$$

Even though the method has smaller total error than the Euler method, it becomes unstable for small enough Δt due to roundoff errors. In particular, the second equation in (4.11) gives the angular velocity as the ratio of two small numbers. The problem is that the numerator is the result of the subtraction of two almost equal numbers. For small enough Δt, this difference has to be computed from the last digits of the finite representation of the numbers θ_{n+1} and θ_n in the computer memory. The accuracy in the determination of $(\theta_{n+1} - \theta_n)$ decreases until it eventually becomes *exactly* zero. For the first equation of (4.11), the term $\alpha_n \Delta t^2$ is smaller by a factor Δt compared to the term $\alpha_n \Delta t$ in Euler's method. At some point, by decreasing Δt, we obtain $\alpha_n \Delta t^2 \ll 2\theta_n - \theta_{n-1}$ and the accuracy of the method vanishes due to the finite representation of real number in the memory of the computer. When the numbers $\alpha_n \Delta t^2$ and $2\theta_n - \theta_{n-1}$ differ from each other by more that approximately sixteen

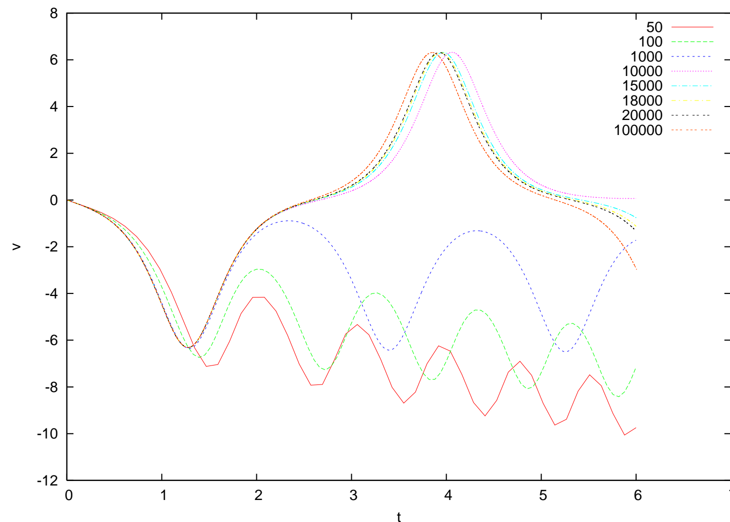

Figure 4.5: Convergence of Euler's method for the simple pendulum like in figure 4.2 for $\theta_0 = 3.0$, $\omega_0 = 0.0$. The behavior of the angular velocity is shown and we notice unstable behavior for `Nt`$\lesssim 1,000$.

orders of magnitude, adding the first one to the second is equivalent to adding zero and the contribution of the acceleration vanishes[4].

Writing programs that implement the methods discussed so far is quite simple. We will write a program that compares the results from all three methods Euler, Euler–Cromer and Euler–Verlet. The main program is mainly a user interface, and the computation is carried out by three functions `euler`, `euler_cromer` and `euler_verlet`. The user must provide the function `accel(x)` which gives the angular acceleration as a function of `x`. The variable `x` in our problem corresponds to the angle $theta$. For starters we take `accel(x)= -10.0 * sin(x)`, the acceleration of the simple pendulum.

The data structure is very simple: Three `double` arrays `T[P]`, `X[P]` and `V[P]` store the times t_n, the angles θ_n and the angular velocities ω_n for $n = 1, \ldots,$ `Nt`. The user determines the time interval for the integration

[4]Numbers of type `double` have approximately sixteen significant digits. The accuracy of the operations described above is determined by the number ϵ, which is the smallest positive number such that $1 + \epsilon > 1$. For variables of type `float`, $\epsilon \approx 1.2 \times 10^{-7}$ and for variables of type `double` $\epsilon \approx 2.2 \times 10^{-16}$.

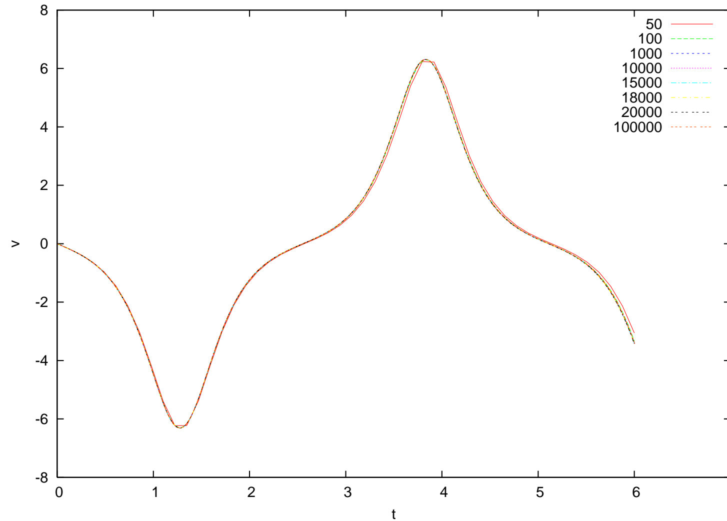

Figure 4.6: Convergence of Euler-Cromer's method, like in figure 4.5. We observe a faster convergence than for Euler's method.

from $t_i = 0$ to $t_f = \texttt{Tfi}$ and the number of discrete times `Nt`. The latter should be less than `P`, the size of the arrays. She also provides the initial conditions $\theta_0 = \texttt{Xin}$ and $\omega_0 = \texttt{Vin}$. After this, we call the main integration functions which take as input the initial conditions, the time interval of the integration and the number of discrete times `Xin,Vin,Tfi,Nt`. The output of the routines is the arrays `T,X,V` which store the results for the time, position and velocity respectively. The results are printed to the files `euler.dat`, `euler_cromer.dat` and `euler_verlet.dat`.

After setting the initial conditions and computing the time step $\Delta t \equiv \texttt{h} = \texttt{Tfi}/(\texttt{Nt} - 1)$, the integration in each of the functions is performed in `for` loops which advance the solution by time Δt. The results are stored at each step in the arrays `T,X,V`. For example, the central part of the program for Euler's method is:

```
T[0] = 0.0;
X[0] = Xin;
V[0] = Vin;
h    = Tfi/(Nt-1);
for(i=1;i<Nt;i++){
```

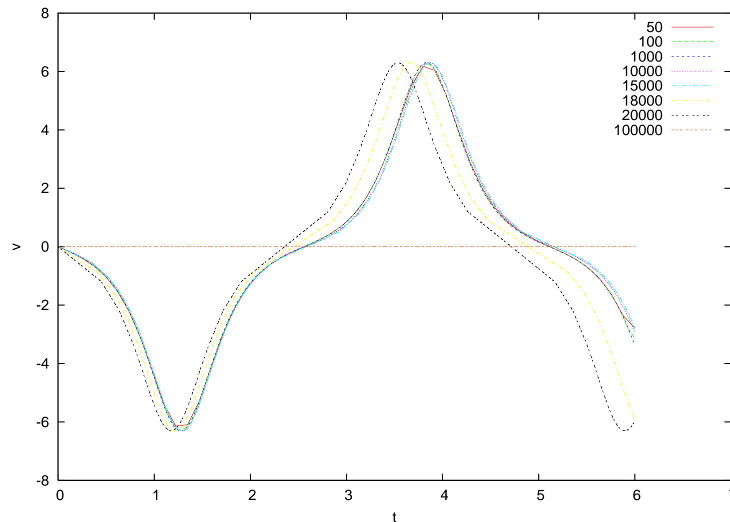

Figure 4.7: Convergence of the Euler-Verlet method, similarly to figure 4.5. We observe a faster convergence compared to Euler's method but that the roundoff errors make the results unstable for `Nt` $\gtrsim 18,000$. In this figure, `float` variables have been used instead of `double` in order to enhance the effect.

```
    T[i] = T[i-1]+h;
    V[i] = V[i-1]+accel(X[i-1])*h;
    X[i] = X[i-1]+V[i]*h;
  }
```

Some care has to be taken in the case of the Euler–Verlet method where one has to initialize the first two steps, as well as take special care for the last step for the velocity:

```
  T[0]      = 0.0;
  X[0]      = Xin;
  V[0]      = Vin;
  X0        = X[0] - V[0] * h + accel(X[0]) * h2 / 2.0;
  T[1]      = h;
  X[1]      = 2.0*X[0] - X0   + accel(X[0]) * h2;
  for(i=2;i<Nt;i++){
    ..............
  }
  V[Nt-1]   = (X[Nt-1] - X[Nt-2])/h;
```

The full program can be found in the file `euler.cpp` and is listed below:

```
//========================================================
//Program to integrate equations of motion for accelerations
//which are functions of x with the method of Euler,
//Euler-Cromer and Euler-Verlet.
//The user sets initial conditions and the functions return
//X[t] and V[t]=dX[t]/dt in arrays
//T[0..Nt-1],X[0..Nt-1],V[0..Nt-1]
//The user provides number of times Nt and the final
//time Tfi. Initial time is assumed to be t_i=0 and the
//integration step h = Tfi/(Nt-1)
//The user programs a real function accel(x) which gives the
//acceleration  dV(t)/dt as function of X.
//NOTE: T[0] = 0 T[Nt-1] = Tfi
//========================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
//--------------------------------------------------------
const int P = 110000;
double T[P], X[P], V[P];
//--------------------------------------------------------
void euler       (const double& Xin, const double& Vin,
                  const double& Tfi, const int    & Nt );
void euler_cromer(const double& Xin, const double& Vin,
                  const double& Tfi, const int    & Nt );
void euler_verlet(const double& Xin, const double& Vin,
                  const double& Tfi, const int    & Nt );
double accel     (const double& x  );
//--------------------------------------------------------
int main(){
  double Xin, Vin, Tfi;
  int    Nt , i;
  string buf;
  //The user provides initial conditions X_0,V_0
  //final time t_f and Nt:
  cout << "Enter X_0,V_0,t_f,Nt (t_i=0):\n";
  cin  >> Xin >> Vin >> Tfi >> Nt; getline(cin,buf);
  //This check is necessary in order to avoid
  //memory access violations:
```

```
  if(Nt>=P){cerr << "Error: Nt>=P\n";exit(1);}
  //Xin= X[0], Vin=V[0], T[0]=0 and the routine gives
  //evolution in T[1..Nt-1], X[1..Nt-1], V[1..Nt-1]
  //which we print in a file
  euler(Xin,Vin,Tfi,Nt);
  ofstream myfile("euler.dat");
  myfile.precision(17);
  for(i=0;i<Nt;i++)
    //Each line in file has time, position, velocity:
    myfile << T[i] << " " << X[i] << " " << V[i] << endl;
  myfile.close();//we close the stream to be reused below
  //------------------------------------
  //We repeat everything for each method
  euler_cromer(Xin,Vin,Tfi,Nt);
  myfile.open("euler_cromer.dat");
  for(i=0;i<Nt;i++)
    myfile << T[i] << " " << X[i] << " " << V[i] << endl;
  myfile.close();
  //------------------------------------
  euler_verlet(Xin,Vin,Tfi,Nt);
  myfile.open("euler_verlet.dat");
  for(i=0;i<Nt;i++)
    myfile << T[i] << " " << X[i] << " " << V[i] << endl;
  myfile.close();
}//main()
//========================================================
//Function which returns the value of acceleration at
//position x used in the integration functions
//euler, euler_cromer and euler_verlet
//========================================================
double accel(const double& x){
  return -10.0 * sin(x);
}
//========================================================
//Driver routine for integrating equations of motion
//using the Euler method
//Input:
//Xin=X[0], Vin=V[0] -- initial condition at t=0,
//Tfi the final time and Nt the number of times
//Output:
//The arrays T[0..Nt-1], X[0..Nt-1], V[0..Nt-1] which
//gives x(t_k)=X[k-1], dx/dt(t_k)=V[k-1], t_k=T[k-1] k=1..Nt
//where for k=1 we have the initial condition.
//========================================================
void euler      (const double& Xin, const double& Vin,
```

```
                    const double& Tfi, const int   & Nt ){
  int    i;
  double h;
  //Initial conditions set here:
  T[0] = 0.0;
  X[0] = Xin;
  V[0] = Vin;
  //h is the time step Dt
  h    = Tfi/(Nt-1);
  for(i=1;i<Nt;i++){
    T[i] = T[i-1]+h;        //time advances by Dt=h
    X[i] = X[i-1]+V[i-1]*h;//advancement and storage of
    V[i] = V[i-1]+accel(X[i-1])*h;//position and velocity
  }
}//euler()
//========================================================
//Driver routine for integrating equations of motion
//using the Euler-Cromer method
//Input:
//Xin=X[0], Vin=V[0] -- initial condition at t=0,
//Tfi the final time and Nt the number of times
//Output:
//The arrays T[0..Nt-1], X[0..Nt-1], V[0..Nt-1] which
//gives x(t_k)=X[k-1], dx/dt(t_k)=V[k-1], t_k=T[k-1] k=1..Nt
//where for k=1 we have the initial condition.
//========================================================
void euler_cromer(const double& Xin, const double& Vin,
                  const double& Tfi, const int   & Nt ){
  int    i;
  double h;
  //Initial conditions set here:
  T[0] = 0.0;
  X[0] = Xin;
  V[0] = Vin;
  //h is the time step Dt
  h    = Tfi/(Nt-1);
  for(i=1;i<Nt;i++){
    T[i] = T[i-1]+h;
    V[i] = V[i-1]+accel(X[i-1])*h;
    X[i] = X[i-1]+V[i]*h; //note difference from Euler
  }

}//euler_cromer()
//========================================================
//Driver routine for integrating equations of motion
```

```
//using the Euler-Verlet method
//Input:
//Xin=X[0], Vin=V[0] -- initial condition at t=0,
//Tfi the final time and Nt the number of times
//Output:
//The arrays T[0..Nt-1], X[0..Nt-1], V[0..Nt-1] which
//gives x(t_k)=X[k-1], dx/dt(t_k)=V[k-1], t_k=T[k-1] k=1..Nt
//where for k=1 we have the initial condition.
//===========================================================
void euler_verlet(const double& Xin, const double& Vin,
                  const double& Tfi, const int    & Nt ){
  int    i;
  double h,h2,X0,o2h;
  //Initial conditions set here:
  T[0]     = 0.0;
  X[0]     = Xin;
  V[0]     = Vin;
  h        = Tfi/(Nt-1); //time step
  h2       = h*h;        //time step squared
  o2h      = 0.5/h;      //  h/2
  //We have to initialize one more step:
  //X0 corresponds to 'X[-1]'
  X0       = X[0] - V[0] * h + accel(X[0]) * h2 / 2.0;
  T[1]     = h;
  X[1]     = 2.0*X[0] - X0   + accel(X[0]) * h2;
  //Now i starts from 2:
  for(i=2;i<Nt;i++){
    T[i]   = T[i-1] + h;
    X[i]   = 2.0*X[i-1] - X[i-2] + accel(X[i-1])*h2;
    V[i-1] = o2h * (X[i]- X[i-2]);
  }
  //we have one more step for the velocity:
  V[Nt-1]  = (X[Nt-1] - X[Nt-2])/h;
}//euler_verlet()
```

Compiling the running the program can be done with the commands:

```
> g++ euler.cpp -o euler
> ./euler
Enter X_0,V_0,t_f,Nt (t_i=0):
0.2 0.0 6.0 1000
> ls  euler*.dat
euler_cromer.dat  euler.dat  euler_verlet.dat
> head -n 5 euler.dat
```

```
0       0.20000  0
0.00600 0.20000 -0.01193
0.01201 0.19992 -0.02386
0.01801 0.19978 -0.03579
0.02402 0.19957 -0.04771
```

The last command shows the first 5 lines of the file `euler.dat`. We see the data for the time, the position and the velocity stored in 3 columns. We can graph the results using `gnuplot`:

```
gnuplot> plot "euler.dat" using 1:2 with lines
gnuplot> plot "euler.dat" using 1:3 with lines
```

These commands result in plotting the positions and the velocities as a function of time respectively. We can add the results of all methods to the last plot with the commands:

```
gnuplot> replot "euler_cromer.dat" using 1:3 with lines
gnuplot> replot "euler_verlet.dat" using 1:3 with lines
```

The results can be seen in figures 4.2–4.7. Euler's method is unstable unless we take a quite small time step. The Euler–Cromer method behaves impressively better. The results converge and remain constant for `Nt`$\sim 100,000$. The Euler–Verlet method converges much faster, but roundoff errors kick in soon. This is more obvious in figure 4.7 where the initial angular position is large[5]. For small angles we can compare with the solution one obtains for the harmonic pendulum ($\sin(\theta)\approx\theta$):

$$\begin{aligned}
\alpha(\theta) &= -\frac{g}{l}\,\theta \equiv -\Omega^2\,\theta \\
\theta(t) &= \theta_0\cos(\Omega t) + (\omega_0/\Omega)\sin(\Omega t) \\
\omega(t) &= \omega_0\cos(\Omega t) - (\theta_0\Omega)\sin(\Omega t)\,. \qquad (4.14)
\end{aligned}$$

In figures 4.2–4.4 we observe that the results agree with the above formulas for the values of Δt where the methods converge. This way we can check our program for bugs. The plot of the functions above can be

[5] In this figure, roundoff errors are enhanced by using `float` variables instead of `double`.

done with the following `gnuplot` commands[6]:

```
gnuplot> set dummy t
gnuplot> omega2  = 10
gnuplot> X0      = 0.2
gnuplot> V0      = 0.0
gnuplot> omega   = sqrt(omega2)
gnuplot> x(t)    = X0 * cos(omega * t) +(V0/omega)*sin(omega*t)
gnuplot> v(t)    = V0 * cos(omega * t) -(omega*X0)*sin(omega*t)
gnuplot> plot x(t), v(t)
```

The results should not be compared only graphically since subtle differences can remain unnoticed. It is more desirable to plot the *differences* of the theoretical values from the numerically computed ones which can be done using the commands:

```
gnuplot> plot "euler.dat" using 1:($2-x($1)) with lines
gnuplot> plot "euler.dat" using 1:($3-v($1)) with lines
```

The command `using 1:($2-x($1))` puts the values found in the first column on the x axis and the value found in the second column minus the value of the function `x(t)` for t equal to the value found in the first column on the y axis. This way, we can make the plots shown in[7] figures 4.11-4.14.

4.3 Runge–Kutta Methods

Euler's method is a one step finite difference method of first order. First order means that the total error introduced by the discretization of the integration interval $[t_i, t_f]$ by N discrete times is of order $\sim \mathcal{O}(h)$, where $h \equiv \Delta t = (t_f - t_i)/N$ is the time step of the integration. In this section we will discuss a generalization of this approach where the total error will be of higher order in h. This is the class of Runge-Kutta methods which are one step algorithms where the total discretization error is of order $\sim \mathcal{O}(h^p)$. The local error introduced at each step is of order $\sim \mathcal{O}(h^{p+1})$

[6] The command `set dummy t` sets the independent variable for functions to be `t` instead of `x` which is the default.

[7] A small modification is necessary in order to plot the *absolute* value of the differences.

leading after $N = (t_f - t_i)/\Delta t$ steps to a maximum error of order

$$\sim \mathcal{O}(h^{p+1}) \times N = \mathcal{O}(h^{p+1}) \times \frac{t_f - t_i}{\Delta t} \sim \mathcal{O}(h^{p+1}) \times \frac{1}{h} = \mathcal{O}(h^p)\,. \qquad (4.15)$$

In such a case we say that we have a Runge-Kutta method of pth order. The price one has to pay for the increased accuracy is the evaluation of the derivatives of the functions in more than one points in the interval $(t, t + \Delta t)$.

Let's consider for simplicity the problem with only one unknown function $x(t)$ which evolves in time according to the differential equation:

$$\frac{dx}{dt} = f(t, x)\,. \qquad (4.16)$$

Consider the first order method first. The most naive approach would

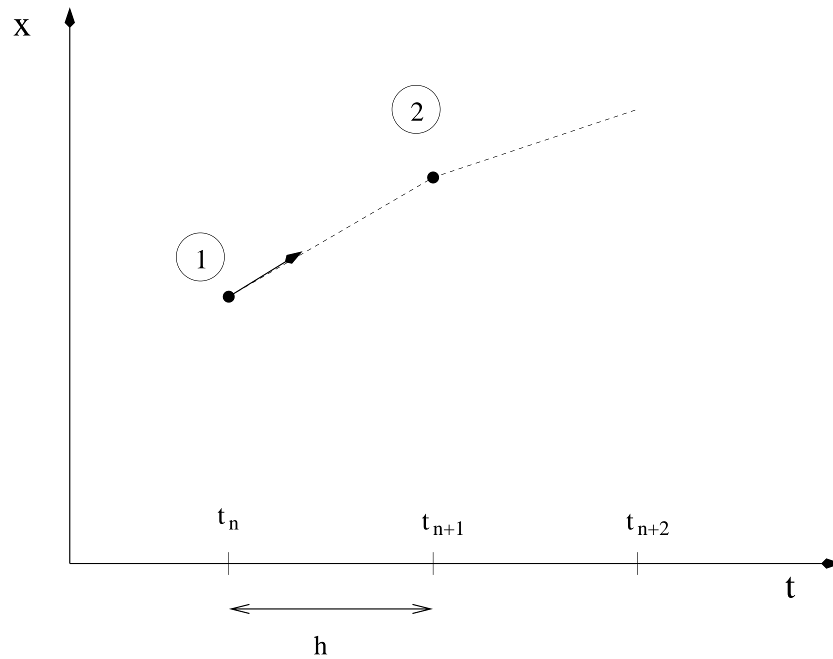

Figure 4.8: The geometry of the step of the Runge-Kutta method of 1st order given by equation (4.17).

be to take the derivative to be given by the finite difference

$$\frac{dx}{dt} \approx \frac{x_{n+1} - x_n}{\Delta t} = f(t_n, x_n) \Rightarrow x_{n+1} = x_n + hf(t_n, x_n)\,. \qquad (4.17)$$

By Taylor expanding, we see that the error at each step is $\mathcal{O}(h^2)$, therefore the error after integrating from $t_i \to t_f$ is $\mathcal{O}(h)$. Indeed,

$$x_{n+1} = x(t_n+h) = x_n + h\frac{dx}{dt}(x_n) + \mathcal{O}(h^2) = x_n + hf(t_n, x_n) + \mathcal{O}(h^2)\,. \quad (4.18)$$

The geometry of the step is shown in figure 4.8. We start from point 1 and by linearly extrapolating in the direction of the derivative $k_1 \equiv f(t_n, x_n)$ we determine the point x_{n+1}.

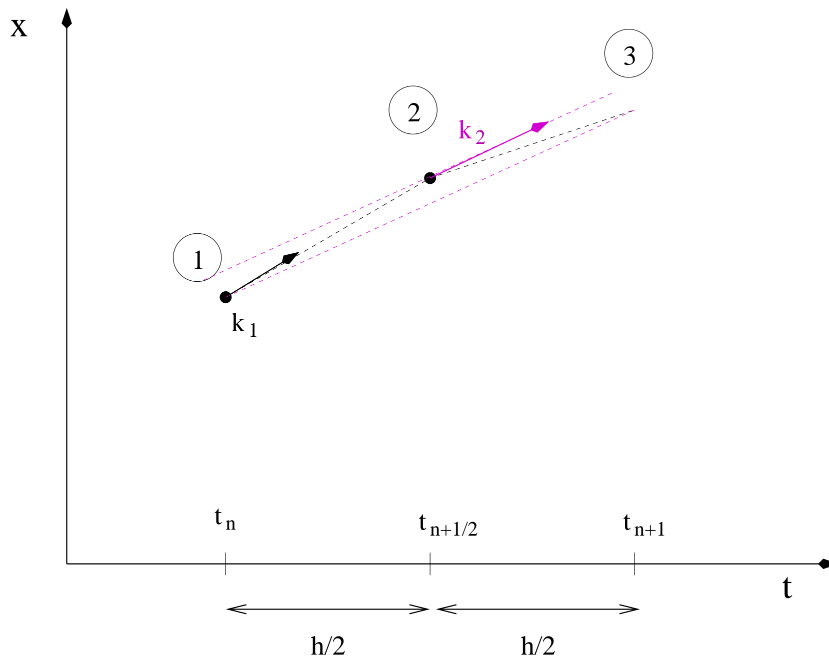

Figure 4.9: The geometry of an integration step of the 2nd order Runge-Kutta method given by equation (4.19).

We can improve the method above by introducing an intermediate point 2. This process is depicted in figure 4.9. We take the point 2 in the middle of the interval (t_n, t_{n+1}) by making a linear extrapolation from x_n in the direction of the derivative $k_1 \equiv f(t_n, x_n)$. Then we use the slope at point 2 as an *estimator* of the derivative within this interval, i.e. $k_2 \equiv f(t_{n+1/2}, x_{n+1/2}) = f(t_n + h/2, x_n + (h/2)k_1)$. We use k_2 to linearly

extrapolate from x_n to x_{n+1}. Summarizing, we have that

$$\begin{aligned} k_1 &= f(t_n, x_n) \\ k_2 &\equiv f(t_n + \frac{h}{2}, x_n + \frac{h}{2} k_1) \\ x_{n+1} &= x_n + h k_2 \,. \end{aligned} \tag{4.19}$$

For the procedure described above we have to evaluate f twice at each step, thereby doubling the computational effort. The error at each step (4.19) becomes $\sim \mathcal{O}(h^3)$, however, giving a total error of $\sim \mathcal{O}(h^2) \sim \mathcal{O}(1/N^2)$. So for given computational time, (4.19) is superior to (4.17).

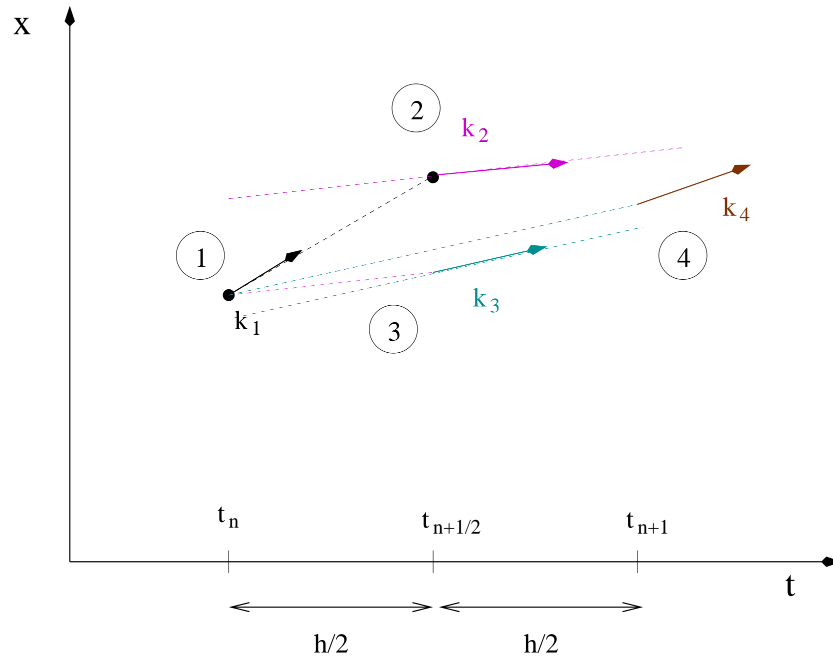

Figure 4.10: The geometry of an integration step of the Runge-Kutta method of 4th order given by equation (4.20).

We can further improve the accuracy gain by using the Runge–Kutta method of 4th order. In this case we have 4 evaluations of the derivative f per step, but the total error becomes now $\sim \mathcal{O}(h^4)$ and the method is superior to that of (4.19)[8]. The process followed is explained geometrically

[8]Not always though! Higher order does not necessarily mean higher accuracy, although this is true in the simple cases considered here.

in figure 4.10. We use 3 intermediate points for evolving the solution from x_n to x_{n+1}. Point 2 is determined by linearly extrapolating from x_n to the midpoint of the interval $(t_n, t_{n+1} = t_n + h)$ by using the direction given by the derivative $k_1 \equiv f(t_n, x_n)$, i.e. $x_2 = x_n + (h/2)k_1$. We calculate the derivative $k_2 \equiv f(t_n + h/2, x_n + (h/2)k_1)$ at the point 2 and we use it in order to determine point 3, also located at the midpoint of the interval (t_n, t_{n+1}). Then we calculate the derivative $k_3 \equiv f(t_n + h/2, x_n + (h/2)k_2)$ at the point 3 and we use it to linearly extrapolate to the *end* of the interval (t_n, t_{n+1}), thereby obtaining point 4, i.e. $x_4 = x_n + hk_3$. Then we calculate the derivative $k_4 \equiv f(t_n + h, x_n + hk_3)$ at the point 4, and we use all four derivative k_1, k_2, k_3 and k_4 as estimators of the derivative of the function in the interval (t_n, t_{n+1}). If each derivative contributes with a particular weight in this estimate, the discretization error can become $\sim \mathcal{O}(h^5)$. Such a choice is

$$
\begin{aligned}
k_1 &= f(t_n, x_n) \\
k_2 &= f(t_n + \frac{h}{2}, x_n + \frac{h}{2}\, k_1) \\
k_3 &= f(t_n + \frac{h}{2}, x_n + \frac{h}{2}\, k_2) \\
k_4 &= f(t_n + h, x_n + h\, k_3) \\
x_{n+1} &= x_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)\,.
\end{aligned}
\tag{4.20}
$$

We note that the second term of the last equation takes an average of the four derivatives with weights 1/6, 1/3, 1/3 and 1/6 respectively. A generic small change in these values will increase the discretization error to worse than h^5.

We remind to the reader the fact that by decreasing h the discretization errors decrease, but that roundoff errors will start showing up for small enough h. Therefore, a careful determination of h that minimizes the total error should be made by studying the dependence of the results as a function of h.

4.3.1 A Program for the 4th Order Runge–Kutta

Consider the problem of the motion of a particle in one dimension. For this, we have to integrate a system of two differential equations (4.5) for

two unknown functions of time $x_1(t) \equiv x(t)$ and $x_2(t) \equiv v(t)$ so that

$$\frac{dx_1}{dt} = f_1(t, x_1, x_2) \qquad \frac{dx_2}{dt} = f_2(t, x_1, x_2) \tag{4.21}$$

In this case, equations (4.20) generalize to:

$$\begin{aligned}
k_{11} &= f_1(t_n, x_{1,n}, x_{2,n}) \\
k_{21} &= f_2(t_n, x_{1,n}, x_{2,n}) \\
k_{12} &= f_1(t_n + \frac{h}{2}, x_{1,n} + \frac{h}{2}\, k_{11}, x_{2,n} + \frac{h}{2}\, k_{21}) \\
k_{22} &= f_2(t_n + \frac{h}{2}, x_{1,n} + \frac{h}{2}\, k_{11}, x_{2,n} + \frac{h}{2}\, k_{21}) \\
k_{13} &= f_1(t_n + \frac{h}{2}, x_{1,n} + \frac{h}{2}\, k_{12}, x_{2,n} + \frac{h}{2}\, k_{22}) \\
k_{23} &= f_2(t_n + \frac{h}{2}, x_{1,n} + \frac{h}{2}\, k_{12}, x_{2,n} + \frac{h}{2}\, k_{22}) \\
k_{14} &= f_1(t_n + h, x_{1,n} + h\, k_{13}, x_{2,n} + h\, k_{23}) \\
k_{24} &= f_2(t_n + h, x_{1,n} + h\, k_{13}, x_{1,n} + h\, k_{23}) \\
x_{1,n+1} &= x_{1,n} + \frac{h}{6}(k_{11} + 2k_{12} + 2k_{13} + k_{14}) \\
x_{2,n+1} &= x_{1,n} + \frac{h}{6}(k_{21} + 2k_{22} + 2k_{23} + k_{24})\,.
\end{aligned} \tag{4.22}$$

Programming this algorithm is quite simple. The main program is an interface between the user and the driver routine of the integration. The user enters the initial and final times $t_i =$ `Ti` and $t_f =$ `Tf` and the number of discrete time points `Nt`. The initial conditions are $x_1(t_i) =$ `X10`, $x_2(t_i) =$ `X20`. The main data structure consists of three global `double` arrays `T[P]`, `X1[P]`, `X2[P]` which store the times $t_i \equiv t_1, t_2, \ldots, t_{\texttt{Nt}} \equiv t_f$ and the corresponding values of the functions $x_1(t_k)$ and $x_2(t_k)$, $k = 1, \ldots,$ `Nt`. The main program calls the driver routine `RK(Ti,Tf,X10,X20,Nt)` which "drives" the heart of the program, the function `RKSTEP(t,x1,x2,dt)` which performs one integration step using equations (4.22). `RKSTEP` evolves the functions `x1`, `x2` at time `t` by one step $h =$ `dt`. The function `RK` stores the calculated values in the arrays `T`, `X1` and `X2` at each step. When `RK` returns the control to the main program, all the results are stored in `T`, `X1` and `X2`, which are subsequently printed in the file `rk.dat`. The full program is listed below and can be found in the file `rk.cpp`:

```
//========================================================
//Program to solve a 2 ODE system using Runge-Kutta Method
//User must supply derivatives
//dx1/dt=f1(t,x1,x2) dx2/dt=f2(t,x1,x2)
//as real functions
//Output is written in file rk.dat
//========================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
//--------------------------------------------------------
const int P = 110000;
double T[P], X1[P], X2[P];
//--------------------------------------------------------
double
f1(const double& t  , const double& x1, const double& x2);
double
f2(const double& t  , const double& x1, const double& x2);
void
RK(const double& Ti , const double& Tf, const double& X10,
   const double& X20, const int   & Nt);
void
RKSTEP(double& t, double& x1, double& x2,
       const double& dt);
//--------------------------------------------------------
int main(){
  double Ti,Tf,X10,X20;
  int    Nt;
  int     i;
  string buf;

  //Input:
  cout << "Runge-Kutta Method for 2-ODEs Integration\n";
  cout << "Enter Nt,Ti,TF,X10,X20:"    << endl;
  cin  >> Nt >> Ti >> Tf >> X10 >> X20;getline(cin,buf);
  cout << "Nt = "                 << Nt  << endl;
  cout << "Time: Initial Ti = " << Ti
       << " Final Tf = "          << Tf  << endl;
  cout << "           X1(Ti)= " << X10
       << " X2(Ti)= "             << X20 << endl;
  if(Nt >= P){cerr << "Error! Nt >= P\n";exit(1);}
  //Calculate:
```

```
  RK(Ti,Tf,X10,X20,Nt);
  //Output:
  ofstream myfile("rk.dat");
  myfile.precision(17);
  for(i=0;i<Nt;i++)
    myfile << T [i] << " "
           << X1[i] << " "
           << X2[i] << '\n';
  myfile.close();
}//main()
//========================================================
//The functions f1,f2(t,x1,x2) provided by the user
//========================================================
double
f1(const double& t, const double& x1, const double& x2){
  return x2;
}
//--------------------------------------------------------
double
f2(const double& t, const double& x1, const double& x2){
  return -10.0*x1; //harmonic oscillator
}
//========================================================
//RK(Ti,Tf,X10,X20,Nt) is the driver
//for the Runge-Kutta integration routine RKSTEP
//Input: Initial and final times Ti,Tf
//       Initial values at t=Ti  X10,X20
//       Number of steps of integration: Nt-1
//Output: values in arrays T[Nt],X1[Nt],X2[Nt] where
//T[0]    = Ti X1[0]   = X10 X2[0] = X20
//             X1[k-1] = X1(at t=T(k))
//             X2[k-1] = X2(at t=T(k))
//T[Nt-1] = Tf
//========================================================
void
RK(const double& Ti , const double& Tf, const double& X10,
   const double& X20, const int   & Nt){
  double dt;
  double TS,X1S,X2S; //time and X1,X2 at given step
  int     i;
  //Initialize variables:
  dt      = (Tf-Ti)/(Nt-1);
  T [0]   = Ti;
  X1[0]   = X10;
  X2[0]   = X20;
```

```
  TS      = Ti;
  X1S     = X10;
  X2S     = X20;
  //Make RK steps: The arguments of RKSTEP are
  //replaced with the new ones!
  for(i=1;i<Nt;i++){
    RKSTEP(TS,X1S,X2S,dt);
    T [i]  = TS;
    X1[i]  = X1S;
    X2[i]  = X2S;
  }
}//RK()
//========================================================
//Function RKSTEP(t,x1,x2,dt)
//Runge-Kutta Integration routine of ODE
//dx1/dt=f1(t,x1,x2) dx2/dt=f2(t,x1,x2)
//User must supply derivative functions:
//real function f1(t,x1,x2)
//real function f2(t,x1,x2)
//Given initial point (t,x1,x2) the routine advances it
//by time dt.
//Input : Inital time t    and function values x1,x2
//Output: Final  time t+dt and function values x1,x2
//Careful!: values of t,x1,x2 are overwritten...
//========================================================
void
RKSTEP(double& t, double& x1, double& x2,
       const double& dt){
  double k11,k12,k13,k14,k21,k22,k23,k24;
  double h,h2,h6;

  h  =dt;     //h =dt, integration step
  h2 =0.5*h; //h2=h/2
  h6 =h/6.0; //h6=h/6

  k11=f1(t,x1,x2);
  k21=f2(t,x1,x2);
  k12=f1(t+h2,x1+h2*k11,x2+h2*k21);
  k22=f2(t+h2,x1+h2*k11,x2+h2*k21);
  k13=f1(t+h2,x1+h2*k12,x2+h2*k22);
  k23=f2(t+h2,x1+h2*k12,x2+h2*k22);
  k14=f1(t+h ,x1+h *k13,x2+h *k23);
  k24=f2(t+h ,x1+h *k13,x2+h *k23);

  t  =t+h;
```

```
  x1 =x1+h6*(k11+2.0*(k12+k13)+k14);
  x2 =x2+h6*(k21+2.0*(k22+k23)+k24);

}//RKSTEP()
```

4.4 Comparison of the Methods

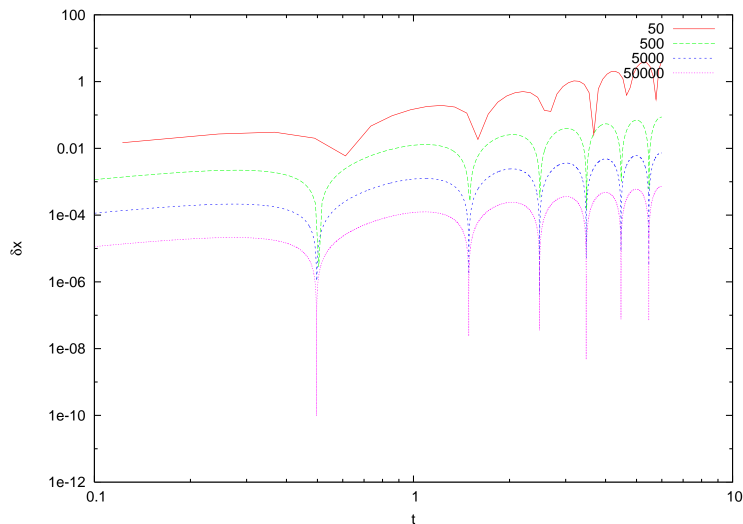

Figure 4.11: The discrepancy of the numerical results of the Euler method from the analytic solution for the simple harmonic oscillator. The parameters chosen are $\omega^2 = 10$, $t_i = 0$, $t_f = 6$, $x(0) = 0.2$, $v(0) = 0$ and the number of steps is $N = 50, 500, 5,000, 50,000$. Observe that the error becomes approximately ten times smaller each time according to the expectation of being of order $\sim \mathcal{O}(\Delta t)$.

In this section we will check our programs for correctness and accuracy w.r.t. discretization and roundoff errors. The simplest test is to check the results against a known analytic solution of a simple model. This will be done for the simple harmonic oscillator. We will change the functions that compute the acceleration of the particle to give $a = -\omega^2 x$. We will take $\omega^2 = 10$ ($T \approx 1.987$). Therefore the relevant part of the program in `euler.cpp` becomes

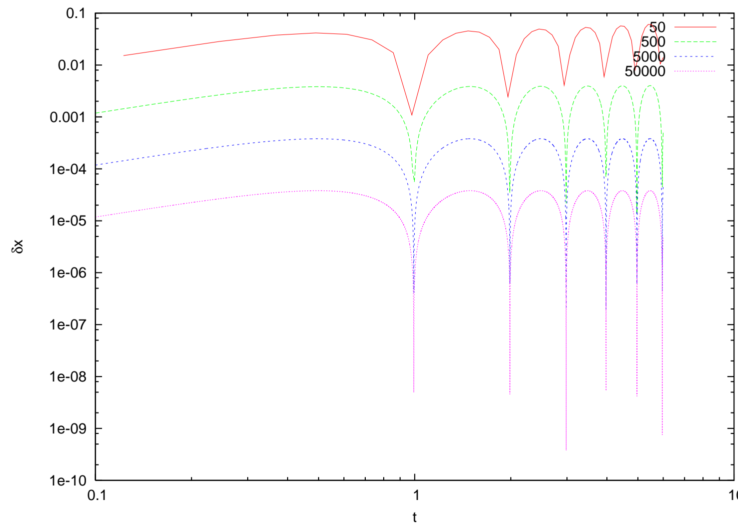

Figure 4.12: Like in figure 4.11 for the Euler-Cromer method. The error becomes approximately ten times smaller each time according to the expectation of being of order $\sim \mathcal{O}(\Delta t)$.

```
double accel(const double& x){
  return -10.0 * x;
}
```

and that of the program in `rk.cpp` becomes

```
double
f2(const double& t, const double& x1, const double& x2){
  return -10.0*x1;
}
```

The programs are run for a given time interval $t_i = 0$ to $t_f = 6$ with the initial conditions $x_0 = 0.2$, $v_0 = 0$. The time step Δt is varied by varying the number of steps `Nt-1`. The computed numerical solution is compared to the well known solution for the simple harmonic oscillator

$$
\begin{aligned}
a(x) &= -\omega^2 x \\
x_h(t) &= x_0 \cos(\omega t) + (v_0/\omega)\sin(\omega t) \\
v_h(t) &= v_0 \cos(\omega t) - (x_0\omega)\sin(\omega t)\,,
\end{aligned}
\tag{4.23}
$$

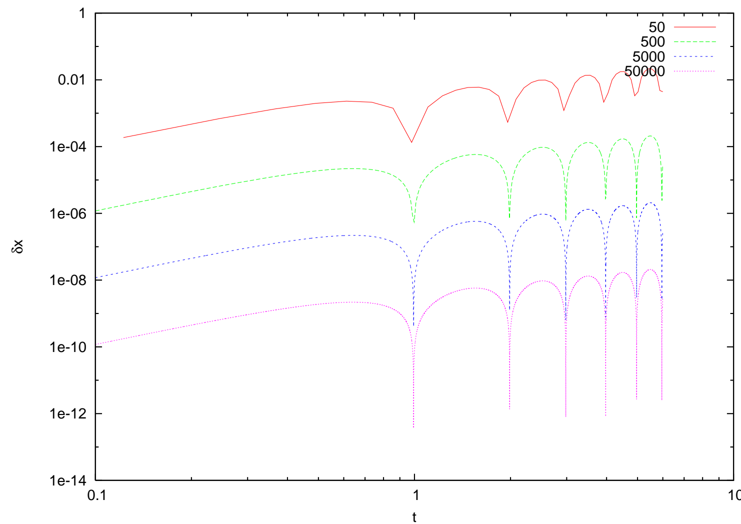

Figure 4.13: Like in figure 4.11 for the Euler-Verlet method. The error becomes approximately 100 times smaller each time according to the expectation of being of order $\sim \mathcal{O}(\Delta t^2)$.

We study the deviation $\delta x(t) = |x(t) - x_h(t)|$ and $\delta v(t) = |v(t) - v_h(t)|$ as a function of the time step Δt. The results are shown in figures 4.11–4.14. We note that for the Euler method and the Euler–Cromer method, the errors are of order $\mathcal{O}(\Delta t)$ as expected. However, the latter has smaller errors compared to the first one. For the Euler–Verlet method, the error turns out to be of order $\mathcal{O}(\Delta t^2)$ whereas for the 4th order Runge–Kutta is of order[9] $\mathcal{O}(\Delta t^4)$.

Another way for checking the numerical results is by looking at a conserved quantity, like the energy, momentum or angular momentum, and study its deviation from its original value. In our case we study the mechanical energy

$$E = \frac{1}{2}mv^2 + \frac{1}{2}m\omega^2 x^2 \tag{4.24}$$

which is computed at each step. The deviation $\delta E = |E - E_0|$ is shown in figures 4.15–4.18.

[9]The reader should confirm these claims, initially by looking at the figures 4.11-4.14 and then by reproducing these results. A particular time t can be chosen and the errors can be plotted against Δt, Δt^2 and Δt^4 respectively.

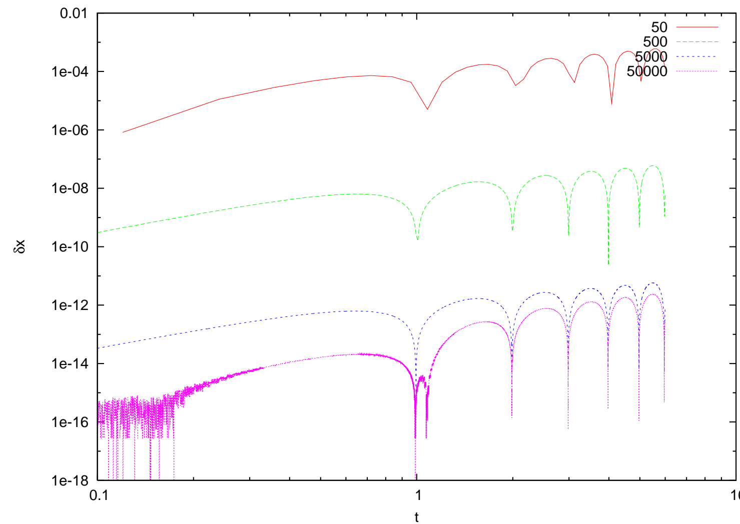

Figure 4.14: Like in figure 4.11 for the 4th order Runge–Kutta method. The error becomes approximately 10^{-4} times smaller each time according to the expectation of being of order $\sim \mathcal{O}(\Delta t^4)$. The roundoff errors become apparent for $50,000$ steps.

4.5 The Forced Damped Oscillator

In this section we will study a simple harmonic oscillator subject to a damping force proportional to its velocity and an external periodic driving force, which for simplicity will be taken to have a sinusoidal dependence in time,

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = a_0 \sin\omega t\,, \tag{4.25}$$

where $F(t) = ma_0 \sin\omega t$ and ω is the angular frequency of the driving force.

Consider initially the system without the influence of the driving force, i.e. with $a_0 = 0$. The real solutions of the differential equation[10] which are finite for $t \to +\infty$ are given by

$$x_0(t) = c_1 \mathrm{e}^{-(\gamma+\sqrt{\gamma^2-4\omega_0^2})t/2} + c_2 \mathrm{e}^{-(\gamma-\sqrt{\gamma^2-4\omega_0^2})t/2}\,, \quad \gamma^2 - 4\omega_0^2 > 0\,, \tag{4.26}$$

[10] These are easily obtained by substituting the ansatz $x(t) = A\mathrm{e}^{-\Omega t}$ and solving for Ω.

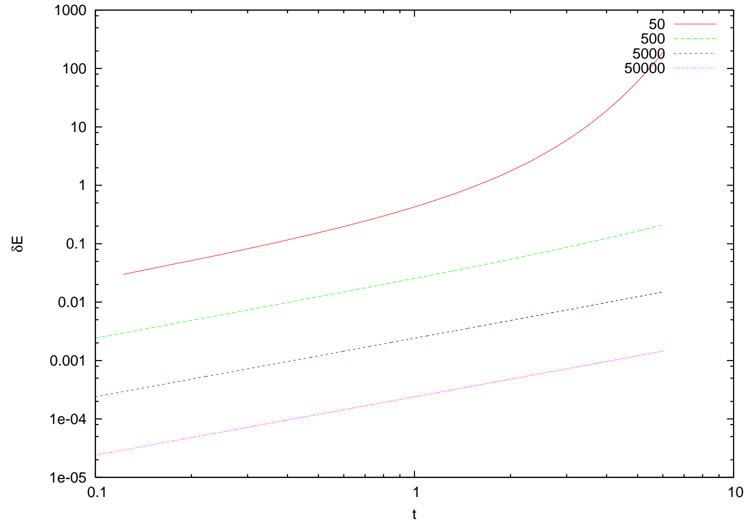

Figure 4.15: Like in figure 4.11 for the case of mechanical energy for the Euler method.

$$x_0(t) = c_1 \mathrm{e}^{-\gamma t/2} + c_2 \mathrm{e}^{-\gamma t/2} t\,, \quad \gamma^2 - 4\omega_0^2 = 0\,, \tag{4.27}$$

$$\begin{aligned} x_0(t) &= c_1 \mathrm{e}^{-\gamma t/2} \cos\left(\sqrt{-\gamma^2 + 4\omega_0^2}\; t/2\right) \\ &\quad + c_2 \mathrm{e}^{-\gamma t/2} \sin\left(\sqrt{-\gamma^2 + 4\omega_0^2}\; t/2\right)\,, \quad \gamma^2 - 4\omega_0^2 < 0\,. \end{aligned} \tag{4.28}$$

In the last case, the solution oscillates with an amplitude decreasing exponentially with time.

In the $a_0 > 0$ case, the general solution is obtained from the sum of a special solution $x_s(t)$ and the solution of the homogeneous equation $x_0(t)$. A special solution can be obtained from the ansatz $x_s(t) = A \sin \omega t + B \cos \omega t$, which when substituted in (4.25) and solved for A and B we find that

$$x_s(t) = \frac{a_0 \left[(\omega_0^2 - \omega^2) \cos \omega t + \gamma\omega \sin \omega t\right]}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}\,, \tag{4.29}$$

and

$$x(t) = x_0(t) + x_s(t)\,. \tag{4.30}$$

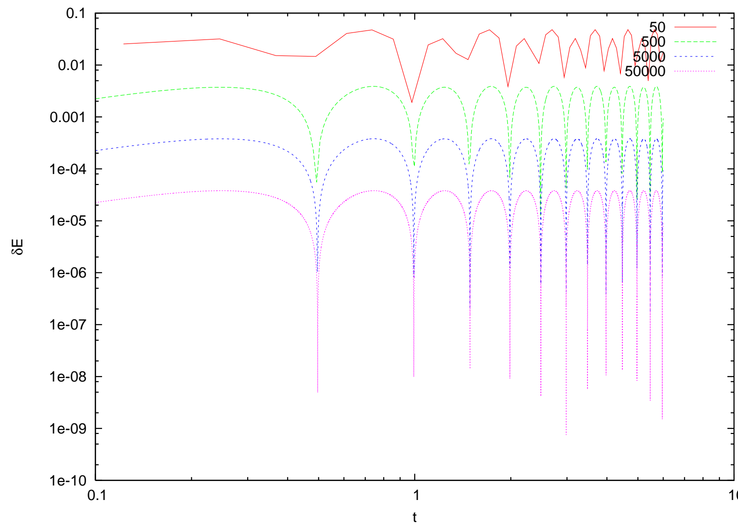

Figure 4.16: Like in figure 4.11 for the case of mechanical energy for the Euler–Cromer method.

The solution $x_0(t)$ decreases exponentially with time and eventually only $x_s(t)$ remains. The only case where this is not true, is when we have resonance without damping for $\omega = \omega_0$, $\gamma = 0$. In that case the solution is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{a_0}{4\omega^2} \left(\cos \omega t + 2(\omega t) \sin \omega t \right) . \tag{4.31}$$

The first two terms are the same as that of the simple harmonic oscillator. The last one increases the amplitude linearly with time, which is a result of the influx of energy from the external force to the oscillator.

Our program will be a simple modification of the program in `rk.cpp`. The main routines `RK(T0,TF,X10,X20,Nt)` and `RKSTEP(t,x1,x2,dt)` remain as they are. We only change the user interface. The basic parameters ω_0, ω, γ, a_0 are entered interactively by the user from the standard input `stdin`. These parameters should be accessible also by the function `f2(t,x1,x2)` and they are declared within the global scope. Another point that needs our attention is the function `f2(t,x1,x2)` which now takes the velocity $v \to$ `x2` in its arguments:

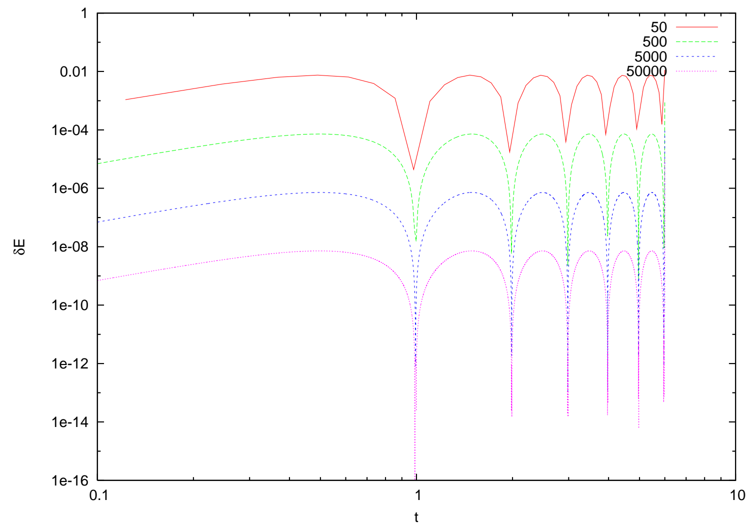

Figure 4.17: Like in figure 4.11 for the case of mechanical energy for the Euler–Verlet method.

```
double
f2(const double& t, const double& x1, const double& x2){
  double a;
  a = a_0*cos(omega*t);
  return -omega_02*x1-gam*x2+a;
}
```

The main program, found in the file `dlo.cpp`, is listed below. The functions `RK`, `RKSTEP` are the same as in `rk.cpp` and should also be included in the same file.

```
//========================================================
//Program to solve Damped Linear Oscillator
//using 4th order Runge-Kutta Method
//Output is written in file dlo.dat
//========================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
```

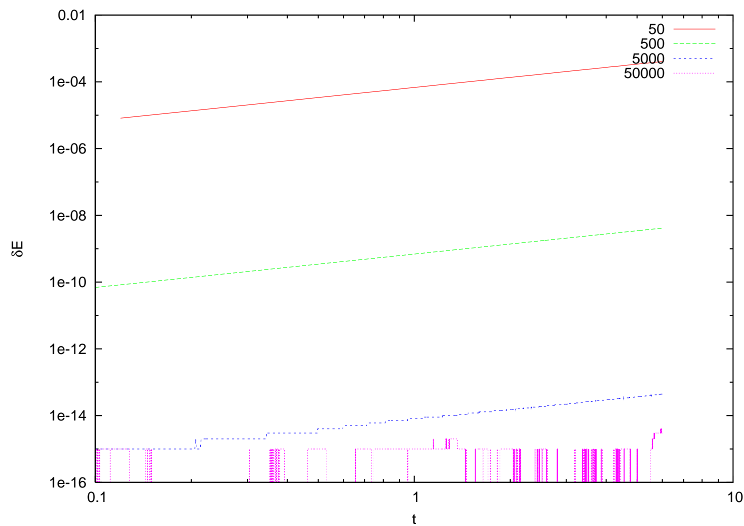

Figure 4.18: Like in figure 4.11 for the case of mechanical energy for the 4th order Runge–Kutta method. Roundoff errors appear for large enough number of steps.

```
//--------------------------------------------------------
const int P = 110000;
double T[P], X1[P], X2[P];
double omega_0,omega,gam,a_0,omega_02,omega2;
//--------------------------------------------------------
double
f1(const double& t  , const double& x1, const double& x2);
double
f2(const double& t  , const double& x1, const double& x2);
void
RK(const double& Ti , const double& Tf, const double& X10,
   const double& X20, const int   & Nt);
void
RKSTEP(double& t, double& x1, double& x2,
       const double& dt);
//--------------------------------------------------------
int main(){
  double Ti,Tf,X10,X20;
  double Energy;
  int    Nt;
  int     i;
  string buf;
```

```
  //Input:
  cout << "Runge-Kutta Method for DLO Integration\n";
  cout << "Enter omega_0, omega, gamma, a_0:\n";
  cin  >> omega_0>> omega>> gam>> a_0;getline(cin,buf);
  omega_02 = omega_0*omega_0;
  omega2   = omega  *omega;
  cout << "omega_0= " << omega_0
       << "  omega= " << omega         << endl;
  cout << "gamma=   " << gamma
       << "  a_0=   " << a_0           << endl;
  cout << "Enter Nt,Ti,TF,X10,X20:"    << endl;
  cin  >> Nt >> Ti >> Tf >> X10 >> X20;getline(cin,buf);
  cout << "Nt = "                 << Nt  << endl;
  cout << "Time: Initial Ti = " << Ti
       << " Final Tf = "          << Tf  << endl;
  cout << "            X1(Ti)= " << X10
       << " X2(Ti)= "             << X20 << endl;
  if(Nt >= P){cerr << "Error! Nt >= P\n";exit(1);}
  //Calculate:
  RK(Ti,Tf,X10,X20,Nt);
  //Output:
  ofstream myfile("dlo.dat");
  myfile.precision(17);
  myfile << "# Damped Linear Oscillator - dlo\n";
  myfile << "# omega_0= " << omega_0 << " omega= " << omega
         << "   gamma= " << gam     << "   a_0= " << a_0 << ↩
            endl;
  for(i=0;i<Nt;i++){
    Energy = 0.5*X2[i]*X2[i]+0.5*omega_02*X1[i]*X1[i];
    myfile << T [i]  << " "
           << X1[i]  << " "
           << X2[i]  << " "
           << Energy << '\n';
  }
  myfile.close();
}//main()
//========================================================
//The functions f1,f2(t,x1,x2) provided by the user
//========================================================
double
f1(const double& t, const double& x1, const double& x2){
  return x2;
}
//--------------------------------------------------------
double
```

```
f2(const double& t, const double& x1, const double& x2){
  double a;
  a = a_0*cos(omega*t);
  return -omega_02*x1-gam*x2+a;
}
```

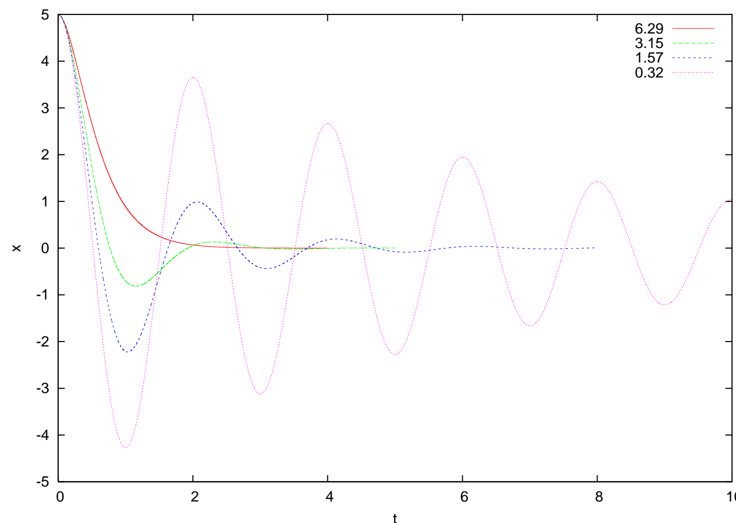

Figure 4.19: The position as a function of time for the damped oscillator for several values of γ and $\omega_0 = 3.145$.

The results are shown in figures 4.19–4.22. Figure 4.19 shows the transition from a damped motion for $\gamma > 2\omega_0$ to an oscillating motion with damping amplitude for $\gamma < 2\omega_0$. The exponential decrease of the amplitude is shown in figure 4.21, whereas the dependence of the period T from the damping coefficient γ is shown in figure 4.22. Motivated by equation (4.28), written in the form

$$4\omega_0^2 - \left(\frac{2\pi}{T}\right) = \gamma^2 \,, \tag{4.32}$$

we construct the plot in figure 4.22. The right hand side of the equation is put on the horizontal axis, whereas the left hand side on the vertical. Equation (4.32) predicts that both quantities are equal and all measurements should lie on a particular line, the diagonal $y = x$. The period T

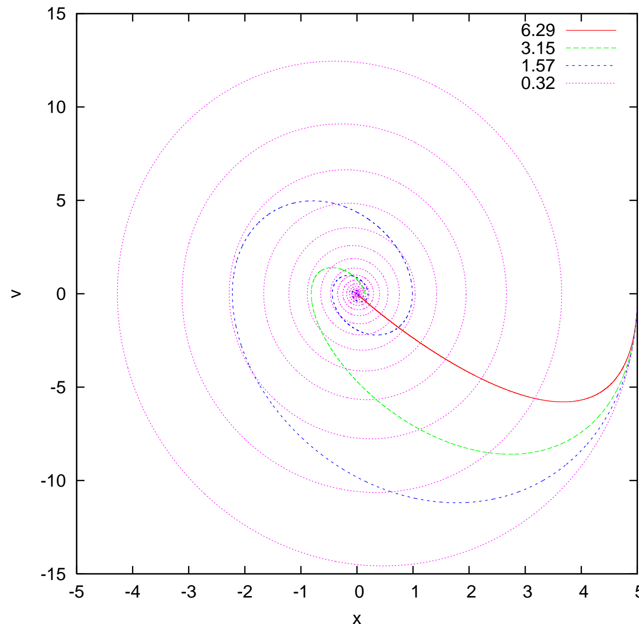

Figure 4.20: The phase space trajectory for the damped oscillator for several values of γ and $\omega_0 = 3.145$. Note the attractor at $(x, v) = (0, 0)$ where all trajectories are "attracted to" as $t \to +\infty$.

can be estimated from the time between two consecutive extrema of $x(t)$ or two consecutive zeros of the velocity $v(t)$ (see figure 4.19).

Finally it is important to study the trajectory of the system in phase space. This can be seen[11] in figure 4.20. A point in this space is a *state* of the system and a trajectory describes the evolution of the system's states in time. We see that all such trajectories end up as $t \to +\infty$ to the point $(0, 0)$, independently of the initial conditions. Such a point is an example of a system's *attractor*.

Next, we add the external force and study the response of the system to it. The system exhibits a *transient behavior* that depends on the initial conditions. For large enough times it approaches a *steady state* that does not depend on (almost all of) the initial conditions. This can be seen in figure 4.23. This is easily understood for our system by looking at equa-

[11]To be precise, phase space is the space of positions-momenta, but in our case the difference is trivial.

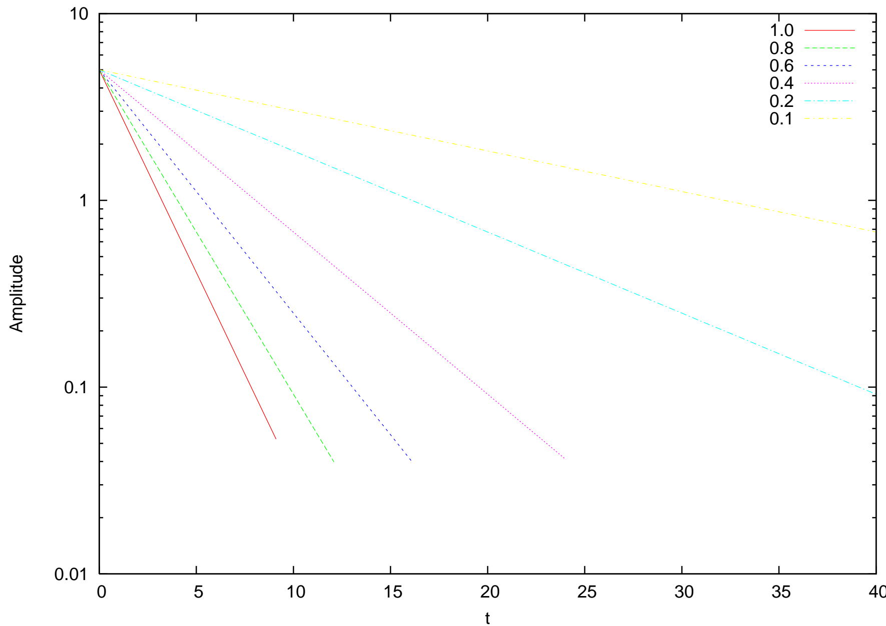

Figure 4.21: The amplitude of oscillation for the damped oscillator for several values of γ and $\omega_0 = 3.145$. Note the exponential damping of the amplitude with time.

tions (4.26)–(4.28). We see that the steady state $x_s(t)$ becomes dominant when the exponentials have damped away. $x_s(t)$ can be written in the form

$$\begin{aligned} x(t) &= x_0(\omega)\cos(\omega t + \delta(\omega)) \\ x_0(\omega) &= \frac{a_0}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}\,, \qquad \tan\delta(\omega) = \frac{\omega\gamma}{\omega^2-\omega_0^2}\,. \end{aligned} \tag{4.33}$$

These equations are verified in figure 4.24 where we study the dependence of the amplitude $x_0(\omega)$ on the angular frequency of the driving force. Finally we study the trajectory of the system in phase space. As we can see in figure 4.20, this time the attractor is an ellipse, which is a one dimensional curve instead of a zero dimensional point. For large enough times, all trajectories approach their attractor asymptotically.

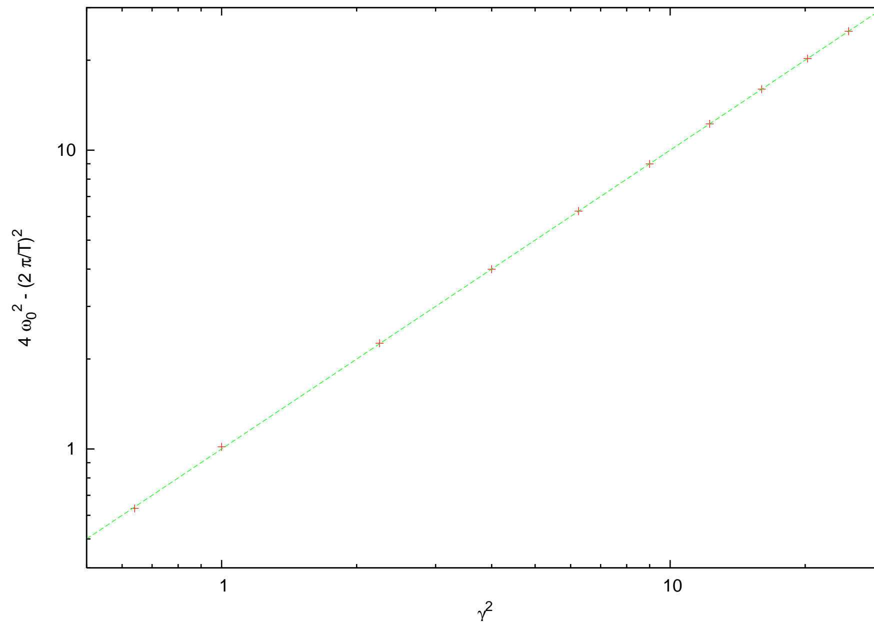

Figure 4.22: The period of oscillation of the damped oscillator for several values of γ and $\omega_0 = 3.145$. The axes are chosen so that equation (4.28) $(2\pi/T)^2 = 4\omega_0^2 - \gamma^2$ can be easily verified. The points in the plot are our measurements whereas the straight line is the theoretical prediction, the diagonal $y = x$

4.6 The Forced Damped Pendulum

In this section we will study a non-linear dynamical system which exhibits interesting chaotic behavior. This is a simple model which, despite its deterministic nature, the prediction of its future behavior becomes intractable after a short period of time. Consider a simple pendulum in a constant gravitational field whose motion is damped by a force proportional to its velocity and it is under the influence of a vertical, harmonic external driving force:

$$\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega_0^2 \sin\theta = -2A\cos\omega t\, \sin\theta\,. \tag{4.34}$$

In the equation above, θ is the angle of the pendulum with the vertical axis, γ is the damping coefficient, $\omega_0^2 = g/L$ is the pendulum's natural angular frequency, ω is the angular frequency of the driving force and

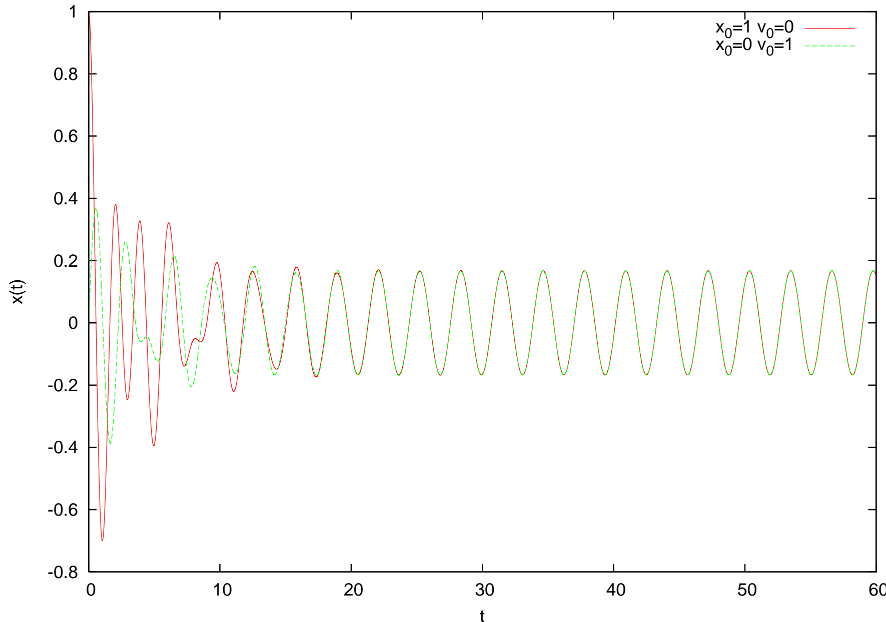

Figure 4.23: The period of oscillation for the forced damped oscillator for different initial conditions. We have chosen $\omega_0 = 3.145$, $\omega = 2.0$, $\gamma = 0.5$ and $a_0 = 1.0$. We note that after the transient behavior the system oscillates harmonically according to the relation $x(t) = x_0(\omega)\cos(\omega t + \delta)$.

$2A$ is the amplitude of the external angular acceleration caused by the driving force.

In the absence of the driving force, the damping coefficient drives the system to the point $(\theta, \dot{\theta}) = (0, 0)$, which is an attractor for the system. This continues to happen for small enough A, but for $A > A_c$ the behavior of the system becomes more complicated.

The program that integrates the equations of motion of the system can be obtained by making trivial changes to the program in the file `dlo.cpp`. This changes are listed in detail below, but we note that `X1` $\leftrightarrow \theta$, `X2` $\leftrightarrow \dot{\theta}$, `a_0` $\leftrightarrow A$. The final program can be found in the file `fdp.cpp`. It is listed below, with the understanding that the commands in between the dots are the same as in the programs found in the files `dlo.cpp, rk.cpp`.

```
//========================================================
```

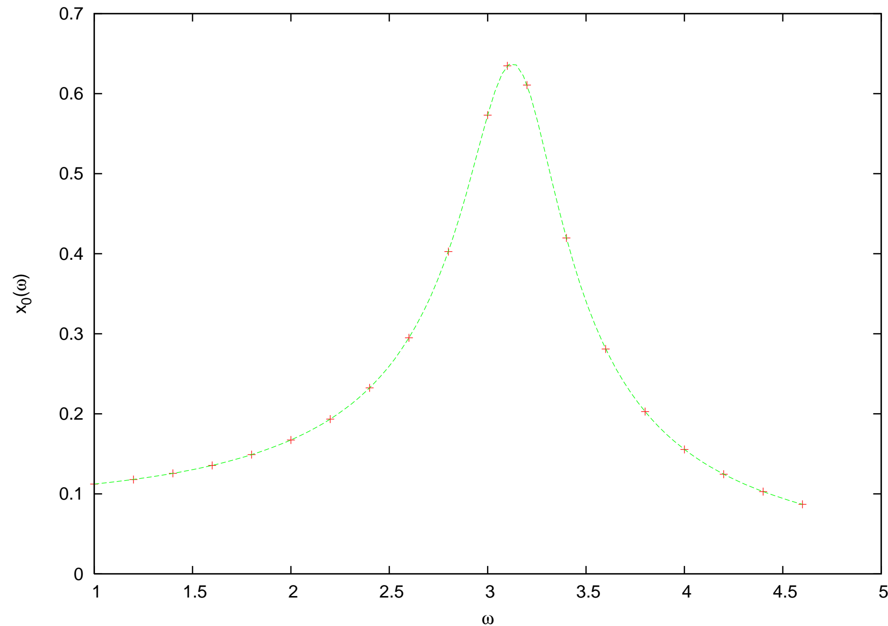

Figure 4.24: The oscillation amplitude $x_0(\omega)$ as a function of ω for the forced damped oscillator, where $\omega_0 = 3.145$, $\gamma = 0.5$ and $a_0 = 1.0$. We observe a resonance for $\omega \approx \omega_0$. The points of the plot are our measurements and the line is the theoretical prediction given by equation (4.33).

```
//Program to solve Forced Damped Pendulum
//using 4th order Runge-Kutta Method
//Output is written in file fdp.dat
//========================================================
 ...............................
const int P = 1010000;
 ...............................
   Energy = 0.5*X2[i]*X2[i]+omega_02*(1.0-cos(X1[i]));
 ...............................
double
f2(const double& t, const double& x1, const double& x2){
  return -(omega_02+2.0*a_0*cos(omega*t))*sin(x1)-gam*x2;
}
 ...............................
void
RKSTEP(double& t, double& x1, double& x2,
       const double& dt){
```

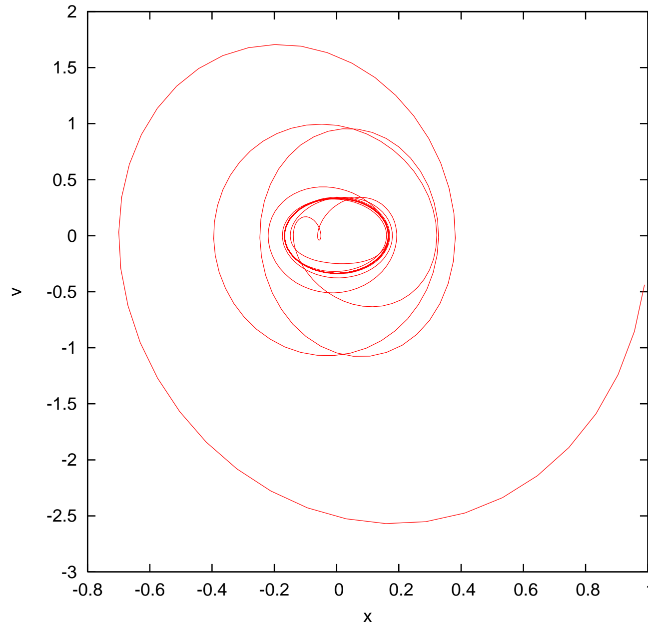

Figure 4.25: A phase space trajectory of the forced damped oscillator with $\omega_0 = 3.145$, $\omega = 2.0$, $\gamma = 0.5$ and $a_0 = 1.0$. The harmonic oscillation which is the steady state of the system is an ellipse, which is an attractor of all the phase space trajectories that correspond to different initial conditions.

```
  ................................
  const double pi = 3.14159265358979324;
  const double pi2= 6.28318530717958648;
  x1 =x1+h6*(k11+2.0*(k12+k13)+k14);
  x2 =x2+h6*(k21+2.0*(k22+k23)+k24);
  if( x1 >  pi ) x1 -= pi2;
  if( x1 < -pi ) x1 += pi2;
}//RKSTEP()
```

The final lines in the program are added so that the angle is kept within the interval $[-\pi, \pi]$.

In order to study the system's properties we will set $\omega_0 = 1$, $\omega = 2$, and $\gamma = 0.2$ unless we explicitly state otherwise. The natural period of the pendulum is $T_0 = 2\pi/\omega_0 = 2\pi \approx 6.28318530717958648$ whereas that of the driving force is $T = 2\pi/\omega = \pi \approx 3.14159265358979324$. For $A < A_c$, with $A_c \approx 0.18$, the point $(\theta, \dot{\theta}) = (0, 0)$ is an attractor, which

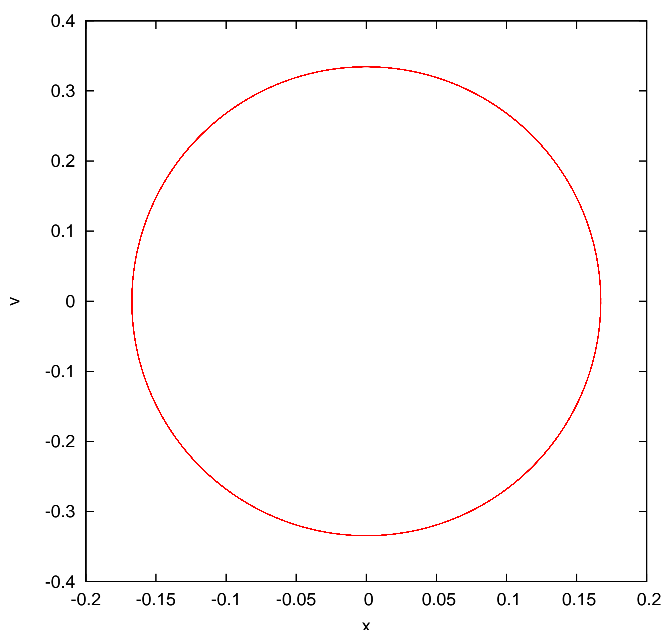

Figure 4.26: The trajectory shown in figure 4.25 for $t > 100$. The trajectory is almost on top of an ellipse corresponding to the steady state motion of the system. This ellipse is an attractor of the system.

means that the pendulum eventually stops at its stable equilibrium point. For $A_c < A < 0.71$ the attractor is a closed curve, which means that the pendulum at its steady state oscillates indefinitely without circling through its unstable equilibrium point at $\theta = \pm\pi$. The period of motion is found to be twice that of the driving force. For $0.72 < A < 0.79$ the attractor is an open curve, because at its steady state the pendulum crosses the $\theta = \pm\pi$ point. The period of the motion becomes equal to that of the driving force. For $0.79 < A \lesssim 1.033$ we have *period doubling* for critical values of A, but the trajectory is still periodic. For even larger values of A the system enters into a chaotic regime where the trajectories are non periodic. For $A \approx 3.1$ we find the system in a periodic steady state again, whereas for $A \approx 3.8 - 4.448$ we have period doubling. For $A \approx 4.4489$ we enter into a chaotic regime again etc. These results can be seen in figures 4.27–4.29. The reader should construct the bifurcation diagram of the system by solving problem 19 of this chapter.

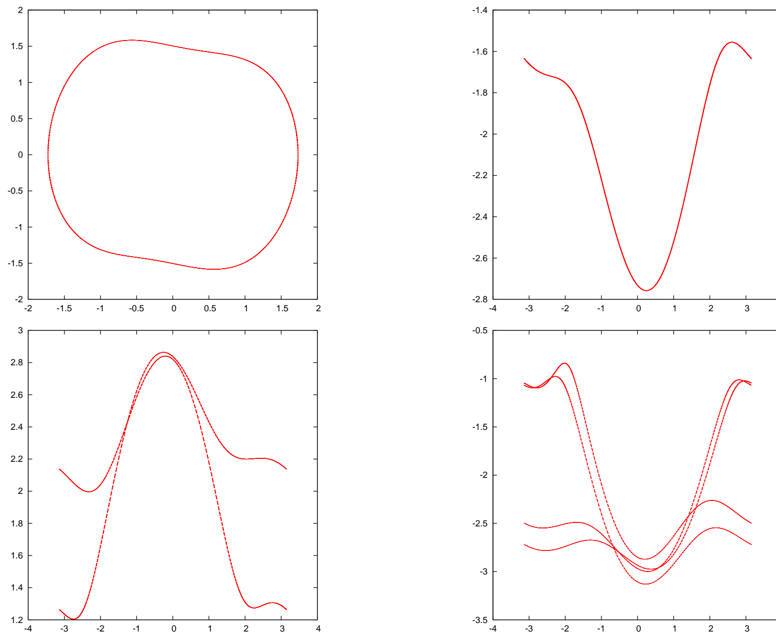

Figure 4.27: A phase space trajectory of the forced damped pendulum. The parameters chosen are $\omega_0 = 1.0$, $\omega = 2.0$, $\gamma = 0.2$ and $A = 0.60, 0.72, 0.85, 1.02$. We observe the phenomenon of period doubling.

We can also use the so called *Poincaré* diagrams in order to study the chaotic behavior of a system. These are obtained by placing a point in phase space when the time is an integer multiple of the period of the driving force. Then, if for example the period of the motion is equal to that of the period of the driving force, the Poincaré diagram consists of only one point. If the period of the motion is an n–multiple of the period of the driving force then the Poincaré diagram consists of only n points. Therefore, in the period doubling regime, the points of the Poincaré diagram double at each period doubling point. In the chaotic regime, the Poincaré diagram consists of an infinite number of points which belong to sets that have interesting fractal structure. One way to construct the Poincaré diagram numerically, is to process the data of the output file `fdp.dat` using `awk`[12]:

[12]The command can be written in one line without the final \ of the first and second lines.

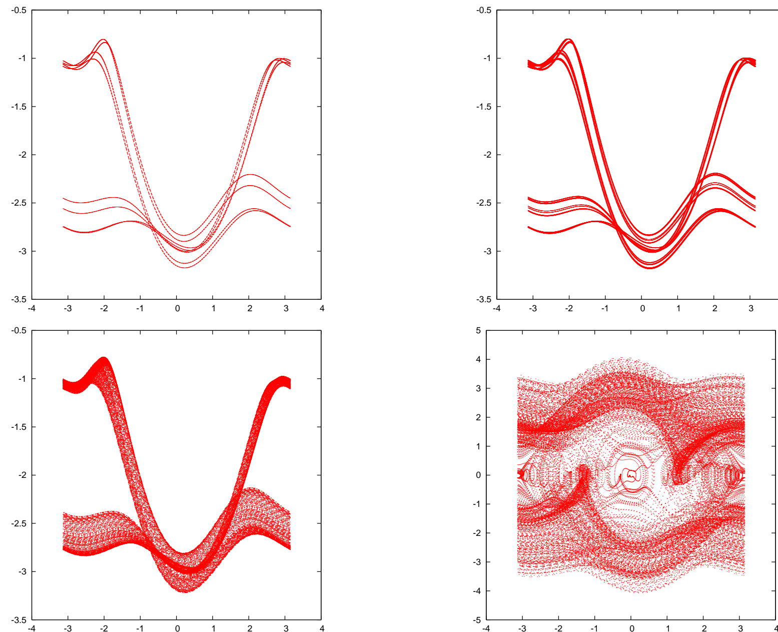

Figure 4.28: A phase space trajectory of the forced damped pendulum. The parameters chosen are $\omega_0 = 1.0$, $\omega = 2.0$, $\gamma = 0.2$ and $A = 1.031, 1.033, 1.04, 1.4$. We observe the chaotic behavior of the system.

```
awk -v o=$omega -v nt=$Nt -v tf=$TF \
 'BEGIN{T=6.283185307179/o;dt=tf/nt;} $1%T<dt{print $2,$3}'\
 fdp.dat
```

where `$omega`, `$Nt`, `$TF` are the values of the angular frequency ω, the number of points of time and the final time t_f. We calculate the period `T` and the time step `dt` in the program. Then we print those lines of the file where the time is an integer multiple of the period[13]. This is accomplished by the modulo operation `$1 % T`. The value of the expression `$1 % T < dt` is true when the remainder of the division of the first column (`$1`) of the file `fdp.dat` with the period `T` is smaller than `dt`. The results in the chaotic regime are displayed in figure 4.30.

We close this section by discussing another concept that helps us in

[13] The accuracy of this condition is limited by `dt`, which makes the points in the Poincaré diagram slightly fuzzy.

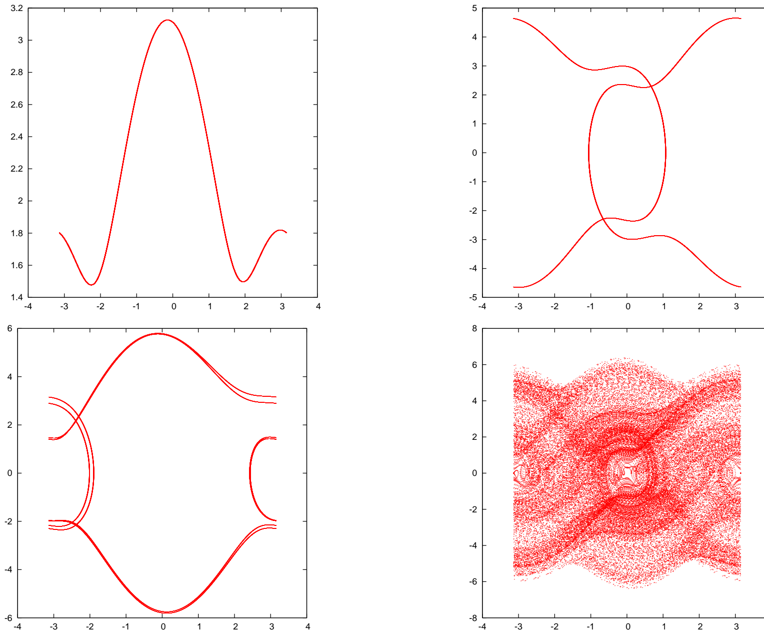

Figure 4.29: A phase space trajectory of the forced damped pendulum. The parameters chosen are $\omega_0 = 1.0$, $\omega = 2.0$, $\gamma = 0.2$ and $A = 1.568, 3.8, 4.44, 4.5$. We observe the system exiting and reentering regimes of chaotic behavior.

the analysis of the dynamical properties of the pendulum. This is the concept of the *basin of attraction* which is the set of initial conditions in phase space that lead the system to a specific attractor. Take for example the case for $A > 0.79$ in the regime where the pendulum at its steady state has a circular trajectory with a positive or negative direction. By taking a large sample of initial conditions and recording the direction of the resulting motion after the transient behavior, we obtain figure 4.31.

4.7 Appendix: On the Euler–Verlet Method

Equations (4.11) can be obtained from the Taylor expansion

$$\begin{aligned}
\theta(t+\Delta t) &= \theta(t) + (\Delta t)\theta'(t) + \frac{(\Delta t)^2}{2!}\theta''(t) + \frac{(\Delta t)^3}{3!}\theta'''(t) + \mathcal{O}((\Delta t)^4) \\
\theta(t-\Delta t) &= \theta(t) - (\Delta t)\theta'(t) + \frac{(\Delta t)^2}{2!}\theta''(t) - \frac{(\Delta t)^3}{3!}\theta'''(t) + \mathcal{O}((\Delta t)^4)\,.
\end{aligned}$$

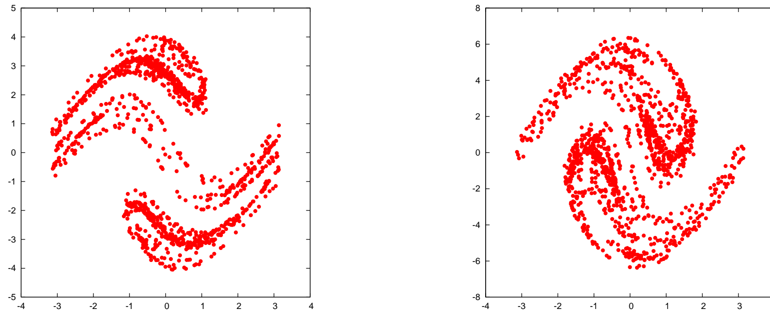

Figure 4.30: A Poincaré diagram for the forced damped pendulum in its chaotic regime. The parameters chosen are $\omega_0 = 1.0$, $\omega = 2.0$, $\gamma = 0.2$ and $A = 1.4, 4.5$.

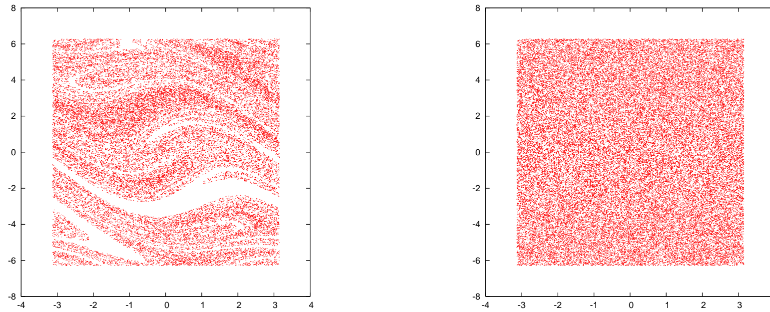

Figure 4.31: Basin of attraction for the forced damped pendulum. The parameters chosen are $\omega_0 = 1.0$, $\omega = 2.0$, $\gamma = 0.2$ and $A = 0.85, 1.4$.

By adding and subtracting the above equations we obtain

$$\begin{aligned}\theta(t+\Delta t)+\theta(t-\Delta t) &= 2\theta(t)+(\Delta t)^2\theta''(t)+\mathcal{O}((\Delta t)^4)\\ \theta(t+\Delta t)-\theta(t-\Delta t) &= 2(\Delta t)\theta'(t)+\mathcal{O}((\Delta t)^3)\end{aligned} \tag{4.35}$$

which give equations (4.11)

$$\begin{aligned}\theta(t+\Delta t) &= 2\theta(t)-\theta(t-\Delta t)+(\Delta t)^2\alpha(t)+\mathcal{O}((\Delta t)^4)\\ \omega(t) &= \frac{\theta(t+\Delta t)-\theta(t-\Delta t)}{2(\Delta t)}+\mathcal{O}((\Delta t)^2)\end{aligned} \tag{4.36}$$

From the first equation and equations (4.9) we obtain:

$$\theta(t+\Delta t)=\theta(t)+\omega(t)(\Delta t)+\mathcal{O}((\Delta t)^2) \tag{4.37}$$

When we perform a numerical integration, we are interested in the total error accumulated after $N-1$ integration steps. In this method, these errors must be studied carefully:

- The error in the velocity $\omega(t)$ does not accumulate because it is given by the *difference* of the positions $\theta(t+\Delta t)-\theta(t-\Delta t)$.
- The accumulation of the errors for the position is estimated as follows: Assume that $\delta\theta(t)$ is the *total* accumulated error from the integration from time t_0 to t. Then according to the expansions (4.36) the error for the first step is $\delta\theta(t_0+\Delta t)=\mathcal{O}((\Delta t)^4)$. Then[14]

$$\begin{aligned}\theta(t_0+2\Delta t) &= 2\theta(t_0+\Delta t)-\theta(t_0)+\Delta t^2\alpha(t_0+\Delta t)+\mathcal{O}((\Delta t)^4)\Rightarrow\\ \delta\theta(t_0+2\Delta t) &= 2\delta\theta(t_0+\Delta t)-\delta\theta(t_0)+\mathcal{O}((\Delta t)^4)\\ &= 2\mathcal{O}((\Delta t)^4)-0+\mathcal{O}((\Delta t)^4)\\ &= 3\mathcal{O}((\Delta t)^4)\,.\end{aligned}$$

For the next steps we obtain

$$\begin{aligned}\theta(t_0+3\Delta t) &= 2\theta(t_0+2\Delta t)-\theta(t_0+\Delta t)+\Delta t^2\alpha(t_0+2\Delta t)+\mathcal{O}((\Delta t)^4)\Rightarrow\\ \delta\theta(t_0+3\Delta t) &= 2\delta\theta(t_0+2\Delta t)-\delta\theta(t_0+\Delta t)+\mathcal{O}((\Delta t)^4)\\ &= 6\mathcal{O}((\Delta t)^4)-\mathcal{O}((\Delta t)^4)+\mathcal{O}((\Delta t)^4)\\ &= 6\mathcal{O}((\Delta t)^4)\,,\end{aligned}$$

$$\begin{aligned}\theta(t_0+4\Delta t) &= 2\theta(t_0+3\Delta t)-\theta(t_0+2\Delta t)+\Delta t^2\alpha(t_0+3\Delta t)+\mathcal{O}((\Delta t)^4)\Rightarrow\\ \delta\theta(t_0+4\Delta t) &= 2\delta\theta(t_0+3\Delta t)-\delta\theta(t_0+2\Delta t)+\mathcal{O}((\Delta t)^4)\\ &= 12\mathcal{O}((\Delta t)^4)-3\mathcal{O}((\Delta t)^4)+\mathcal{O}((\Delta t)^4)\\ &= 10\mathcal{O}((\Delta t)^4)\,.\end{aligned}$$

Then, inductively, if $\delta\theta(t_0+(n-1)\Delta t)=\frac{(n-1)n}{2}\mathcal{O}((\Delta t)^4)$, we obtain

$$\begin{aligned}\theta(t_0+n\Delta t) &= 2\theta(t_0+(n-1)\Delta t)-\theta(t_0+(n-2)\Delta t)+\Delta t^2\alpha(t_0+(n-1)\Delta t)\\ &\quad+\mathcal{O}((\Delta t)^4)\Rightarrow\\ \delta\theta(t_0+n\Delta t) &= 2\delta\theta(t_0+(n-1)\Delta t)-\delta\theta(t_0+(n-2)\Delta t)+\mathcal{O}((\Delta t)^4)\\ &= 2\frac{(n-1)n}{2}\mathcal{O}((\Delta t)^4)-\frac{(n-2)(n-1)}{2}\mathcal{O}((\Delta t)^4)+\mathcal{O}((\Delta t)^4)\\ &= \frac{n(n+1)}{2}\mathcal{O}((\Delta t)^4)\,.\end{aligned}$$

[14] Remember that the acceleration $\alpha(t)$ is given, therefore $\delta\alpha(t)=0$.

Finally

$$\delta\theta(t_0 + n\Delta t) = \frac{n(n+1)}{2}\mathcal{O}((\Delta t)^4) \sim \frac{1}{\Delta t^2}\mathcal{O}((\Delta t)^4) \sim \mathcal{O}((\Delta t)^2)\,. \tag{4.38}$$

Therefore the total error is $\mathcal{O}((\Delta t)^2)$.

We also mention the Velocity Verlet method or the Leapfrog method. In this case we use the velocity explicitly:

$$\begin{aligned}
\theta_{n+1} &= \theta_n + \omega_n \Delta t + \frac{1}{2}\alpha_n \Delta t^2 \\
\omega_{n+\frac{1}{2}} &= \omega_n + \frac{1}{2}\alpha_n \Delta t \\
\omega_{n+1} &= \omega_{n+\frac{1}{2}} + \frac{1}{2}\alpha_{n+1}\Delta t\,.
\end{aligned} \tag{4.39}$$

The last step uses the acceleration α_{n+1} which should depend only on the position θ_{n+1} and not on the velocity.

The Verlet methods are popular in *molecular dynamics* simulations of many body systems. One of their advantages is that the constraints of the system of particles are easily encoded in the algorithm.

4.8 Appendix: 2nd order Runge–Kutta Method

In this appendix we will show how the choice of the intermediate point 2 in equation (4.17) reduces the error by a power of h. This choice is special, since by choosing another point (e.g. $t = t_n + 0.4h$) the result would have not been the same. Indeed, from the relation

$$\frac{dx}{dt} = f(t,x) \Rightarrow x_{n+1} = x_n + \int_{t_n}^{t_{n+1}} f(t,x)\,dx\,. \tag{4.40}$$

By Taylor expanding around the point $(t_{n+1/2}, x_{n+1/2})$ we obtain

$$f(t,x) = f(t_{n+1/2}, x_{n+1/2}) + (t - t_{n+1/2})\frac{df}{dt}(t_{n+1/2}) + \mathcal{O}(h^2)\,. \tag{4.41}$$

Therefore

$$
\begin{aligned}
&\int_{t_n}^{t_{n+1}} f(t,x)\,dx \\
&= f(t_{n+1/2},x_{n+1/2})(t_{n+1}-t_n) + \frac{df}{dt}(t_{n+1/2})\left.\frac{(t-t_{n+1/2})^2}{2}\right|_{t_n}^{t_{n+1}} \\
&\quad +\mathcal{O}(h^2)(t_{n+1}-t_n) \\
&= f(t_{n+1/2},x_{n+1/2})h + \frac{df}{dt}(t_{n+1/2})\left\{\frac{(t_{n+1}-t_{n+1/2})^2}{2} - \frac{(t_n-t_{n+1/2})^2}{2}\right\} \\
&\quad +\mathcal{O}(h^2)h \\
&= f(t_{n+1/2},x_{n+1/2})h + \frac{df}{dt}(t_{n+1/2})\left\{\frac{h^2}{2} - \frac{(-h)^2}{2}\right\} + \mathcal{O}(h^3) \\
&= f(t_{n+1/2},x_{n+1/2})h + \mathcal{O}(h^3)\,. \qquad (4.42)
\end{aligned}
$$

Note that for the vanishing of the $\mathcal{O}(h)$ term it is necessary to place the intermediate point at time $t_{n+1/2}$.

This is not a unique choice. This can be most easily seen by a different analysis of the Taylor expansion. Expanding around the point (t_n, x_n) we obtain

$$
\begin{aligned}
x_{n+1} &= x_n + (t_{n+1}-t_n)\frac{dx_n}{dt} + \frac{1}{2}(t_{n+1}-t_n)^2\frac{d^2x_n}{dt^2} + \mathcal{O}(h^3) \\
&= x_n + hf_n + \frac{h^2}{2}\frac{df_n}{dt} + \mathcal{O}(h^3) \\
&= x_n + hf_n + \frac{h^2}{2}\left(\frac{\partial f_n}{\partial t} + \frac{\partial f_n}{\partial x}\frac{dx_n}{dt}\right) + \mathcal{O}(h^3) \\
&= x_n + hf_n + \frac{h^2}{2}\left(\frac{\partial f_n}{\partial t} + \frac{\partial f_n}{\partial x}f_n\right) + \mathcal{O}(h^3)\,, \qquad (4.43)
\end{aligned}
$$

where we have set $f_n \equiv f(t_n, x_n)$, $\frac{dx_n}{dt} \equiv \frac{dx}{dt}(x_n)$ etc. We define

$$
\begin{aligned}
k_1 &= f(t_n, x_n) = f_n \\
k_2 &= f(t_n + ah, x_n + bhk_1) \\
x_{n+1} &= x_n + h(c_1k_1 + c_2k_2)\,. \qquad (4.44)
\end{aligned}
$$

and we will determine the conditions so that the terms $\mathcal{O}(h^2)$ of the last equation in the error are identical with those of equation (4.43). By

expanding k_2 we obtain

$$\begin{aligned}
k_2 &= f(t_n + ah, x_n + bhk_1) \\
&= f(t_n, x_n + bhk_1) + ha\frac{\partial f}{\partial t}(t_n, x_n + bhk_1) + \mathcal{O}(h^2) \\
&= f(t_n, x_n) + hbk_1\frac{\partial f}{\partial x}(t_n, x_n) + ha\frac{\partial f}{\partial t}(t_n, x_n) + \mathcal{O}(h^2) \\
&= f_n + h\left\{a\frac{\partial f_n}{\partial t} + bk_1\frac{\partial f_n}{\partial x}\right\} + \mathcal{O}(h^2) \\
&= f_n + h\left\{a\frac{\partial f_n}{\partial t} + bf_n\frac{\partial f_n}{\partial x}\right\} + \mathcal{O}(h^2) \qquad (4.45)
\end{aligned}$$

Substituting in (4.44) we obtain

$$\begin{aligned}
x_{n+1} &= x_n + h(c_1k_1 + c_2k_2) \\
&= x_n + h\left\{c_1f_n + c_2f_n + c_2h\left(a\frac{\partial f_n}{\partial t} + bf_n\frac{\partial f_n}{\partial x}\right) + \mathcal{O}(h^2)\right\} \\
&= x_n + h(c_1 + c_2)f_n + \frac{h^2}{2}\left((2c_2a)\frac{\partial f_n}{\partial t} + (2c_2b)f_n\frac{\partial f_n}{\partial x}\right) \\
&\quad + \mathcal{O}(h^3)\,. \qquad (4.46)
\end{aligned}$$

All we need is to choose

$$\begin{aligned}
c_1 + c_2 &= 1 \\
2c_2a &= 1 \\
2c_2b &= 1\,. \qquad (4.47)
\end{aligned}$$

The choice $c_1 = 0$, $c_2 = 1$, $a = b = 1/2$ leads to equation (4.19). Some other choices in the bibliography are $c_2 = 1/2$ and $c_2 = 3/4$.

4.9 Problems

4.1 Prove that the total error in the Euler–Cromer method is of order Δt.

4.2 Reproduce the results in figures 4.11–4.18

4.3 Improve your programs so that there is no accumulation of roundoff error in the calculation of time when `h` is very small for the methods Euler, Euler-Cromer, Euler-Verlet and Runge-Kutta. Repeat the analysis of the previous problem.

4.4 Compare the results obtained from the Euler, Euler-Cromer, Euler-Verlet, Runge-Kutta methods for the following systems where the analytic solution is known:

(a) Particle falling in a constant gravitational field. Consider the case $v(0) = 0$, $m = 1$, $g = 10$.

(b) Particle falling in a constant gravitational field moving in a fluid from which exerts a force $F = -kv$ on the particle. Consider the case $v(0) = 0$, $m = 1$, $g = 10$ $k = 0.1, 1.0, 2.0$. Calculate the limiting velocity of the particle numerically and compare the value obtained to the theoretical expectation.

(c) Repeat for the case of a force of resistance of magnitude $|F| = kv^2$.

4.5 Consider the damped harmonic oscillator

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0\,. \tag{4.48}$$

Take $\omega_0 = 3.145$, $\gamma = 0.5$ and calculate its mechanical energy as a function of time. Is it monotonic? Why? (show that $d(E/m)/dt = -\gamma v^2$). Repeat for $\gamma = 4, 5, 6, 7, 8$. When is the system oscillating and when it's not? Calculate numerically the critical value of γ for which the system passes from a non oscillating to an oscillating regime. Compare your results with the theoretical expectations.

4.6 Reproduce the results of figures 4.19–4.22.

4.7 Reproduce the results of figures 4.23–4.26. Calculate the phase $\delta(\omega)$ numerically and compare with equation (4.33).

4.8 Consider a simple model for a swing. Take the damped harmonic oscillator and a driving force which periodically exerts a momentary push with angular frequency ω. Define "momentary" to be an impulse given by the acceleration a_0 by an appropriately small time interval Δt. The acceleration is 0 for all other times. Calculate the amplitude $x_0(\omega)$ for $\omega_0 = 3.145$ and $\gamma = 0.5$.

4.9 Consider a "half sine" driving force on a damped harmonic oscillator

$$a(t) = \begin{cases} a_0 \cos\omega t & \cos\omega t > 0 \\ 0 & \cos\omega t \leq 0 \end{cases}$$

Study the transient behavior of the system for several initial conditions and calculate its steady state motion for $\omega_0 = 3.145$ and $\gamma = 0.5$. Calculate the amplitude $x_0(\omega)$.

4.10 Consider the driving force on a damped oscillator given by

$$a(t) = \frac{1}{\pi} + \frac{1}{2}\cos\omega + \frac{2}{3\pi}\cos 2\omega t - \frac{2}{15\pi}\cos 4\omega t$$

Study the transient behavior of the system for several initial conditions and calculate its steady state motion for $\omega_0 = 3.145$ and $\gamma = 0.5$. Calculate the amplitude $x_0(\omega)$. Compare your results with those of the previous problem and comment about.

4.11 Write a program that simulates N identical, independent harmonic oscillators. Take $N = 20$ and choose random initial conditions for each one of them. Study their trajectories in phase space and check whether they cross each other. Comment on your results.

4.12 Place the $N = 20$ harmonic oscillators of the previous problem in a small square in phase space whose center is at the origin of the axes. Consider the evolution of the system in time. Does the shape of the rectangle change in time? Does the area change in time? Explain...

4.13 Repeat the previous problem when each oscillator is damped with $\gamma = 0.5$. Take $\omega_0 = 3.145$.

4.14 Consider the forced damped oscillator with $\omega = 2$, $\omega_0 = 1.0$, $\gamma = 0.2$. Study the transient behavior of the system in the plots of $\theta(t)$, $\dot{\theta}(t)$ for $A = 0.1, 0.5, 0.79, 0.85, 1.03, 1.4$.

4.15 Consider the forced damped pendulum with $\omega = 2$, $\omega_0 = 1.0$, $\gamma = 0.2$ and study the phase space trajectories for $A =$ 0.1, 0.19, 0.21, 0.25, 0.5, 0.71, 0.79, 0.85, 1.02, 1.031, 1.033, 1.05, 1.08, 1.1, 1.4, 1.8, 3.1, 3.5, 3.8, 4.2, 4.42, 4.44, 4.445, 4.447, 4.4488. Consider both the transient behavior and the steady state motion.

4.16 Reproduce the results in figures 4.30.

4.17 Reproduce the results in figures 4.31.

4.18 Consider the forced damped oscillator with

$$\omega_0 = 1\,, \quad \omega = 2\,, \quad \gamma = 0.2$$

After the transient behavior, the motion of the system for $A = 0.60$, $A = 0.75$ and $A = 0.85$ is periodic. Measure the period of the motion with an accuracy of three significant digits and compare it with the natural period of the pendulum and with the period of the driving force. Take as initial conditions the following pairs: $(\theta_0, \dot{\theta}_0) = (3.1, 0.0)$, $(2.5, 0.0)$, $(2.0, 0.0)$, $(1.0, 0.0)$, $(0.2, 0.0)$, $(0.0, 1.0)$, $(0.0, 3.0)$, $(0.0, 6.0)$. Check if the period is independent of the initial conditions.

4.19 Consider the forced damped pendulum with

$$\omega_0 = 1\,, \quad \omega = 2\,, \quad \gamma = 0.2$$

Study the motion of the pendulum when the amplitude A takes values in the interval $[0.2, 5.0]$. Consider specific discrete values of A by splitting the interval above in subintervals of width equal to $\delta A = 0.002$. For each value of A, record in a file the value of A, the angular position and the angular velocity of the pendulum when $t_k = k\pi$ with $k = k_{trans}, k_{trans}+1, k_{trans}+2, \ldots, k_{max}$:

$$A \qquad \theta(t_k) \qquad \dot{\theta}(t_k)$$

The choice of k_{trans} is made so that the transient behavior will be discarded and study only the steady state of the pendulum. You

may take $k_{max} = 500$, $k_{trans} = 400$, $t_i = 0$, $t_f = 500\pi$, and split the intervals $[t_k, t_k + \pi]$ to 50 subintervals. Choose $\theta_0 = 3.1$, $\dot{\theta}_0 = 0$.

(a) Construct the bifurcation diagram by plotting the points $(A, \theta(t_k))$.

(b) Repeat by plotting the points $(A, \dot{\theta}(t_k))$.

(c) Check whether your results depend on the choice of θ_0, $\dot{\theta}_0$. Repeat your analysis for $\theta_0 = 0$, $\dot{\theta}_0 = 1$.

(d) Study the onset of chaos: Take $A \in [1.0000, 1.0400]$ with $\delta A = 0.0001$ and $A \in [4.4300, 4.4500]$ with $\delta A = 0.0001$ and compute with the given accuracy the value A_c where the system enters into the chaotic behavior regime.

(e) The plot the points $(\theta(t_k), \dot{\theta}(t_k))$ for $A = 1.034$, 1.040, 1.080, 1.400, 4.450, 4.600. Put 2000 points for each value of A and commend on the strength of the chaotic behavior of the pendulum.

Chapter 5

Planar Motion

In this chapter we will study the motion of a particle moving on the plane under the influence of a dynamical field. Special emphasis will be given to the study of the motion in a central field, like in the problem of planetary motion and scattering. We also study the motion of two or more interacting particles moving on the plane, which requires the solution of a larger number of dynamical equations. These problems can be solved numerically by using Runge–Kutta integration methods, therefore this chapter extends and applies the numerical methods studied in the previous chapter.

5.1 Runge–Kutta for Planar Motion

In two dimensions, the initial value problem that we are interested in, is solving the system of equations (4.6)

$$\begin{aligned} \frac{dx}{dt} &= v_x & \frac{dv_x}{dt} &= a_x(t, x, v_x, y, v_y) \\ \frac{dy}{dt} &= v_y & \frac{dv_y}{dt} &= a_y(t, x, v_x, y, v_y)\,. \end{aligned} \tag{5.1}$$

The 4th order Runge-Kutta method can be programmed by making small modifications of the program in the file `rk.cpp`. In order to facilitate the study of many different dynamical fields, for each field we put the code of the respective acceleration in a different file. The code which is common for all the forces, namely the user interface and the implementation of the Runge–Kutta method, will be put in the file `rk2.cpp`.

The program that computes the acceleration will be put in a file named `rk_XXX.cpp`, where `XXX` is a string of characters that identifies the force. For example, the file `rk2_hoc.cpp` contains the program computing the acceleration of the simple harmonic oscillator, the file `rk2_g.cpp` the acceleration of a constant gravitational field $\vec{g} = -g\,\hat{y}$ etc.

Different force fields will require the use of one or more coupling constants which need to be accessible to the code in the main program and some subroutines. For this reason, we will provide two variables `k1, k2` in the global scope which will be accessed by the acceleration functions `f3` and `f4`, the function `energy` and the main program where the user will enter their. The initial conditions are stored in the variables `X10` $\leftrightarrow x_0$, `X20` $\leftrightarrow y_0$, `V10` $\leftrightarrow v_{x0}$, `V20` $\leftrightarrow v_{y0}$, and the values of the functions of time will be stored in the arrays `X1[P]` $\leftrightarrow x(t)$, `X2[P]` $\leftrightarrow y(t)$, `V1[P]` $\leftrightarrow v_x(t)$, `V2[P]` $\leftrightarrow v_y(t)$. The integration is performed by a call to the function `RK(Ti,Tf,X10,X20,V10,V20,Nt)` The results are written to the file `rk2.dat`. Each line in this file contains the time, position, velocity and the total mechanical energy, where the energy is calculated by the function `energy(t,x1,x2,v1,v2)`. The code for the function `energy`, which is different for each force field, is written in the same file with the acceleration. The code for the function `RKSTEP(t,x1,x2,x3,x4,dt)` should be extended in order to integrate four instead of two functions. The full code is listed below:

```
//========================================================
//Program to solve a 4 ODE system using Runge-Kutta Method
//User must supply derivatives
//dx1/dt=f1(t,x1,x2,x3,x4) dx2/dt=f2(t,x1,x2,x3,x4)
//dx3/dt=f3(t,x1,x2,x3,x4) dx4/dt=f4(t,x1,x2,x3,x4)
//as double functions
//Output is written in file rk2.dat
//========================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
//--------------------------------------------------------
const int P = 1010000;
double T[P], X1[P], X2[P], V1[P], V2[P];
```

```
double k1,k2;
//--------------------------------------------------------------
double
f1(const double& t  , const double& x1, const double& x2,
   const double& v1 , const double& v2);
double
f2(const double& t  , const double& x1, const double& x2,
   const double& v1 , const double& v2);
double
f3(const double& t  , const double& x1, const double& x2,
   const double& v1 , const double& v2);
double
f4(const double& t  , const double& x1, const double& x2,
   const double& v1 , const double& v2);
double
energy
  (const double& t  , const double& x1, const double& x2,
   const double& v1 , const double& v2);
void
RK(const double& Ti , const double& Tf ,
   const double& X10, const double& X20,
   const double& V10, const double& V20,
   const int    & Nt);
void
RKSTEP(double& t ,
       double& x1, double& x2,
       double& x3, double& x4,
       const       double& dt);
//--------------------------------------------------------------
int main(){
  string buf;
  double Ti,Tf,X10,X20,V10,V20;
  int    Nt,i ;
  double E0,EF,DE;
  //Input:
  cout << "Runge-Kutta Method for 4-ODEs Integration\n";
  cout << "Enter coupling constants:\n";
  cin  >> k1 >> k2;getline(cin,buf);
  cout << "k1= " << k1 << " k2= " << k2 << endl;
  cout << "Enter Nt,Ti,Tf,X10,X20,V10,V20:\n";
  cin  >> Nt >> Ti >> Tf>> X10 >> X20 >> V10 >> V20;
  getline(cin,buf);
  cout << "Nt = " << Nt << endl;
  cout << "Time: Initial Ti = " << Ti
       << " Final Tf= "          << Tf  << endl;
```

```
  cout << "           X1(Ti)= " << X10
       << " X2(Ti)="            << X20 << endl;
  cout << "           V1(Ti)= " << V10
       << " V2(Ti)="            << V20 << endl;
  //Calculate:
  RK(Ti,Tf,X10,X20,V10,V20,Nt);
  ofstream myfile("rk2.dat");
  myfile.precision(17);
  for(i=0;i<Nt;i++)
    myfile << T [i] << " "
           << X1[i] << " " << X2[i] << " "
           << V1[i] << " " << V2[i] << " "
           << energy(T[i],X1[i],X2[i],V1[i],V2[i])
           << endl;
  myfile.close();
  //Rutherford scattering angles:
  cout.precision(17);
  cout <<"v-angle: "<< atan2(V2[Nt-1],V1[Nt-1])  << endl;
  cout <<"b-angle: "<< 2.0*atan(k1/(V10*V10*X20))<< endl;
  E0=energy(Ti     ,X10     ,X20     ,V10     ,V20     );
  EF=energy(T[Nt-1],X1[Nt-1],X2[Nt-1],V1[Nt-1],V2[Nt-1]);
  DE = abs(0.5*(EF-E0)/(EF+E0));
  cout << "E0,EF, DE/E= " << E0
       << " "             << EF
       << " "             << DE << endl;
}//main()
//========================================================
//The velocity functions f1,f2(t,x1,x2,v1,v2)
//========================================================
double
f1(const double& t , const double& x1, const double& x2,
   const double& v1, const double& v2){
  return v1;
}
//--------------------------------------------------------
double
f2(const double& t , const double& x1, const double& x2,
   const double& v1, const double& v2){
  return v2;
}
//========================================================
//RK(Ti,Tf,X10,X20,V10,V20,Nt) is the driver
//for the Runge-Kutta integration routine RKSTEP
//Input: Initial and final times Ti,Tf
//       Initial values at t=Ti  X10,X20,V10,V20
```

```
//        Number of steps of integration: Nt-1
//        Size of arrays T,X1,X2,V1,V2
//Output: real arrays T[Nt],X1[Nt],X2[Nt],
//                         V1[Nt],V2[Nt] where
//T[0] = Ti X1[0] = X10 X2[0] = X20 V1[0] = V10 V2[0] = V20
//           X1[k] = X1(at t=T[k]) X2[k] = X2(at t=T[k])
//           V1[k] = V1(at t=T[k]) V2[k] = V2(at t=T[k])
//T[Nt-1]= Tf
//========================================================
void
RK(const double& Ti , const double& Tf ,
   const double& X10, const double& X20,
   const double& V10, const double& V20,
   const int    & Nt){

  double dt;
  double TS,X1S,X2S; //values of time and X1,X2 at given step
  double    V1S,V2S;
  int i;
  //Initialize:
  dt     = (Tf-Ti)/(Nt-1);
  T [0]  = Ti;
  X1[0]  = X10; X2[0] = X20;
  V1[0]  = V10; V2[0] = V20;
  TS     = Ti;
  X1S    = X10; X2S   = X20;
  V1S    = V10; V2S   = V20;
  //Make RK steps: The arguments of RKSTEP are
  //replaced with the new ones
  for(i=1;i<Nt;i++){
    RKSTEP(TS,X1S,X2S,V1S,V2S,dt);
    T [i] = TS;
    X1[i] = X1S; X2[i] = X2S;
    V1[i] = V1S; V2[i] = V2S;
  }
}//RK()
//========================================================
//Subroutine RKSTEP(t,x1,x2,dt)
//Runge-Kutta Integration routine of ODE
//dx1/dt=f1(t,x1,x2,x3,x4) dx2/dt=f2(t,x1,x2,x3,x4)
//dx3/dt=f3(t,x1,x2,x3,x4) dx4/dt=f4(t,x1,x2,x3,x4)
//User must supply derivative functions:
//real function f1(t,x1,x2,x3,x4)
//real function f2(t,x1,x2,x3,x4)
//real function f3(t,x1,x2,x3,x4)
```

```
//real function f4(t,x1,x2,x3,x4)
//Given initial point (t,x1,x2) the routine advances it
//by time dt.
//Input : Inital time t    and function values x1,x2,x3,x4
//Output: Final  time t+dt and function values x1,x2,x3,x4
//Careful: values of t,x1,x2,x3,x4 are overwritten...
//=========================================================
void
RKSTEP(double& t ,
       double& x1, double& x2,
       double& x3, double& x4,
       const       double& dt){
  double k11,k12,k13,k14,k21,k22,k23,k24;
  double k31,k32,k33,k34,k41,k42,k43,k44;
  double h,h2,h6;

  h =dt;      // h  = dt, integration step
  h2=0.5*h;   // h2 = h/2
  h6=h/6.0;   // h6 = h/6

  k11=f1(t,x1,x2,x3,x4);
  k21=f2(t,x1,x2,x3,x4);
  k31=f3(t,x1,x2,x3,x4);
  k41=f4(t,x1,x2,x3,x4);

  k12=f1(t+h2,x1+h2*k11,x2+h2*k21,x3+h2*k31,x4+h2*k41);
  k22=f2(t+h2,x1+h2*k11,x2+h2*k21,x3+h2*k31,x4+h2*k41);
  k32=f3(t+h2,x1+h2*k11,x2+h2*k21,x3+h2*k31,x4+h2*k41);
  k42=f4(t+h2,x1+h2*k11,x2+h2*k21,x3+h2*k31,x4+h2*k41);

  k13=f1(t+h2,x1+h2*k12,x2+h2*k22,x3+h2*k32,x4+h2*k42);
  k23=f2(t+h2,x1+h2*k12,x2+h2*k22,x3+h2*k32,x4+h2*k42);
  k33=f3(t+h2,x1+h2*k12,x2+h2*k22,x3+h2*k32,x4+h2*k42);
  k43=f4(t+h2,x1+h2*k12,x2+h2*k22,x3+h2*k32,x4+h2*k42);

  k14=f1(t+h ,x1+h *k13,x2+h *k23,x3+h *k33,x4+h *k43);
  k24=f2(t+h ,x1+h *k13,x2+h *k23,x3+h *k33,x4+h *k43);
  k34=f3(t+h ,x1+h *k13,x2+h *k23,x3+h *k33,x4+h *k43);
  k44=f4(t+h ,x1+h *k13,x2+h *k23,x3+h *k33,x4+h *k43);

  t =t+h;
  x1=x1+h6*(k11+2.0*(k12+k13)+k14);
  x2=x2+h6*(k21+2.0*(k22+k23)+k24);
  x3=x3+h6*(k31+2.0*(k32+k33)+k34);
  x4=x4+h6*(k41+2.0*(k42+k43)+k44);
```

```
}//RKSTEP()
```

5.2 Projectile Motion

Consider a particle in the constant gravitational field near the surface of the earth which moves with constant acceleration $\vec{g} = -g\,\hat{y}$ so that

$$\begin{array}{llll} x(t) & = x_0 + v_{0x}t & , \quad y(t) & = y_0 + v_{0y}t - \frac{1}{2}gt^2 \\ v_x(t) & = v_{0x} & , \quad v_y(t) & = v_{0y} - gt \\ a_x(t) & = 0 & , \quad a_y(t) & = -g \end{array} \tag{5.2}$$

The particle moves on a parabolic trajectory that depends on the initial conditions

$$\begin{aligned} (y - y_0) &= \left(\frac{v_{0y}}{v_{0x}}\right)(x - x_0) - \frac{1}{2}\frac{g}{v_{0x}^2}(x - x_0)^2 \\ &= \tan\theta\,(x - x_0) - \frac{\tan^2\theta}{4h_{\mathrm{max}}}(x - x_0)^2\,, \end{aligned} \tag{5.3}$$

where $\tan\theta = v_{0y}/v_{0x}$ is the direction of the initial velocity and h_{max} is the maximum height of the trajectory.

The acceleration $a_x(t) = 0$ $a_y(t) = -g$ ($a_x \leftrightarrow$ `f3` , $a_y \leftrightarrow$ `f4`) and the mechanical energy is coded in the file `rk2_g.cpp`:

```
//========================================================
//The acceleration functions f3,f4(t,x1,x2,v1,v2) provided
//by the user
//========================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
extern double k1,k2;
//--------------------------------------------------------
//Free fall in constant gravitational field with
//g = -k2
double
```

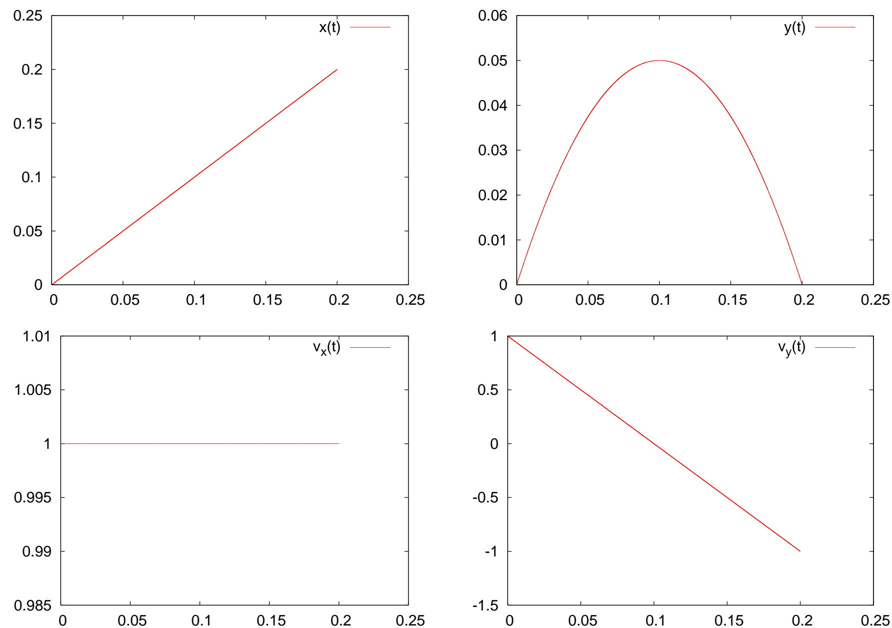

Figure 5.1: Plots of $x(t)$, $y(t)$, $v_x(t)$, $v_y(t)$ for a projectile fired in a constant gravitational field $\vec{g} = -10.0\,\hat{y}$ with initial velocity $\vec{v}_0 = \hat{x} + \hat{y}$.

```
f3(const double& t , const double& x1, const double& x2,
   const double& v1, const double& v2){
  return 0.0;  // dx3/dt=dv1/dt=a1
}
//--------------------------------------------------------
double
f4(const double& t , const double& x1, const double& x2,
   const double& v1, const double& v2){
  return -k1;  // dx4/dt=dv2/dt=a2
}
//--------------------------------------------------------
double
energy
  (const double& t , const double& x1, const double& x2,
   const double& v1, const double& v2){
  return 0.5*(v1*v1+v2*v2) + k1*x2;
}
```

In order to calculate a projectile's trajectory you may use the following

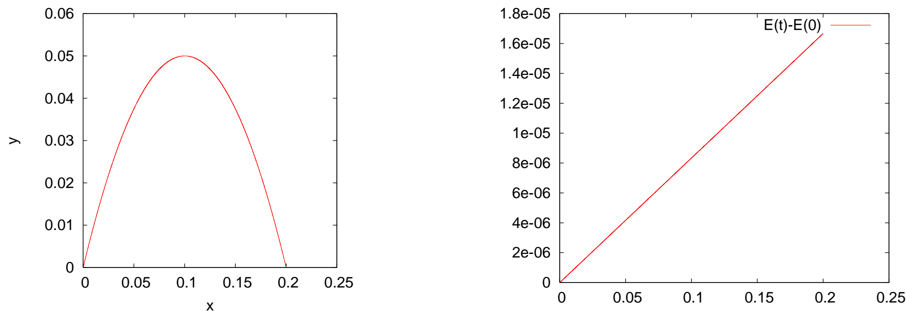

Figure 5.2: (Left) The parabolic trajectory of a projectile fired in a constant gravitational field $\vec{g} = -10.0\,\hat{y}$ with initial velocity $\vec{v}_0 = \hat{x} + \hat{y}$. (Right) The deviation of the projectile's energy from its initial value is due to numerical errors.

commands:

```
> g++ -O2 rk2.cpp rk2_g.cpp -o rk2
> ./rk2
Runge-Kutta Method for 4-ODEs Integration
Enter coupling constants:
10.0 0.0
k1= 10 k2= 0
Enter Nt,Ti,Tf,X10,X20,V10,V20:
20000 0.0 0.2 0.0 0.0 1.0 1.0
Nt = 20000
Time: Initial Ti = 0 Final Tf= 0.2
            X1(Ti)= 0 X2(Ti)=0
            V1(Ti)= 1 V2(Ti)=1
```

The analysis of the results contained in the file `rk2.dat` can be done using `gnuplot`:

```
gnuplot> set terminal x11 1
gnuplot> plot "rk2.dat" using 1:2 with lines title "x(t)"
gnuplot> set terminal x11 2
gnuplot> plot "rk2.dat" using 1:3 with lines title "y(t)"
gnuplot> set terminal x11 3
gnuplot> plot "rk2.dat" using 1:4 with lines title "vx(t)"
gnuplot> set terminal x11 4
gnuplot> plot "rk2.dat" using 1:5 with lines title "vy(t)"
gnuplot> set terminal x11 5
```

```
gnuplot> plot "rk2.dat" using 1:($6-1.0) w lines t "E(t)E-(0)"
gnuplot> set terminal x11 6
gnuplot> set size square
gnuplot> set title "Trajectory"
gnuplot> plot "rk2.dat" using 2:3 with lines notit
```

The results can be seen in figures 5.1 and 5.2. We note a small increase in the mechanical energy which is due to the accumulation of numerical errors.

We can animate the trajectory by writing a script of `gnuplot` commands in a file `rk2_animate.gpl`

```
icount = icount+skip
plot "<cat -n rk2.dat" \
  using 3:($1<= icount ? $4: 1/0) with lines notitle
# pause 1
if(icount < nlines ) reread
```

Before calling the script, the user must set the values of the variables `icount`, `skip` and `nlines`. Each time `gnuplot` reads the script, it plots `icount` number of lines from `rk2.dat`. Then the script is read again and a new plot is made with `skip` lines more than the previous one, unless `icount < nlines`. The plotted "file" `"<cat -n rk2.dat"` is the standard output (`stdout`) of the command `cat -n rk2.dat` which prints to the `stdout` the contents of the file `rk2.dat` line by line, together with the line number. Therefore the `plot` command reads data which are the line number, the time, the coordinate x, the coordinate y etc. The keyword `using` in

```
  using 3:($1<= icount ? $4: 1/0)
```

instructs the plot command to use the 3rd column on the horizontal axis and if the first column is less than `icount` (`$1<= icount`) put on the vertical axis the value of the 4th column if the first column is less than `icount`. Otherwise (`$1 > icount`) it prints an undefined number (`1/0`) which makes `gnuplot` print nothing at all. You may also uncomment the command `pause` if you want to make the animation slower. In order to run the script from gnuplot, issue the commands

```
gnuplot> icount = 10
gnuplot> skip   = 200
gnuplot> nlines = 20000
gnuplot> load "rk2_animate.gpl"
```

The scripts shown above can be found in the accompanying software. More scripts can be found there that automate many of the boring procedures. The usage of two of these is explained below. The first one is in the file `rk2_animate.csh`:

```
> ./rk2_animate.csh -h
Usage: rk2_animate.csh -t [sleep time] -d [skip points] <file>
Default file is rk2.dat
Other options:
   -x: set lower value in xrange
   -X: set lower value in xrange
   -y: set lower value in yrange
   -Y: set lower value in yrange
   -r: automatic determination of x-y range
> ./rk2_animate.csh -r -d 500 rk2.dat
```

The last line is a command that animates a trajectory read from the file `rk2.dat`. Each animation frame contains 500 more points than the previous one. The option `-r` calculates the plot range automatically. The option `-h` prints a short help message.

A more useful script is in the file `rk2.csh`.

```
> ./rk2.csh -h
Usage: rk2.csh -f <force> k1 k2 x10 x20 v10 v20 STEPS t0 tf
Other Options:
 -n Do not animate trajectory
Available forces (value of <force>):
1: ax=-k1               ay= -k2 y           Harmonic oscillator
2: ax= 0                ay= -k1             Free fall
3: ax= -k2     vx       ay= -k2     vy - k1 Free fall + \
                                            air resistance ~ v
4: ax= -k2 |v| vx       ay= -k2 |v|vy - k1  Free fall + \
                                            air resistance ~ v^2
5: ax= k1*x1/r^3        ay= k1*x2/r^3       Coulomb Force
....
```

The option `-h` prints operating instructions. A menu of forces is available,

and a choice can be made using the option `-f`. The rest of the command line consists of the parameters read by the program in `rk2.cpp`, i.e. the coupling constants `k1, k2`, the initial conditions `x10, x20, v10, v20` and the integration parameters `STEPS, t0` and `tf`. For example, the commands

```
> rk2.csh -f 2 -- 10.0 0.0 0.0 0.0 1.0 1.0 20000 0.0 0.2
> rk2.csh -f 1 -- 16.0 1.0 0.0 1.0 1.0 0.0 20000 0.0 6.29
> rk2.csh -f 5 -- 10.0 0.0 -10 0.2 10. 0.0 20000 0.0 3.00
```

compute the trajectory of a particle in the constant gravitational field discussed above, the trajectory of an anisotropic harmonic oscillator (`k1` = $a_x = -\omega_1^2 x$, `k2` = $a_y = -\omega_2^2 y$) and the scattering of a particle in a Coulomb field – try them! I hope that you will have enough curiosity to look "under the hood" of the scripts and try to modify them or create new ones. Some advise to the lazy guys: If you need to program your own force field follow the recipe: Write the code of your acceleration field in a file named e.g. `rk2_myforce.cpp` as we did with `rk2_g.cpp`. Edit the file `rk2.csh` and modify the line

```
set forcecode = (hoc g vg v2g cb)
```

to

```
set forcecode = (hoc g vg v2g cb myforce)
```

(the variable `$forcecode` may have more entries than the ones shown above). Count the order of the string `myforce`, which is 6 in our case. In order to access this force field from the command line, use the option `-f` 6:

```
> rk2.csh -f 6 -- .......
```

Now, we will study the effect of the air resistance on the motion of the projectile. For small velocities this is a force proportional to the velocity $\vec{F}_r = -mk\vec{v}$, therefore

$$\begin{aligned} a_x &= -kv_x \\ a_y &= -kv_y - g\,. \end{aligned} \tag{5.4}$$

By taking

$$
\begin{aligned}
x(t) &= x_0 + \frac{v_{0x}}{k}\left(1 - e^{-kt}\right) \\
y(t) &= y_0 + \frac{1}{k}\left(v_{0y} + \frac{g}{k}\right)\left(1 - e^{-kt}\right) - \frac{g}{k}t \\
v_x(t) &= v_{0x}e^{-kt} \\
v_y(t) &= \left(v_{0y} + \frac{g}{k}\right)e^{-kt} - \frac{g}{k}\,, \qquad (5.5)
\end{aligned}
$$

we obtain the motion of a particle with terminal velocity $v_y(+\infty) = -g/k$ ($x(+\infty) = \text{const.}$, $y(+\infty) \sim t$).

The acceleration caused by the air resistance is programmed in the file (`k1` $\leftrightarrow g$, `k2` $\leftrightarrow k$) `rk2_vg.cpp`:

```
//========================================================
//The acceleration functions f3,f4(t,x1,x2,v1,v2) provided
//by the user
//========================================================
//Free fall in constant gravitational filled with
//ax = -k2 vx    ay = -k2 vy - k1
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
extern double k1,k2;
//--------------------------------------------------------
double
f3(const double& t , const double& x1, const double& x2,
   const double& v1, const double& v2){
  return -k2*v1;  // dx3/dt=dv1/dt=a1
}
//--------------------------------------------------------
double
f4(const double& t , const double& x1, const double& x2,
   const double& v1, const double& v2){
  return -k2*v2-k1;  // dx4/dt=dv2/dt=a2
}
//--------------------------------------------------------
double
energy
```

```
  (const double& t , const double& x1, const double& x2,
   const double& v1, const double& v2){
  return 0.5*(v1*v1+v2*v2) + k1*x2;
}
```

The results are shown in figure 5.3 where we see the effect of an increasing air resistance on the particle trajectory. The effect of a resistance force of the form $\vec{F}_r = -mkv^2\hat{v}$ is shown in figure 5.4.

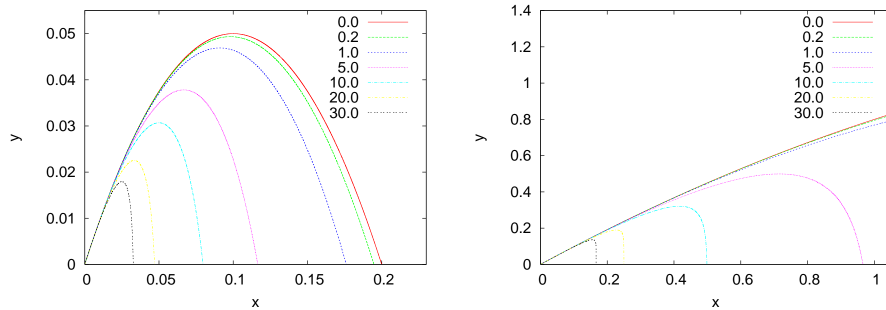

Figure 5.3: The trajectory of a projectile moving in a constant gravitational field $\vec{g} = -10\,\hat{y}$ with air resistance causing acceleration $\vec{a}_r = -k\vec{v}$ for $k = 0, 0.2, 1, 5, 10, 20, 30$. The left plot has $\vec{v}(0) = \hat{x} + \hat{y}$ and the right plot has $\vec{v}(0) = 5\hat{x} + 5\hat{y}$.

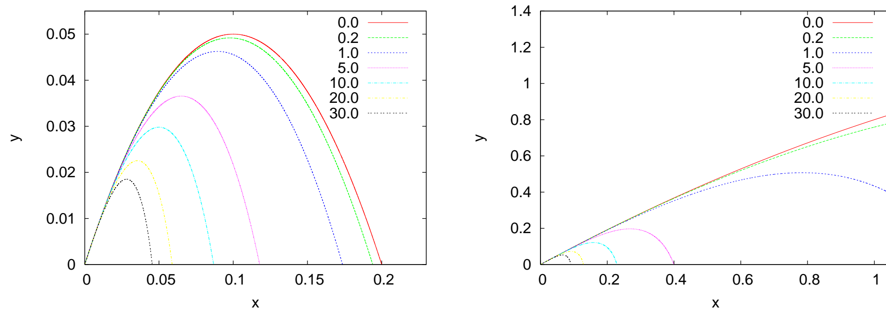

Figure 5.4: The trajectory of a projectile moving in a constant gravitational field $\vec{g} = -10\,\hat{y}$ with air resistance causing acceleration $\vec{a}_r = -kv^2\hat{v}$ for $k = 0, 0.2, 1, 5, 10, 20, 30$. The left plot has $\vec{v}(0) = \hat{x} + \hat{y}$ and the right plot has $\vec{v}(0) = 5\hat{x} + 5\hat{y}$.

5.3 Planetary Motion

Consider the simple planetary model of a "sun" of mass M and a planet "earth" at distance r from the sun and mass m such that $m \ll M$. According to Newton's law of gravity, the earth's acceleration is

$$\vec{a} = \vec{g} = -\frac{GM}{r^2}\hat{r} = -\frac{GM}{r^3}\vec{r}, \tag{5.6}$$

where $G = 6.67 \times 10^{-11}\frac{\text{m}^3}{\text{kgr}\cdot\text{sec}^2}$, $M = 1.99 \times 10^{30}$kgr, $m = 5.99 \times 10^{24}$kgr. When the hypothesis $m \ll M$ is not valid, the two body problem is reduced to that of the one body problem with the mass replaced by the reduced mass μ

$$\frac{1}{\mu} = \frac{1}{m} + \frac{1}{M}\,.$$

The force of gravity is a central force. This implies conservation of the angular momentum $\vec{L} = \vec{r} \times \vec{p}$ with respect to the center of the force, which in turn implies that the motion is confined on one plane. We choose the z axis so that

$$\vec{L} = L_z\hat{k} = m(xv_y - yv_x)\hat{k}\,. \tag{5.7}$$

The force of gravity is conservative and the mechanical energy

$$E = \frac{1}{2}mv^2 - \frac{GmM}{r} \tag{5.8}$$

is conserved. If we choose the origin of the coordinate axes to be the center of the force, the equations of motion (5.6) become

$$\begin{aligned} a_x &= -\frac{GM}{r^3}\,x \\ a_y &= -\frac{GM}{r^3}\,y\,, \end{aligned} \tag{5.9}$$

where $r^2 = x^2 + y^2$. This is a system of two coupled differential equations for the functions $x(t)$, $y(t)$. The trajectories are conic sections which are either an ellipse (bound states - "planet"), a parabola (e.g. escape to infinity when the particle starts moving with speed equal to the escape velocity) or a hyperbola (e.g. scattering).

Kepler's third law of planetary motion states that the orbital period T of a planet satisfies the equation

$$T^2 = \frac{4\pi^2}{GM} a^3 \,, \tag{5.10}$$

where a is the semi-major axis of the elliptical trajectory. The eccentricity is a measure of the deviation of the trajectory from being circular

$$e = \sqrt{1 - \frac{b^2}{a^2}} \,, \tag{5.11}$$

where b is the semi-minor axis. The eccentricity is 0 for the circle and tends to 1 as the ellipse becomes more and more elongated. The foci F_1 and F_2 are located at a distance ea from the center of the ellipse. They have the property that for every point on the ellipse

$$PF_1 + PF_2 = 2a \,. \tag{5.12}$$

The acceleration given to the particle by Newton's force of gravity is programmed in the file `rk2_cb.cpp`:

```
//========================================================
//The acceleration functions f3,f4(t,x1,x2,v1,v2) provided
//by the user
//========================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
extern double k1,k2;
//--------------------------------------------------------
//Motion in Coulombic potential:
//ax= k1*x1/r^3 ay= k1*x2/r^3
double
f3(const double& t , const double& x1, const double& x2,
   const double& v1, const double& v2){
  double r2,r3;
  r2=x1*x1+x2*x2;
  r3=r2*sqrt(r2);
  if(r3>0.0)
```

```
    return k1*x1/r3; // dx3/dt=dv1/dt=a1
  else
    return 0.0;
}
//-----------------------------------------------------------
double
f4(const double& t , const double& x1, const double& x2,
   const double& v1, const double& v2){
  double r2,r3;
  r2=x1*x1+x2*x2;
  r3=r2*sqrt(r2);
  if(r3>0.0)
    return k1*x2/r3; // dx4/dt=dv4/dt=a4
  else
    return 0.0;
}
//-----------------------------------------------------------
double
energy
  (const double& t , const double& x1, const double& x2,
   const double& v1, const double& v2){
  double r;
  r=sqrt(x1*x1+x2*x2);
  if( r > 0.0)
    return 0.5*(v1*v1+v2*v2) + k1/r;
  else
    return 0.0;
}
```

We set `k1=` $-GM$ and take special care to avoid hitting the center of the force, the singular point at $(0,0)$. The same code can be used for the electrostatic Coulomb field with `k1=` $qQ/4\pi\epsilon_0 m$.

At first we study trajectories which are bounded. We set $GM = 10$, $x(0) = 1.0$, $y(0) = 0$, $v_{0x} = 0$ and vary v_{0y}. We measure the period T and the length of the semi axes of the resulting ellipse. The results can be found in table 5.1. Some of the trajectories are shown in figure 5.5. There we can see the dependence of the size of the ellipse on the period. Figure 5.6 confirms Kepler's third law of planetary motion given by equation (5.10).

In order to confirm Kepler's third law of planetary motion numeri-

v_{0x}	$T/2$	$2a$
3.2	1.030	2.049
3.4	1.281	2.370
3.6	1.682	2.841
3.8	2.396	3.597
4.0	3.927	5.000
4.1	5.514	6.270
4.2	8.665	8.475
4.3	16.931	13.245
4.3	28.088	18.561
4.38	42.652	24.522
4.40	61.359	31.250
4.42	99.526	43.141

Table 5.1: The results for the period T and the length of the semi-major axis a of the trajectory of planetary motion for $GM = 10$, $x(0) = 1.0$, $y(0) = 0$, $v_{0y} = 0$.

cally, we take the logarithm of both sides of equation (5.10)

$$\ln T = \frac{3}{2}\ln a + \frac{1}{2}\ln\left(\frac{4\pi^2}{GM}\right) . \tag{5.13}$$

Therefore, the points $(\ln a, \ln T)$ lie on a straight line. Using a linear least squares fit we calculate the slope and the intercept which should be equal to $\frac{3}{2}$ and $1/2\ln(4\pi^2/GM)$ respectively. This is left as an exercise.

In the case where the initial velocity of the particle becomes larger than the escape velocity v_e, the particle escapes from the influence of the gravitational field to infinity. The escape velocity corresponds to zero mechanical energy, which gives

$$v_e^2 = \frac{2GM}{r} . \tag{5.14}$$

When $GM = 10$, $x(0) = 1.0$, $y(0) = 0$, we obtain $v_e \approx 4.4721\ldots$. The numerical calculation of v_e is left as an exercise.

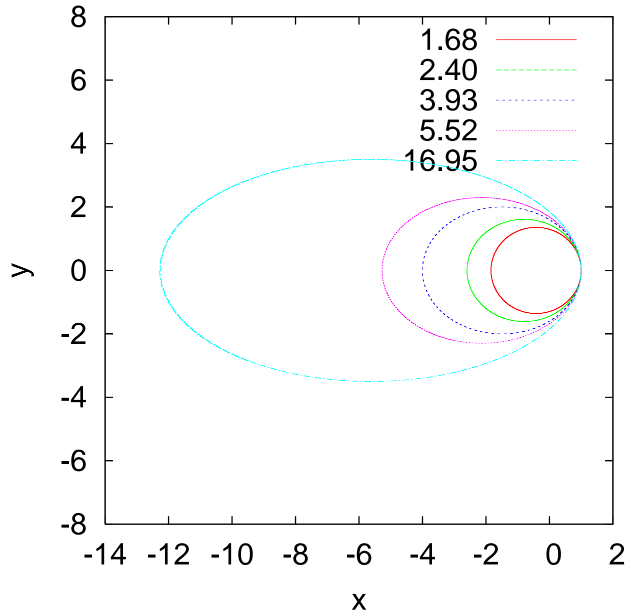

Figure 5.5: Planetary trajectories for $GM = 10$, $x(0) = 1.0$, $y(0) = 0$, $v_{0y} = 0$ and $v_{0x} = 3.6, 3.8, 4.0, 4.1, 4.3$. The numbers are the corresponding half periods.

5.4 Scattering

In this section we consider scattering of particles from a central potential[1]. We assume particles that follow unbounded trajectories that start from infinity and move almost free from the influence of the force field towards its center. When they approach the region of interaction they get deflected and get off to infinity in a new direction. We say that the particles have been scattered and that the angle between their original and final direction is the scattering angle θ. Scattering problems are interesting because we can infer to the properties of the scattering potential from the distribution of the scattering angle. This approach is heavily used in today's particle accelerators for the study of fundamental interactions between elementary particles.

First we will discuss scattering of small hard spheres of radius r_1 by

[1]We refer the reader to [40], chapter 4.

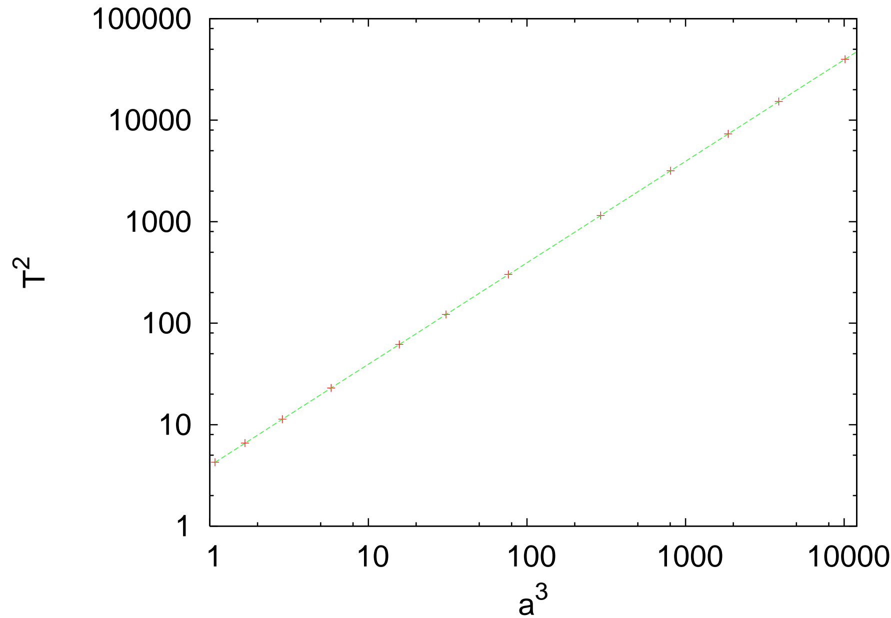

Figure 5.6: Kepler's third law of planetary motion for $GM = 10$. The points are the measurements taken from table 5.1. The solid line is the known analytic solution (5.10).

other hard spheres or radius R_2. The interaction potential[2] is given by

$$V(r) = \begin{cases} 0 & r > R_2 + r_1 \\ \infty & r < R_2 + r_1 \end{cases} , \tag{5.15}$$

where r is the distance between the center of r_1 from the center of R_2. Assume that the particles in the beam do not interact with each other and that there is only one collision per scattering. Let J be the intensity of the beam[3] and A its cross sectional area. Assume that the target has n particles per unit area. The cross sectional area of the interaction is $\sigma = \pi(r_1 + R_2)^2$ where r_1 and R_2 are the radii of the scattered particles and targets respectively (see figure (5.8)): All the spheres of the beam which lie outside this area are not scattered by the particular target. The

[2]The so called hard core potential.

[3]The number of particles crossing a surface perpendicular to the beam per unit time and unit area.

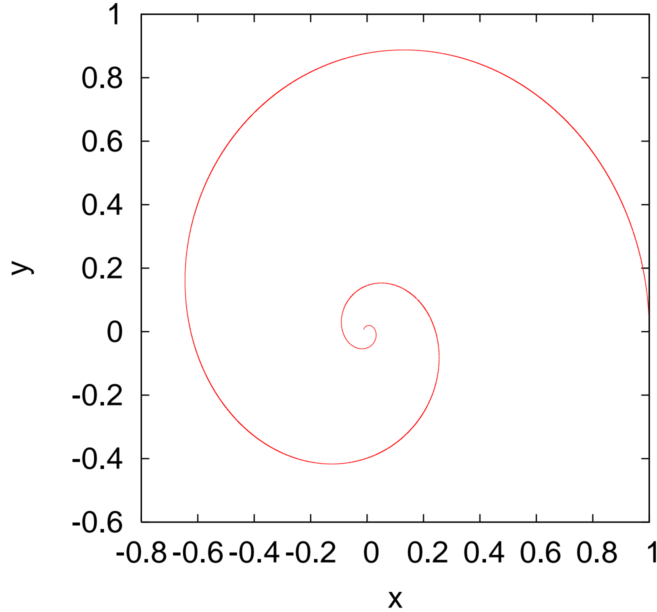

Figure 5.7: The spiral orbit of a particle moving under the influence of a central force $\vec{F} = -k/r^3\hat{r}$.

total interaction cross section is

$$\Sigma = nA\sigma, \tag{5.16}$$

where nA is the total number of target spheres which lie within the beam. On the average, the scattering rate is

$$N = J\Sigma = JnA\sigma\,. \tag{5.17}$$

The above equation is the definition of the total scattering cross section σ of the interaction. The differential cross section $\sigma(\theta)$ is defined by the relation

$$dN = JnA\sigma(\theta)\,d\Omega\,, \tag{5.18}$$

where dN is the number of particles per unit time scattered within the solid angle $d\Omega$. The total cross section is

$$\sigma_{tot} = \int_\Omega \sigma(\theta)\,d\Omega = \int \sigma(\theta)\sin\theta\,d\theta d\phi = 2\pi\int \sigma(\theta)\sin\theta\,d\theta\,. \tag{5.19}$$

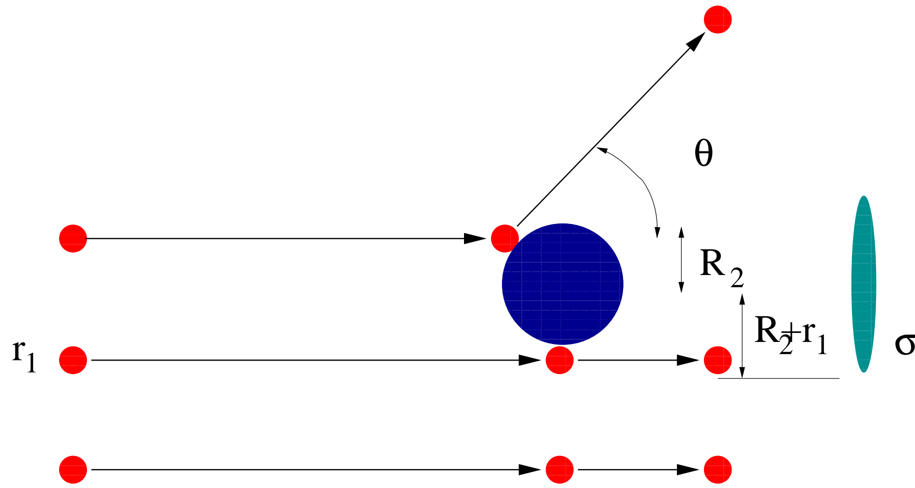

Figure 5.8: Scattering of hard spheres. The scattering angle is θ. The cross sectional area σ is shown to the right.

In the last relation we used the cylindrical symmetry of the interaction with respect to the axis of the collision. Therefore

$$\sigma(\theta) = \frac{1}{nAJ}\frac{dN}{2\pi\sin\theta\, d\theta}\,. \tag{5.20}$$

This relation can be used in experiments for the measurement of the differential cross section by measuring the rate of detection of particles within the space contained in between two cones defined by the angles θ and $\theta + d\theta$. This is the relation that we will use in the numerical calculation of $\sigma(\theta)$.

Generally, in order to calculate the differential cross section we shoot a particle at a target as shown in figure 5.9. The scattering angle θ depends on the impact parameter b. The part of the beam crossing the ring of radius $b(\theta)$, thickness db and area $2\pi b\, db$ is scattered in angles between θ and $\theta + d\theta$. Since there is only one particle at the target we have that $nA = 1$. The number of particles per unit time crossing the ring is $J 2\pi b\, db$, therefore

$$2\pi b(\theta)\, db = -2\pi\sigma(\theta)\sin\theta\, d\theta \tag{5.21}$$

(the $-$ sign is because as b increases, θ decreases). From the potential we

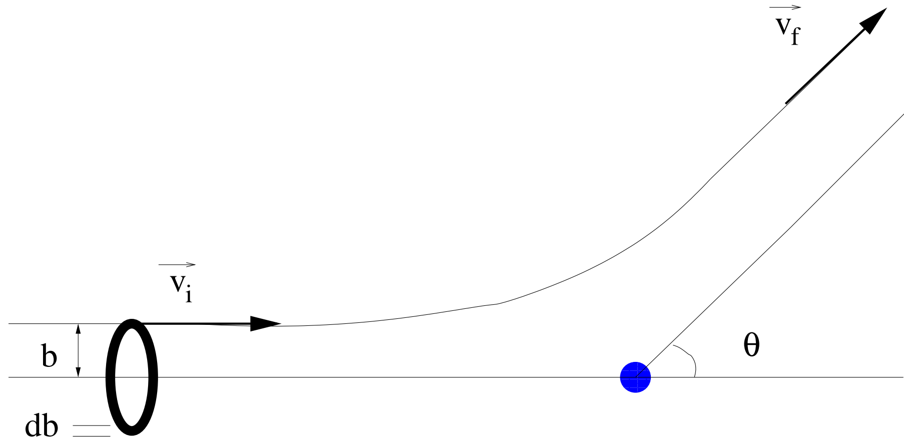

Figure 5.9: Beam particles passing through the ring $2\pi bdb$ are scattered within the solid angle $d\Omega = 2\pi sin\theta\, d\theta$.

can calculate $b(\theta)$ and from $b(\theta)$ we can calculate $\sigma(\theta)$. Conversely, if we measure $\sigma(\theta)$, we can calculate $b(\theta)$.

5.4.1 Rutherford Scattering

The scattering of a charged particle with charge q ("electron") in a Coulomb potential of a much heavier charge Q ("nucleus") is called Rutherford scattering. In this case, the interaction potential is given by

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}\,, \tag{5.22}$$

which accelerates the particle with acceleration

$$\vec{a} = \frac{qQ}{4\pi\epsilon_0 m}\frac{\hat{r}}{r^2} \equiv \alpha\frac{\vec{r}}{r^3}\,. \tag{5.23}$$

The energy of the particle is $E = \frac{1}{2}mv^2$ and the magnitude of its angular momentum is $l = mvb$, where $v \equiv |\vec{v}|$. The dependence of the impact parameter on the scattering angle is [40]

$$b(\theta) = \frac{\alpha}{v^2}\cot\frac{\theta}{2}\,. \tag{5.24}$$

Using equation (5.21) we obtain

$$\sigma(\theta) = \frac{\alpha^2}{4}\frac{1}{v^4}\sin^{-4}\frac{\theta}{2}\,. \qquad (5.25)$$

Consider the scattering trajectories. The results for same charges are

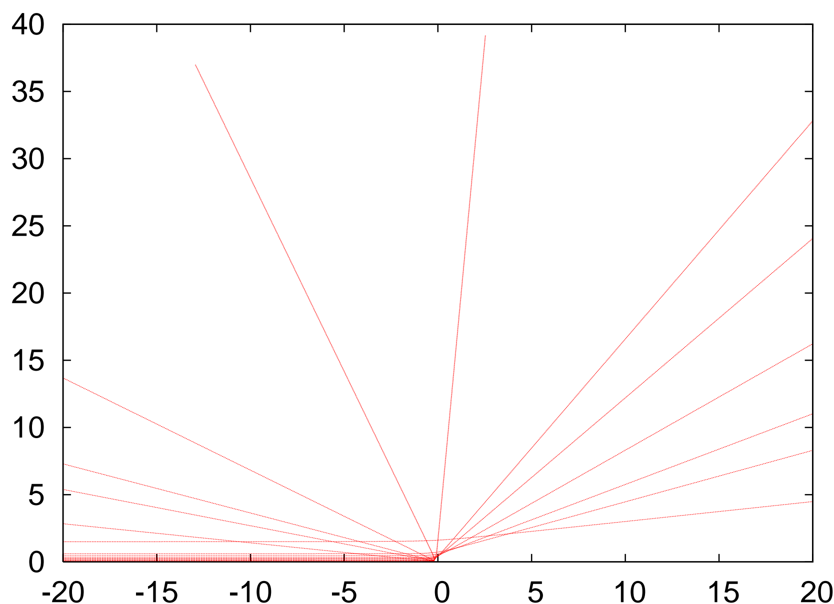

Figure 5.10: Rutherford scattering trajectories. We set $\mathtt{k1} \equiv \frac{qQ}{4\pi\epsilon_0 m} = 1$ (see code in the file `rk2_cb.cpp`) and $b = 0.08,\ 0.015,\ 0.020,\ 0.035,\ 0.080,\ 0.120,\ 0.200,\ 0.240,\ 0.320,\ 0.450,\ 0.600,\ 1.500$. The initial position of the particle is at $x(0) = -50$ and its initial velocity is $v = 3$ in the x direction. The number of integration steps is 1000, the initial time is 0 and the final time is 30.

shown in figure 5.10. A similar figure is obtained in the case of opposite charges. In the latter case we have to take special care for small impact parameters $b < 0.2$ where the scattering angle is ≈ 1. A large number of integration steps is needed in order to obtain the desired accuracy. A useful monitor of the accuracy of the calculation is the measurement of the energy of the particle which should be conserved. The results are shown in table 5.2. We will now describe a method for calculating the cross section by using equation (5.20). Alternatively we could have used

b	θ_n	θ_a	$\Delta E/E$	`Nt`
0.008	2.9975	2.9978	$2.8\,10^{-9}$	5000
0.020	2.7846	2.7854	$2.7\,10^{-9}$	5000
0.030	2.6131	2.6142	$2.5\,10^{-9}$	5000
0.043	2.4016	2.4031	$2.3\,10^{-9}$	5000
0.056	2.2061	2.2079	$2.0\,10^{-9}$	5000
0.070	2.0152	2.0172	$1.7\,10^{-9}$	5000
0.089	1.7887	1.7909	$1.4\,10^{-9}$	5000
0.110	1.5786	1.5808	$1.0\,10^{-9}$	5000
0.130	1.4122	1.4144	$0.8\,10^{-9}$	5000
0.160	1.2119	1.2140	$0.5\,10^{-9}$	5000
0.200	1.0123	1.0142	$0.3\,10^{-9}$	5000
0.260	0.8061	0.8077	$0.1\,10^{-9}$	5000
0.360	0.5975	0.5987	$2.9\,10^{-11}$	5000
0.560	0.3909	0.3917	$0.3\,10^{-11}$	5000
1.160	0.1905	0.1910	$5.3\,10^{-14}$	5000

Table 5.2: Scattering angles of Rutherford scattering. We set `k1` $\equiv \frac{qQ}{4\pi\epsilon_0 m} = 1$ (see file `rk2_cb.cpp`) and study the resulting trajectories for the values of b shown in column 1. θ_n is the numerically calculated scattering angle and θ_a is the one calculated from equation (5.24). The ratio $\Delta E/E$ shows the change in the particle's energy due to numerical errors. The last column is the number of integration steps. The particle's initial position is at $x(0) = -50$ and initial velocity $\vec{v} = 3\hat{x}$.

equation (5.21) and perform a numerical calculation of the derivatives. This is left as an exercise for the reader. Our calculation is more like an experiment. We place a "detector" that "detects" particles scattered within angles θ and $\theta+\delta\theta$. For this reason we split the interval $[0,\pi]$ in N_b bins so that $\delta\theta = \pi/N_b$. We perform "scattering experiments" by varying $b \in [b_m, b_M]$ with step δb. Due to the symmetry of the problem we fix ϕ to be a constant, therefore a given θ corresponds to a cone with an opening angle θ and an apex at the center of scattering. For given b we measure the scattering angle θ and record the number of particles per unit time $\delta N \propto b\delta b$. The latter is proportional to the area of the ring of radius b. All we need now is the beam intensity J which is the total number of particles per unit time $J \propto \sum_i b\delta b$ (note than in the ratio $\delta N/J$ the proportionality constant and δb cancel) and the solid angle $2\pi\sin(\theta)\,\delta\theta$.

b	θ_n	θ_a	$\Delta E/E$	STEPS
0.020	2.793	2.785	0.02	1 000 000
0.030	2.620	2.614	$8.2\,10^{-3}$	300 000
0.043	2.405	2.403	$7.2\,10^{-4}$	150 000
0.070	2.019	2.017	$3.2\,10^{-7}$	150 000
0.089	1.793	1.791	$8.2\,10^{-7}$	60 000
0.110	1.583	1.581	$1.2\,10^{-6}$	30 000
0.130	1.417	1.414	$9.4\,10^{-7}$	20 000
0.160	1.216	1.214	$6.0\,10^{-5}$	5 000
0.200	1.016	1.014	$4.1\,10^{-6}$	5 000
0.260	0.8093	0.8077	$2.2\,10^{-7}$	5 000
0.360	0.6000	0.5987	$7.6\,10^{-9}$	5 000
0.560	0.3926	0.3917	$1.2\,10^{-10}$	5 000
1.160	0.1913	0.1910	$2.9\,10^{-13}$	5 000

Table 5.3: Rutherford scattering of opposite charges with $\frac{qQ}{4\pi\epsilon_0 m} = -1$. The table is similar to table 5.2. We observe the numerical difficulty for small impact parameters.

Finally we can easily use equation (5.19) in order to calculate the total cross section σ_{tot}. The program that performs this calculation is in the file `scatter.cpp` and it is a simple modification of the program in `rk2.cpp`:

```
//========================================================
//Program that computes scattering cross-section of a central
//force on the plane. The user should first check that the
//parameters used, lead to a free state in the end.
//   **  X20 is the impact parameter b  **
//A 4 ODE system is solved using Runge-Kutta Method
//User must supply derivatives
//dx1/dt=f1(t,x1,x2,x3,x4) dx2/dt=f2(t,x1,x2,x3,x4)
//dx3/dt=f3(t,x1,x2,x3,x4) dx4/dt=f4(t,x1,x2,x3,x4)
//as real(8) functions
//Output is written in file scatter.dat
//========================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
```

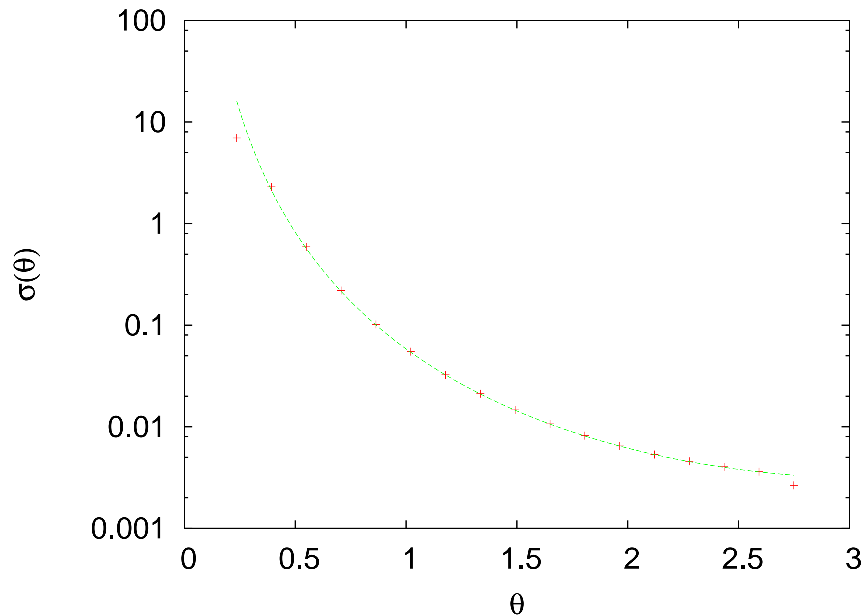

Figure 5.11: Differential cross section of the Rutherford scattering. The solid line is the function (5.25) for $\alpha = 1$, $v = 3$. We set $\frac{qQ}{4\pi\epsilon_0 m} = 1$. The particle's initial position is $x(0) = -50$ and its initial velocity is $\vec{v} = 3\hat{x}$. We used 5000 integration steps, initial time equal to 0 and final time equal to 30. The impact parameter varies between 0.02 and 1 with step equal to 0.0002.

```
using namespace std;
//--------------------------------------------------------------------
const int P = 1010000;
double T[P], X1[P], X2[P], V1[P], V2[P];
double k1,k2;
//--------------------------------------------------------------------
double
f1(const double& t   , const double& x1, const double& x2,
   const double& v1  , const double& v2);
double
f2(const double& t   , const double& x1, const double& x2,
   const double& v1  , const double& v2);
double
f3(const double& t   , const double& x1, const double& x2,
   const double& v1  , const double& v2);
double
```

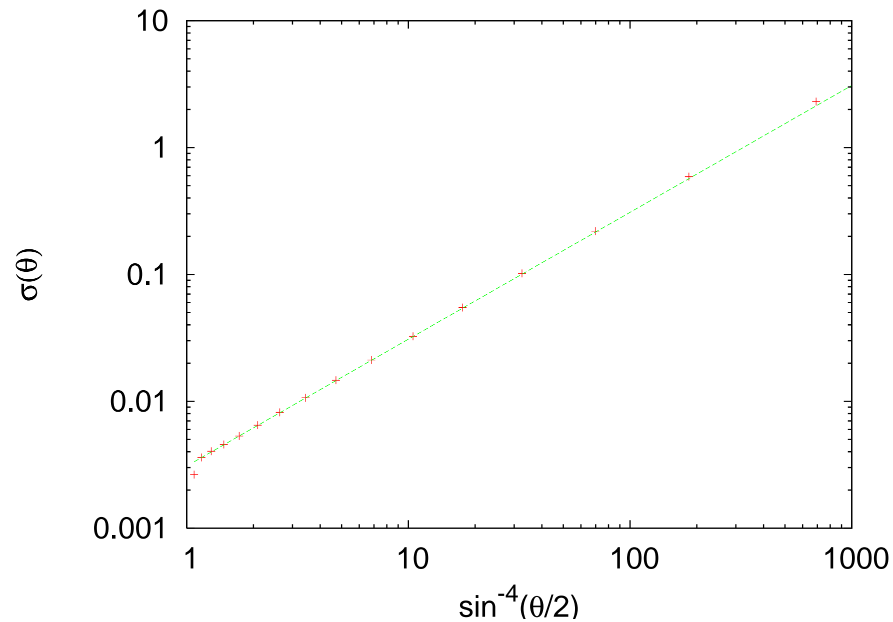

Figure 5.12: Differential cross section of the Rutherford scattering like in figure 5.11. The solid line is the function $1/(4\times3^4)x$ from which we can deduce the functional form of $\sigma(\theta)$.

```
f4(const double& t  , const double& x1, const double& x2,
   const double& v1 , const double& v2);
double
energy
  (const double& t  , const double& x1, const double& x2,
   const double& v1 , const double& v2);
void
RK(const double& Ti , const double& Tf ,
   const double& X10, const double& X20,
   const double& V10, const double& V20,
   const int    & Nt);
void
RKSTEP(double& t ,
       double& x1, double& x2,
       double& x3, double& x4,
       const       double& dt);
//------------------------------------------------------------
int main(){
```

```
  string buf;
  double Ti,Tf,X10,X20,V10,V20;
  double X20F,dX20; //max impact parameter and step
  int    Nt,i ;
  const int Nbins=20;
  int index;
  double angle,bins[Nbins],Npart;
  const double PI     =3.14159265358979324;
  const double rad2deg=180.0/PI;
  const double dangle =PI/Nbins;
  double R,density,dOmega,sigma,sigmatot;

  //Input:
  cout << "Runge-Kutta Method for 4-ODEs Integration\n";
  cout << "Enter coupling constants:\n";
  cin  >> k1 >> k2;getline(cin,buf);
  cout << "k1= " << k1 << " k2= " << k2 << endl;
  cout << "Enter Nt,Ti,Tf,X10,X20,V10,V20:\n";
  cin  >> Nt >> Ti >> Tf>> X10 >> X20 >> V10 >> V20;
  getline(cin,buf);
  cout << "Enter final impact parameter X20F and step dX20:\n";
  cin  >> X20F >> dX20;
  cout << "Nt = " << Nt << endl;
  cout << "Time: Initial Ti = " << Ti
       << " Final Tf= "         << Tf   << endl;
  cout << "            X1(Ti)= " << X10
       << " X2(Ti)="            << X20  << endl;
  cout << "            V1(Ti)= " << V10
       << " V2(Ti)="            << V20  << endl;
  cout << "Impact par X20F  ="  << X20F
       << " dX20  ="            << dX20 << endl;
  ofstream myfile("scatter.dat");
  myfile.precision(17);
  for(i=0;i<Nbins;i++) bins[i] = 0.0;
  //Calculate:
  Npart  = 0.0;
  X20    = X20 + dX20/2.0; //starts in middle of first interval
  while( X20 < X20F ){
    RK(Ti,Tf,X10,X20,V10,V20,Nt);
    //Take absolute value due to symmetry:
    angle = abs(atan2(V2[Nt-1],V1[Nt-1]));
    //Output: The final angle. Check if almost constant
    myfile << "@ " << X20 << " " <<  angle
           << "  " << abs(atan2(V2[Nt-51],V1[Nt-51]))
           << "  " << k1/(V10*V10)/tan(angle/2.0) << endl;
```

```
    //Update histogram:
    index       = int(angle/dangle);
    //Number of incoming particles per unit time
    //is proportional to radius of ring
    //of radius X20, the impact parameter:
    bins[index] += X20 ; // db is cancelled from density
    Npart       += X20 ; // <-- i.e. from here
    X20         += dX20;
  }//while( X20 < X20F )
  //Print scattering cross section:
  R          = X20;              //beam radius
  density    = Npart/(PI*R*R);   //beam flux density J
  sigmatot   = 0.0;              //total cross section
  for(i=0;i<Nbins;i++){
    angle    = (i+0.5)*dangle;
    dOmega   = 2.0*PI*sin(angle)*dangle; //d(Solid Angle)
    sigma    = bins[i]/(density*dOmega);
    if(sigma>0.0)
      myfile << "ds= " << angle
             << " "    << angle*rad2deg
             << " "    << sigma << endl;
    sigmatot += sigma*dOmega;
  }//for(i=0;i<Nbins;i++)
  myfile << "sigmatot= " << sigmatot << endl;
  myfile.close();
}//main()
```

The results are recorded in the file `scatter.dat`. An example session that reproduces figures 5.11 and 5.12 is

```
> g++ scatter.cpp rk2_cb.cpp -o scatter
> ./scatter
Runge-Kutta Method for 4-ODEs Integration
Enter coupling constants:
1.0 0.0
k1= 1 k2= 0
Enter Nt,Ti,Tf,X10,X20,V10,V20:
5000 0 30 -50 0.02 3 0
Enter final impact parameter X20F and step dX20:
1 0.0002
Nt = 5000
Time: Initial Ti = 0 Final Tf= 30
           X1(Ti)= -50 X2(Ti)=0.02
           V1(Ti)= 3   V2(Ti)=0
```

```
Impact par X20F   =1 dX20   =0.0002
```

The results can be plotted with the `gnuplot` commands:

```
gnuplot> set log
gnuplot> plot [:1000] "<grep ds= scatter.dat" \
 u ((sin($2/2))**(-4)):($4) notit,\
          (1./(4.*3.**4))*x notit
gnuplot> unset log
gnuplot> set log y
gnuplot> plot [:] "<grep ds= scatter.dat" u 2:4 notit, \
 (1./(4.*3.**4))*(sin(x/2))**(-4) notit
```

The results are in a very good agreement with the theoretical ones given by (5.25). The next step will be to study other central potentials whose solution is not known analytically.

5.4.2 More Scattering Potentials

Consider scattering from a force field

$$\vec{F} = f(r)\,\hat{r}\,, \qquad f(r) = \begin{cases} \frac{1}{r^2} - \frac{r}{a^3} & r \leq a \\ 0 & r > a \end{cases} . \tag{5.26}$$

This is a very simple classical model of the scattering of a positron e^+ by the hydrogen atom. The positron has positive charge $+e$ and the hydrogen atom consists of a positively charged proton with charge $+e$ in an electron cloud of opposite charge $-e$. We set the scales so that $m_{e^+} = 1$ and $e^2/4\pi\epsilon_0 = 1$. We will perform a numerical calculation of $b(\theta)$, $\sigma(\theta)$ and σ_{tot}.

The potential energy is given by

$$f(r) = -\frac{dV(r)}{dr} \Rightarrow V(r) = \frac{1}{r} + \frac{r^2}{2a^2} - \frac{3}{2a}\,. \tag{5.27}$$

where $V(r) = 0$ for $r \geq a$. The program containing the calculation of the acceleration caused by this force can be found in the file `rk_hy.cpp`:

```
//========================================================
//The acceleration functions f3,f4(t,x1,x2,v1,v2) provided
```

```
//by the user
//=========================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
extern double k1,k2;
//--------------------------------------------------------
//Motion in hydrogen atom + positron:
//f(r) = 1/r^2-r/k1^3
//ax= f(r)*x1/r ay= f(r)*x2/r
double
f3(const double& t , const double& x1, const double& x2,
   const double& v1, const double& v2){
  double r2,r,fr;
  r2=x1*x1+x2*x2;
  r =sqrt(r2);
  if(r <= k1 && r2 > 0.0)
    fr = 1.0/r2-r/(k1*k1*k1);
  else
    fr = 0.0;

  if(fr > 0.0 && r > 0.0)
    return fr*x1/r; // dx3/dt=dv1/dt=a1
  else
    return 0.0;

}
//--------------------------------------------------------
double
f4(const double& t , const double& x1, const double& x2,
   const double& v1, const double& v2){
  double r2,r,fr;
  r2=x1*x1+x2*x2;
  r =sqrt(r2);
  if(r <= k1 && r2 > 0.0)
    fr = 1.0/r2-r/(k1*k1*k1);
  else
    fr = 0.0;

  if(fr > 0.0 && r > 0.0)
    return fr*x2/r; // dx4/dt=dv2/dt=a2
  else
```

```
    return 0.0;

}
//------------------------------------------------------------------------
double
energy
  (const double& t , const double& x1, const double& x2,
   const double& v1, const double& v2){
  double r,Vr;
  r=sqrt(x1*x1+x2*x2);
  if(r <= k1 && r > 0.0)
    Vr = 1/r + 0.5*r*r/(k1*k1*k1) - 1.5 / k1;
  else
    Vr = 0.0;
  return 0.5*(v1*v1+v2*v2) + Vr;
}
```

The results are shown in figures 5.13–5.14. We find that $\sigma_{tot} = \pi a^2$ (see problem 5.10).

Another interesting dynamical field is given by the Yukawa potential. This is a phenomenological model of nuclear interactions:

$$V(r) = k\frac{\mathrm{e}^{-r/a}}{r}\,. \tag{5.28}$$

This field can also be used as a model of the effective interaction of electrons in metals (Thomas–Fermi) or as the Debye potential in a classic plasma. The resulting force is

$$\vec{F}(r) = f(r)\,\hat{r}\,, \qquad f(r) = k\frac{\mathrm{e}^{-r/a}}{r^2}\left(1+\frac{r}{a}\right) \tag{5.29}$$

The program of the resulting acceleration can be found in the file `rk2_yu.cpp`. The results are shown in figures 5.15–5.16.

5.5 More Particles

In this section we will generalize the discussion of the previous paragraphs in the case of a dynamical system with more degrees of freedom. The number of dynamical equations that need to be solved depends on the number of degrees of freedom and we have to write a program that

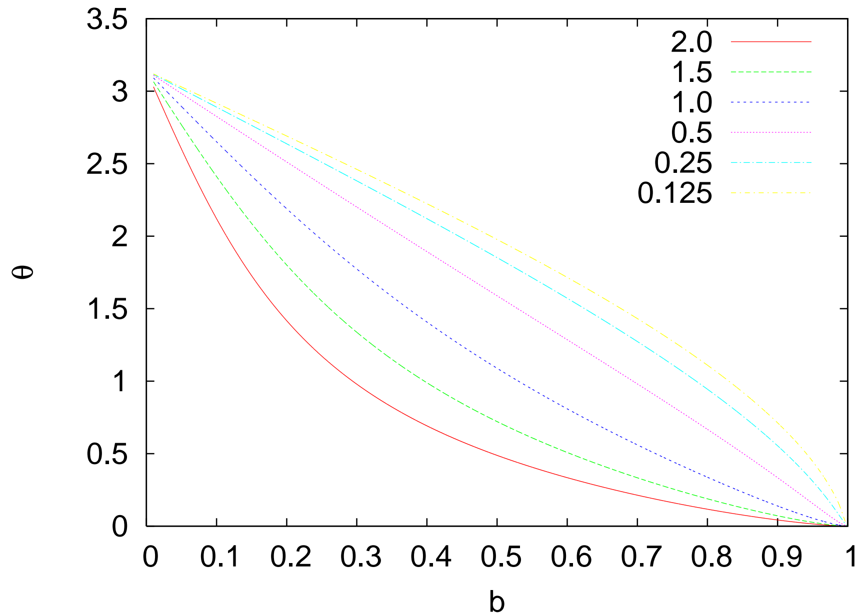

Figure 5.13: The impact parameter $b(\theta)$ for the potential given by equation (5.27) for different values of the initial velocity v. We set $a = 1$, $x(0) = -5$ and made 4000 integration steps from $t_i = 0$ to $t_f = 40$.

implements the 4th order Runge–Kutta method for an arbitrary number of equations NEQ. We will explain how to allocate memory *dynamically*, in which case the necessary memory storage space, which depends on NEQ, is allocated at the time of running the program and not at compilation time.

Until now, memory has been allocated *statically*. This means that arrays have sizes which are known at compile time. For example, in the program rk2.cpp the integer parameter P had a given value which determined the size of all arrays using the declarations:

```
const int P = 1010000;
double T[P], X1[P], X2[P], V1[P], V2[P];
```

Changing P after compilation is impossible and if this becomes necessary we have to edit the file, change the value of P and recompile. Dynamical memory allocation allows us to read in Nt and NEQ at execution time and

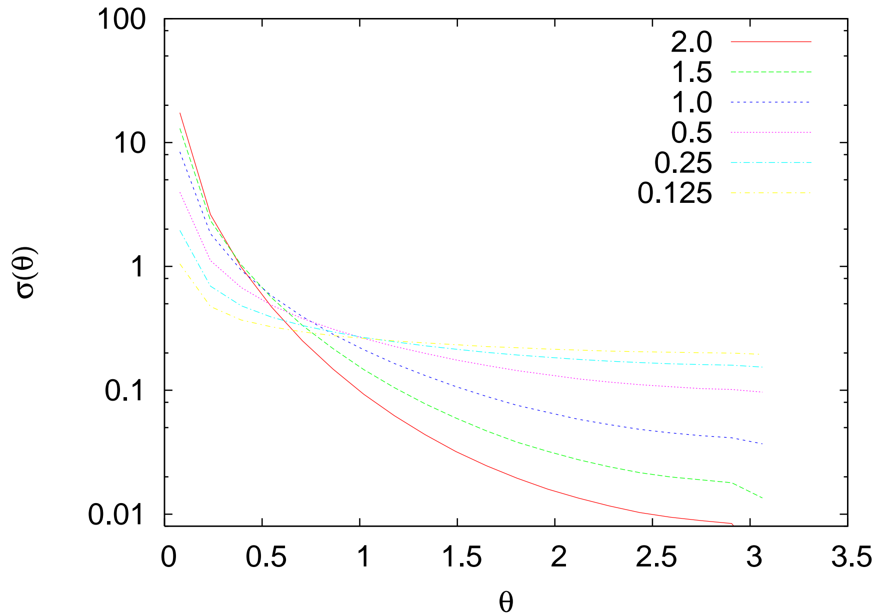

Figure 5.14: The function $\sigma(\theta)$ for the potential given by equation (5.27) for different values of the initial velocity v. We set $a=1$, $x(0)=-5$ and the integration is performed by making 4000 steps from $t_i=0$ to $t_f=40$.

then ask from the operating system to allocate the necessary memory. The needed memory can be asked for at execution time by using the `new` operator. Here is an example:

```
double  *T; //Declare 1-Dim arrays as pointers
double **X; //Declare 2-Dim arrays as pointers to pointers
int NEQ,Nt; //Variables of array sizes
//--------------------------------------------------------------
finit(NEQ); //function that sets NEQ at run time
cin  >> Nt; //read Nt at run time
//--------------------------------------------------------------
//allocates 1-Dim array of Nt doubles
T  = new double [Nt];
//allocates 1-Dim array of Nt pointers to doubles
X  = new double*[Nt];
//for each i, allocate an array of NEQ doubles at X[i]:
for(int i=0;i<Nt;i++) X[i] = new double[NEQ];
```

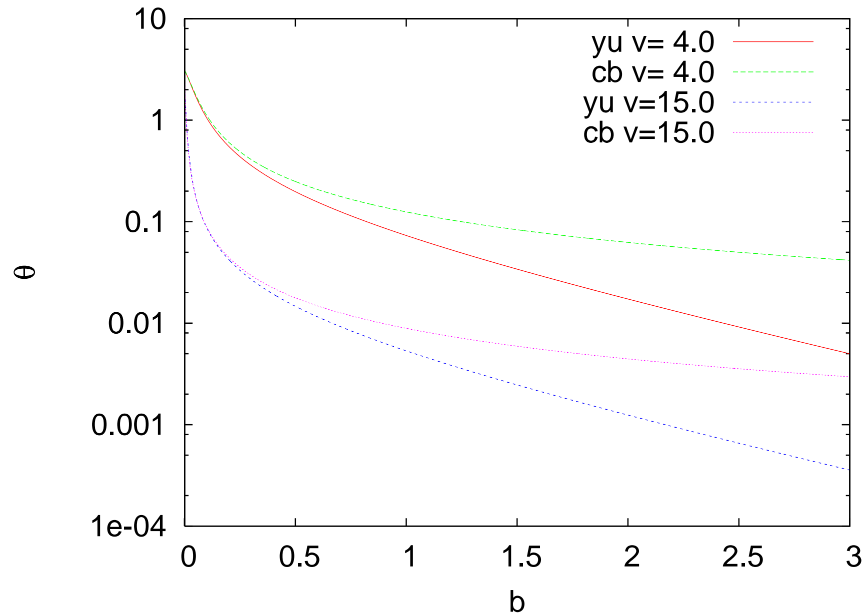

Figure 5.15: The function $b(\theta)$ for the Yukawa scattering for several values of the initial velocity v. We set $a = 1$, $k = 1$, $x(0) = -50$ and the integration is performed with 5000 steps from $t_i = 0$ to $t_f = 30$. The lines marked as `cb` are equation (5.24) of the Rutherford scattering.

```
.....
(compute with T[Nt], X[Nt][NEQ])
.....
//return allocated memory back to the system:
delete [] T; // deallocate an array
//for each i, deallocate the array of doubles X[i][NEQ]
for(int i=0;i<NEQ;i++) delete [] X[i];
//and then deallocate the array of pointers to doubles X[Nt]:
delete[] X ;
.....................
void finit(int& NEQ){//function that sets value of NEQ
  NEQ = 4;
}
```

In this program, we should remember the fact that in C++, the name `T` of an array `T[Nt]` is a pointer to `T[0]`, which is denoted by `&T[0]`. This is the address of the first element of the array in the memory. Therefore,

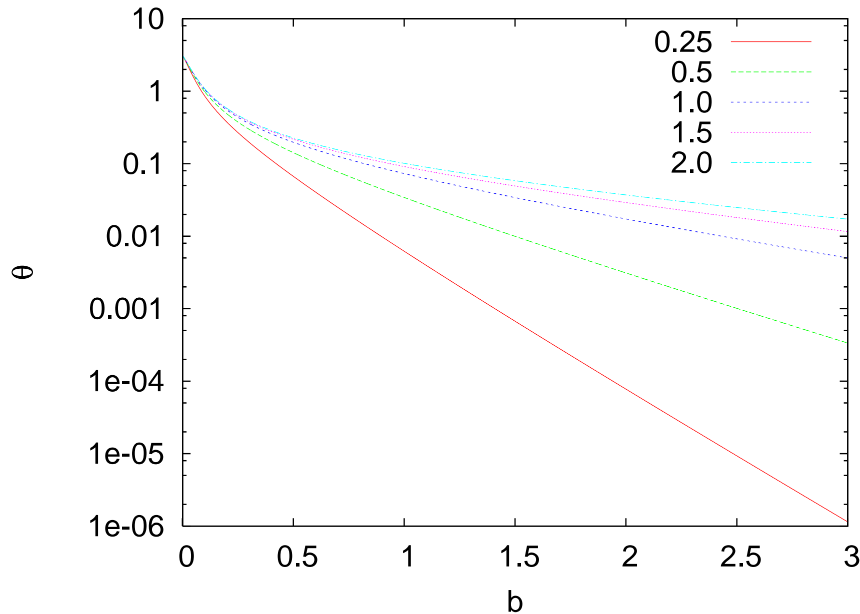

Figure 5.16: The function $b(\theta)$ for the Yukawa scattering for several values of the range a of the force. We set $v = 4.0$, $k = 1$, $x(0) = -50$ and the integration is performed with 5000 steps from $t_i = 0$ to $t_f = 30$.

if the array is `double T[Nt]`, then `T` is a pointer to a double: `double *T`. Then `T+i` is a pointer to the address of the `(i+1)`-th element of the array `T[i]` and

```
T+i = &T[i]
```

as well as

```
T[i] = *(T+i)
```

We can use pointers with the same notation as we do with arrays: If we declare a pointer to a `double *T1`, and assign `T1=T`, then `T1[0], T1[1], ...` are the same as `T[0], T[1], ....` For example, `T1[0] = 2.0` assigns also `T[0]` and vice-versa. Conversely, we can declare a pointer to a `double *T`, as we do in our program, and make it point to a region in the memory where we have reserved space for `Nt` doubles. This is what

the operator `new` does. It asks for the memory for `Nt` doubles and returns a pointer to it. Then we assign this pointer to `T`:

```
double *T;
T = new double[Nt];
```

Then we can use `T[0], T[1], ..., T[Nt-1]` as we do with ordinary arrays.

Two dimensional arrays are slightly trickier: For a two dimensional array, `double X[Nt][NEQ]`, `X` is a pointer to the value `X[0][0]`, which is `&X[0][0]`. Then `X[i]` is a *pointer* to the one dimensional array `X[i][NEQ]`, therefore `X` is a pointer to a pointer of a double!

```
X[i][jeq] is:  double
X[i]      is:  double *
X         is:  double **
```

Conversely, we can declare a `double **X`, and use the operator `new` to return a pointer to an array of `Nt` pointers to doubles, and then for each element of the array, use `new` to return a pointer to `NEQ` doubles:

```
double **X
X       = new double*[Nt];
X[0]    = new double[NEQ];
X[1]    = new double[NEQ];
...
X[Nt-1] = new double[NEQ];
```

Then we can use the notation `T[i][jeq]`, `i= 0, 1, ..., Nt-1` and `jeq= 0, 1, ..., NEQ-1`, as we do with statically defined arrays.

The memory that we ask to be allocated dynamically is a finite resource that can easily be exhausted (heard of *memory leaks*?). Therefore, we should be careful to return unused memory to the system, so that it can be recycled. This should happen especially within functions that we call many times, which allocate large memory dynamically. The operator `delete` can be used to deallocate memory that has been allocated with the operator `new`. For one dimensional arrays, this is particularly simple:

```
double *T;
```

```
T = new double[Nt];
....
(use T[i])
...
delete [] T;
...
(cannot use T after delete)
```

For "two dimensional arrays" that have been allocated as we described above, first we have to delete the arrays pointed by `X[i]` for `i=0, ... , Nt-1`, and then the arrays of pointers pointed by `X`:

```
X  = new double*[Nt ];
for(int i=0;i<Nt;i++) X[i] = new double[NEQ];
.... use X[i][jeq] ....
for(int i=0;i<NEQ;i++) delete [] X[i]; //delete all arrays X[i]
delete[] X ;                            //delete array of pointers
```

The main program will be written in the file `rkA.cpp`, whereas the force-dependent part of the code will be written in files with names of the form `rkA_XXX.cpp`. In the latter, the user must program a *function* `f(t,X,dXdt)` which takes as input the time `t` and the values of the functions `X[NEQ]` and outputs the values of their derivatives `dXdt[NEQ]` at time `t`. The function `finit(NEQ)` sets the number of functions in `f` and it is called once during the initialization phase of the program.

The program in the file `rkA.cpp` is listed below:

```
//========================================================
//Program to solve an ODE system using the
//4th order Runge-Kutta Method
//NEQ: Number of equations
//User supplies two functions:
//f(t,x,xdot): with double  t,x[NEQ],xdot[NEQ] which
//given the time t and current values of functions x[NEQ]
//it returns the values of derivatives: xdot = dx/dt
//The values of two coupling constants k1,k2 may be used
//in f which are read in the main program
//finit(NEQ) : sets the value of NEQ
//
//User Interface:
//double k1,k2:  coupling constants in global scope
//Nt,Ti,Tf: Nt-1 integration steps, initial/final time
```

```
//double X0[NEQ]: initial conditions
//Output:
//rkA.dat with Nt lines consisting of: T[Nt],X[Nt][NEQ]
//=========================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
//---------------------------------------------------------
double  *T;  // T[Nt]      stores the values of times
double **X;  // X[Nt][NEQ] stores the values of functions
double k1,k2;
//---------------------------------------------------------
void RK(const double& Ti, const double& Tf, double* X0,
        const int   & Nt, const int   & NEQ);
void RKSTEP(  double& t ,        double* x ,
        const double& dt, const int   & NEQ);
//The following functions are defined in rkA_XX.cpp:
void finit(   int   & NEQ); //Sets number of equations
//Derivative and energy functions:
void   f     (const double& t, double* X,double* dXdt);
double energy(const double& t, double* X);
//---------------------------------------------------------
int main(){
  string  buf;
  int     NEQ,Nt;
  double*  X0;
  double Ti,Tf;
  //Get number of equations and allocate memory for X0:
  finit(NEQ);
  X0 = new double [NEQ];
  //Input:
  cout << "Runge-Kutta Method for ODE Integration.\n";
  cout << "NEQ= " << NEQ << endl;
  cout << "Enter coupling constants:\n";
  cin  >> k1 >> k2;getline(cin,buf);
  cout << "k1= " << k1 << " k2= " << k2 << endl;
  cout << "Enter Nt, Ti, Tf, X0:\n";
  cin  >>       Nt>>Ti>>Tf;
  for(int i=0;i<NEQ;i++) cin >> X0[i];getline(cin,buf);
  cout << "Nt = " << Nt << endl;
  cout << "Time: Initial Ti =" << Ti << " "
       << "Final Tf="          << Tf << endl;
```

```
  cout << "                X0 =";
  for(int i=0;i<NEQ;i++) cout  << X0[i] << " ";
  cout << endl;
  //Allocate memory for data arrays:
  T  = new double [Nt];
  X  = new double*[Nt];
  for(int i=0;i<Nt;i++) X[i] = new double[NEQ];
  //The Calculation:
  RK(Ti,Tf,X0,Nt,NEQ);
  //Output:
  ofstream  myfile("rkA.dat");
  myfile.precision(16);
  for(int i=0;i<Nt;i++){
    myfile   << T[i]      << " ";
    for(int jeq=0;jeq<NEQ;jeq++)
      myfile << X[i][jeq] << " ";
    myfile   << energy(T[i],X[i]) << '\n';
  }
  myfile.close();
  //------------------------------------------------------------
  //Cleaning up dynamic memory: delete[] each array
  //created with the new operator (Not necessary in this
  //program,it is done at the end of the program anyway)
  delete[] X0;
  delete[] T ;
  for(int i=0;i<NEQ;i++) delete [] X[i];
  delete[] X ;
}//main()
//=============================================================
//Driver of the RKSTEP routine
//=============================================================
void RK(const double& Ti, const double& Tf, double* X0,
        const int   & Nt, const int   & NEQ){
  double  dt;
  double  TS;
  double* XS;

  XS     = new double[NEQ];
  //Initialize variables:
  dt     = (Tf-Ti)/(Nt-1);
  T [0]  = Ti;
  for(int ieq=0;ieq<NEQ;ieq++) X[0][ieq]=X0[ieq];
  TS     = Ti;
  for(int ieq=0;ieq<NEQ;ieq++) XS  [ieq]=X0[ieq];
  //Make RK steps: The arguments of RKSTEP are
```

```
  //replaced with the new ones
  for(int i=1;i<Nt;i++){
    RKSTEP(TS,XS,dt,NEQ);
    T[i] = TS;
    for(int ieq=0;ieq<NEQ;ieq++) X[i][ieq] = XS[ieq];
  }
  // Clean up memory:
  delete [] XS;
}//RK()
//========================================================
//Function RKSTEP(t,X,dt)
//Runge-Kutta Integration routine of ODE
//========================================================
void RKSTEP(double& t, double* x, const double& dt ,
                                  const int   & NEQ){
  double  tt;
  double *k1, *k2, *k3, *k4, *xx;
  double  h,h2,h6;

  k1 = new double[NEQ];
  k2 = new double[NEQ];
  k3 = new double[NEQ];
  k4 = new double[NEQ];
  xx = new double[NEQ];

  h =dt;      // h =dt, integration step
  h2=0.5*h;   // h2=h/2
  h6=h/6.0;   // h6=h/6
  //1st step:
  f(t ,x ,k1);
  //2nd step:
  for(int ieq=0;ieq<NEQ;ieq++)
    xx[ieq] = x[ieq] + h2*k1[ieq];
  tt =t+h2;
  f(tt,xx,k2);
  //3rd step:
  for(int ieq=0;ieq<NEQ;ieq++)
    xx[ieq] = x[ieq] + h2*k2[ieq];
  tt =t+h2;
  f(tt,xx,k3);
  //4th step:
  for(int ieq=0;ieq<NEQ;ieq++)
    xx[ieq] = x[ieq] + h *k3[ieq];
  tt=t+h ;
  f(tt,xx,k4);
```

```
  //Update:
  t += h;
  for(int ieq=0;ieq<NEQ;ieq++)
    x[ieq] += h6*(k1[ieq]+2.0*(k2[ieq]+k3[ieq])+k4[ieq]);
  //Clean up memory:
  delete[] k1;
  delete[] k2;
  delete[] k3;
  delete[] k4;
  delete[] xx;
}//RKSTEP()
```

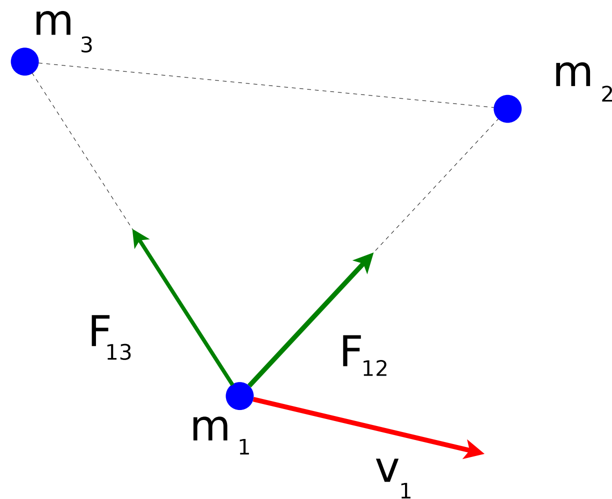

Figure 5.17: Three particles of equal mass interact via their mutual gravitational attraction. The problem is solved numerically using the program in the files `rkA.cpp`, `rkA_3pcb.cpp`. The same program can be used in order to study the motion of three equal charges under the influence of their attractive or repulsive electrostatic force.

Consider three particles of equal mass exerting a force of gravitational attraction on each other[4] like the ones shown in figure 5.17. The forces exerting on each other are given by

$$\vec{F}_{ij} = \frac{mk_1}{r_{ij}^3}\vec{r}_{ij}\,, \quad i,j = 1,2,3\,, \tag{5.30}$$

[4]The same program can be used for three equal charges exerting an electrostatic force on each other, which can be either attractive or repulsive.

where $k_1 = -Gm$ and the equations of motion become $(i = 1, 2, 3)$

$$
\begin{aligned}
\frac{dx_i}{dt} &= v_{ix} \qquad \frac{dv_{ix}}{dt} = k_1 \sum_{j=1, j\neq i}^{3} \frac{x_i - x_j}{r_{ij}^3} \\
\frac{dy_i}{dt} &= v_{iy} \qquad \frac{dv_{iy}}{dt} = k_1 \sum_{j=1, j\neq i}^{3} \frac{y_i - y_j}{r_{ij}^3} \,, \qquad (5.31)
\end{aligned}
$$

where $r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2$. The total energy of the system is

$$
E/m = \frac{1}{2}(v_1^2 + v_2^2) + \sum_{i,j=1, j<i}^{3} \frac{k_1}{r_{ij}} \,. \qquad (5.32)
$$

The relations shown above are programmed in the file `rkA_3pcb.cpp` listed below:

```
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
//--------------------------------------------------------------------
extern double k1,k2;
//--------------------------------------------------------------------
//Sets number of equations:
void finit(int& NEQ){
  NEQ = 12;
}
//================================
//Three particles of the same
//mass on the plane interacting
//via Coulombic force
//================================
void f(const double& t, double* X,double* dXdt){
  double x11,x12,x21,x22,x31,x32;
  double v11,v12,v21,v22,v31,v32;
  double r12,r13,r23;
  //-----------------
  x11 = X[0];x21 = X[4];x31 = X[8];
  x12 = X[1];x22 = X[5];x32 = X[9];
```

```
  v11 = X[2];v21 = X[6];v31 = X[10];
  v12 = X[3];v22 = X[7];v32 = X[11];
  //--------------------
  r12 = pow((x11-x21)*(x11-x21)+(x12-x22)*(x12-x22),-3.0/2.0);
  r13 = pow((x11-x31)*(x11-x31)+(x12-x32)*(x12-x32),-3.0/2.0);
  r23 = pow((x21-x31)*(x21-x31)+(x22-x32)*(x22-x32),-3.0/2.0);
  //--------------------
  dXdt[0]  = v11;
  dXdt[1]  = v12;
  dXdt[2]  = k1*(x11-x21)*r12+k1*(x11-x31)*r13; // a11=dv11/dt
  dXdt[3]  = k1*(x12-x22)*r12+k1*(x12-x32)*r13; // a12=dv12/dt
  //--------------------
  dXdt[4]  = v21;
  dXdt[5]  = v22;
  dXdt[6]  = k1*(x21-x11)*r12+k1*(x21-x31)*r23; // a21=dv21/dt
  dXdt[7]  = k1*(x22-x12)*r12+k1*(x22-x32)*r23; // a22=dv22/dt
  //--------------------
  dXdt[8]  = v31;
  dXdt[9]  = v32;
  dXdt[10] = k1*(x31-x11)*r13+k1*(x31-x21)*r23; // a31=dv31/dt
  dXdt[11] = k1*(x32-x12)*r13+k1*(x32-x22)*r23; // a32=dv32/dt
}
//================================
double energy(const double& t, double* X){
  double x11,x12,x21,x22,x31,x32;
  double v11,v12,v21,v22,v31,v32;
  double r12,r13,r23;
  double e;
  //--------------------
  x11 = X[0];x21 = X[4];x31 = X[8];
  x12 = X[1];x22 = X[5];x32 = X[9];
  v11 = X[2];v21 = X[6];v31 = X[10];
  v12 = X[3];v22 = X[7];v32 = X[11];
  //--------------------
  r12 = pow((x11-x21)*(x11-x21)+(x12-x22)*(x12-x22),-0.5);
  r13 = pow((x11-x31)*(x11-x31)+(x12-x32)*(x12-x32),-0.5);
  r23 = pow((x21-x31)*(x21-x31)+(x22-x32)*(x22-x32),-0.5);
  //--------------------
  e   = 0.5*(v11*v11+v12*v12+v21*v21+v22*v22+v31*v31+v32*v32);
  e  += k1*(r12+r13+r23);
  return e;
}
```

In order to run the program and see the results, look at the commands in the shell script in the file `rkA_3pcb.csh`. In order to run the script use

the command

```
> rkA_3pcb.csh -0.5 4000 1.5 -1 0.1 1 0 1 -0.1 -1 0 0.05 1 0 -1
```

which will run the program setting $k_1 = -0.5$, $\vec{r}_1(0) = -\hat{x}+0.1\hat{y}$, $\vec{v}_1(0) = \hat{x}$, $\vec{r}_2(0) = \hat{x} - 0.1\hat{y}$, $\vec{v}_2(0) = -\hat{x}$, $\vec{r}_3(0) = 0.05\hat{x} + \hat{y}$, $\vec{v}_3(0) = -\hat{y}$, Nt$= 4000$ and $t_f = 1.5$.

5.6 Problems

5.1 Reproduce the results shown in figures 5.3 and 5.4. Compare your results to the known analytic solution.

5.2 Write a program for the force on a charged particle in a constant magnetic field $\vec{B} = B\hat{k}$ and compute its trajectory for $\vec{v}(0) = v_{0x}\hat{x} + v_{0y}\hat{y}$. Set $x(0) = 1, y(0) = 0, v_{0y} = 0$ and calculate the resulting radius of the trajectory. Plot the relation between the radius and v_{0x}. Compare your results to the known analytic solution. (assume non relativistic motion)

5.3 Consider the anisotropic harmonic oscillator $a_x = -\omega_1^2 x$, $a_y = -\omega_2^2 y$. Construct the Lissajous curves by setting $x(0) = 0, y(0) = 1, v_x(0) = 1, v_y(0) = 0$, $t_f = 2\pi$, $\omega_2^2 = 1$, $\omega_1^2 = 1, 2, 4, 9, 16, \ldots$. What happens when $\omega_1^2 \neq n\omega_2^2$?

5.4 Reproduce the results displayed in table 5.1 and figures 5.5 and 5.6. Plot $\ln a$ vs $\ln T$ and calculate the slope of the resulting straight line by using the linear least squares method. Is it what you expect? Calculate the intercept and compare your result with the expected one.

5.5 Calculate the angular momentum with respect to the center of the force at each integration step of the planetary motion and check whether it is conserved. Show analytically that conservation of angular momentum implies that the position vector sweeps areas at constant rate.

5.6 Calculate the escape velocity of a planet v_e for $GM = 10.0$, $y(0) = 0.0$, $x_0 = x(0) = 1$ using the following steps: First show that $v_0^2 = -GM(1/a) + v_e^2$. Then set $v_x(0) = 0$, $v_y(0) = v_0$. Vary $v_y(0) = v_0$ and measure the resulting semi-major axis a. Determine the intercept of the resulting straight line in order to calculate v_e.

5.7 Repeat the previous problem for $x_0 = 0.5$, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0. From the $v_e = f(1/x_0)$ plot confirm the relation (5.14).

5.8 Check that for the bound trajectory of a planet with $GM = 10.0$, $x(0) = 1$, $y(0) = 0.0$, $v_x(0) = 0$, $v_y(0) = 4$ you obtain that $F_1 P +$

$F_2P = 2a$ for each point P of the trajectory. The point F_1 is the center of the force. After determining the semi-major axis a numerically, the point F_2 will be taken symmetric to F_1 with respect to the center of the ellipse.

5.9 Consider the planetary motion studied in the previous problem. Apply a momentary push in the tangential direction after the planet has completed 1/4 of its elliptical orbit. How stable is the particle trajectory (i.e. what is the dependence of the trajectory on the magnitude and the duration of the push?)? Repeat the problem when the push is in the vertical direction.

5.10 Consider the scattering potential of the positron-hydrogen system given by equation (5.26). Plot the functions $f(r)$ and $V(r)$ for different values of a. Calculate the total cross section σ_{tot} numerically and show that it is equal to πa^2.

5.11 Consider the Morse potential of diatomic molecules:

$$V(r) = D\left(\exp(-2\alpha r) - 2\exp(-\alpha r)\right) \tag{5.33}$$

where $D, \alpha > 0$. Compute the solutions of the problem numerically in one dimension and compare them to the known analytic solutions when $E < 0$:

$$x(t) = \frac{1}{\alpha}\ln\left\{\frac{D - \sqrt{D(D-|E|)}\sin(\alpha t\sqrt{2|E|/m} + C)}{|E|},\right\} \tag{5.34}$$

where the integration constant as a function of the initial position and energy is given by

$$C = \sin^{-1}\left[\frac{D - |E|e^{\alpha x_0}}{\sqrt{D(D-|E|)}}\right]. \tag{5.35}$$

We obtain a periodic motion with an energy dependent period $= (\pi/\alpha)\sqrt{2m/|E|}$. For $E > 0$ we obtain

$$x(t) = \frac{1}{\alpha}\ln\left\{\frac{\sqrt{D(D+E)}\cosh(\alpha t\sqrt{2E/m} + C) - D}{|E|}\right\} \tag{5.36}$$

whereas for $E = 0$

$$x(t) = \frac{1}{\alpha} \ln \left\{ \frac{1}{2} + \frac{D\alpha^2}{m} (t + C)^2 \right\} . \tag{5.37}$$

In these equations, the integration constant C is given by a different relation and not by equation (5.35). Compute the motion in phase space $(x, \dot{x})$ and study the transition from open to closed trajectories.

5.12 Consider the effective potential term $V_{eff}(r) = l^2/2mr^2$ ($l \equiv |\vec{L}|$) in the previous problem. Plot the function $V_{tot}(r) = V(r) + V_{eff}(r)$ for $D = 20$, $\alpha = 1$, $m = 1$, $l = 1$, and of course for $r > 0$. Determine the equilibrium position and the ionization energy.

Calculate the solutions $x(t)$, $y(t)$, $y(x)$, $r(t)$ on the plane for $E > 0$, $E = 0$, and $E < 0$ numerically. In the $E < 0$ case consider the scattering problem and calculate the functions $b(\theta)$, $\sigma(\theta)$ and the total cross section σ_{tot}.

5.13 Consider the potential of the molecular model given by the force $\vec{F}(r) = f(r)\,\hat{r}$ where $f(r) = 24(2/r^{13} - 1/r^7)$. Calculate the potential $V(r)$ and plot the function $V_{tot}(r) = V(r) + V_{eff}(r)$. Determine the equilibrium position and the ionization energy.

Consider the problem of scattering and calculate $b(\theta)$, $\sigma(\theta)$ and σ_{tot} numerically. How much do your results depend on the minimum scattering angle?

5.14 Compute the trajectories of a particle under the influence of a force $\vec{F} = -k/r^3\hat{r}$. Determine appropriate initial conditions that give a spiral trajectory.

5.15 Compute the total cross section σ_{tot} for the Rutherford scattering both analytically and numerically. What happens to your numerical results as you vary the integration limits?

5.16 Write a program that computes the trajectory of a particle that moves on the plane in the static electric field of N static point charges.

5.17 Solve the three body problem described in the text in the case of three different electric charges by making the appropriate changes to the program in the file `rkA_3cb.cpp`.

5.18 Two charged particles of equal mass and charge are moving on the xy plane in a constant magnetic field $\vec{B} = B\hat{z}$. Solve the equations of motion using a 4th order Runge–Kutta Method. Plot the resulting trajectories for the initial conditions that you will choose.

5.19 Three particles of equal mass m are connected by identical springs. The springs' spring constant is equal to k and their equilibrium length is equal to l. The particles move without friction on a horizontal plane. Solve the equations of motion of the system numerically by using a 4th order Runge–Kutta Method. Plot the resulting trajectories for the initial conditions that you will choose. (Hint: Look in the files `rkA_3hoc.cpp`, `rkA_3hoc.csh`.)

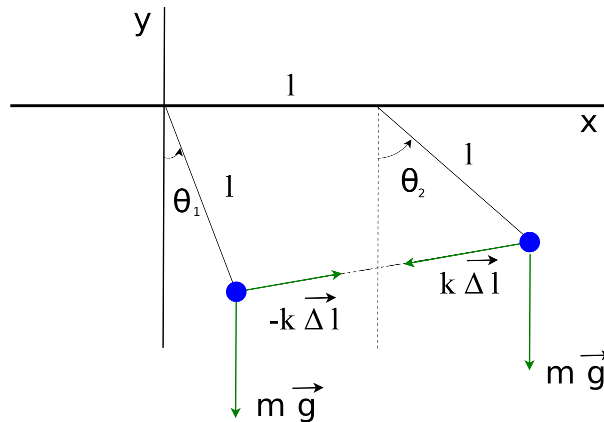

Figure 5.18: Two identical particles are attached to thin weightless rods of length l and they are connected by an ideal weightless spring with spring constant k and equilibrium length l. The rods are hinged to the ceiling at points whose distance is l. (Problem 5.20).

5.20 Two identical particles are attached to thin weightless rods of length l and they are connected by an ideal weightless spring with spring constant k and equilibrium length l. The rods are hinged to the ceiling at points whose distance is l (see figure 5.18). Compute the Lagrangian of the system and the equations of motion for the degrees of freedom θ_1 and θ_2. Solve these equations numerically by using a 4th order Runge–Kutta method. Plot the positions of

the particles in a Cartesian coordinate system and the resulting trajectory. Study the normal modes for small angles $\theta_1 \lesssim 0.1$ and compute the deviation of the solutions from the small oscillation approximation as the angles become larger. (Hint: Look in the files `rk_cpend.cpp`, `rk_cpend.csh`)

5.21 Repeat the previous problem when the hinges of the rods slide without friction on the x axis.

5.22 Repeat problem 5.20 by adding a third pendulum to the right at distance l.

Chapter 6

Motion in Space

In this chapter we will study the motion of a particle in space (three dimensions). We will also discuss the case of the relativistic motion, which is important if one wants to consider the motion of particles moving with speeds comparable to the speed of light. This will be an opportunity to use an *adaptive stepsize* Runge-Kutta method for the numerical solution of the equations of motion. We will use the open source code `rksuite`[1] available at the Netlib[2] repository. Netlib is an open source, high quality repository for numerical analysis software. The software it contains is used by many researchers in their high performance computing programs and it is a good investment of time to learn how to use it. Most of it is code written in Fortran and, in order to use it, you should learn how to link a program written in C++ with functions written in a different programming language.

The main technical skill that you will develop in this chapter is looking for solutions to your numerical problems provided by software written by others. It is important to be able to locate the optimal solution to your problem, find the relevant functions, read the software's documentation carefully and filter out the necessary information in order to call and link the functions to your program.

[1] R.W. Brankin, I. Gladwell, and L.F. Shampine, RKSUITE: a suite of Runge-Kutta codes for the initial value problem for ODEs, Softreport 92-S1, Department of Mathematics, Southern Methodist University, Dallas, Texas, U.S.A, 1992.

[2] `www.netlib.org`

6.1 Adaptive Stepsize Control for RK Methods

The three dimensional equation of motion of a particle is an initial value problem given by the equations (4.6)

$$
\begin{aligned}
\frac{dx}{dt} &= v_x & \frac{dv_x}{dt} &= a_x(t,x,v_x,y,v_y,z,v_z)\\
\frac{dy}{dt} &= v_y & \frac{dv_z}{dt} &= a_y(t,x,v_x,y,v_y,z,v_z)\\
\frac{dz}{dt} &= v_z & \frac{dv_z}{dt} &= a_z(t,x,v_x,y,v_y,z,v_z)\,.
\end{aligned}
\tag{6.1}
$$

For its numerical solution we will use an adaptive stepsize Runge–Kutta algorithm for increased performance and accuracy. Adaptive stepsize is used in cases where one needs to minimize computational effort for given accuracy goal. The method frequently changes the time step during the integration process, so that it is set to be large through smooth intervals and small when there are abrupt changes in the values of the functions. This is achieved by exercising *error control*, either by monitoring a conserved quantity or by computing the same solution using two different methods. In our case, two Runge-Kutta methods are used, one of order p and one of order $p+1$, and the difference of the results is used as an estimate of the truncation error. If the error needs to be reduced, the step size is reduced and if it is satisfactorily small the step size is increased. For the details we refer the reader to [33]. Our goal is not to analyze and understand the details of the algorithm, but to learn how to find and use appropriate and high quality code written by others.

6.1.1 The `rksuite` Suite of RK Codes

The link `http://www.netlib.org/ode/` reads

```
lib  rksuite
alg  Runge−Kutta
for  initial value problem for first order ordinary ↩
   differential
     equations. A suite of codes for solving IVPs in ODEs. A
     choice of RK methods, is available. Includes an error
     assessment facility and a sophisticated stiffness checker.
```

```
     Template programs and example results provided.
     Supersedes RKF45, DDERKF, D02PAF.
ref  RKSUITE, Softreport 92-S1, Dept of Math, SMU, Dallas, ↩
   Texas
by   R.W. Brankin (NAG), I. Gladwell and L.F. Shampine (SMU)
lang Fortran
prec double
```

There, we learn that the package provides code for Runge–Kutta methods, whose source is open and written in the Fortran language. We also learn that the code is written for double precision variables, which is suitable for our problem. Last, but not least, we are also happy to learn that it is written by highly reputable people! We download[3] the files `rksuite.f`, `rksuite.doc`, `details.doc`, `templates`, `readme`.

In order to link the subroutines provided by the suite to our program we need to read the documentation carefully. In the general case, documentation is available on the web (html, pdf, ...), bundled files with names like `README` and `INSTALL`, in whole directories with names like `doc/`, online help in `man` and/or `info` pages and, finally, in good old fashioned printed manuals. Good quality software is also well documented inside the source code files, something that is true for the software at hand.

In order to link the suite's subroutines to our program we need the following basic information:

- INPUT DATA: This is the necessary information that the program needs in order to perform the calculation. In our case, the minimal such information is the initial conditions, the integration time interval and the number of integration steps. The user should also provide the functions on the right hand side of (6.1). It might also be necessary to provide information about the desired accuracy goal, the scale of the problem, the hardware etc.

- OUTPUT DATA: This is the information on how we obtain the results of the calculation for further analysis. Information whether the calculation was successful and error free could also be provided.

[3] For the convenience of the reader, these files can be found bundled in the accompanied software in a subdirectory `rksuite`.

- WORKSPACE: This is information on how we provide the necessary memory space used in the intermediate calculations. Such space needs to be provided by the user in programming languages where dynamical memory allocation is not possible, like in `Fortran 77`, and the size of workspace depends on the parameters of the calling program.

It is easy to install the software. All the necessary code is in one file `rksuite.f`. The file `rksuite.doc`[4] contains the documentation. There we read that we need to inform the program about the hardware dependent accuracy of floating point numbers. We need to set the values of three variables:

```
...
RKSUITE requires three environmental constants OUTCH, MCHEPS,
DWARF. When you use RKSUITE, you may need to know their
values. You can obtain them by calling the subroutine ENVIRN
in the suite:

  CALL ENVIRN(OUTCH,MCHPES,DWARF)

returns values

  OUTCH  - INTEGER
           Standard output channel on the machine being used.
  MCHEPS - DOUBLE PRECISION
           The unit of roundoff, that is, the largest
           positive number such that 1.0D0 + MCHEPS = 1.0D0.
  DWARF  - DOUBLE PRECISION
           The smallest positive number on the machine being
           used.
...
************************** Installation Details ************

 All machine-dependent aspects of the suite have been
 isolated in the subroutine ENVIRN in the rksuite.f file.
...
```

The variables `OUTCH`, `MCHEPS`, `DWARF` are defined in the subroutine `ENVIRN`. They are given generic default values but the programmer is free to

[4]This is a simple text file which you can read with the command `less rksuite.doc` or with `emacs`.

change them by editing `ENVIRN`. We should identify the routine in the file `rksuite.f` and read the comments in it[5]:

```
...
      SUBROUTINE ENVIRN(OUTCH,MCHEPS,DWARF)
...
C  The following six statements are to be Commented out
C  after verification that the machine and installation
C  dependent quantities are specified correctly.
...
      WRITE(*,*) ' Before using RKSUITE, you must verify that the  '
      WRITE(*,*) ' machine- and installation-dependent quantities  '
      WRITE(*,*) ' specified in the subroutine ENVIRN are correct, '
      WRITE(*,*) ' and then Comment these WRITE statements and the '
      WRITE(*,*) ' STOP statement out of ENVIRN.                   '
      STOP
...
C  The following values are appropriate to IEEE
C  arithmetic with the typical standard output channel.
C
      OUTCH = 6
      MCHEPS = 1.11D-16
      DWARF = 2.23D-308
```

All we need to do is to comment out the `WRITE` and `STOP` commands since we will keep the default values of the `OUTCH`, `MCHEPS`, `DWARF` variables:

```
...
C     WRITE(*,*) ' Before using RKSUITE, you must verify that the  '
C     WRITE(*,*) ' machine- and installation-dependent quantities  '
C     WRITE(*,*) ' specified in the subroutine ENVIRN are correct, '
C     WRITE(*,*) ' and then Comment these WRITE statements and the '
C     WRITE(*,*) ' STOP statement out of ENVIRN.                   '
C     STOP
...
```

In order to check whether the default values are satisfactory, we can use the C++ template `numeric_limits`, which is part of the C++ Standard Library. In the file `numericLimits.cpp`, we write a small test program[6]:

```cpp
#include <iostream>
#include <limits>
using namespace std;
```

[5]These are lines that begin with a `C`, as this is old fixed format Fortran code.

[6]The file in the accompanying software, shows you how to compute numeric limits for several types of variables.

```
int main(){
  double MCHEPS,DWARF;

  MCHEPS = numeric_limits<double>::epsilon();
  DWARF  = numeric_limits<double>::min    ();
  cout << "MCHEPS = " << MCHEPS/2.0 << endl;
  cout << "DWARF  = " << DWARF      << endl;
}
```

We compile and run the above program as follows:

```
> g++ numericLimits.cpp -o n
> ./n
MCHEPS = 1.11022e-16
DWARF  = 2.22507e-308
```

We conclude that our choices are satisfactory.

Next, we need to learn how to use the subroutines in the suite. By carefully reading `rksuite.doc` we learn the following: The interface to the adaptive stepsize Runge–Kutta algorithm is the routine `UT` (`UT` = "Usual Task"). The routine can use a 2nd-3rd (RK23) order Runge-Kutta pair for error control (`METHOD=1`), a 4th-5th (RK45) order pair (`METHOD=2`) or a 7th-8th (RK78) order pair (`METHOD=3`). We will set `METHOD=2` (RK45). The routine `SETUP` must be called before `UT` for initialization. The user should provide a function `F` that calculates the derivatives of the functions we integrate for, i.e. the right hand side of 6.1.

The fastest way to learn how to use the above routines is "by example". The suite include a templates package which can be unpacked by executing the commands in the file `templates` using the `sh` shell:

```
> sh templates
tmpl1.out
tmpl1a.f
...
```

The file `tmpl1a.f` contains the solution of the simple harmonic oscillator and has many explanatory comments in it. The code is in Fortran, but it is not so hard to read. You may compile it and run it with the commands:

```
> cd rksuite/templates
> gfortran tmpl1a.f ../rksuite.f -o example1
> ./example1
```

We encourage the reader to study it carefully, run it and test its results.

6.1.2 Interfacing C++ Programs with Fortran

Next, we have to learn how to link the Fortran code in `rksuite.f` to a C++ program. There is a lot of relevant information that you can find with a simple search in the web, we will concentrate on what is relevant to the program that we need to write. The first thing that we need to learn is how to call a function written in Fortran from a C++ program. A simple “Hello World” program can teach us how to do it. Suppose that a Fortran function/subroutine `HELLO()` is coded in a file `helloF.f90`:

```
SUBROUTINE  HELLO()

 PRINT *, 'Hello World!'

END SUBROUTINE HELLO
```

Then, we write in the file `hello.cpp`:

```
#include <iostream>
using namespace std;
extern "C" void  hello_();
int main(){
  hello_();
}
```

The first thing that we notice is that we call the function by lowering all letters in its name: `HELLO` → `hello`. In Fortran, lowercase and uppercase letters are equivalent, and the compiler creates names with lowercase letters only. Next, we find that we need to append an underscore _ to the function's name[7]: `hello` → `hello_`. The Fortran function needs to be declared in the “`"C"` language linkage”:

[7]Read your compiler's manual, there could be options that you can use at compile time so that you can avoid it. But the advice is to stick with this convention, so that your code will be portable and ... long lived!

```
extern "C" void  hello_();
```

This is something one has to do both for functions written in Fortran, as well as for functions written in plain C[8].

In order to compile and run the code we have to run the commands:

```
> gfortran -c helloF.f90
> g++ hello.cpp helloF.o -o hello -lgfortran
> ./hello
 Hello World!
```

Compilation is done in two steps: We first need to compile the Fortran program using the Fortran compiler[9] `gfortran`. The flag `-c` forces the compiler to perform compilation but not linking. It produces an *object file*, whose extension is `.o`, `helloF.o`. These are files which contain compiled code in non-readable text form, but they are not autonomous executable programs. The functions that they contain, can be *linked* to other compiled programs that call them. In the second step, g++ is called to *compile* the C++ source code in `hello.cpp` and *link* it to the functions in `helloF.o`. The flag `-lgfortran` is necessary in order to link to the standard Fortran functions. If you use a different compiler, you should read its manual in order to find the correct linking options.

Another subtle point that needs to be considered is that, in Fortran, variables are passed to functions *by reference* and not *by value*. Therefore, we have to pass variables as pointers or as reference to variables. For example, the following Fortran function[10]

```
REAL (8) FUNCTION SQUAREMYDOUBLE(X)
 REAL(8) x

 X = 2.0*X
```

[8]Language linkage encapsulates the set of requirements necessary in order to *link* with a function written in another programming language, and it should be done for all function types, names and variable names. There could be linkage to other programming languages, but only the C and C++ linkage is guaranteed to be available.

[9]Available for free download from `gcc.gnu.org/fortran/`. On debian based Linux systems, install with `sudo apt-get install gfortran`.

[10]A Fortran function returns the value stored in a variable with the same name as the name of the function, here `SQUAREMYDOUBLE`.

```
 SQUAREMYDOUBLE = X*X

END      FUNCTION SQUAREMYDOUBLE
```

must be declared and called as:

```
extern "C" double squaremydouble_(double& x);
...
  double x,x2;
  x2 = squaremydouble_(x);
```

For example, by modifying the `hello.cpp` program as

```
#include <iostream>
using namespace std;
extern "C" void   hello_();
extern "C" double squaremydouble_(double& x);
int main(){
  double x,x2;
  hello_();
  x  = 2.0;
  x2 = squaremydouble_(x);
  cout << "x   = "          << 2.0 << endl;
  cout << "2 x = "          << x   << endl;
  cout << "(2 x)*(2 x) = " << x2  << endl;
}
```

we obtain the output:

```
> gfortran -c helloF.f90;
> g++ hello.cpp helloF.o -o hello -lgfortran
> ./hello
 Hello World!
x   = 2
2 x = 4
(2 x)*(2 x) = 16
```

Notice that the value of x is modified in the calling program.

The final issue that we will consider, it how to pass arrays to Fortran functions. One dimensional arrays are quite easy to handle. In order to pass an array `double v[N]` to a Fortran function, we only need to declare it and pass it as one of its arguments. If the Fortran program is

```
real(8) function make_array1(v,N)
 integer   N
 real(8) v(N)
 ... compute v(1) ... v(N) ...
end     function make_array1
```

then the corresponding C++ program should be:

```
const int N=4;
extern "C" double make_array1_(double v[N], const int& N);
int main(){
  double v[N], x;
  ...
  x = make_array1_(v,N);
  ... use v[0] ... v[N-1] ...
}
```

The only point we need to stress is that the array `real(8) v(N)` is indexed from `1` to `N` in the Fortran program, whereas the array `double v[N]` is indexed from `0` to `N-1` in the C++ program. The correspondence of the values stored in memory is `v(1)` → `v[0]`, ... , `v(i)` → `v[i-1]`, ... `v(N)` → `v[N-1]`.

Two dimensional arrays need more attention. In C++, arrays are in *row-major* mode, whereas in Fortran in *column-major* mode. The contents of a two dimensional array are stored linearly in memory. In C++, the elements of an array `double A[N][M]` are stored in the sequence `A[0][0]`, `A[0][1]`, `A[0][2]`, ... , `A[0][M-1]`, `A[1][0]`, `A[1][1]`, ... , `A[1][M-1]`, ... `A[N-1][M-1]`. In Fortran, the elements of an array `real(8) A(N,M)` are stored in the sequence `A(1,1)`, `A(2,1)`, `A(3,1)`, ... , `A(1,M)`, `A(2,1)`, `A(2,2)`, ... , `A(2,M)`, ... , `A(N,M)`. Therefore, when we pass an array from the C++ program to a Fortran function, we have to keep in mind that the Fortran function will use it with its indices transposed. For example, if the Fortran code defines

```
   A(i,j) = i + j/10.0
```

which results into `A(i,j) = i.j` in decimal notation[11] (e.g. `A(2,3)=2.3`), then the value of `A[i][j]` will be `(j+1).(i+1)` (e.g. `A[2][3] =`

[11]Of course, we assume `i,j<10`.

4.3, A[2][1] = 2.3)!

All of the above are summarized in the Fortran program in the file CandFortranF.f90:

```
real(8) function make_array1(v,N)
 implicit none
 integer N,i
 real(8) v(N)

 do i=1,N
  v(i) = i
 end do

 make_array1 = -11.0 ! a return value

end function make_array1
!----------------------------------------------------
real(8) function make_array2(A,N,M)
 implicit none
 integer N,M,i,j
 real(8) A(M,N) ! Careful: N and M are interchanged!!

 do i=1,M
  do j=1,N
  !A(i,j) = i.j, e.g. A(2,3)=2.3
   A(i,j) = i + j/10.0
  end do
 end do

 make_array2 = -22.0 ! a return value

end function make_array2
```

and the C++ program in the file CandFortranC.cpp:

```
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>

using namespace std;
```

```
const int N=4, M=3;
extern "C" {
  double make_array1_(double v[N]    , const int& N);
  double make_array2_(double A[N][M], const int& N,
                      const int& M);
}
int main(){
  double A[N][M], v[N];
  double x;

  //Make a  1D array using a fortran function:
  x = make_array1_(v,N);
  cout << "1D array: Return value x= " << x << endl;
  for(int i=0;i<N;i++)
    cout << v[i] << " ";
  cout << "\n-------------------------\n";
  //Make an 2D array using a fortran function:
  x = make_array2_(A,N,M);
  cout << "2D array: Return value x= " << x << endl;
  for(int i=0;i<N;i++){
    for(int j=0;j<M;j++)   //A is ... transposed!
      //A[i][j] = (j+1).(i+1), e.g. A[1][2] = 3.1
      cout  << A[i][j] << " ";
    cout << '\n';
  }
}
```

Note that the array `A[N][M]` is defined as `A(M,N)` in the Fortran function, and the roles of `N` and `M` are interchanged. You can run the code and see the output with the commands:

```
> gfortran -c CandFortranF.f90
> g++ CandFortranC.cpp CandFortranF.o -o CandFortran -lgfortran
> ./CandFortran
1D array: Return value x= -11
1 2 3 4
-------------------------
2D array: Return value x= -22
1.1 2.1 3.1
1.2 2.2 3.2
1.3 2.3 3.3
1.4 2.4 3.4
```

Note that the values of the array `A(M,N)` are transposed when printed as

rows in the C++ program.

6.1.3 The rksuite Driver

After we become wise enough, we can write the driver for the integration routine `UT` (called by `ut_` from our C++ program), which can be found in the file `rk3.cpp`:

```
//========================================================
//Program to solve a 6 ODE system using Runge-Kutta Method
//Output is written in file rk3.dat
//========================================================
// Compile with the commands:
// gfortran -c rksuite/rksuite.f;
// g++ rk3.cpp rk3_g.cpp rksuite.o -o rk3 -lgfortran
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
#include "rk3.h"
using namespace std;
//--------------------------------------------------------
double k1,k2,k3,k4;
double energy(const double& t, double* Y);
void f(double& t,double* Y, double* YP);
extern "C" {
  void setup_(const int& NEQ,
              double& TSTART,double* YSTART,double& TEND,
              double& TOL   ,double* THRES  ,
              const int& METHOD, const char& TASK,
              bool  & ERRASS,double& HSTART,double* WORK,
              const int& LENWRK,      bool& MESSAGE);
  void ut_( void f(double& t,double* Y, double* YP),
            double& TWANT, double& TGOT, double* YGOT,
            double* YPGOT, double* YMAX, double* WORK,
            int& UFLAG);
}
//--------------------------------------------------------
int main(){
  string buf;
  double T0,TF,X10,X20,X30,V10,V20,V30;
  double t,dt,tstep;
```

```
int     STEPS, i;
// rksuite variables:
double TOL,THRES[NEQ],WORK[LENWRK],HSTART;
double Y[NEQ],YMAX[NEQ],YP[NEQ],YSTART[NEQ];
bool    ERRASS, MESSAGE;
int     UFLAG;
const char TASK = 'U';
//Input:
cout << "Runge-Kutta Method for 6-ODEs Integration\n";
cout << "Enter coupling constants k1,k2,k3,k4:\n";
cin  >> k1 >> k2 >> k3 >> k4;getline(cin,buf);
cout << "Enter STEPS,T0,TF,X10,X20,X30,V10,V20,V30:\n";
cin  >> STEPS >> T0  >> TF
     >> X10   >> X20 >> X30
     >> V10   >> V20 >> V30;getline(cin,buf);
cout << "No. Steps= " << STEPS << endl;
cout << "Time: Initial T0 =" << T0
     << " Final TF="         << TF  << endl;
cout << "            X1(T0)=" << X10
     << " X2(T0)="           << X20
     << " X3(T0)="           << X30 << endl;
cout << "            V1(T0)=" << V10
     << " V2(T0)="           << V20
     << " V3(T0)="           << V30 << endl;
//Initial Conditions:
dt          = (TF-T0)/STEPS;
YSTART[0] = X10;
YSTART[1] = X20;
YSTART[2] = X30;
YSTART[3] = V10;
YSTART[4] = V20;
YSTART[5] = V30;
//Set control parameters:
TOL = 5.0e-6;
for( i = 0; i < NEQ; i++)
  THRES[i] = 1.0e-10;
MESSAGE = true;
ERRASS  = false;
HSTART  = 0.0;
//Initialization:
setup_(NEQ,T0,YSTART,TF,TOL,THRES,METHOD,TASK,
       ERRASS,HSTART,WORK,LENWRK,MESSAGE);
ofstream  myfile("rk3.dat");
myfile.precision(16);
myfile << T0<< " "
```

```
        << YSTART[0] << " " << YSTART[1] << " "
        << YSTART[2] << " " << YSTART[3] << " "
        << YSTART[4] << " " << YSTART[5] << " "
        << energy(T0,YSTART)<< '\n';
  //The calculation:
  for(i=1;i<=STEPS;i++){
    t = T0 + i*dt;
    ut_(f,t,tstep,Y,YP,YMAX,WORK,UFLAG);
    if(UFLAG > 2) break; //error: break the loop and exit
    myfile << tstep << " "
           << Y[0]  << " " << Y[1] << " "
           << Y[2]  << " " << Y[3] << " "
           << Y[4]  << " " << Y[5] << " "
           << energy(T0,Y) << '\n';
  }
  myfile.close();
}// main()
```

All common parameters and variables are declared in an include file `rk3.h`. This is done so that they are accessible by the function `f` which calculates the derivatives:

```
const int NEQ    = 6;
const int LENWRK = 32*NEQ;
const int METHOD = 2;
extern double k1,k2,k3,k4;
```

The number of differential equations is set equal to `NEQ=6`. The integration method is set by the choice `METHOD=2`. The variable `LENWRK` sets the size of the workspace needed by the suite for the intermediate calculations.

The declaration of functions needs some care. The functions `energy()` and `f()` are defined in the C++ program and are declared in the global scope (the function `f()` will also be passed on to the Fortran function `UT`). The functions `setup_()` and `ut_()` are defined in the Fortran program in the file `rksuite.f` as `SETUP()` and `UT()`. Therefore, they are declared within the `extern "C"` language linkage, defined using lowercase letters and an underscore is appended to their names. All arguments are passed by reference. Scalar doubles are passed as references to `double&`, the double precision arrays are declared to be pointers `double *`, the Fortran `logical` variables are declared to be references to `bool&`, and the Fortran

`character*` variables[12] as simple references to `char&`.

The main program starts with the user interface. The initial state of the particle is stored in the array `YSTART` in the positions $0 \ldots 5$. The first three positions are the coordinates of the initial position and the last three the components of the initial velocity. Then, we set some variables that determine the behavior of the integration program (see the file `rksuite.doc` for details) and call the subroutine `SETUP`. The main integration loop is:

```
for(i=1;i<=STEPS;i++){
  t = T0 + i*dt;
  ut_(f,t,tstep,Y,YP,YMAX,WORK,UFLAG);
  if(UFLAG > 2) break; //error: break the loop and exit
  myfile << ... << energy(T0,Y) << '\n';
}
```

The function `f` calculates the derivatives and it will be programmed by us later. The variable `t` stores the *desired* moment of time at which we want to calculate the functions. Because of the adaptive stepsize, it can be different than the one returned by the Fortran subroutine `UT`. The actual value of time that the next step lands[13] on is `tstep`. The array `Y` stores the values of the functions. We choose the data structure to be such that x= `Y[0]`, y= `Y[1]`, z= `Y[2]` and v_x= `Y[3]`, v_y= `Y[4]`, v_z= `Y([5]` (the same sequence as in the array `YSTART`). The function `energy(t,Y)` returns the value of the mechanical energy of the particle and its code will be written in the same file as that of `f`. Finally, the variable `UFLAG` indicates the error status of the calculation by `UT` and if `UFLAG`>2 we end the calculation.

Our test code will be on the study of the motion of a projectile in a constant gravitational field, subject also to the influence of a dissipative force $\vec{F}_r = -mk\vec{v}$. The program is in the file `rk3_g.cpp`. We choose the parameters `k1` and `k2` so that $\vec{g} =$ `-k1` $\hat{k}$ and $k =$ `k2`.

```
#include <iostream>
#include <fstream>
```

[12] The Fortran program uses only their first character, so we don't need to use strings.

[13] When `UGLAG` ≤ 2, `tstep=t` and we will not worry about them being different with our program.

```
#include <cstdlib>
#include <string>
#include <cmath>
#include "rk3.h"
using namespace std;
void f(double& t,double* Y, double* YP){
  double x1,x2,x3,v1,v2,v3;
  x1 = Y[0]; v1 = Y[3];
  x2 = Y[1]; v2 = Y[4];
  x3 = Y[2]; v3 = Y[5];
  //Velocities:   dx_i/dt = v_i
  YP[0] = v1;
  YP[1] = v2;
  YP[2] = v3;
  //Acceleration: dv_i/dt = a_i
  YP[3] = -k2*v1;
  YP[4] = -k2*v2;
  YP[5] = -k2*v3-k1;
}
//--------------------------------------------
double energy(const double& t, double* Y){
  double e;
  double x1,x2,x3,v1,v2,v3;
  x1 = Y[0]; v1 = Y[3];
  x2 = Y[1]; v2 = Y[4];
  x3 = Y[2]; v3 = Y[5];
  //Kinetic   Energy:
  e  = 0.5*(v1*v1+v2*v2+v3*v3);
  //Potential Energy:
  e += k1*x3;

  return e;
}
```

For convenience we "translated" the values in the array `Y[NEQ]` into user-friendly variable names. If the file `rksuite.f` is in the directory `rksuite/`, then the compilation, running and visualization of the results can be done with the commands:

```
> gfortran -c rksuite/rksuite.f
> g++ rk3.cpp rk3_g.cpp rksuite.o -o rk3 -lgfortran
> ./rk3
Runge-Kutta Method for 6-ODEs Integration
Enter coupling constants k1,k2,k3,k4:
```

```
10 0 0 0
Enter STEPS,T0,TF,X10,X20,X30,V10,V20,V30:
10000 0 3 0 0 0 1 1 1
No. Steps= 10000
Time: Initial T0 =0 Final TF=3
           X1(T0)=0 X2(T0)=0 X3(T0)=0
           V1(T0)=1 V2(T0)=1 V3(T0)=1
> gnuplot
gnuplot> plot "rk3.dat"  using 1:2 with lines title "x1(t)"
gnuplot> plot "rk3.dat"  using 1:3 with lines title "x2(t)"
gnuplot> plot "rk3.dat"  using 1:4 with lines title "x3(t)"
gnuplot> plot "rk3.dat"  using 1:5 with lines title "v1(t)"
gnuplot> plot "rk3.dat"  using 1:6 with lines title "v2(t)"
gnuplot> plot "rk3.dat"  using 1:7 with lines title "v3(t)"
gnuplot> plot "rk3.dat"  using 1:8 with lines title "E(t)"
gnuplot> set title "trajectory"
gnuplot> splot "rk3.dat" using 2:3:4 with lines  notitle
```

All the above commands can be executed together using the shell script in the file rk3.csh. The script uses the animation script rk3_animate.csh. The following command executes all the commands shown above:

```
./rk3.csh -f 1 -- 10 0. 0 0 0 0 0 1 1 1 10000 0 3
```

6.2 Motion of a Particle in an EM Field

In this section we study the non-relativistic motion of a charged particle in an electromagnetic (EM) field. The particle is under the influence of the Lorentz force:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})\,. \tag{6.2}$$

Consider the constant EM field of the form $\vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$, $\vec{B} = B\hat{z}$. The components of the acceleration of the particle are:

$$\begin{aligned} a_x &= (qE_x/m) + (qB/m)v_y \\ a_y &= (qE_y/m) - (qB/m)v_x \\ a_z &= (qE_z/m)\,. \end{aligned} \tag{6.3}$$

This field is programmed in the file rk3_B.cpp. We set k1 $= qB/m$, k2 $= qE_x/m$, k3 $= qE_y/m$ and k4 $= qE_z/m$:

```
//========================================================
//   Particle in constant magnetic and electric field
//   q B/m = k1 z    q E/m = k2 x + k3 y + k4 z
//========================================================
#include "sr.h"
void f(double& t,double* Y, double* YP){
  double x1,x2,x3,v1,v2,v3,p1,p2,p3;
  x1 = Y[0]; p1 = Y[3];
  x2 = Y[1]; p2 = Y[4];
  x3 = Y[2]; p3 = Y[5];
  velocity(p1,p2,p3,v1,v2,v3);
  //now we can use all x1,x2,x3,p1,p2,p3,v1,v2,v3
  YP[0] = v1;
  YP[1] = v2;
  YP[2] = v3;
  //Acceleration:
  YP[3] = k2 + k1 * v2;
  YP[4] = k3 - k1 * v1;
  YP[5] = k4;
}
//--------------------------------------------------------
double energy(const double& t, double* Y){
  double e;
  double x1,x2,x3,v1,v2,v3,p1,p2,p3,psq;
  x1 = Y[0]; p1 = Y[3];
  x2 = Y[1]; p2 = Y[4];
  x3 = Y[2]; p3 = Y[5];
  psq= p1*p1+p2*p2+p3*p3;
  //Kinetic  Energy:
  e  = sqrt(1.0+psq)-1.0;
  //Potential Energy/m_0
  e += - k2*x1 - k3*x2 - k4*x3;

  return e;
}
```

We can also study space-dependent fields in the same way. The fields must satisfy Maxwell's equations. We can study the confinement of a particle in a region of space by a magnetic field by taking $\vec{B} = B_y\hat{y} + B_z\hat{z}$ with $qB_y/m = -k_2 y$, $qB_z/m = k_1 + k_2 z$ and $qB_y/m = k_3 z$, $qB_z/m = k_1 + k_2 y$. Note that $\vec{\nabla}\cdot\vec{B} = 0$. You may also want to calculate the current density from the equation $\vec{\nabla}\times\vec{B} = \mu_0\vec{j}$.

The results are shown in figures 6.1–6.4.

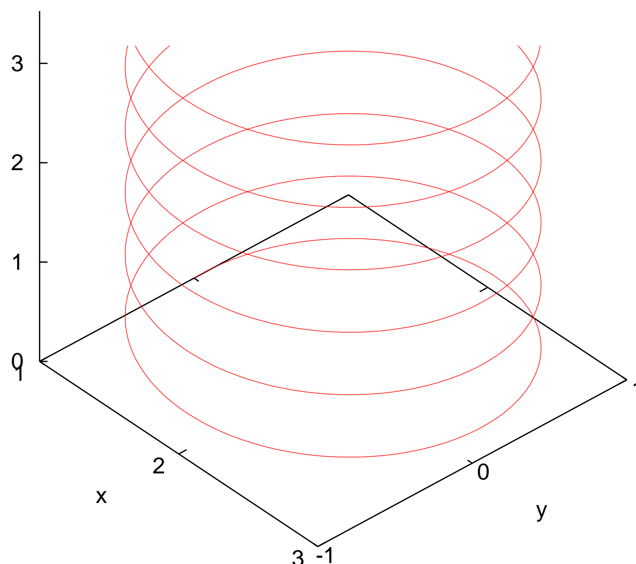

Figure 6.1: The trajectory of a charged particle in a constant magnetic field $\vec{B} = B\hat{z}$, where $qB/m = 1.0$, $\vec{v}(0) = 1.0\hat{y} + 0.1\hat{z}$, $\vec{r}(0) = 1.0\hat{x}$. The integration of the equations of motion is performed using the RK45 method from $t_0 = 0$ to $t_f = 40$ with 1000 steps.

6.3 Relativistic Motion

Consider a particle of non zero rest mass moving with speed comparable to the speed of light. In this case, it is necessary to study its motion using the equations of motion given by special relativity[14]. In the equations below we set $c = 1$. The particle's rest mass is $m_0 > 0$, its mass is $m = m_0/\sqrt{1 - v^2}$ (where $v < 1$), its momentum is $\vec{p} = m\vec{v}$ and its energy is $E = m = \sqrt{p^2 + m_0^2}$. Then the equations of motion in a dynamic field $\vec{F}$ are given by:

$$\frac{d\vec{p}}{dt} = \vec{F}\,. \tag{6.4}$$

[14] Of course for lower speeds, the special relativity equations of motion are a better approximation to the particle's motion, but the corrections to the non relativistic equations of motion are negligible.

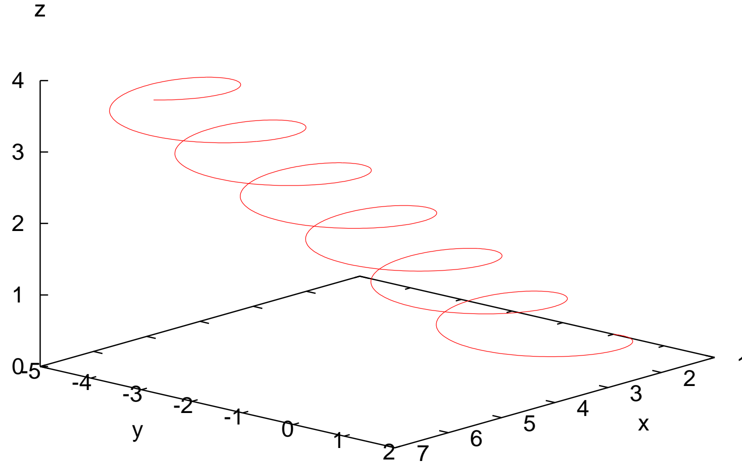

Figure 6.2: The trajectory of a charged particle in a constant magnetic field $\vec{B} = B\hat{z}$, where $qB/m = 1.0$ and a constant electric field $\vec{E} = E_x\hat{x} + E_y\hat{y}$ με $qE_x/m = qE_y/m = 0.1$. $\vec{v}(0) = 1.0\hat{y} + 0.1\hat{z}$, $\vec{r}(0) = 1.0\hat{x}$. The integration of the equations of motion is performed using the RK45 method from $t_0 = 0$ to $t_f = 40$ with 1000 steps. Each axis is on a different scale.

In order to write a system of first order equations, we use the relations

$$\vec{v} = \frac{\vec{p}}{m} = \frac{\vec{p}}{E} = \frac{\vec{p}}{\sqrt{p^2 + m_0^2}}\,. \tag{6.5}$$

Using $\vec{v} = d\vec{r}/dt$ we obtain

$$\begin{aligned}
\frac{dx}{dt} &= \frac{(p_x/m_0)}{\sqrt{1+(p/m_0)^2}}\,, \qquad & \frac{d(p_x/m_0)}{dt} &= \frac{F_x}{m_0} \\
\frac{dy}{dt} &= \frac{(p_y/m_0)}{\sqrt{1+(p/m_0)^2}}\,, \qquad & \frac{d(p_y/m_0)}{dt} &= \frac{F_y}{m_0} \\
\frac{dz}{dt} &= \frac{(p_z/m_0)}{\sqrt{1+(p/m_0)^2}}\,, \qquad & \frac{d(p_z/m_0)}{dt} &= \frac{F_z}{m_0}\,,
\end{aligned} \tag{6.6}$$

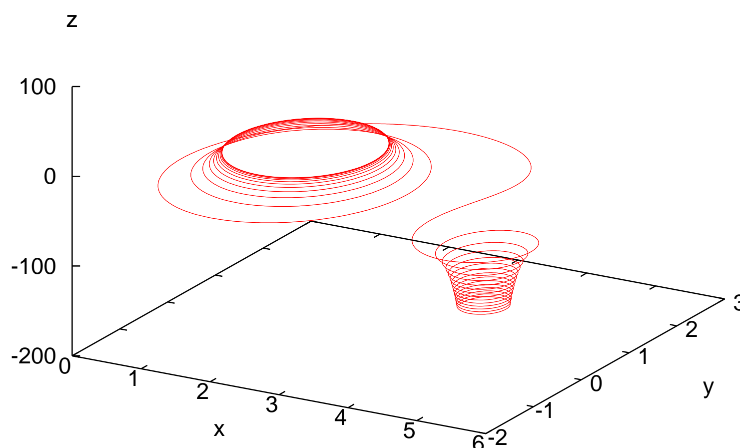

Figure 6.3: The trajectory of a charged particle in a magnetic field $\vec{B} = B_y\hat{y} + B_z\hat{z}$ with $qB_y/m = -0.02y$, $qB_z/m = 1 + 0.02z$, $\vec{v}(0) = 1.0\hat{y} + 0.1\hat{z}$, $\vec{r}(0) = 1.0\hat{x}$. The integration of the equations of motion is performed using the RK45 method from $t_0 = 0$ to $t_f = 500$ with 10000 steps. Each axis is on a different scale.

which is a system of first order differential equations for the functions $(x(t), y(t), z(t), (p_x/m_0)(t), (p_y/m_0)(t), (p_z/m_0)(t))$. Given the initial conditions $(x(0), y(0), z(0), (p_x/m_0)(0), (p_y/m_0)(0), (p_z/m_0)(0))$ their solution is unique and it can be computed numerically using the 4th-5th order Runge–Kutta method according to the discussion of the previous section. By using the relations

$$
\begin{aligned}
(p_x/m_0) &= \frac{v_x}{\sqrt{1-v^2}} \qquad & v_x &= \frac{(p_x/m_0)}{\sqrt{1+(p/m_0)^2}} \\
(p_y/m_0) &= \frac{v_y}{\sqrt{1-v^2}} \qquad & v_y &= \frac{(p_y/m_0)}{\sqrt{1+(p/m_0)^2}} \\
(p_z/m_0) &= \frac{v_z}{\sqrt{1-v^2}} \qquad & v_z &= \frac{(p_z/m_0)}{\sqrt{1+(p/m_0)^2}}\,,
\end{aligned}
\tag{6.7}
$$

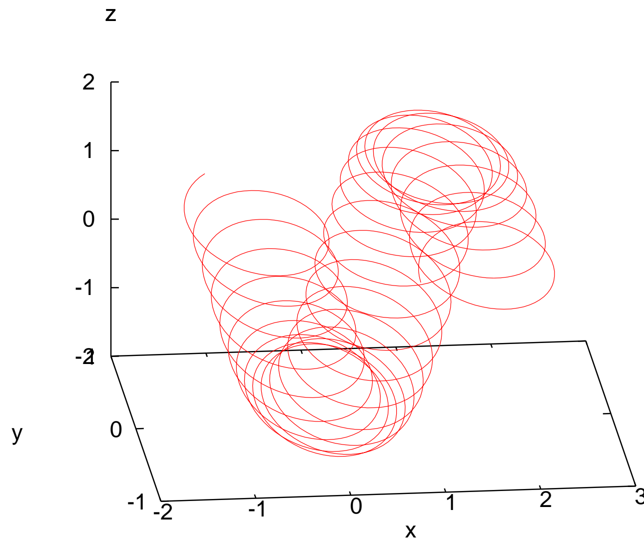

Figure 6.4: The trajectory of a charged particle in a magnetic field $\vec{B} = B_y\hat{y} + B_z\hat{z}$ with $qB_y/m = 0.08z$, $qB_z/m = 1.4 + 0.08y$, $\vec{v}(0) = 1.0\hat{y} + 0.1\hat{z}$, $\vec{r}(0) = 1.0\hat{x}$. The integration of the equations of motion is performed using the RK45 method from $t_0 = 0$ to $t_f = 3000$ with 40000 steps. Each axis is on a different scale.

we can use the initial conditions $(x(0),\ y(0),\ z(0),\ v_x(0),\ v_y(0),\ v_z(0))$ instead. Similarly, from the solutions $(x(t),\ y(t),\ z(t),\ (p_x/m_0)(t),\ (p_y/m_0)(t),\ (p_z/m_0)(t))$ we can calculate $(x(t),\ y(t),\ z(t),\ v_x(t),\ v_y(t),\ v_z(t))$. We always have to check that

$$v^2 = (v_x)^2 + (v_y)^2 + (v_z)^2 < 1\,. \tag{6.8}$$

Since half of the functions that we integrate for are the momentum instead of the velocity components, we need to make some modifications to the program in the file `rk3.cpp`. The main program can be found in the file `sr.cpp`:

```
//============================================================
//Program to solve a 6 ODE system using Runge-Kutta Method
//Output is written in file sr.dat
//============================================================
```

```
// Compile with the commands:
// gfortran -c rksuite/rksuite.f
// g++ sr.cpp sr_B.cpp rksuite.o -o rk3 -lgfortran
//--------------------------------------------------------
#include "sr.h"
double k1,k2,k3,k4;
extern "C" {
  void setup_(const int& NEQ,
              double& TSTART,double* YSTART,double& TEND,
              double& TOL    ,double* THRES  ,
              const int& METHOD, const char& TASK,
              bool  & ERRASS,double& HSTART,double* WORK,
              const int& LENWRK,        bool& MESSAGE);
  void ut_( void f(double& t,double* Y, double* YP),
            double& TWANT, double& TGOT, double* YGOT,
            double* YPGOT, double* YMAX, double* WORK,
            int& UFLAG);
}
//--------------------------------------------------------
int main(){
  string buf;
  double T0,TF,X10,X20,X30,V10,V20,V30;
  double P10,P20,P30;
  double P1,P2,P3,V1,V2,V3;
  double t,dt,tstep;
  int    STEPS, i;
  // rksuite variables:
  double TOL,THRES[NEQ],WORK[LENWRK],HSTART;
  double Y[NEQ],YMAX[NEQ],YP[NEQ],YSTART[NEQ];
  bool   ERRASS, MESSAGE;
  int    UFLAG;
  const char TASK = 'U';
  //Input:
  cout << "Runge-Kutta Method for 6-ODEs Integration\n";
  cout << "Special Relativistic Particle:\n";
  cout << "Enter coupling constants k1,k2,k3,k4:\n";
  cin  >> k1 >> k2 >> k3 >> k4;getline(cin,buf);
  cout << "Enter STEPS,T0,TF,X10,X20,X30,V10,V20,V30:\n";
  cin  >> STEPS >> T0  >> TF
       >> X10   >> X20 >> X30
       >> V10   >> V20 >> V30;getline(cin,buf);
  momentum(V10,V20,V30,P10,P20,P30);
  cout << "No. Steps= " << STEPS << endl;
  cout << "Time: Initial T0 =" << T0
       << " Final TF="         << TF  << endl;
```

```
cout << "          X1(T0)=" << X10
     << " X2(T0)="          << X20
     << " X3(T0)="          << X30 << endl;
cout << "          V1(T0)=" << V10
     << " V2(T0)="          << V20
     << " V3(T0)="          << V30 << endl;
cout << "          P1(T0)=" << P10
     << " P2(T0)="          << P20
     << " P3(T0)="          << P30 << endl;
//Initial Conditions:
dt        = (TF-T0)/STEPS;
YSTART[0] = X10;
YSTART[1] = X20;
YSTART[2] = X30;
YSTART[3] = P10;
YSTART[4] = P20;
YSTART[5] = P30;
//Set control parameters:
TOL = 5.0e-6;
for( i = 0; i < NEQ; i++)
  THRES[i] = 1.0e-10;
MESSAGE = true;
ERRASS  = false;
HSTART  = 0.0;
//Initialization:
setup_(NEQ,T0,YSTART,TF,TOL,THRES,METHOD,TASK,
       ERRASS,HSTART,WORK,LENWRK,MESSAGE);
ofstream  myfile("sr.dat");
myfile.precision(16);
myfile << T0        << " "
       << YSTART[0] << " " << YSTART[1] << " "
       << YSTART[2] << " "
       << V1        << " " << V2        << " "
       << V3        << " "
       << energy(T0,YSTART)             << " "
       << YSTART[3] << " " << YSTART[4] << " "
       << YSTART[5] << '\n';
//The calculation:
for(i=1;i<=STEPS;i++){
  t = T0 + i*dt;
  ut_(f,t,tstep,Y,YP,YMAX,WORK,UFLAG);
  if(UFLAG > 2) break; //error: break the loop and exit
  velocity(Y[3],Y[4],Y[5],V1,V2,V3);
  myfile << tstep << " "
         << Y[0]  << " " << Y[1] << " "
```

```
           << Y[2]  << " "
           << V1    << " " << V2   << " "
           << V3    << " "
           << energy(T0,Y)         << " "
           << Y[3]  << " " << Y[4] << " "
           << Y[5]  << " " << '\n';
  }
  myfile.close();
}// main()
//========================================================
//momentum -> velocity  transformation
//========================================================
void velocity(const double& p1,const double& p2,
              const double& p3,
                    double& v1,      double& v2,
                    double& v3){
  double psq;
  psq = p1*p1+p2*p2+p3*p3;

  v1  = p1/sqrt(1.0+psq);
  v2  = p2/sqrt(1.0+psq);
  v3  = p3/sqrt(1.0+psq);
}
//========================================================
//velocity -> momentum transformation
//========================================================
void momentum(const double& v1,const double& v2,
              const double& v3,
                    double& p1,      double& p2,
                    double& p3){
  double vsq;
  vsq = v1*v1+v2*v2+v3*v3;
  if(vsq >= 1.0){cerr << "momentum: vsq>=1\n";exit(1);}
  p1  = v1/sqrt(1.0-vsq);
  p2  = v2/sqrt(1.0-vsq);
  p3  = v3/sqrt(1.0-vsq);


}
```

The functions `momentum` and `velocity` compute the transformations (6.7). In the function `momentum` we check whether the condition (6.8) is satisfied. These functions are also used in the function `F` that computes the derivatives of the functions.

Common declarations are now in an include file `sr.h`:

```
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
//--------------------------------------------------------
const int NEQ    = 6;
const int LENWRK = 32*NEQ;
const int METHOD = 2;
extern double k1,k2,k3,k4;
//--------------------------------------------------------
double energy(const double& t,        double* Y);
void   f(double& t, double* Y,        double* YP);
void velocity(const double& p1,const double& p2,
              const double& p3,
                    double& v1,       double& v2,
                    double& v3);
void momentum(const double& v1,const double& v2,
              const double& v3,
                    double& p1,       double& p2,
                    double& p3);
```

The test drive of the above program is the well known relativistic motion of a charged particle in a constant EM field. The acceleration of the particle is given by equations (6.3). The relativistic kinetic energy of the particle is

$$T = \left(\frac{1}{\sqrt{1-v^2}} - 1\right) m_0 = \left(\sqrt{1+(p/m_0)^2} - 1\right) m_0 \qquad (6.9)$$

These relations are programmed in the file `sr_B.cpp`. The contents of the file `sr_B.cpp` are:

```
//========================================================
//  Particle in constant Magnetic and electric field
//  q B/m = k1 z    q E/m = k2 x + k3 y + k4 z
//========================================================
#include "sr.h"
void f(double& t,double* Y, double* YP){
  double x1,x2,x3,v1,v2,v3,p1,p2,p3;
  x1 = Y[0]; p1 = Y[3];
  x2 = Y[1]; p2 = Y[4];
```

```
  x3 = Y[2]; p3 = Y[5];
  velocity(p1,p2,p3,v1,v2,v3);
  //now we can use all x1,x2,x3,p1,p2,p3,v1,v2,v3
  YP[0] = v1;
  YP[1] = v2;
  YP[2] = v3;
  //Acceleration:
  YP[3] = k2 + k1 * v2;
  YP[4] = k3 - k1 * v1;
  YP[5] = k4;
}
//--------------------------------------------------------
double energy(const double& t, double* Y){
  double e;
  double x1,x2,x3,v1,v2,v3,p1,p2,p3,psq;
  x1 = Y[0]; p1 = Y[3];
  x2 = Y[1]; p2 = Y[4];
  x3 = Y[2]; p3 = Y[5];
  psq= p1*p1+p2*p2+p3*p3;
  //Kinetic   Energy:
  e  = sqrt(1.0+psq)-1.0;
  //Potential Energy/m_0
  e += - k2*x1 - k3*x2 - k4*x3;

  return e;
}
```

The results are shown in figures 6.5–6.6.

Now we can study a more interesting problem. Consider a simple model of the Van Allen radiation belt. Assume that the electrons are moving within the Earth's magnetic field which is modeled after a magnetic dipole field of the form:

$$\vec{B} = B_0 \left(\frac{R_E}{r}\right)^3 \left[3(\hat{d}\cdot\hat{r})\,\hat{r} - \hat{d}\right] , \tag{6.10}$$

where $\vec{d} = d\hat{d}$ is the magnetic dipole moment of the Earth's magnetic field and $\vec{r} = r\hat{r}$. The parameter values are approximately equal to $B_0 = 3.5\times 10^{-5}T$, $r \sim 2R_E$, where R_E is the radius of the Earth. The typical energy of the moving particles is ~ 1 MeV which corresponds to velocities of magnitude $v/c = \sqrt{E^2 - m_0^2}/E \approx \sqrt{1-0.512^2}/1 = 0.86$. We choose the coordinate axes so that $\hat{d} = \hat{z}$ and we measure distance in R_E units[15].

[15]Since $c = 1$, the unit of time is the time that the light needs to travel distance equal

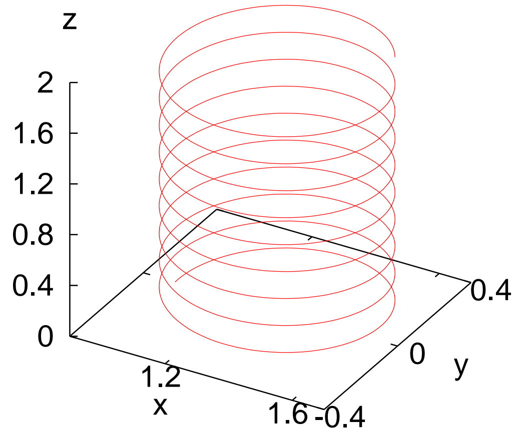

Figure 6.5: The trajectory of a relativistic charged particle in a magnetic field $\vec{B} = B_z\hat{z}$ with $qB_z/m_0 = 10.0$, $\vec{v}(0) = 0.95\hat{y} + 0.10\hat{z}$, $\vec{r}(0) = 1.0\hat{x}$. The integration is performed by using the RK45 method from $t_0 = 0$ to $t_f = 20$ with 1000 steps. Each axis is on a different scale.

Then we obtain:

$$
\begin{aligned}
B_x &= B_0\frac{3xz}{r^5} \\
B_y &= B_0\frac{3yz}{r^5} \\
B_z &= B_0\left(\frac{3zz}{r^5} - \frac{1}{r^3}\right)
\end{aligned}
\qquad (6.11)
$$

The magnetic dipole field is programmed in the file `sr_Bd.cpp`:

```
//============================================================
//   Particle in Magnetic dipole field:
//   q B_1/m = k1 (3 x1 x3)/r^5
```

to R_E in the vacuum.

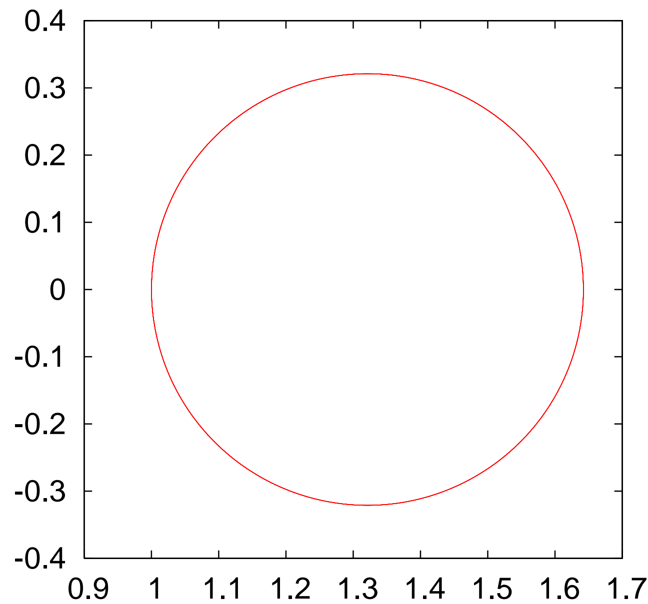

Figure 6.6: Projection of the trajectory of a relativistic charged particle in a magnetic field $\vec{B} = B_z\hat{z}$ with $qB_z/m_0 = 10.0$, on the xy plane. $\vec{v}(0) = 0.95\hat{y} + 0.10\hat{z}$, $\vec{r}(0) = 1.0\hat{x}$. The integration is performed by using the RK45 method from $t_0 = 0$ to $t_f = 20$ with 1000 steps. Each axis is on a different scale.

```
//   q B_2/m = k1 (3 x2 x3)/r^5
//   q B_3/m = k1[(3 x3 x3)/r^5-1/r^3]
//=====================================================
#include "sr.h"
void f(double& t,double* Y, double* YP){
  double x1,x2,x3,v1,v2,v3,p1,p2,p3;
  double B1,B2,B3;
  double r,r5,r3;
  x1 = Y[0]; p1 = Y[3];
  x2 = Y[1]; p2 = Y[4];
  x3 = Y[2]; p3 = Y[5];
  velocity(p1,p2,p3,v1,v2,v3);
  //now we can use all x1,x2,x3,p1,p2,p3,v1,v2,v3
  YP[0]   = v1;
  YP[1]   = v2;
  YP[2]   = v3;
  //Acceleration:
```

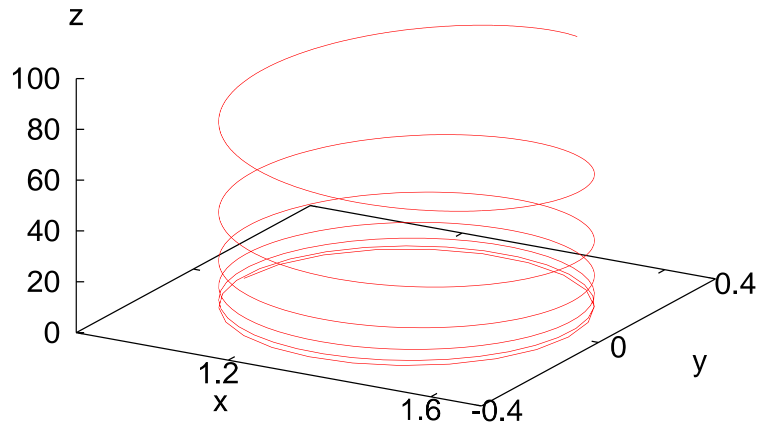

Figure 6.7: The influence of an additional electric field $q\vec{E}/m_0 = 1.0\hat{z}$ on the trajectory shown in figure 6.5.

```
  r       = sqrt(x1*x1+x2*x2+x3*x3);
  r3      = r*r*r;
  r5      = r*r*r3;
  if( r > 0.0){
    B1    = k1*( 3.0*x1*x3)/r5;
    B2    = k1*( 3.0*x2*x3)/r5;
    B3    = k1*((3.0*x3*x3)/r5-1/r3);
    YP[3] = v2*B3-v3*B2;
    YP[4] = v3*B1-v1*B3;
    YP[5] = v1*B2-v2*B1;
  }else{
    YP[3] = 0.0;
    YP[4] = 0.0;
    YP[5] = 0.0;
  }
}
//--------------------------------------------------------
double energy(const double& t, double* Y){
  double e;
```

```
  double x1,x2,x3,v1,v2,v3,p1,p2,p3,psq;
  x1 = Y[0]; p1 = Y[3];
  x2 = Y[1]; p2 = Y[4];
  x3 = Y[2]; p3 = Y[5];
  psq= p1*p1+p2*p2+p3*p3;
  //Kinetic   Energy:
  e  = sqrt(1.0+psq)-1.0;

  return e;
}
```

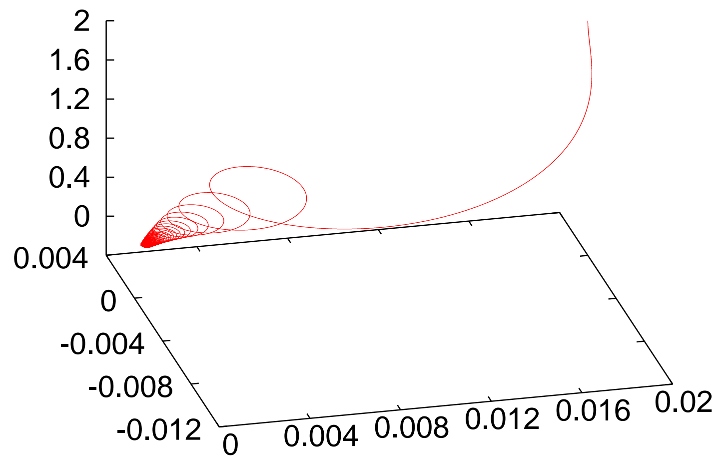

Figure 6.8: The trajectory of a charged particle in a magnetic dipole field given by equation (6.11). We used $B_0 = 1000$, $\vec{r} = 0.02\hat{x} + 2.00\hat{z}$, $\vec{v} = -0.99999\hat{z}$. The integration was done from $t_0 = 0$ to $t_f = 5$ in 10000 steps.

The results are shown in figure 6.8. The parameters have been exaggerated in order to achieve an aesthetically pleasant result. In reality, the electrons are moving in very thin spirals and the reader is encouraged to use more realistic values for the parameters $\vec{v}_0$, B_0, $\vec{r}_0$. The problem of why the effect is not seen near the equator is left as an exercise.

6.4 Problems

6.1 Compute the trajectory of a projectile moving in space in a constant gravitational field and under the influence of an air resistance proportional to the square of its speed.

6.2 Two point charges are moving with non relativistic speeds in a constant magnetic field $\vec{B} = B\hat{z}$. Assume that their interaction is given by the Coulomb force only. Write a program that computes their trajectory numerically using the RK45 method.

6.3 Write a program that computes the trajectory of the anisotropic harmonic oscillator $\vec{F} = -k_x x\hat{x} - k_y y\hat{y} - k_z z\hat{z}$. Compute the three dimensional Lissajous curves which appear for appropriate values of the angular frequencies $\omega_x = \sqrt{k_x/m}$, $\omega_y = \sqrt{k_y/m}$, $\omega_z = \sqrt{k_z/m}$.

6.4 Two particles of mass M are at the fixed positions $\vec{r}_1 = a\hat{z}$ and $\vec{r}_2 = -a\hat{z}$. A third particle of mass m interacts with them via a Newtonian gravitational force and moves at non relativistic speeds. Compute the particle's trajectory and find initial conditions that result in a planar motion.

6.5 Solve problem 5.19 of page 302 using the RK45 method. Choose initial conditions so that the system executes only translational motion. Next, choose initial conditions so that the system executes small vibrations and its center of mass remains stationary. Find the normal modes of the system and choose appropriate initial conditions that put the system in each one of them.

6.6 Solve the previous problem by putting the system in a box $|x| \leq L$ and $|y| \leq L$.

6.7 Solve the problem 5.20 in page 302 by using the RK45 method.

6.8 Solve the problem 5.21 in page 303 by using the RK45 method.

6.9 The electric field of an electric dipole $\vec{p} = p\hat{z}$ is given by:

$$\begin{aligned}
\vec{E} &= E_\rho \hat{\rho} + E_z \hat{z} \\
E_\rho &= \frac{1}{4\pi\epsilon_0}\frac{3p\sin\theta\cos\theta}{r^3} \\
E_z &= \frac{1}{4\pi\epsilon_0}\frac{p(3\cos^2\theta - 1)}{r^3}
\end{aligned} \tag{6.12}$$

where $\rho = \sqrt{x^2+y^2} = r\sin\theta$, $E_x = E_\rho\cos\phi$, $E_y = E_\rho\sin\phi$ and (r,θ,ϕ) are the polar coordinates of the point where the electric field is calculated. Calculate the trajectory of a test charge moving in this field at non relativistic speeds. Calculate the deviation between the relativistic and the non relativistic trajectories when the initial speed is $0.01c, 0.1c, 0.5c, 0.9c$ respectively (ignore radiation effects).

6.10 Consider a linear charge distribution with constant linear charge density λ. The electric field is given by

$$\vec{E} = E_\rho \hat{\rho} = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{\rho}\hat{\rho}$$

Calculate the trajectories of two equal negative test charges that move at non relativistic speeds in this field. Consider only the electrostatic Coulomb forces and ignore anything else.

6.11 Consider a linear charge distribution on four straight lines parallel to the z axis. The linear charge density is λ and it is constant. The four straight lines intersect the xy plane at the points $(0,0)$, $(0,a)$, $(a,0)$, (a,a). Calculate the trajectory of a non relativistic charge in this field. Next, compute the relativistic trajectories (ignore all radiation effects).

6.12 Three particles of mass m interact via their Newtonian gravitational force. Compute their (non relativistic) trajectories in space.

6.13 There is a C++ "translation" of `rksuite`. Download it from `netlib .org/ ode/ rksuite` and teach yourself how to use it. The documentation is not as explicit as for the Fortran version, part of it is in the source code file `rksuite.cpp`. You can teach yourself how to use it by reading the example file `RksuiteTest.cpp` and the methods of

the class `RKSUITE` in the file `rksuite.h`. Write a program to study the motion of the non relativistic electron in a constant magnetic field. Then repeat for the relativistic electron.

Chapter 7

Electrostatics

In this chapter we will study the electric field generated by a static charge distribution. First we will compute the electric field lines and the equipotential surfaces of the electric field generated by a static point charge distribution on the plane. Then we will study the electric field generated by a continuous charge distribution on the plane. This requires the numerical solution of an *elliptic* boundary value problem which will be done using successive over-relaxation (SOR) methods.

7.1 Electrostatic Field of Point Charges

Consider N point charges Q_i which are located at fixed positions on the plane given by their position vectors $\vec{r}_i$, $i = 1, \ldots, N$. The electric field is given by Coulomb's law

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{Q_i}{|\vec{r} - \vec{r}_i|^2}\, \hat{\rho}_i \tag{7.1}$$

where $\hat{\rho}_i = (\vec{r} - \vec{r}_i)/|\vec{r} - \vec{r}_i|$ is the unit vector in the direction of $\vec{r} - \vec{r}_i$. The components of the field are

$$\begin{aligned} E_x(x,y) &= \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{Q_i(x - x_i)}{\left((x - x_i)^2 + (y - y_i)^2\right)^{3/2}} \\ E_y(x,y) &= \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{Q_i(y - y_i)}{\left((x - x_i)^2 + (y - y_i)^2\right)^{3/2}}\,, \end{aligned} \tag{7.2}$$

The electrostatic potential at $\vec{r}$ is

$$V(\vec{r}) = V(x,y) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{Q_i}{\left((x-x_i)^2+(y-y_i)^2\right)^{1/2}}\,, \tag{7.3}$$

and we have that

$$\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})\,. \tag{7.4}$$

The electric field lines are the integral curves of the vector field $\vec{E}$, i.e. the curves whose tangent lines at each point are parallel to the electric field at that point. The magnitude of the electric field is proportional to the density of the field lines (the number of field lines per perpendicular area). This means that the electric flux $\Phi_E = \int_{\mathcal{S}} \vec{E}\cdot d\vec{A}$ through a surface $\mathcal{S}$ is proportional to the number of field lines that cross the surface. Electric field lines of point charge distributions start from positive charges (sources), end in negative charges (sinks) or extend to infinity.

The equipotential surfaces are the loci of the points of space where the electrostatic potential takes fixed values. They are closed surfaces. Equation (7.4) tells us that a strong electric field at a point is equivalent to a strong spatial variation of the electric potential at this point, i.e. to dense equipotential surfaces. The direction of the electric field is perpendicular to the equipotential surfaces at each point[1], which is the direction of the strongest spatial variation of V, and it points in the direction of decreasing V. The planar cross sections of the equipotential surfaces are closed curves which are called equipotential lines.

The computer cannot solve a problem in the continuum and we have to consider a finite discretization of a field line. A continuous curve is approximated by a large but finite number of small line segments. The basic idea is illustrated in figure 7.1: The small line segment Δl is taken in the direction of the electric field and we obtain

$$\Delta x = \Delta l\,\frac{E_x}{E}\,,\qquad \Delta y = \Delta l\,\frac{E_y}{E}\,, \tag{7.5}$$

where $E \equiv |\vec{E}| = \sqrt{E_x^2+E_y^2}$.

---

[1]Since for every small displacement $d\vec{r}$ along an equipotential surface the potential stays constant ($dV=0$), we have that $0 = dV = \vec{\nabla}V\cdot d\vec{r} = -\vec{E}\cdot d\vec{r}$, which implies $\vec{E}\perp d\vec{r}$.

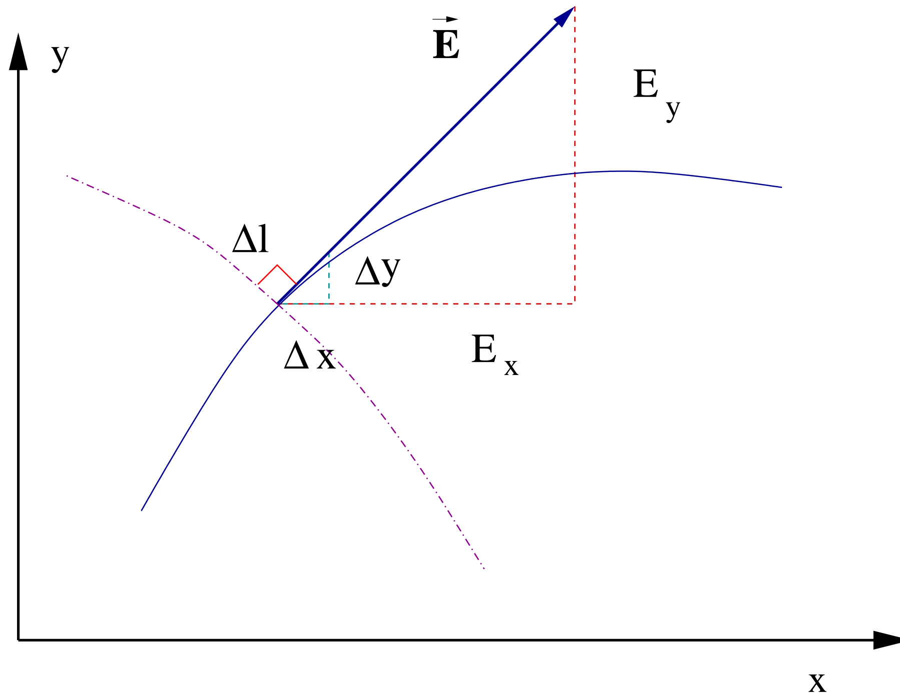

Figure 7.1: The electric field is tangent at each point of an electric field line and perpendicular to an equipotential line. By approximating the continuous curve by the line segment Δl, we have that $\Delta y/\Delta x = E_y/E_x$.

In order to calculate the equipotential lines we use the property that they are perpendicular to the electric field at each point. Therefore, if $(\Delta x, \Delta y)$ is in the tangential direction of a field line, then $(-\Delta y, \Delta x)$ is in the perpendicular direction since $(\Delta x, \Delta y)\cdot(-\Delta y, \Delta x) = -\Delta x\Delta y + \Delta y\Delta x = 0$. Therefore the equations that give the equipotential lines are

$$\Delta x = -\Delta l\ \frac{E_y}{E}\,, \qquad \Delta y = \Delta l\ \frac{E_x}{E}\,. \tag{7.6}$$

The algorithm that will allow us to perform an approximate calculation of the electric field lines and the equipotential lines is the following: Choose an initial point that belongs to the (unique) line that you want to draw. The electric field can be calculated from the known electric charge distribution and equation (7.2). By using a small enough step Δl we

move in the direction $(\Delta x, \Delta y)$ to the new position

$$x \to x + \Delta x\,, \quad y \to y + \Delta y\,, \tag{7.7}$$

where we use equations (7.5) or (7.6). The procedure is repeated until the drawing is finished. The programmer sets a criterion for that, e.g. when the field line steps out of the drawing area or approaches a charge closer than a minimum distance.

7.2 The Program – Appetizer and ... Desert

The hurried, but slightly experienced reader may skip the details of this section and go directly to section 7.4. There she can find the final form of the program and brief usage instructions.

In order to program the algorithm described in the previous section, we will separate the algorithmic procedures into four different but well defined tasks:

- Main program: The data structure, which is given by the position of the charges stored in the arrays `X[P]`, `Y[P]` and the charges stored in the array `Q[P]`, is defined. It also contains the user interface which consists of reading data entered by the user, like the number of charges `N`, their positions and magnitude. Then the calculation of a group of field or equipotential lines is performed by calling the routines `eline` or `epotline` respectively.

- `void function eline(xin,yin,X,Y,Q,N)`: Calculates the electric field line passing through the point `xin,yin`. On entry, the user inputs the point `xin,yin` and the data `N`, `X[N]`, `Y[N]`, `Q[N]`. On exit, the function prints to the `stdout` the coordinates of the approximate electric field line. The line extends up to a point that is either too close to one of the point charges or until the line leaves the drawing area[2]. It calls the functions `efield` for the calculation of the electric field and `mdist` for the calculation of the minimum and maximum distance of a point on the field line from all the point charges.

[2]Remember that field lines start at sources, end at sinks or extend to infinity.

- `void function epotline(xin,yin,X,Y,Q,N)`: Calculates the equipotential line passing through the point `xin,yin`. On entry, the user inputs the point `xin,yin` and the data `N, X[N], Y[N], Q[N]`. On exit, the function prints to the `stdout` the coordinates of the approximate equipotential line. The function stops calculating an equipotential line when it comes back close enough to the original point[3] `xin,yin` or when it leaves the drawing area. It calls the functions `efield` for the calculation of the electric field and `mdist` for the calculation of the minimum and maximum distance of a point on the equipotential line from all the point charges.

- `void function efield(x0,y0,X,Y,Q,N,Ex,Ey)`: Calculates the electric field `Ex, Ey` at position `x0, y0`. On entry, the user provides the number of charges `N`, the position of charges `X[N], Y[N]`, the charges `Q[N]` and the position `x0, y0`. On exit, the routine provides the values `Ex, Ey`.

- `void function mdist(x0,y0,X,Y,N,rmin,rmax)`: Calculates the maximum and minimum distance of the point `x0, y0` from all charges located at `X[N], Y[N]`. On entry, the user provides the number of charges `N`, the position of charges `X[N], Y[N]` and the point `x0, y0`. On exit, the routine provides the minimum and maximum distances `rmin,rmax`.

In the main program, the variables `N, X[N], Y[N]` and `Q[N]` must be set. These can be hard coded by the programmer or entered by the user interactively. The first choice is coded in the program listed below, which can be found in the file ELines.cpp. We list a simple version of the `main()` function below:

```
int main(){
  string buf;
  const int      P = 20;  //max number of charges
  double         X[P], Y[P], Q[P];
  int            N;
  //------------- SET CHARGE DISTRIBUTION ----
  N    =  2;
  X[0] =  1.0;
```

[3]Remember that the equipotential lines are closed.

```
  Y[0] =  0.0;
  Q[0] =  1.0;
  X[1] = -1.0;
  Y[1] =  0.0;
  Q[1] = -1.0;

  //------------ DRAWING LINES ------------
  eline(0.0, 0.5,X,Y,Q,N);
  eline(0.0, 1.0,X,Y,Q,N);
  eline(0.0, 1.5,X,Y,Q,N);
  eline(0.0, 2.0,X,Y,Q,N);
  eline(0.0,-0.5,X,Y,Q,N);
  eline(0.0,-1.0,X,Y,Q,N);
  eline(0.0,-1.5,X,Y,Q,N);
  eline(0.0,-2.0,X,Y,Q,N);
}//main()
```

The statements

```
  N    =  2;
  X[0] =  1.0;
  Y[0] =  0.0;
  Q[0] =  1.0;
  X[1] = -1.0;
  Y[1] =  0.0;
  Q[1] = -1.0;
```

define two opposite charges `Q[0]= -Q[1]= 1.0` located at $(1,0)$ and $(-1,0)$ respectively. The next lines call the function `eline` in order to perform the calculation of 8 field lines passing through the points $(0,\pm 1/2)$, $(0,\pm 1)$, $(0,\pm 3/2)$, $(0,\pm 2)$:

```
  eline(0.0, 0.5,X,Y,Q,N);
  eline(0.0, 1.0,X,Y,Q,N);
  eline(0.0, 1.5,X,Y,Q,N);
  eline(0.0, 2.0,X,Y,Q,N);
  eline(0.0,-0.5,X,Y,Q,N);
  eline(0.0,-1.0,X,Y,Q,N);
  eline(0.0,-1.5,X,Y,Q,N);
  eline(0.0,-2.0,X,Y,Q,N);
```

These commands print the coordinates of the field lines to the `stdout` and the user can analyze them further.

The program for calculating the equipotential lines is quite similar. The calls to the function `eline` are substituted by calls to `epotline`.

For the program to be complete, we must program the functions `eline`, `efield`, `mdist`. This will be done later, and you can find the full code in the file `ELines.cpp`. For the moment, let's copy the main program[4] listed above into the file `Elines.cpp` and compile and run it with the commands:

```
> g++ ELines.cpp -o el
> ./el > el.out
```

The `stdout` of the program is redirected to the file `el.out`. We can plot the results with `gnuplot`:

```
gnuplot> plot "el.out" with dots
```

The result is shown in figure 7.2.

Let's modify the program so that the user can enter the charge distribution, as well as the number and position of the field lines that she wants to draw, interactively. The part of the code that we need to change is:

```
  //---------------- SET CHARGE DISTRIBUTION -----
  cout << "# Enter number of charges:"        << endl;
  cin  >> N;                              getline(cin,buf);
  cout << "# N= "          << N               << endl;
  for(i=0;i<N;i++){
    cout << "# Charge: " << i+1                << endl;
    cout << "# Position and charge: (X,Y,Q):" << endl;
    cin  >> X[i] >> Y[i] >> Q[i];     getline(cin,buf);
    cout <<          "# (X,Y)= "
         << X[i] << " "
         << Y[i] << " Q= "
         << Q[i]                               << endl;
  }
```

The first line asks the user to enter the number of charges in the distribution. It proceeds with reading it from the `stdin` and prints the result

[4] See the file `ELines_version0.cpp`.

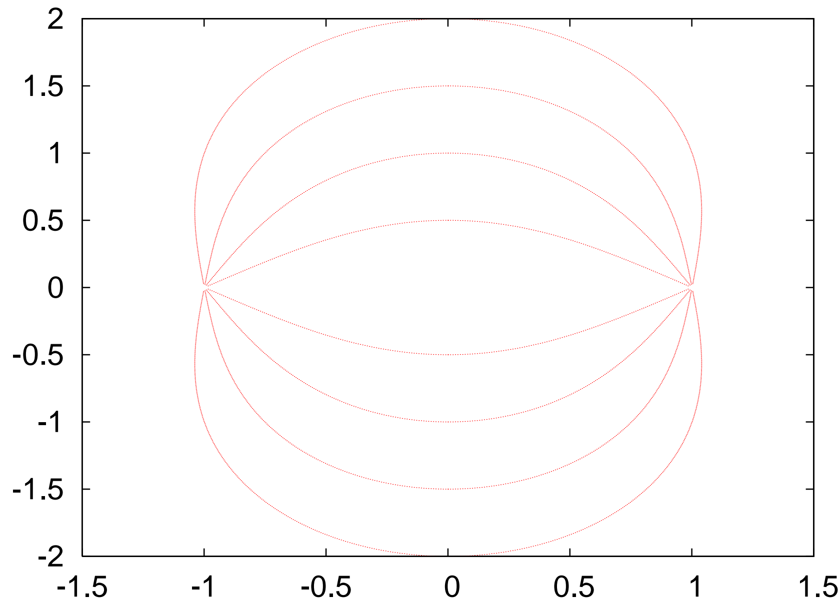

Figure 7.2: Some electric field lines of the electric field of two opposite charges calculated by the program `ELines.cpp` (version 1!).

to the `stdout`. The following loop reads the positions and charges and stores them at the position `i` of the arrays `X[i]`, `Y[i]`, `Q[i]`. The results are printed to the `stdout` so that the user can check the values read by the program.

The drawing of the field lines is now done by modifying the code so that:

```
//---------------- DRAWING LINES ----------------
cout << "# How many lines to draw?\n";
cin  >> draw;                            getline(cin,buf);
for(i=1;i<=draw;i++){
  cout << "# Initial point (x0,y0):\n";
  cin  >> x0 >> y0;                      getline(cin,buf);
  eline(x0,y0,X,Y,Q,N);
}
```

As a test case, we run the program for one charge $q = 1.0$ located at the origin and we draw one field line passing through the point $(0.1, 0.1)$.

```
>  g++ ELines.cpp -o el
> ./el
# Enter number of charges:
1
# N= 1
# Charge: 1
# Position and charge: (X,Y,Q):
0.0 0.0 1.0
# (X,Y)= 0 0 Q= 1
# How many lines to draw?
1
# Initial point (x0,y0):
0.1 0.1
0.10000000000000001  0.10000000000000001
0.092928932188134528 0.092928932188134528
0.08585786437626905  0.08585786437626905
....
```

For charge distributions with a large number of point charges, use an editor to record the charges, their positions and the points where the field lines should go through.

```
2                     N: Number of Charges
 1.0 0.0  1.0         (X,Y,Q): Position and charge
-1.0 0.0 -1.0         (X,Y,Q): Position and charge
8                     Number of lines to draw
0.0  0.5              x0,y0: Initial point of line
0.0  1.0              x0,y0: Initial point of line
0.0  1.5              x0,y0: Initial point of line
0.0  2.0              x0,y0: Initial point of line
0.0 -0.5              x0,y0: Initial point of line
0.0 -1.0              x0,y0: Initial point of line
0.0 -1.5              x0,y0: Initial point of line
0.0 -2.0              x0,y0: Initial point of line
```

If the data listed above is written into a file, e.g. `Input`, then the command

```
./el < Input > el.out
```

reads the data from the file `Input` and redirects the data printed to the `stdout` to the file `el.out`. This way you can create a "library" of charge

distributions and the field lines of their respective electric fields. The `main()` function (version 2) is listed below:

```
int main(){
  string buf;
  const int     P = 20;  //max number of charges
  double        X[P], Y[P], Q[P];
  int           N;
  int           i,j,draw;
  double        x0,y0;
  //------------- SET CHARGE DISTRIBUTION -----
  cout << "# Enter number of charges:"        << endl;
  cin  >> N;                            getline(cin,buf);
  cout << "# N= "         << N                << endl;
  for(i=0;i<N;i++){
    cout << "# Charge: " << i+1                << endl;
    cout << "# Position and charge: (X,Y,Q):" << endl;
    cin  >> X[i] >> Y[i] >> Q[i];       getline(cin,buf);
    cout <<         "# (X,Y)= "
         << X[i] << " "
         << Y[i] << " Q= "
         << Q[i]                               << endl;
  }
  //------------- DRAWING LINES -------------
  cout << "# How many lines to draw?\n";
  cin  >> draw;                         getline(cin,buf);
  for(i=1;i<=draw;i++){
    cout << "# Initial point (x0,y0):\n";
    cin  >> x0 >> y0;                   getline(cin,buf);
    eline(x0,y0,X,Y,Q,N);
  }
}
```

If you did the exercises described above, you should have already realized that in order to draw a nice representative picture of the electric field can be time consuming. For field lines, one can use simple physical intuition in order to automate the procedure. For distances close enough to a point charge the electric field is approximately isotropic. The number of field lines crossing a small enough curve which contains only the charge is proportional to the charge (Gauss's law). Therefore we can draw a small circle centered around each charge and choose initial points isotropically distributed on the circle as initial points of the field lines.

The code listed below (version 3) implements the idea for charges that are equal in magnitude. For charges different in magnitude, the program is left as an exercise to the reader.

```
int main(){
  string buf;
  const double PI = 2.0*atan2(1.0,0.0);
  const int     P = 20;  //max number of charges
  double        X[P], Y[P], Q[P];
  int           N;
  int           i,j,nd;
  double        x0,y0,theta;
  //-------------- SET CHARGE DISTRIBUTION ----
  cout << "# Enter number of charges:"        << endl;
  cin  >> N;                            getline(cin,buf);
  cout << "# N= "          << N                << endl;
  for(i=0;i<N;i++){
    cout << "# Charge: " << i+1                << endl;
    cout << "# Position and charge: (X,Y,Q):" << endl;
    cin  >> X[i] >> Y[i] >> Q[i];     getline(cin,buf);
    cout <<          "# (X,Y)= "
         << X[i] << " "
         << Y[i] << " Q= "
         << Q[i]                               << endl;
  }
  //-------------- DRAWING LINES --------------
  //We draw 2*nd field lines around each charge
  nd = 6;
  for(i=0;i<N;i++)
    for(j=1;j<=(2*nd);j++){
      theta = (PI/nd)*j;
      x0    = X[i] + 0.1 * cos(theta);
      y0    = Y[i] + 0.1 * sin(theta);
      eline(x0,y0,X,Y,Q,N);
    }
}
```

We set the number of field lines around each charge to be equal to 12 (`nd=6`). The initial points are taken on the circle whose center is `(X[i],Y[i])` and its radius is `0.1`. The `2*nd` points are determined by the angle `theta=(PI/nd)*j`.

We record the data of a charge distribution in a file, e.g. `Input`. We list the example of four equal charges $q_i = \pm 1$ below, located at the vertices

of a square:

```
4               N: Number of charges
 1   1  -1      (X,Y,Q): Position and charge
-1   1   1      (X,Y,Q): Position and charge
 1  -1   1      (X,Y,Q): Position and charge
-1  -1  -1      (X,Y,Q): Position and charge
```

Then we give the commands:

```
> g++ ELines.cpp -o el
> ./el < Input > el.out
> gnuplot
gnuplot> plot "el.out" with dots
```

The results are shown in figures 7.3 and 7.4. The reader should determine the charge distributions that generate those fields and reproduce the figures as an exercise.

For the computation of the equipotential lines we can work in a similar way. We will follow a quick and dirty way which will not produce an accurate picture of the electric field and choose the initial points evenly spaced on an square lattice. For a better choice see problem 5. The function `main()` from the file `EPotential.cpp` is listed below:

```
int main(){
  string buf;
  const int       P = 20;  //max number of charges
  double          X[P], Y[P], Q[P];
  int             N;
  int             i,j,nd;
  double          x0,y0,rmin,rmax,L;
  //---------------  SET CHARGE DISTRIBUTION ----
  cout << "# Enter number of charges:"          << endl;
  cin  >> N;                             getline(cin,buf);
  cout << "# N= "         << N                  << endl;
  for(i=0;i<N;i++){
    cout << "# Charge: " << i+1                 << endl;
    cout << "# Position and charge: (X,Y,Q):" << endl;
    cin  >> X[i] >> Y[i] >> Q[i];     getline(cin,buf);
    cout <<              "# (X,Y)= "
         << X[i] << " "
```

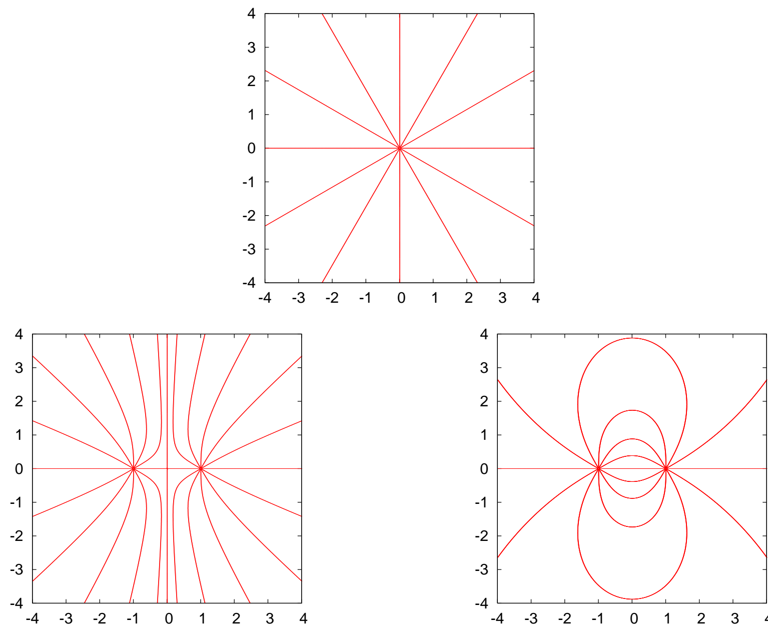

Figure 7.3: Field lines of a static charge distribution of point charges generated by the program ELines.cpp.

```
        << Y[i] << " Q= "
        << Q[i]                              << endl;
 }
 //----------- DRAWING LINES -----------
 //We draw lines passing through an equally
 //spaced lattice of N=(2*nd+1)x(2*nd+1) points
 //in the square -L<= x <= L, -L<= y <= L.
 nd = 6; L = 1.0;
 for(i=-nd;i<nd;i++)
   for(j=-nd;j<=nd;j++){
     x0  = i*(L/nd);
     y0  = j*(L/nd);
     cout << "# @ "
          << i << " " << j<< " " << L/nd    << " "
          << x0<< " " << y0                 << endl;
     mdist(x0,y0,X,Y,N,rmin,rmax);
     //we avoid getting too close to a charge:
     if(rmin > L/(nd*10) ) epotline(x0,y0,X,Y,Q,N);
   }
```

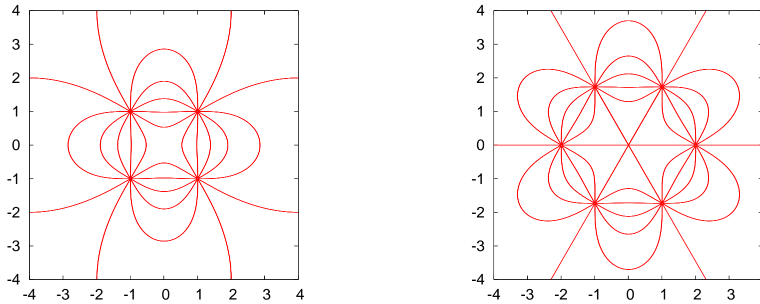

Figure 7.4: Field lines of a static charge distribution of point charges generated by the program `ELines.cpp`.

```
}
```

The first and second part of the code is identical to the previous one. In the third part we call the function `epotline` for drawing an equipotential line for each initial point. The initial points are on a square lattice with `(2*nd+1)*(2*nd+1)= 81` points (`nd=4`). The lattice extends within the limits set by the square $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$ (`L=1.0`). For each point `(x0,y0)` we calculate the equipotential line that passes through it. We check that this point is not too close to one of the charges by calling the function `mdist`. The call determines the minimum distance `rmin` of the point from all the charges which should be larger than `L/(nd*10)`. You can run the program with the commands:

```
> g++ EPotential.cpp -o ep
> ./ep < Input > ep.out
> gnuplot
gnuplot> plot "ep.out" with dots
```

Some of the results are shown in figure 7.5.

7.3 The Program – Main Dish

In this section we look under the hood and give the details of the inner parts of the program: The functions `eline` and `epotline` that calculate the field and equipotential lines, the function `efield` that calculates the

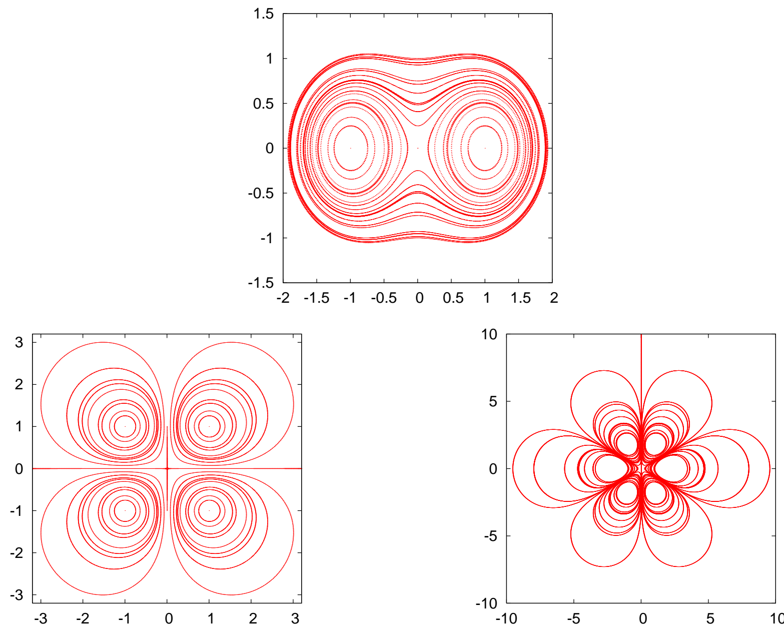

Figure 7.5: Equipotential lines of the electric field generated by a point charge distribution on the plane calculated by the program in `EPotential.cpp`. Beware: the density of the lines is not correctly calculated and it is not proportional to the magnitude of the electric field. See problem 7.5.

electric field at a point and the function `mdist` that calculates the minimum and maximum distances of a point from the point charges.

The function `eline` is called by the statement:

```
eline(x0,y0,X,Y,Q,N);
```

The input to the routine is the initial point `(x0,y0)`, the number of charges `N`, the positions of the charges `(X[N],Y[N])` and the charges `Q[N]`. The routine needs some parameters in order to draw the field line. These are “hard coded”, i.e. set to fixed values by the programmer that cannot be changed by the user that calls the routine in her program. One of them is the step Δl of equation (7.5) which sets the discretization step of the field line. It also sets the minimum distance of approaching to a charge equal to $2\Delta l$. The size of the drawing area of the curve is set by the parameter `max_dist=20.0`. We should also provide a check in

the program that checks whether the electric field is zero, in which case the result of the calculation in equation (7.5) becomes indeterminate. By taking $\Delta l > 0$, the motion is in the direction of the electric field, which ends on a negative charge or outside the drawing area (why?). In order to draw the line in both directions, set $\Delta l < 0$ and repeat the calculation.

The code is listed below:

```
void
eline (double  xin,double  yin,
       double*   X,double*   Y,double*  Q, const int N){
  const double step    =0.01;
  const double max_dist=20.0;
  int          i, direction;
  double       x0,y0;
  double       rmin,rmax,r,dx,dy,dl;
  double       Ex,Ey,E;

  cout.precision(17);
  for(direction=-1;direction<=1;direction+=2){
    dl = direction * step;
    x0 = xin;
    y0 = yin;
    dx = 0.0;
    dy = 0.0;
    mdist(x0,y0,X,Y,N,rmin,rmax);
    while(rmin > (2.0*step) && rmax < max_dist){
      cout << x0 << " " << y0 << '\n';
      //We evaluate the E-field at the midpoint:
      //This reduces systematic errors
      efield(x0+0.5*dx,y0+0.5*dy,X,Y,Q,N,Ex,Ey);
      E  = sqrt(Ex*Ex+Ey*Ey);
      if( E <= 1.0e-10 ) break;
      dx = dl*Ex/E;
      dy = dl*Ey/E;
      x0 = x0 + dx;
      y0 = y0 + dy;
      mdist(x0,y0,X,Y,N,rmin,rmax);
    }//while(rmin > (2.0*step) && rmax < max_dist)
  }//for(direction=-1;direction<=1;direction+=2)
}
```

In the first part of the code we have the variable declarations. We note that the values of the parameters `step` and `max_dist` are "hard coded"

into our program and the user cannot change them:

```
const double step     =0.01;
const double max_dist=20.0;
```

Their values should be the result of a careful study by the programmer since they determine the accuracy of the calculation.

The outmost loop

```
for(direction=-1;direction<=1;direction+=2){
  dl = direction * step;
  ...
}
```

sets the direction of motion on the field line (i.e. the sign of Δl). The loop is executed twice with `direction` taking the two values -1 and 1.

The commands `x0 = xin, y0 = yin` define the initial point on the field line. `(x0, y0)` is the current point on the field line which is printed to the `stdout`. The variables `(dx, dy)` set the step `(x0, y0)` $\to$ `(x0+dx, y0+dy)`. The drawing of the field line is done in the inner loop

```
  mdist(x0,y0,X,Y,N,rmin,rmax);
  while(rmin > (2.0*step) && rmax < max_dist){
    cout << x0 << " " << y0 << '\n';
    ...
    mdist(x0,y0,X,Y,N,rmin,rmax);
  }
```

which is executed provided that the logical expression `(rmin > (2.0*step) && rmax < max_dist)` is `true` This happens as long as the current point is at a distance greater than `2.0*step` and the maximum distance from all charges is less than max_dist[5]. The minimum and maximum distances are calculated by calling the function `mdist`.

The electric field, needed in equation (7.5), is calculated by a call to `efield(x0+0.5*dx,y0+0.5*dy,X,Y,Q,N,Ex,Ey)`. The first two arguments give the point at which we want to calculate the electric field, which is chosen to be the *midpoint* `(x0+dx/2,y0+dy/2)` instead of `(x0,y0)`. This improves the stability and the accuracy of the algorithm.

[5]The choice is not unique of course, you may also try e.g. `rmin < max_dist`.

Equation (7.5) is coded in the commands

```
    E  = sqrt(Ex*Ex+Ey*Ey);
    dx = dl*Ex/E;
    dy = dl*Ey/E;
    x0 = x0 + dx;
    y0 = y0 + dy;
```

We also perform checks for the cases `E=0.0` and `dx=dy=0.0`:

```
    if( E <= 1.0e-10 ) break;
```

When the magnitude of the electric field becomes too small we stop the calculation by exiting the loop with the command `break`. The reader can improve the code by adding more checks of singular cases.

The function `epotline` is programmed in a similar way. The `main()` function is listed below:

```
int main(){
  string buf;
  const int     P = 20;  //max number of charges
  double        X[P], Y[P], Q[P];
  int           N;
  int           i,j,nd;
  double        x0,y0,rmin,rmax,L;
  //------------- SET CHARGE DISTRIBUTION ----
  cout << "# Enter number of charges:"       << endl;
  cin  >> N;                          getline(cin,buf);
  cout << "# N= "          << N            << endl;
  for(i=0;i<N;i++){
    cout << "# Charge: " << i+1            << endl;
    cout << "# Position and charge: (X,Y,Q):" << endl;
    cin  >> X[i] >> Y[i] >> Q[i];     getline(cin,buf);
    cout <<         "# (X,Y)= "
         << X[i] << " "
         << Y[i] << " Q= "
         << Q[i]                           << endl;
  }
  //------------- DRAWING LINES  -------------
  //We draw lines passing through an equally
  //spaced lattice of N=(2*nd+1)x(2*nd+1) points
  //in the square -L<= x <= L, -L<= y <= L.
```

```
  nd = 6; L = 1.0;
  for(i=-nd;i<nd;i++)
    for(j=-nd;j<=nd;j++){
      x0  = i*(L/nd);
      y0  = j*(L/nd);
      cout << "# @ "
           << i << " " << j<< " " << L/nd    << " "
           << x0<< " " << y0                 << endl;
      mdist(x0,y0,X,Y,N,rmin,rmax);
      //we avoid getting too close to a charge:
      if(rmin > L/(nd*10) ) epotline(x0,y0,X,Y,Q,N);
    }
}
```

The differences are minor: The equipotential lines are closed curves, therefore we only need to transverse them in one direction. The criterion for ending the calculation is to approach the initial point close enough or leave the drawing area:

```
  while(r > (0.9*dl) && r < max_dist){
    ...
  }
```

The values of `dx`, `dy` are calculated according to equation (7.6):

```
    dx =  dl*Ey/E;
    dy = -dl*Ex/E;
```

The function `efield` is an application of equations[6] (7.2):

```
void
efield(double   x0,double   y0,
       double*   X,double*   Y,double*  Q, const int N,
       double&  Ex,double&  Ey){
  int    i;
  double r3,xi,yi;
  Ex  = 0.0;
  Ey  = 0.0;
  for(i=0;i<N;i++){
    xi = x0-X[i];
```

[6] You may improve the program by checking whether $r_i = 0$.

```
    yi = y0-Y[i];
    r3 = pow(xi*xi+yi*yi,-1.5);
    Ex = Ex + Q[i]*xi*r3;
    Ey = Ey + Q[i]*yi*r3;
  }
}
```

Finally, the function `mdist` calculates the minimum and maximum distance `rmin` and `rmax` of a point `(x0,y0)` from all the point charges in the distribution:

```
void
mdist (double    x0,double    y0,
       double*    X,double*    Y,       const int N,
       double&  rmin,double&  rmax){
  int    i;
  double r;
  rmax = 0.0;
  rmin = 1000.0;
  for(i=0;i<N;i++){
    r = sqrt((x0-X[i])*(x0-X[i]) + (y0-Y[i])*(y0-Y[i]));
    if(r > rmax) rmax = r;
    if(r < rmin) rmin = r;
  }
}
```

The initial value of `rmin` depends of the limits of the drawing area (why?).

7.4 The Program - Conclusion

In this section we list the programs discussed in the previous sections and provide short usage information for compiling, running and analyzing your results. You can jump into this section without reading the previous ones and go back to them if you need to clarify some points that you find hard to understand.

First we list the contents of the file `ELines.cpp`:

```
#include <iostream>
#include <fstream>
#include <cstdlib>
```

```
#include <string>
#include <cmath>
using namespace std;
//------------------------------------------------------------------------
void
eline (double  xin,double  yin,
       double*   X,double*   Y,double*  Q, const int N);
void
efield(double   x0,double   y0,
       double*   X,double*   Y,double*  Q, const int N,
       double&  Ex,double&  Ey);
void
mdist (double   x0,double   y0,
       double*   X,double*   Y,             const int N,
       double&  rm,double&  rM);
//------------------------------------------------------------------------
int main(){
  string buf;
  const double PI = 2.0*atan2(1.0,0.0);
  const int      P = 20;  //max number of charges
  double         X[P], Y[P], Q[P];
  int            N;
  int            i,j,nd;
  double         x0,y0,theta;
  //-------------- SET CHARGE DISTRIBUTION ----
  cout << "# Enter number of charges:"          << endl;
  cin  >> N;                           getline(cin,buf);
  cout << "# N= "          << N                  << endl;
  for(i=0;i<N;i++){
    cout << "# Charge: " << i+1                  << endl;
    cout << "# Position and charge: (X,Y,Q):" << endl;
    cin  >> X[i] >> Y[i] >> Q[i];     getline(cin,buf);
    cout <<           "# (X,Y)= "
         << X[i] << " "
         << Y[i] << " Q= "
         << Q[i]                                  << endl;
  }
  //-------------- DRAWING LINES  --------------
  //We draw 2*nd field lines around each charge
  nd = 6;
  for(i=0;i<N;i++)
    for(j=1;j<=(2*nd);j++){
      theta = (PI/nd)*j;
      x0    = X[i] + 0.1 * cos(theta);
      y0    = Y[i] + 0.1 * sin(theta);
```

```
      eline(x0,y0,X,Y,Q,N);
    }
}//main()
//------------------------------------------------------------
void
eline (double  xin,double  yin,
       double*   X,double*   Y,double*  Q, const int N){
  const double step    =0.01;
  const double max_dist=20.0;
  int          i, direction;
  double       x0,y0;
  double       rmin,rmax,r,dx,dy,dl;
  double       Ex,Ey,E;

  cout.precision(17);
  for(direction=-1;direction<=1;direction+=2){
    dl = direction * step;
    x0 = xin;
    y0 = yin;
    dx = 0.0;
    dy = 0.0;
    mdist(x0,y0,X,Y,N,rmin,rmax);
    while(rmin > (2.0*step) && rmax < max_dist){
      cout << x0 << " " << y0 << '\n';
      //We evaluate the E-field at the midpoint:
      //This reduces systematic errors
      efield(x0+0.5*dx,y0+0.5*dy,X,Y,Q,N,Ex,Ey);
      E  = sqrt(Ex*Ex+Ey*Ey);
      if( E <= 1.0e-10 ) break;
      dx = dl*Ex/E;
      dy = dl*Ey/E;
      x0 = x0 + dx;
      y0 = y0 + dy;
      mdist(x0,y0,X,Y,N,rmin,rmax);
    }//while(rmin > (2.0*step) && rmax < max_dist)
  }//for(direction=-1;direction<=1;direction+=2)
}//eline()
//------------------------------------------------------------
void
efield(double   x0,double   y0,
       double*   X,double*   Y,double*  Q, const int N,
       double&  Ex,double&  Ey){
  int    i;
  double r3,xi,yi;
  Ex  = 0.0;
```

```
  Ey  = 0.0;
  for(i=0;i<N;i++){
    xi = x0-X[i];
    yi = y0-Y[i];
    r3 = pow(xi*xi+yi*yi,-1.5);
    Ex = Ex + Q[i]*xi*r3;
    Ey = Ey + Q[i]*yi*r3;
  }
}//efield()
//--------------------------------------------------------
void
mdist (double     x0,double     y0,
       double*     X,double*     Y,       const int N,
       double&  rmin,double&  rmax){
  int    i;
  double r;
  rmax = 0.0;
  rmin = 1000.0;
  for(i=0;i<N;i++){
    r = sqrt((x0-X[i])*(x0-X[i]) + (y0-Y[i])*(y0-Y[i]));
    if(r > rmax) rmax = r;
    if(r < rmin) rmin = r;
  }
}//mdist()
```

Then we list the contents of the file `EPotential.cpp`:

```
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
//--------------------------------------------------------
void
epotline(double  xin,double  yin,
         double*   X,double*   Y,double*  Q, const int N);
void
efield  (double   x0,double   y0,
         double*   X,double*   Y,double*  Q, const int N,
         double&  Ex,double&  Ey);
void
mdist   (double   x0,double   y0,
         double*   X,double*   Y,            const int N,
```

```
           double&  rm,double&  rM);
//--------------------------------------------------------------
int main(){
  string buf;
  const int      P = 20;  //max number of charges
  double         X[P], Y[P], Q[P];
  int            N;
  int            i,j,nd;
  double         x0,y0,rmin,rmax,L;
  //-------------- SET CHARGE DISTRIBUTION ----
  cout << "# Enter number of charges:"        << endl;
  cin  >> N;                           getline(cin,buf);
  cout << "# N= "          << N                << endl;
  for(i=0;i<N;i++){
    cout << "# Charge: " << i+1                << endl;
    cout << "# Position and charge: (X,Y,Q):" << endl;
    cin  >> X[i] >> Y[i] >> Q[i];      getline(cin,buf);
    cout <<          "# (X,Y)= "
         << X[i] << " "
         << Y[i] << " Q= "
         << Q[i]                               << endl;
  }
  //-------------- DRAWING LINES --------------
  //We draw lines passing through an equally
  //spaced lattice of N=(2*nd+1)x(2*nd+1) points
  //in the square -L<= x <= L, -L<= y <= L.
  nd = 6; L = 1.0;
  for(i=-nd;i<nd;i++)
    for(j=-nd;j<=nd;j++){
      x0  = i*(L/nd);
      y0  = j*(L/nd);
      cout << "# @ "
           << i << " " << j<< " " << L/nd    << " "
           << x0<< " " << y0                 << endl;
      mdist(x0,y0,X,Y,N,rmin,rmax);
      //we avoid getting too close to a charge:
      if(rmin > L/(nd*10) ) epotline(x0,y0,X,Y,Q,N);
    }
}//main()
//--------------------------------------------------------------
void
epotline(double xin,double  yin,
         double*  X,double* Y,double* Q, const int N){
  const  double step    =0.02;
  const  double max_dist=20.0;
```

```
  int            i;
  double         x0,y0;
  double         r,dx,dy,dl;
  double         Ex,Ey,E;

  cout.precision(17);
  dl = step;
  x0 = xin;
  y0 = yin;
  dx = 0.0;
  dy = 0.0;
  r  = step;
  while(r > (0.9*dl) && r < max_dist){
    cout << x0 << " " << y0 << '\n';
    //We evaluate the E-field at the midpoint:
    //This reduces systematic errors
    efield(x0+0.5*dx,y0+0.5*dy,X,Y,Q,N,Ex,Ey);
    E  = sqrt(Ex*Ex+Ey*Ey);
    if( E <= 1.0e-10 ) break;
    dx =  dl*Ey/E;
    dy = -dl*Ex/E;
    x0 =  x0 + dx;
    y0 =  y0 + dy;
    r  = sqrt((x0-xin)*(x0-xin)+(y0-yin)*(y0-yin));
  }//while()
}//epotline()
...
```

where ... are the functions `efield` and `mdist` which are identical to the ones in the file `ELines.cpp`.

In order to compile the program use the commands:

```
> g++ ELines.cpp     -o el
> g++ EPotential.cpp -o ep
```

Then, edit a file and name it e.g. `Input` and write the data that define a charge distribution. For example:

```
4             N: Number of charges
 1  1  -1     (X,Y,Q): Position and charge
-1  1   1     (X,Y,Q): Position and charge
 1 -1   1     (X,Y,Q): Position and charge
-1 -1  -1     (X,Y,Q): Position and charge
```

The results are obtained with the commands:

```
> ./el < Input > el.dat
> ./ep < Input > ep.dat
> gnuplot
gnuplot> plot "el.dat" with dots
gnuplot> plot "ep.dat" with dots
```

Have fun!

7.5 Electrostatic Field in the Vacuum

Consider a time independent electric field in an area of space which is empty of electric charge. Maxwell's equations are reduced to Gauss's law

$$\vec{\nabla}\cdot\vec{E}(x,y,z) = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0\,, \tag{7.8}$$

together with the equation that defines the electrostatic potential[7]

$$\vec{E}(x,y,z) = -\vec{\nabla}V(x,y,z)\,. \tag{7.9}$$

Equations (7.8) and (7.9) give the Laplace equation for the function $V(x,y,z)$:

$$\nabla^2 V(x,y,z) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0\,. \tag{7.10}$$

The solution of the equation above is a boundary value problem: We are looking for the potential $V(x,y,z)$ in a region of space $\mathcal{S}$ bounded by a closed surface $\partial\mathcal{S}$. When the potential is known on $\partial\mathcal{S}$ the solution to (7.10) is unique and the potential and the electric field is determined everywhere in $\mathcal{S}$.

For simplicity consider the problem confined on a plane, therefore $V = V(x,y)$. In this case the last term in equation (7.10) vanishes, the region $\mathcal{S}$ is a compact subset of the plane and $\partial\mathcal{S}$ is a closed curve.

For the numerical solution of the problem, we approximate $\mathcal{S}$ by a discrete, square lattice. The potential is defined on the N sites of the lattice. We take $\mathcal{S}$ to be bounded by a square with sides of length l. The

[7]Equivalent to the equation $\vec{\nabla}\times\vec{E} = 0$.

distance between the nearest lattice points is called the lattice constant a. Then $l = (L-1)a$, where $L = \sqrt{N}$ is the number of lattice points on each side of the square. The continuous solution is approximated by the solution on the lattice, and the approximation becomes exact in the $N \to \infty$ and $a \to 0$ limits, so that the length $l = (L-1)a$ remains constant. The curve $\partial\mathcal{S}$ is approximated by the lattice sites that are located on the perimeter of the square and the loci in the square where the potential takes constant values. This is a simple model of a set of conducting surfaces (points where $V = \text{const.} \neq 0$) in a compact region whose boundary is grounded (points where $V = 0$). An example is depicted in figure 7.6.

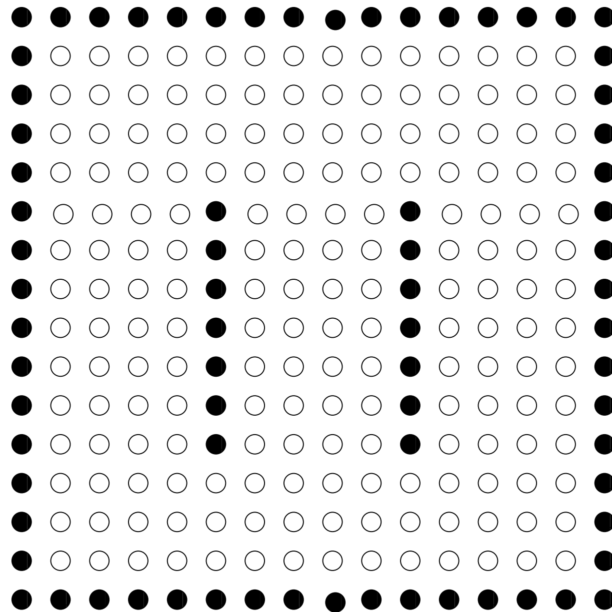

Figure 7.6: A lattice which corresponds to a cross section of two parallel conducting planes inside a grounded cubic box. The black lattice sites are the points of constant, fixed potential whereas the white ones are sites in the vacuum.

In order to derive a finite difference equation which approximates equation (7.10), we Taylor expand around a point (x, y) according to the

equations:

$$\begin{aligned}
V(x+\delta x, y) &= V(x,y) + \frac{\partial V}{\partial x}\delta x + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(\delta x)^2 + \ldots \\
V(x-\delta x, y) &= V(x,y) - \frac{\partial V}{\partial x}\delta x + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(\delta x)^2 + \ldots \\
V(x, y+\delta y) &= V(x,y) + \frac{\partial V}{\partial y}\delta y + \frac{1}{2}\frac{\partial^2 V}{\partial y^2}(\delta y)^2 + \ldots \\
V(x, y-\delta y) &= V(x,y) - \frac{\partial V}{\partial y}\delta y + \frac{1}{2}\frac{\partial^2 V}{\partial y^2}(\delta y)^2 + \ldots .
\end{aligned}$$

By summing both sides of the above equations, taking $\delta x = \delta y$ and ignoring the terms implied by $\ldots$, we obtain

$$\begin{aligned}
& V(x+\delta x, y) + V(x-\delta x, y) + V(x, y+\delta y) + V(x, y-\delta y) \\
& \quad = 4V(x,y) + (\delta x)^2 (\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}) + \ldots \\
& \quad \approx 4V(x,y)\,, \qquad (7.11)
\end{aligned}$$

The second term in the second line was eliminated by using equation (7.10).

We map the coordinates of the lattice points to integers (i,j) such that $x_i = (i-1)a$ and $y_j = (j-1)a$ where $i,j = 1,\ldots,L$. By taking $\delta x = \delta y = a$ so that $x_i \pm \delta x = x_i \pm a = (i-1\pm 1)a = x_{i\pm 1}$ and $y_j \pm \delta y = y_j \pm a = (j-1\pm 1)a = y_{j\pm 1}$, equation (7.11) becomes:

$$V(i,j) = \frac{1}{4}(V(i-1,j) + V(i+1,j) + V(i,j-1) + V(i,j+1))\,. \qquad (7.12)$$

The equation above states that the potential at the position (i,j) is the arithmetic mean of the potential of the nearest neighbors. We will describe an algorithm which belongs to the class of "successive overrelaxation methods" (SOR) whose basic steps are:

1. Set the size L of the square lattice.

2. Flag the sites that correspond to "conductors", i.e. the sites where the potential remains fixed to the boundary conditions values.

3. Choose an initial trial function for $V(x,y)$ on the vacuum sites. Of course it is not the solution we are looking for. A good choice will lead to fast convergence of the algorithm to the true solution. A bad choice may lead to slow convergence, no convergence or even convergence to the wrong solution. In our case the problem is easy and the simple choice $V(x,y) = 0$ will do.

4. Sweep the lattice and enforce equation (7.12) on each visited vacuum site. This defines the new value of the potential at this site.

5. Sweep the lattice repeatedly until two successive sweeps result in a very small change in the function $V(x,y)$.

A careful study of the above algorithm requires to test different criteria of "very small change" and test that different choices of the initial function $V(x,y)$ result in the same solution.

We write a program that implements this algorithm in the case of a system which is the projection of two parallel conducting planes inside a grounded cubic box on the plane. The lattice is depicted in figure 7.6, where the black dots correspond to the conductors. All the points of the box have $V = 0$ and the two conductors are at constant potential V_1 and V_2 respectively. The user enters the values V_1 and V_2, the lattice size L and the required accuracy interactively. The latter is determined by a small number ϵ. The convergence criterion that we set is that the maximum difference between the values of the potential between two successive sweeps should be less than ϵ.

The data structure is very simple. We use an array `double V[L][L]` in order to store the values of the potential at each lattice site. A logical array `bool isConductor[L][L]` flags each site as a "conductor site" (=`true`) or as a "vacuum site" (=`false`). Both arrays are put in the global scope and are accessible by all functions.

The main program reads in the data entered by the user and then calls three functions:

1. `initialize_lattice(V1,V2)`:
   The routine needs at its input the values of the potential `V1` and `V2` on the left and right plate respectively. On exit it provides the initial values of the potential `V[L][L]` and the flags `isConductor[L][L]`. The geometry of the setting is hard coded and the user needs to

change this function each time that she wants to study a different geometry.

2. `laplace(epsilon)`:
This is the heart of the program. On entry we provide the desired accuracy `epsilon`. On exit we obtain the final solution `V[L][L]`. This function calculates the arithmetic mean of the potential of the nearest neighbors `Vav` and the value `V[i][j]=Vav` is changed immediately[8]. The maximum change in the new value of the potential `Vav` from the old one `V[i][j]` is stored in the variable `error`. When `error` becomes smaller than `epsilon` we assume that convergence has been achieved.

3. `print_results()`:
This function prints the potential `V[L][L]` to the file `data`. Each line contains the integers `i, j` and the value of the potential `V[i][j]`. We note that each time that the index `i` changes, the function prints an extra empty line. This is done so that the output can be read easily by the three dimensional plotting function `splot` of `gnuplot`.

The full program is listed below:

```
//************************************************************
//PROGRAM LAPLACE_EM
//Computes the electrostatic potential around conductors.
//The computation is performed on a square lattice of linear
//dimension L. A relaxation method is used to converge to the
//solution of Laplace equation for the potential.
//DATA STRUCTURE:
//double V[L][L]: Value of the potential on the lattice sites
//bool isConductor[L][L]: If true  site has fixed potential
//                        If false site is empty space
//double epsilon: Determines the accuracy of the solution
//The maximum difference of the potential on each site
//between two consecutive sweeps should be less than epsilon.
//PROGRAM STRUCTURE
//main program:
// . Data Input
```

[8]A different choice would have been to store the value `Vav` in a temporary array `Vnew[i][j]`. After the sweep, the potential `V[i][j]=Vnew[i][j]` is changed to the new values. Which method do you expect to have better convergence properties? Try...

```
// . call functions for initialization , computation and
//   printing of results
//   function initialize_lattice:
// . Initilization of V[L][L] and isConductor[L][L]
//   function laplace:
// . Solves laplace equation using a relaxation method
//   function print_results:
// . Prints results for V[L][L] in a file. Uses format
//   compatible with splot of gnuplot.
//*************************************************************
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
//-------------------------------------------------------------
const  int L = 31;
bool   isConductor[L][L];
double V          [L][L];
//-------------------------------------------------------------
void initialize_lattice(const double& V1,const double& V2);
void laplace           (const double& epsilon);
void print_results     ();
//-------------------------------------------------------------
int main(){
  string buf;
  double V1,V2,epsilon;

  cout << "Enter V1,V2:"                       << endl;
  cin  >> V1 >> V2;               getline(cin,buf);
  cout << "Enter epsilon:"                     << endl;
  cin  >> epsilon;                getline(cin,buf);
  cout << "Starting Laplace:"                  << endl;
  cout << "Grid Size= "                  << L  << endl;
  cout << "Conductors set at V1= " << V1
       << " V2= "                        << V2 << endl;
  cout << "Relaxing with accuracy epsilon= "
       << epsilon << endl;
  //The arrays V and isConductor are initialized
  initialize_lattice(V1,V2);
  //On entry, V,isConductor is initialized.
  //On exit the routine gives the solution V
  laplace(epsilon);
  //We print V in a file.
```

```
  print_results();
}//main()
//************************************************************
//function initialize_lattice
//Initializes arrays V[L][L] and isConductor[L][L].
//V[L][L]= 0.0  and isConductor[L][L]=  false  by default
//isConductor[i][j]= true on boundary of lattice where V=0
//isConductor[i][j]= true on sites with i=  L/3+1,5<=j<= L-5
//isConductor[i][j]= true on sites with i=2*L/3+1,5<=j<= L-5
//V[i][j] = V1 on all sites with i=  L/3+1,5<=j<= L-5
//V[i][j] = V2 on all sites with i=2*L/3+1,5<=j<= L-5
//V[i][j] = 0  on boundary (i=1,L and j=1,L)
//V[i][j] = 0  on interior sites with isConductor[i][j]=false
//INPUT:
//integer L: Linear size of lattice
//double V1,V2: Values of potential on interior conductors
//OUTPUT:
//double V[L][L]: Array provided by user. Values of potential
//bool   isConductor[L][L]: If true  site has fixed potential
//                          If false site is empty space
//************************************************************
void initialize_lattice(const double& V1,const double& V2){

  //Initialize to 0 and false (default values for
  //boundary and interior sites).
  for(int i=0;i<L;i++)
    for(int j=0;j<L;j++){
      V          [i][j  ] = 0.0;
      isConductor[i][j  ] = false;
    }
  //We set the boundary to be a conductor: (V=0 by default)
  for(int i=0;i<L;i++){
    isConductor[0  ][i  ] = true;
    isConductor[i  ][0  ] = true;
    isConductor[L-1][i  ] = true;
    isConductor[i  ][L-1] = true;
  }
  //We set two conductors at given potential V1 and V2
  for(int i=4;i<L-5;i++){
    V          [  L/3][i] = V1;
    isConductor[  L/3][i] = true;
    V          [2*L/3][i] = V2;
    isConductor[2*L/3][i] = true;
  }
}//initialize_lattice()
```

```
//********************************************************
//function laplace
//Uses a relaxation method to compute the solution of the
//Laplace equation for the electrostatic potential on a
//2 dimensional squarelattice of linear size L.
//At every sweep of the lattice we compute the average
//Vav of the potential at each site (i,j) and we immediately
//update V[i][j]
//The computation continues until Max |Vav-V[i][j]| < epsilon
//INPUT:
//integer L: Linear size of lattice
//double V[L][L]: Value of the potential at each site
//bool   isConductor[L][L]: If true    potential is fixed
//                          If false   potential is updated
//double epsilon: if Max |Vav-V[i][j]| < epsilon return to
//callingprogram.
//OUTPUT:
//double V[L][L]: The computed solution for the potential
//********************************************************
void laplace           (const double& epsilon){
  int    icount;
  double Vav,error,dV;

  icount = 0;
  while(icount < 10000){
    icount++;
    error = 0.0;
    for(  int i = 1;i<L-1;i++){
      for(int j = 1;j<L-1;j++){
        //We change V only for non conductors:
        if( ! isConductor[i][j]){
          Vav = 0.25*(V[i-1][j]+V[i+1][j]+V[i][j-1]+V[i][j+1]);
          dV  = abs(V[i][j]-Vav);
          if(error < dV) error = dV; //maximum error
          V[i][j] = Vav;
        }//if( ! isConductor[i][j])
      }//for(int j = 1;j<L-1;j++)
    }//for(  int i = 1;i<L-1;i++)
    cout << icount << " err= " << error << endl ;
    if(error < epsilon) return;
  }//while(icount < 10000)
  cerr << "Warning: laplace did not converge.\n";
}//laplace()
//********************************************************
//function  print_results
```

```
//Prints the array V[L][L] in file "data"
//The format of the output is appropriate for the splot
//function of gnuplot: Each time i changes an empty line
// is printed.
//**************************************************************
void print_results(){
  ofstream  myfile("data");
  myfile.precision(16);
  for(  int i = 0; i < L ; i++){
    for(int j = 0; j < L ; j++){
      myfile << i+1 << " " << j+1 << " " << V[i][j] << endl;
    }
    //print empty line for gnuplot, separate isolines:
    myfile   << " "                                  << endl;
  }
  myfile.close();
}//print_results()
```

7.6 Results

The program in the previous section is written in the file `LaplaceEq.cpp`. Compiling and running is done with the commands:

```
> g++ LaplaceEq.cpp -o lf
> ./lf
Enter V1,V2:
100 -100
Enter epsilon:
0.01
Starting Laplace:
Grid Size= 31
Conductors set at V1= 100 V2= -100
Relaxing with accuracy epsilon= 0.01
1   err= 33.3333
2   err= 14.8148
3   err= 9.87654
..............
110 err= 0.0106861
111 err= 0.0101182
112 err= 0.00958049
```

In the example above, the program performs 112 sweeps until the error becomes `0.00958 < 0.01`. The results are stored in the file `data`. We can make a three dimensional plot of the function $V(i,j)$ with the `gnuplot` commands:

```
gnuplot> set pm3d
gnuplot> set hidden3d
gnuplot> set size ratio 1
gnuplot> splot "data" with lines
```

The results are shown in figure 7.7

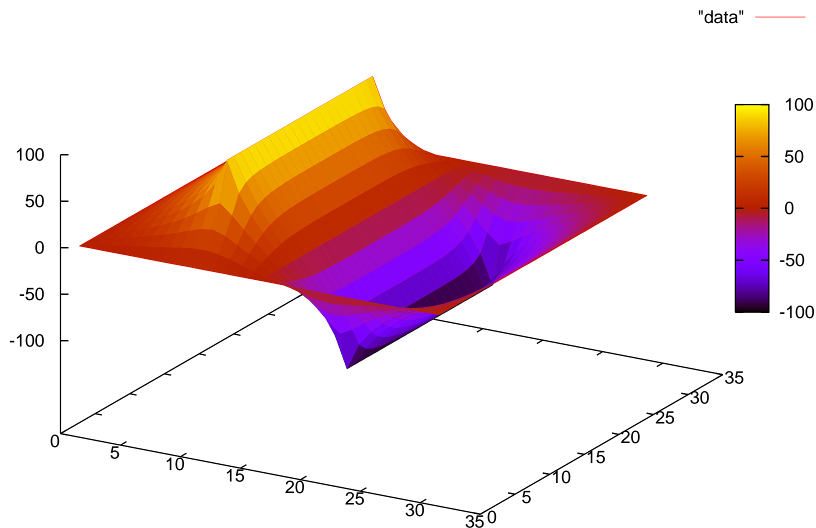

Figure 7.7: The solution of the equation (7.10) computed by the program `LaplaceEq.cpp` for `L= 31, V1=100, V2=-100, epsilon=0.01`.

7.7 Poisson Equation

This section contains a short discussion of the case where the space is filled with a continuous static charge distribution given by the charge

density function $\rho(\vec{r})$. In this case the Laplace equation becomes the Poisson equation:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -4\pi\rho(x, y, z) \tag{7.13}$$

The equation on the lattice becomes

$$V(i,j) = \frac{1}{4}(V(i-1,j)+V(i+1,j)+V(i,j-1)+V(i,j+1)+\tilde{\rho}(i,j)), \tag{7.14}$$

where[9] $\tilde{\rho}(i,j) = 4\pi a^2 \rho(i,j)$.

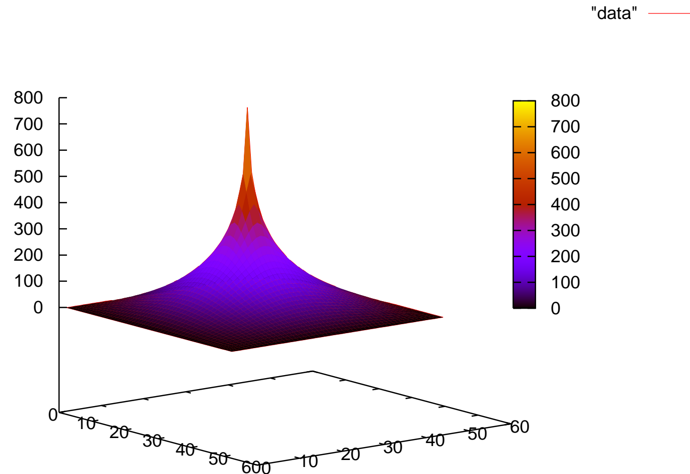

Figure 7.8: The solution of the equation (7.13) by the program in the file `Poisson.cpp` for `L= 51`, `V= 0` on the boundary and the charge $4\pi Q = 1000$ all concentrated at one point.

The program in the file `PoissonEq.cpp` solves equation (7.14) for a uniform charge distribution (figure 7.10), where we have set $a = 1$. The reader is asked to reproduce this figure together with figures 7.8 and 7.9.

[9]Since $Q = \int \rho dA \approx \sum_{i,j} \rho a^2 = (1/4\pi) \sum_{i,j} \tilde{\rho}$. Therefore $\sum_{i,j} \tilde{\rho} \approx 4\pi Q$.

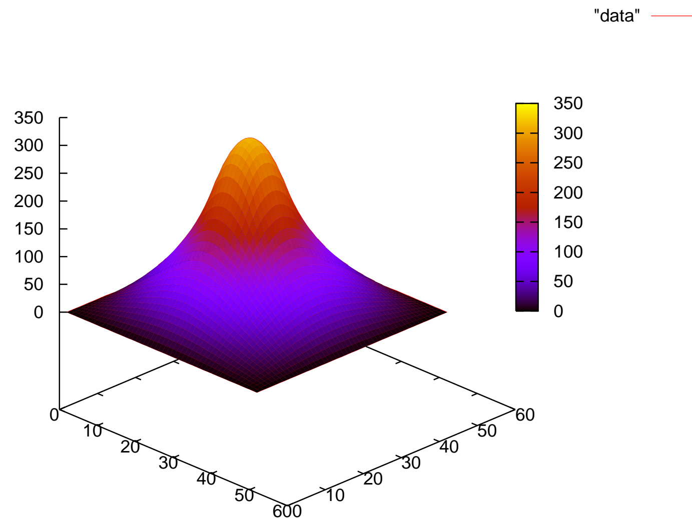

Figure 7.9: The solution of equation (7.13) by the program in the file `Poisson.cpp` for `L= 51`, `V= 0` on the boundary and the charge $4\pi Q = 1000$ uniformly distributed in a small square with sides made of 10 lattice sites.

```
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
//--------------------------------------------------------
const  int L = 51;
bool   isConductor[L][L];
double V          [L][L];
double rho        [L][L];
//--------------------------------------------------------
void initialize_lattice(const double& V1,const double& V2,
                        const double& V3,const double& V4,
                        const double&  Q);
void laplace           (const double& epsilon);
void print_results     ();
//--------------------------------------------------------
```

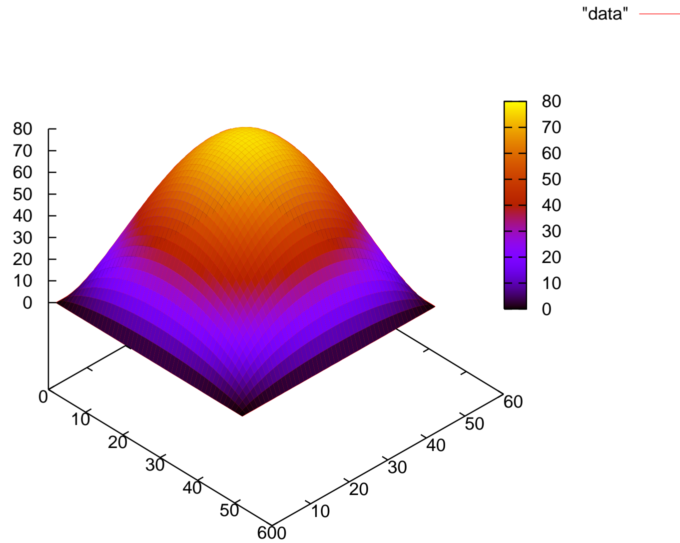

Figure 7.10: The solution of equation (7.13) by the program in the file `Poisson.cpp` for `L= 51`, `V= 0` on the boundary and the charge $4\pi Q = 1000$ uniformly distributed on all internal lattice sites.

```
int main(){
  string buf;
  double V1,V2,V3,V4,Q,epsilon;

  cout << "Enter V1,V2,V3,V4:"           << endl;
  cin  >> V1 >> V2 >> V3 >> V4; getline(cin,buf);
  cout << "Enter 4*PI*Q:";
  cin  >> Q;                    getline(cin,buf);
  cout << "Enter epsilon:"               << endl;
  cin  >> epsilon;              getline(cin,buf);
  cout << "Starting Laplace:"            << endl;
  cout << "Grid Size= "           << L  << endl;
  cout << "Conductors set at V1= " << V1
       << " V2= "                  << V2
       << " V3= "                  << V3
       << " V4= "                  << V4
       << " Q = "                  << Q  << endl;
  cout << "Relaxing with accuracy epsilon= "
       << epsilon << endl;
```

```
  initialize_lattice(V1,V2,V3,V4,Q);

  laplace(epsilon);

  print_results();

}//main()
//*************************************************************
void initialize_lattice(const double& V1,const double& V2,
                        const double& V3,const double& V4,
                        const double&  Q ){
  int    L1,L2;
  double Area;
  //Initialize to 0 and .FALSE (default values
  //for boundary and interior sites).
  for(int i=0;i<L;i++)
    for(int j=0;j<L;j++){
      V          [i][j  ] = 0.0;
      isConductor[i][j  ] = false;
      rho        [i][j  ] = 0.0;
    }
  //We set the boundary to be a conductor: (V=0 by default)
  for(int i=0;i<L;i++){
    isConductor[0  ][i  ] = true;
    isConductor[i  ][0  ] = true;
    isConductor[L-1][i  ] = true;
    isConductor[i  ][L-1] = true;
    V          [0  ][i  ] = V1;
    V          [i  ][L-1] = V2;
    V          [L-1][i  ] = V3;
    V          [i  ][0  ] = V4;
  }
  //We set the points with non-zero charge
  //A uniform distribution at a center square
  L1 = (L/2) - 5;
  L2 = (L/2) + 5;
  if(L1< 0){cerr <<"Array out of bounds. L1< 0\n";exit(1);}
  if(L2>=L){cerr <<"Array out of bounds. L2>=0\n";exit(1);}
  Area = (L2-L1+1)*(L2-L1+1);

  for(  int i=L1;i<=L2;i++)
    for(int j=L1;j<=L2;j++)
      rho[i][j] = Q/Area;
```

```
}//initialize_lattice()
//*********************************************************
void laplace          (const double& epsilon){
  int     icount;
  double Vav,error,dV;

  icount = 0;
  while(icount < 10000){
    icount++;
    error = 0.0;
    for(  int i = 1;i<L-1;i++){
      for(int j = 1;j<L-1;j++){
        //We change V only for non conductors:
        if( ! isConductor[i][j]){
          Vav = 0.25*(V[i-1][j]+V[i+1][j]+V[i][j-1]+V[i][j+1]
                      +rho[i][j]);
          dV  = abs(V[i][j]-Vav);
          if(error < dV) error = dV; //maximum error
          V[i][j] = Vav;
        }//if( ! isConductor[i][j])
      }//for(int j = 1;j<L-1;j++)
    }//for(  int i = 1;i<L-1;i++)
    cout << icount << " err= " << error << endl ;
    if(error < epsilon) return;
  }//while(icount < 10000)
  cerr << "Warning: laplace did not converge.\n";
}//laplace()
//*********************************************************
void print_results(){
  ofstream  myfile("data");
  myfile.precision(16);
  for(  int i = 0; i < L ; i++){
    for(int j = 0; j < L ; j++){
      myfile << i+1 << " " << j+1 << " " << V[i][j] << endl;
    }
    //print empty line for gnuplot, separate isolines:
    myfile   << " "                               << endl;
  }
  myfile.close();
}//print_results()
```

In the bibliography the algorithm described above is called the Gauss–Seidel method. In this method, the right hand side of equation (7.14) uses the updated values of the potential in the calculation of $V(i,j)$ and $V(i,j)$ is immediately updated. In contrast, the Jacobi method uses the

old values of the potential in the right hand side of (7.14) and the new value computed is stored in order to be used in the next sweep. The Gauss–Seidel method is superior to the Jacobi method as far as speed of convergence is concerned. We can generalize Jacobi's method by defining the residual $R_{i,j}$ of equation (7.14)

$$R_{i,j} = V(i+1,j) + V(i-1,j) + V(i,j+1) + V(i,j-1) - 4V(i,j) + \tilde{\rho}(i,j)\,, \tag{7.15}$$

which vanishes when $V(i,j)$ is a solution of equation (7.14). Then, using $R_{i,j}$, Jacobi's method can be formulated as

$$V^{(n+1)}(i,j) = V^{(n)}(i,j) + \frac{1}{4}R_{i,j}^{(n)}\,, \tag{7.16}$$

where the quantities with index (n) refer to the values of the potential during the n-th sweep. The successive overrelaxation (SOR) method is given by:

$$V^{(n+1)}(i,j) = V^{(n)}(i,j) + \frac{\omega}{4}R_{i,j}^{(n)}\,. \tag{7.17}$$

When $\omega < 1$ we have "underrelaxation" and we obtain slower convergence than the Jacobi method. When $1 < \omega < 2$ we have "overrelaxation" and an appropriate choice of ω can lead to an improvement compared to the Jacobi method. When $\omega > 2$ SOR diverges. Further study of the SOR methods is left as an exercise to the reader.

7.8 Problems

7.1 Reproduce the figures with the electric field lines and equipotential lines shown in section 7.2.

7.2 Take the charge distributions that you used in the previous problems, make all the charges to be positive and remake the figures of the field lines and the equipotential lines. Then repeat by taking half of the charges to be twice in magnitude than the others.

7.3 The program `ELines.cpp` gets stuck when you apply it on a charge distribution of four equal charges located at the vertices of a square. How can you correct this pathology?

7.4 Make the necessary changes to the program in the file `ELines.cpp` so that the number of field lines starting near a charge q is proportional to q.

7.5 Improve the program in `EPotential.cpp` so that the equipotential lines are drawn with a density proportional to the magnitude of the electric field.
Hint:

(a) Write a subroutine that calculates the potential $V(x, y)$ at the point (x, y).

(b) From each point charge draw a line in the radial direction and calculate the potential on points that are at small distance Δl from each other.

(c) Calculate the maximum/minimum value of the potential V_{max}/V_{min} and use them in order to choose the values of the potential on the equipotential lines that you plan to draw. If e.g. you choose to draw 5 equipotential lines, take $\delta V = (V_{max} - V_{min})/4$ and $V_i = V_{min} + i\delta V \;\; i = 0, \ldots, 4$.

(d) Repeat the second step. When the potential at a point takes approximately one of the values V_i chosen in the previous step, draw an equipotential line from that point.

7.6 Compute the electric potential using the program in the file `LaplaceEq.cpp` for

(a) `L= 31, V1=100, V2=100`

(b) `L= 31, V1=100, V2=0`

and construct the corresponding plot for $V(i,j)$.

7.7 Compute the electric potential using the program in the file `LaplaceEq.cpp` for

(a) `V1=100, V2=100`

(b) `V1=100, V2=100`

(c) `V1=100, V2=0`

for `L=31,61,121,241,501` and construct the corresponding plot for $V(i,j)$. Vary `epsilon=0.1, 0.01, 0.001, 0.0001, 0.00001, 0.000001`. What is the dependence of the number of sweeps N on `epsilon`? Make the plot of N(`epsilon`). Put the points and curves of N(`epsilon`) for all values of `L` on the same plot.

7.8 Compute the electrostatic potential of a square conductor when the potential on each side is `V1, V2, V3, V4`. Repeat what you did in the previous problem for

(a) `V1=10, V2=5, V3=10, V4= 5`

(b) `V1=10, V2=0, V3=0, V4= -10`

(c) `V1=10, V2=0, V3=0, V4= 0`

7.9 Compute the electrostatic potential of a system of square conductors where the one is inside the other as shown in figure 7.11. The side of each conductor has `L1, L2` sites respectively and the value of the potential is `V1,V2` respectively. Take `L2= L1/5` and repeat the steps in the previous problem for `V1=10, V2=-10` and `L1= 25, 50, 100, 200`.

7.10 Perform a numerical computation of the capacitance $C = Q/V$ of the system of conductors of the previous problem when $V_1 = V$, $V_2 = -V$. In order to calculate the charge Q, compute the surface charge density σ using the equation

$$\sigma = \frac{E_n}{4\pi}\,,$$

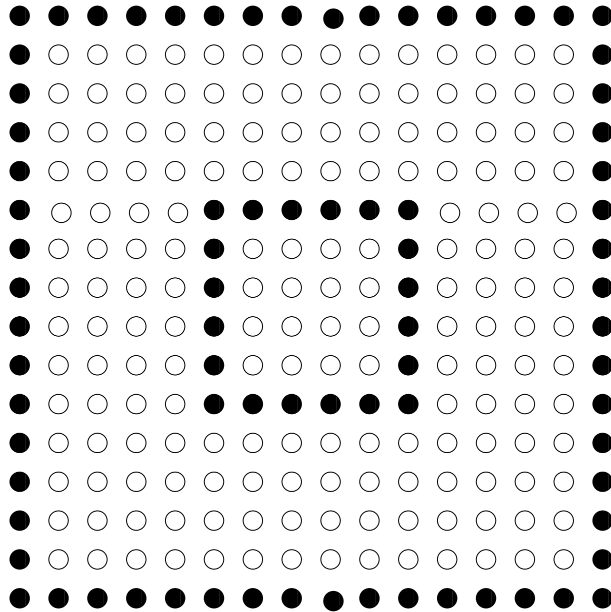

Figure 7.11: The square conductors described in problem 7.9.

where E_n is the perpendicular component of the electric field on the surface. Use the approximation

$$E_n = -\frac{\delta V}{\delta r}\,,$$

where δV is the potential difference between a point on the conductor and its nearest neighbor. By integrating (i.e. summing) you can estimate the total charge on each conductor. If these are opposite and their absolute value is Q, then the capacitance can be calculated from the equation $C = Q/V$. Perform the calculation described above for $V = 10$ and `L1=25, 75`.

7.11 In the system of the previous problem compute the function $Q(V)$. Verify that the capacitance is independent of V. Use `L1=25,50`, `V1= -V2 =1, 2, 5, 10, 15, 20, 25`.

7.12 Reproduce figures 7.8, 7.9 and 7.10. Compare the result of the first case with the known solution of a point charge in empty space.

7.13 Introduce the lattice spacing a in the corresponding equations in the program in the file `PoissonEq.cpp`. Set the length of each side to be $l = 1$ and print the results in the file `data` as $(x_i, y_i, V(x_i, y_i))$ instead of $(i, j, V(i, j))$. Take `L=51,101,151,201,251` and plot $V(x, y)$ in the square $0 < x < 1$, $0 < y < 1$. Study the convergence of the solutions by plotting the section $V(x, 1/2)$ for each `L`.

7.14 Write a program that implements the SOR algorithm given by equation (7.16) for the problem solved in `LaplaceEq.cpp`. Compare the speed of convergence of SOR with that of the Gauss-Seidel method for $L = 51$, $\omega = 1.0,\ 0.9,\ 0.8,\ 0.6,\ 0.4,\ 0.2$. What happens when $\omega > 1$?

7.15 Write a program that implements the SOR algorithm given by equation (7.16) for the problem solved in `PoissonEq.cpp`. Compare the speed of convergence of SOR with that of the Gauss-Seidel method for $L = 51$, $\omega = 1.0,\ 0.9,\ 0.8,\ 0.6,\ 0.4,\ 0.2$. What happens when $\omega > 1$?

Chapter 8

Diffusion Equation

8.1 Introduction

The diffusion equation is related to the study of random walks. Consider a particle moving on a line (one dimension) performing a random walk. The motion is stochastic and the kernel

$$K(x, x_0; t), \tag{8.1}$$

is interpreted as the probability density to observe the particle at position x at time t if the particle is at x_0 at $t = 0$. The equation that determines $K(x, x_0; t)$ is

$$\frac{\partial K(x, x_0; t)}{\partial t} = D \frac{\partial^2 K(x, x_0; t)}{\partial x^2}, \tag{8.2}$$

which is the diffusion equation. The coefficient D depends on the details of the system that is studied. For example, for the Brownian motion of a dust particle in a fluid which moves under the influence of random collisions with the fluid particles, we have that $D = kT/\gamma$, where T is the (absolute) temperature of the fluid, γ is the friction coefficient[1] of the particle in the fluid and k is the Boltzmann constant.

Usually the initial conditions are chosen so that at $t = 0$ the particle is localized at one point x_0, i.e.[2]

$$K(x, x_0; 0) = \delta(x - x_0). \tag{8.3}$$

[1]For a spherical particle of radius R in a Newtonian liquid with viscosity η we have that $\gamma = 6\pi\eta R$.

[2] $\delta(x - x_0)$ is the Dirac delta "function". It can be defined from the requirement

The interpretation of $K(x, x_0; t)$ as a probability density implies that for *every* t we should have that[3]

$$\int_{-\infty}^{+\infty} K(x, x_0; t)\, dx = 1\,. \tag{8.4}$$

It is not obvious that this relation can be imposed for every instant of time. Even if $K(x, x_0; t)$ is normalized so that (8.4) holds for $t = 0$, the time evolution of $K(x, x_0; t)$ is governed by equation (8.2) which can spoil equation (8.4) at later times.

If we impose equation (8.4) at $t = 0$, then it will hold at all times if

$$\frac{d}{dt}\int_{-\infty}^{+\infty} K(x, x_0; t) dx = 0\,. \tag{8.5}$$

By taking into account that $\frac{d}{dt}\int_{-\infty}^{+\infty} K(x, x_0; t)dx = \int_{-\infty}^{+\infty} \frac{\partial K(x,x_0;t)}{\partial t} dx$ and that $\frac{\partial K(x,x_0;t)}{\partial t} = D\frac{\partial^2 K(x,x_0;t)}{\partial x^2}$ we obtain

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{+\infty} & K(x, x_0; t)dx = D\int_{-\infty}^{+\infty} \frac{\partial}{\partial x}\left(\frac{\partial K(x, x_0; t)}{\partial x}\right) dx \\ &= D\left.\frac{\partial K(x, x_0; t)}{\partial x}\right|_{x\to+\infty} - D\left.\frac{\partial K(x, x_0; t)}{\partial x}\right|_{x\to-\infty}\,.\end{aligned} \tag{8.6}$$

The above equation tells us that for functions for which the right hand side vanishes, the normalization condition will be valid for all $t > 0$.

A careful analysis of equation (8.2) gives that the asymptotic behavior of $K(x, x_0; t)$ for small times is

$$K(x, x_0; t) \sim \frac{\mathrm{e}^{-\frac{|x-x_0|^2}{4Dt}}}{t^{d/2}} \sum_{i=0}^{\infty} a_i(x, x_0) t^i\,. \tag{8.7}$$

This relation shows that diffusion is isotropic (the same in all directions) and that the probability of detecting the particle drops exponentially with

---

that for every function $f(x)$ we have that $\int_{-\infty}^{+\infty} f(x)\delta(x - x_0)\, dx = f(x_0)$. Obviously we also have that $\int_{-\infty}^{+\infty} \delta(x - x_0)\, dx = 1$. Intuitively one can think of it as a function that is almost zero everywhere except in an infinitesimal neighborhood of x_0.

[3] Alternatively, if $K(x, x_0; t)$ is interpreted as e.g. the mass density of a drop of ink of mass m_{ink} inside a transparent liquid, we will have that $\int_{-\infty}^{+\infty} K(x, x_0; t)\, dx = m_{ink}$ and $K(x, x_0; 0) = m_{ink}\delta(x - x_0)$.

the distance squared from the initial position of the particle. This relation cannot hold for all times, since for large enough times the probability of detecting the particle will be the same everywhere[4].

The return probability of the particle to its initial position is

$$P_R(t) = K(x_0, x_0; t) \sim \frac{1}{t^{d/2}} \sum_{i=0}^{\infty} a_i(x_0, x_0) t^i \,. \qquad (8.8)$$

The above relation defines the *spectral dimension* d of space. $d = 1$ in our case.

The expectation value of the distance squared of the particle at time t is easily calculated[5]

$$\langle r^2 \rangle = \langle (x - x_0)^2 \rangle(t) = \int_{-\infty}^{+\infty} (x - x_0)^2 K(x, x_0; t)\, dx \sim 2Dt \,. \qquad (8.9)$$

This equation is very important. It tells us that the random walk (Brownian motion) is not a classical motion but it can only be given a stochastic description: A classical particle moving with constant velocity v so that $x - x_0 \sim vt$ results in $r^2 \sim t^2$.

In the following sections we take[6] $D = 1$ and define

$$u(x, t) \equiv K(x - x_0, x_0; t) \,. \qquad (8.10)$$

8.2 Heat Conduction in a Thin Rod

Consider a thin rod of length L and let $T(x, t)$ be the temperature distribution within the rod at time t. The two ends of the rod are kept at constant temperature $T(0, t) = T(L, t) = T_0$. If the initial temperature distribution in the rod is $T(x, 0)$, then the temperature distribution at all times is determined by the diffusion equation

$$\frac{\partial T(x, t)}{\partial t} = \alpha \frac{\partial^2 T(x, t)}{\partial x^2} \,, \qquad (8.11)$$

[4] Remember the analogy of an ink drop diffusing in a transparent liquid. After long enough time, the ink is homogeneously dissolved in the liquid.

[5] $\int_0^\infty dr\, r^n e^{-r^2/4Dt} = 2^n \Gamma(\frac{n+1}{2}) (Dt)^{\frac{n+1}{2}}$.

[6] According to equation (8.2) this amounts to taking $t \to Dt$.

where $\alpha = k/(c_p\rho)$ is the thermal diffusivity, k is the thermal conductivity, ρ is the density and c_p is the specific heat of the rod.

Define

$$u(x,t) = \frac{T(xL, \frac{L^2}{\alpha}t) - T_0}{T_0}\,, \tag{8.12}$$

where $x \in [0,1]$. The function $u(x,t)$, giving the fraction of the temperature difference to the temperature at the ends of the rod, is dimensionless and

$$u(0,t) = u(1,t) = 0\,. \tag{8.13}$$

These are called Dirichlet boundary conditions[7].

Equation (8.11) becomes

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} \tag{8.14}$$

Equation (8.6) becomes

$$\frac{d}{dt}\int_0^1 u(x,t)dx = \left.\frac{\partial u}{\partial x}\right|_{x=1} - \left.\frac{\partial u}{\partial x}\right|_{x=0} \tag{8.15}$$

The relation above cannot be equal to zero at all times due to the boundary conditions (8.13). This can be easily understood with an example. Suppose that

$$u(x,0) = \sin(\pi x)\,, \tag{8.16}$$

then it is easy to confirm that the boundary conditions are satisfied and that the function

$$u(x,t) = \sin(\pi x)\mathrm{e}^{-\pi^2 t}\,, \tag{8.17}$$

is the solution to the diffusion equation. It is easy to see that

$$\int_0^1 u(x,t)dx = \frac{2}{\pi}\mathrm{e}^{-\pi^2 t}$$

drops exponentially with time and that

$$\frac{d}{dt}\int_0^1 u(x,t)dx = -2\pi\mathrm{e}^{-\pi^2 t}\,,$$

[7] If the derivative $\partial u/\partial x$ was given as a boundary condition instead, then we would have Neumann boundary conditions.

which is in agreement with equations (8.15).

The exponential drop of the magnitude of $u(x,t)$ is in agreement with the expectation that the rod will have constant temperature at long times, which will be equal to the temperature at its ends ($\lim_{t\to+\infty} u(x,t)=0$).

8.3 Discretization

The numerical solution of equation (8.14) will be computed in the interval $x\in[0,1]$ for $t\in[0,t_f]$. The problem will be defined on a two dimensional discrete lattice and the differential equation will be approximated by finite difference equations.

The lattice is defined by N_x spatial points $x_i\in[0,1]$

$$x_i = 0 + (i-1)\Delta x \quad i = 1,\ldots,N_x\,, \tag{8.18}$$

where the N_x-1 intervals have the same width

$$\Delta x = \frac{1-0}{N_x-1}\,, \tag{8.19}$$

and by the N_t time points $t_j\in[0,t_f]$

$$t_j = 0 + (j-1)\Delta t \quad j = 1,\ldots,N_t\,, \tag{8.20}$$

where the N_t-1 time intervals have the same duration

$$\Delta t = \frac{t_f-0}{N_t-1}\,. \tag{8.21}$$

We note that the ends of the intervals correspond to

$$x_1 = 0\,,\quad x_{N_x} = 1\,,\quad t_1 = 0\,,\quad t_{N_t} = t_f\,. \tag{8.22}$$

The function $u(x,t)$ is approximated by its values on the $N_x\times N_t$ lattice

$$u_{i,j} \equiv u(x_i,t_j)\,. \tag{8.23}$$

The derivatives are replaced by the finite differences

$$\frac{\partial u(x,t)}{\partial t} \approx \frac{u(x_i,t_j+\Delta t)-u(x_i,t_j)}{\Delta t} \equiv \frac{1}{\Delta t}\left(u_{i,j+1}-u_{i,j}\right)\,, \tag{8.24}$$

$$\begin{aligned}\frac{\partial^2 u(x,t)}{\partial x^2} &\approx \frac{u(x_i+\Delta x,t_j)-2u(x_i,t_j)+u(x_i-\Delta x,t_j)}{(\Delta x)^2} \\ &\equiv \frac{1}{(\Delta x)^2}\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right) . \end{aligned} \quad (8.25)$$

By equating both sides of the above relations according to (8.14), we obtain the dynamic evolution of $u_{i,j}$ in time

$$u_{i,j+1} = u_{i,j} + \frac{\Delta t}{(\Delta x)^2}\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right) . \quad (8.26)$$

This is a one step iterative relation in time. This is very convenient, because one does not need to store the values $u_{i,j}$ for all j in the computer memory.

The second term (the "second derivative") in (8.26) contains only the nearest neighbors $u_{i\pm 1,j}$ of the lattice point $u_{i,j}$ at a given time slice t_j. Therefore it can be used for all $i = 2,\ldots,N_x-1$. The relations (8.26) are not needed for the points $i=1$ and $i=N_x$ since the values $u_{1,j} = u_{N_x,j} = 0$ are kept constant.

The parameter

$$\frac{\Delta t}{(\Delta x)^2} \quad (8.27)$$

determines the time evolution in the algorithm. It is called the Courant parameter and in order to have a time evolution without instabilities it is necessary to have

$$\frac{\Delta t}{(\Delta x)^2} < \frac{1}{2} . \quad (8.28)$$

This condition will be checked in our analysis empirically.

8.4 The Program

The fact that equation (8.26) is a one time step iterative relation, leads to a substantial simplification of the structure of the program. Because of this, at each time step, it is sufficient to store the values of the second term (the "second derivative") in one array. This array will be used in order to update the values of $u_{i,j}$. Therefore we will define only two arrays in order to store the values $u_{i,j}$ and $\Delta t/(\Delta x)^2(u_{i+1,j}-2u_{i,j}+u_{i-1,j})$ at time t_j.

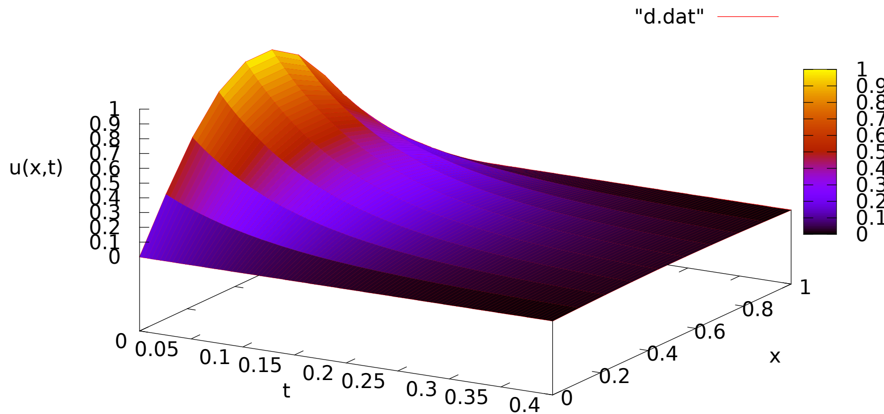

Figure 8.1: The function $u(x,t)$ for `Nx=10, Nt=100, tf= 0.4`.

In the program listed below, the names of these arrays are `u[P]` and `d2udx2[P]`. Some care must be exercised because of the array indexing in C++. The data is stored in the array positions `u[0] ... u[Nx-1]` and `d2udx2[0] ... d2udx2[Nx-1]` and the parameter `P` is taken large enough so that `Nx` is always smaller than `P`.

The user enters the $N_x =$ `Nx`, $N_t =$`Nt` and $t_f =$`tf` interactively. The values of Δx, Δt and $\Delta t/\Delta x^2 =$ `courant` are calculated during the initialization.

On exit, we obtain the results in the file `d.dat` which contains $(t_j, x_i, u_{i,j})$ in three columns. When a time slice is printed, the program prints an empty line so that the output is easily read by the three dimensional plotting function `splot` of `gnuplot`.

The program is in the file `diffusion.cpp` and is listed below:

```
//========================================================
// 1-dimensional Diffusion Equation with simple
// Dirichlet boundary conditions u(0,t)=u(1,t)=0
// 0<= x <= 1 and 0<= t <= tf
//
// We set initial condition u(x,t=0) that satisfies
```

```
// the given boundary conditions.
// Nx is the number of points in spatial lattice:
// x = 0 + i*dx, i=0,...,Nx-1 and dx = (1-0)/(Nx-1)
// Nt is the number of points in temporal lattice:
// t = 0 + j*dt, j=0,...,Nt-1 and dt = (tf-0)/(Nt-1)
//
// u(x,0) = sin(pi*x) tested against analytical solution
// u(x,t) = sin(pi*x)*exp(-pi*pi*t)
//
//=======================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
//-------------------------------------------------------
int main(){
  const int    P  = 100000;
  const double PI = 2.0*atan2(1.0,0.0);
  double u[P], d2udx2[P];
  double t,x,dx,dt,tf,courant;
  int    Nx,Nt,i,j;
  string buf;
  //Input:
  cout  << "# Enter: Nx, Nt, tf: (P= " << P
        << "  Nx must be < P)"         << endl;
  cin   >> Nx >> Nt >> tf;    getline(cin,buf);
  if(Nx >= P){cerr  << "Nx >= P\n";    exit(1);}
  if(Nx <= 3){cerr  << "Nx <= 3\n";    exit(1);}
  if(Nt <= 2){cerr  << "Nx <= 2\n";    exit(1);}
  //Initialize:
  dx     = 1.0/(Nx -1);
  dt     = tf /(Nt -1);
  courant= dt /(dx*dx);
  cout  << "# 1d Diffusion Equation: 0<=x<=1, 0<=t<=tf\n";
  cout  << "# dx= " << dx << " dx= " << dt
        << "  tf= " << tf                   << endl;
  cout  << "# Nx= " << Nx << " Nt= " << Nt << endl;
  cout  << "# Courant Number= " << courant << endl;
  if(courant > 0.5) cout  << "# WARNING: courant > 0.5\n";
  ofstream  myfile("d.dat");
  myfile.precision(17);
  //-----------------------------------------------------
  // Initial condition at t=0
```

```
  // u(x,0) = sin( pi x)
  for(i=0;i<Nx;i++){
    x       = i*dx;
    u[i]    = sin(PI*x);
  }
  u[0    ]  = 0.0;
  u[Nx-1]   = 0.0;
  for(i=0;i<Nx;i++){
    x       = i*dx;
    myfile << 0.0 << " " << x << " " << u[i] << '\n';
  }
  myfile   << " \n";
  //------------------------------------------------------
  // Calculate time evolution:
  for(j=1;j<Nt;j++){
    t = j*dt;
    // Second derivative:
    for(i=1;i<Nx-1;i++)
      d2udx2[i] = courant*(u[i+1]-2.0*u[i]+u[i-1]);
    // Update:
    for(i=1;i<Nx-1;i++)
      u[i]     += d2udx2[i];
    for(i=0;i<Nx;i++){
      x         = i*dx;
      myfile << t << " " << x << " " << u[i] << '\n';
    }
    myfile   << " \n";
  }//for(j=1;j<Nt;j++)
  myfile.close();
}//main()
```

8.5 Results

The compilation and running of the program can be done with the commands:

```
> g++ diffusion.cpp -o d
> echo "10 100 0.4" | ./d
# Enter: Nx, Nt, tf: (P= 100000  Nx must be < P)
# 1d Diffusion Equation: 0<=x<=1, 0<=t<=tf
# dx= 0.111111 dx= 0.0040404  tf= 0.4
# Nx= 10 Nt= 100
```

```
# Courant Number= 0.327273
```

The input to the program `./d` is read from the `stdin` and it is given by the `stdout` of the command `echo` through a pipe, as shown in the second line in the listing above. The lines that follow are the standard output `stdout` of the program.

The three dimensional plot of the function $u(x,t)$ can be made with the `gnuplot` commands:

```
gnuplot> set pm3d
gnuplot> set hidden3d
gnuplot> splot "d.dat" with lines
gnuplot> unset pm3d
```

In order to make the plot of $u(x,t)$ for a fixed value of t we first note that an empty line in the file `d.dat` marks a change in time. The following `awk` program counts the empty lines of `d.dat` and prints only the lines when the number of empty lines that have been encountered so far is equal to 3. The counter `n=0, 1, ..., Nt-1` determines the value of $t_j = t_{n-1}$. We save the results in the file `tj` which can be plotted with `gnuplot`. We repeat as many times as we wish:

```
> awk 'NF<3{n++}n==3  {print}' d.dat > tj
gnuplot> plot "tj" using 2:3 with lines
```

The above task can be completed without creating the intermediate file `tj` by using the `awk` filter within `gnuplot`. For example, the commands

```
gnuplot> ! echo "10 800 2" | ./d
gnuplot> plot   "<awk 'NF<3{n++}n==3  {print}' d.dat" u 2:3 w l
gnuplot> replot "<awk 'NF<3{n++}n==6  {print}' d.dat" u 2:3 w l
gnuplot> replot "<awk 'NF<3{n++}n==10 {print}' d.dat" u 2:3 w l
gnuplot> replot "<awk 'NF<3{n++}n==20 {print}' d.dat" u 2:3 w l
gnuplot> replot "<awk 'NF<3{n++}n==30 {print}' d.dat" u 2:3 w l
gnuplot> replot "<awk 'NF<3{n++}n==50 {print}' d.dat" u 2:3 w l
gnuplot> replot "<awk 'NF<3{n++}n==100{print}' d.dat" u 2:3 w l
```

run the program for `Nx=10, Nt=800, tf= 2` and construct the plot in figure 8.2

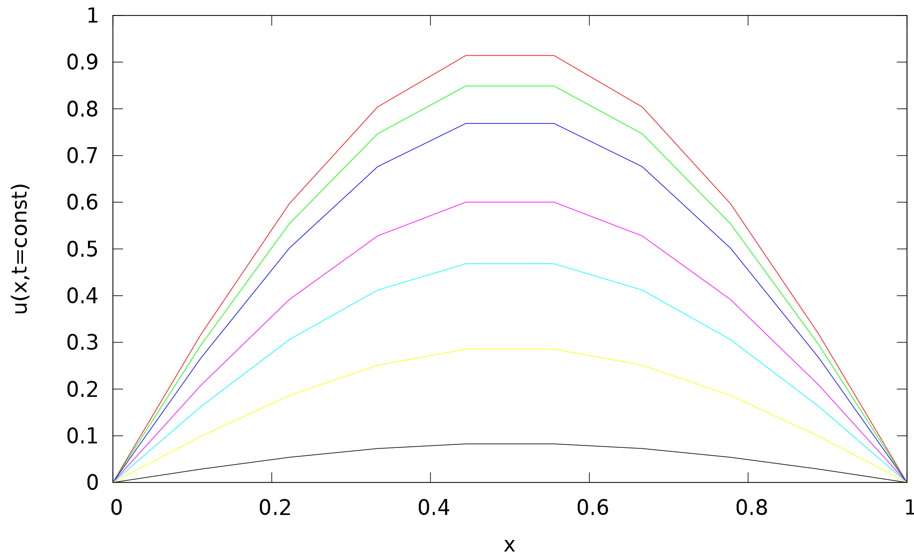

Figure 8.2: The function $u(x,t)$ for `Nx=10, Nt=800, tf= 2` for different values of the time t_j. We take $j = 4, 7, 11, 21, 31, 51, 101$ and observe that the function $u(x,t)$ decreases then j increases.

It is instructive to compare the results with the known solution $u(x,t) = \sin(\pi x)\mathrm{e}^{-\pi^2 t}$. We compute the relative error

$$\frac{u_{i,j} - u(x_i, t_j)}{u_{i,j}},$$

which can be done within `gnuplot` with the commands:

```
gnuplot> du(x,y,z) = (z - sin(pi*x)*exp(-pi*pi*y))/z
gnuplot> plot "<awk 'NF<3{n++}n==2  ' d.dat" u 2:(du($2,$1,$3))
gnuplot> plot "<awk 'NF<3{n++}n==6  ' d.dat" u 2:(du($2,$1,$3))
gnuplot> plot "<awk 'NF<3{n++}n==20 ' d.dat" u 2:(du($2,$1,$3))
gnuplot> plot "<awk 'NF<3{n++}n==200' d.dat" u 2:(du($2,$1,$3))
gnuplot> plot "<awk 'NF<3{n++}n==600' d.dat" u 2:(du($2,$1,$3))
gnuplot> plot "<awk 'NF<3{n++}n==780' d.dat" u 2:(du($2,$1,$3))
```

The results can be seen in figure 8.3.

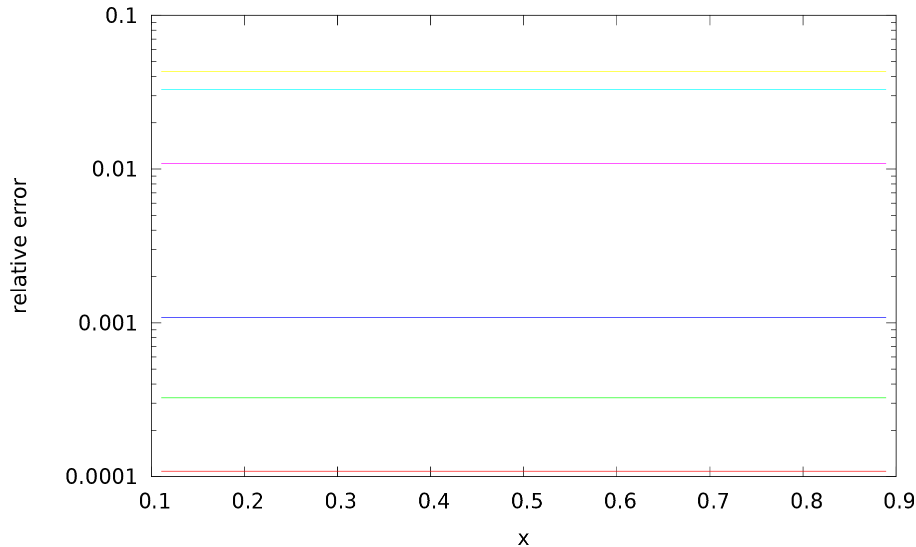

Figure 8.3: The absolute value of the relative error of the numerical computation for `Nx=10, Nt=800, tf= 2` for different times t_j. We take $j = 3, 7, 21, 201, 601, 781$ and observe that the relative error increases with j.

8.6 Diffusion on the Circle

In order to study the kernel $K(x, x_0; t)$ for the diffusion, or random walk, problem, we should impose the normalization condition (8.4) for all times. In the case of the function $u(x, t)$ defined for $x \in [0, 1]$ the relation becomes

$$\int_0^1 u(x,t)\, dx = 1\,. \tag{8.29}$$

In order to maintain this relation at all times, it is necessary that the right hand side of equation (8.15) is equal to 0. One way to impose this condition is to study the diffusion problem on the circle. If we parametrize the circle using the variable $x \in [0, 1]$, then the points $x = 0$ and $x = 1$ are identified and we obtain

$$u(0,t) = u(1,t)\,, \qquad \frac{\partial u(0,t)}{\partial x} = \frac{\partial u(1,t)}{\partial x}\,. \tag{8.30}$$

The second relation in the above equations makes the right hand side of equation (8.15) to vanish. Therefore if $\int_0^1 u(x, 0)\, dx = 1$, we obtain $\int_0^1 u(x, t)\, dx = 1$, $\forall t > 0$.

Using the above assumptions, the discretization of the differential equation is done exactly as in the problem of heat conduction. Instead of keeping the values $u(0,t) = u(1,t) = 0$, we apply equation (8.26) also for the points x_1, x_{N_x}. In order to take into account the cyclic topology we take

$$u_{1,j+1} = u_{1,j} + \frac{\Delta t}{(\Delta x)^2}\left(u_{2,j} - 2u_{1,j} + u_{N_x,j}\right) , \tag{8.31}$$

and

$$u_{N_x,j+1} = u_{i,j} + \frac{\Delta t}{(\Delta x)^2}\left(u_{1,j} - 2u_{N_x,j} + u_{N_x-1,j}\right) , \tag{8.32}$$

since the neighbor to the right of the point x_{N_x} is the point x_1 and the neighbor to the left of the point x_1 is the point x_{N_x}. For the rest of the points $i = 2, \ldots, N_x - 1$ equation (8.26) is applied normally.

The program that implements the problem described above can be found in the file diffusionS1.cpp. At a given time t_j, the boundary conditions (8.30) are enforced in the lines

```
for(i=0;i<Nx;i++){
  nnr = i+1;
  if(nnr > Nx-1) nnr = 0;
  nnl = i-1;
  if(nnl <    0) nnl = Nx-1;
  d2udx2[i] = courant*(u[nnr]-2.0*u[i]+u[nnl]);
}
```

The initial conditions at $t = 0$ are chosen so that the particle is located at $x_{N_x/2}$. For each instant of time we perform measurements in order to verify the equations (8.4) and (8.9) and the fact that $\lim_{t\to+\infty} u(x,t) =$ const.

The variable prob $= \sum_{i=1}^{N_x} u_{i,j}$ and we should check that its value is conserved and is always equal to 1.

The variable r2 $= \sum_{i=1}^{N_x}(x_i - x_{N_x/2})^2 u_{i,j}$ is a discrete estimator of the expectation value of the distance squared from the initial position. For small enough times it should follow the law given by equation (8.9).

These variables are written to the file e.dat together with the values $u_{N_x/2,j}$, $u_{N_x/4,j}$ and $u_{1,j}$. The latter are measured in order to check if for large enough times they obtain the *same* constant value according to the expectation $\lim_{t\to+\infty} u(x,t) =$ const.

The full code is listed below:

```
//========================================================
// 1-dimensional Diffusion Equation with
// periodic boundary conditions u(0,t)=u(1,t)
// 0<= x <= 1 and 0<= t <= tf
//
// We set initial condition u(x,t=0) that satisfies
// the given boundary conditions.
// Nx is the number of points in spatial lattice:
// x = 0 + i*dx, i=0,...,Nx-1 and dx = (1-0)/(Nx-1)
// Nt is the number of points in temporal lattice:
// t = 0 + j*dt, j=0,...,Nt-1 and dt = (tf-0)/(Nt-1)
//
// u(x,0) = \delta_{x,0.5}
//
//========================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
//--------------------------------------------------------
int main(){
  const int    P  = 100000;
  const double PI = 2.0*atan2(1.0,0.0);
  double u[P], d2udx2[P];
  double t,x,dx,dt,tf,courant;
  double prob,r2,x0;
  int    Nx,Nt,i,j,nnl,nnr;
  string buf;
  //Input:
  cout  << "# Enter: Nx, Nt, tf: (P= " << P
        << "  Nx must be < P)"          << endl;
  cin   >> Nx >> Nt >> tf;    getline(cin,buf);
  if(Nx >= P){cerr  << "Nx >= P\n";    exit(1);}
  if(Nx <= 3){cerr  << "Nx <= 3\n";    exit(1);}
  if(Nt <= 2){cerr  << "Nx <= 2\n";    exit(1);}
  //Initialize:
  dx      = 1.0/(Nx -1);
  dt      = tf /(Nt -1);
  courant= dt /(dx*dx);
  cout  << "# 1d Diffusion Equation: 0<=x<=1, 0<=t<=tf\n";
  cout  << "# dx= " << dx << " dx= " << dt
        << "  tf= " << tf                << endl;
  cout  << "# Nx= " << Nx << " Nt= " << Nt << endl;
```

```
cout  << "# Courant Number= " << courant << endl;
if(courant > 0.5) cout  << "# WARNING: courant > 0.5\n";
ofstream  myfile("d.dat");
myfile.precision(17);
ofstream  efile ("e.dat");
efile.precision(17);
//------------------------------------------------------------
// Initial condition at t=0
for(i=0;i<Nx;i++){
  x       = i*dx;
  u[i]    = 0.0;
}
u[Nx/2-1] = 1.0;
for(i=0;i<Nx;i++){
  x      = i*dx;
  myfile << 0.0 << " " << x << " " << u[i] << '\n';
}
myfile   << " \n";
//------------------------------------------------------------
// Calculate time evolution:
for(j=1;j<Nt;j++){
  t = j*dt;
  // Second derivative:
  for(i=0;i<Nx;i++){
    nnr = i+1;
    if(nnr > Nx-1) nnr = 0;
    nnl = i-1;
    if(nnl <    0) nnl = Nx-1;
    d2udx2[i] = courant*(u[nnr]-2.0*u[i]+u[nnl]);
  }
  // Update:
  prob = 0.0;
  r2   = 0.0;
  x0   = ((Nx/2)-1)*dx; //original position
  for(i=0;i<Nx;i++){
    x     = i*dx;
    u[i] += d2udx2[i];
    prob += u[i];
    r2   += u[i]*(x-x0)*(x-x0);
  }
  for(i=0;i<Nx;i++){
    x         = i*dx;
    myfile << t << " " << x << " " << u[i] << '\n';
  }
  myfile   << " \n";
```

```
    efile    << "pu "
             << t          << " " << prob      << " "
             << r2         << " "
             << u[Nx/2-1] << " " << u[Nx/4-1] << " "
             << u[0]       << '\n';
  }//for(j=1;j<Nt;j++)
  myfile.close();
  efile.close ();
}//main()
```

8.7 Analysis

For each moment of time, the program writes the following quantities to the file `e.dat`:

$$U_j = \sum_{i=1}^{N_x} u_{i,j} \tag{8.33}$$

which is an estimator of (8.29) and we expect to obtain $U_j = 1$ for all j,

$$\langle r^2 \rangle_j = \sum_{i=1}^{N_x} u_{i,j}(x_i - x_{N_x/2})^2 \tag{8.34}$$

which is an estimator of (8.9) for which we expect to obtain

$$\langle r^2 \rangle_j \sim 2t_j \,, \tag{8.35}$$

for small times as well as the values of $u_{N_x/2,j}$, $u_{N_x/4,j}$, $u_{1,j}$.

The values of t_j, U_j, $\langle r^2 \rangle_j$, $u_{N_x/2,j}$, $u_{N_x/4,j}$, $u_{1,j}$ are found in columns 2, 3, 4, 5, 6 and 7 respectively of the file `e.dat`. The `gnuplot` commands

```
gnuplot> ! g++ diffusionS1.cpp -o d
gnuplot> ! echo "10 100 0.4" | ./d
```

compile and run the program within `gnuplot`. They set $N_x = 10$, $N_t = 100$, $t_f = 0.4$, $\Delta x \approx 0.111$, $\Delta t \approx 4.0404$, $\Delta t/\Delta x^2 \approx 0.327$. The `gnuplot` commands

```
gnuplot> plot   "e.dat" u 2:5 w l
gnuplot> replot "e.dat" u 2:6 w l
gnuplot> replot "e.dat" u 2:7 w l
```

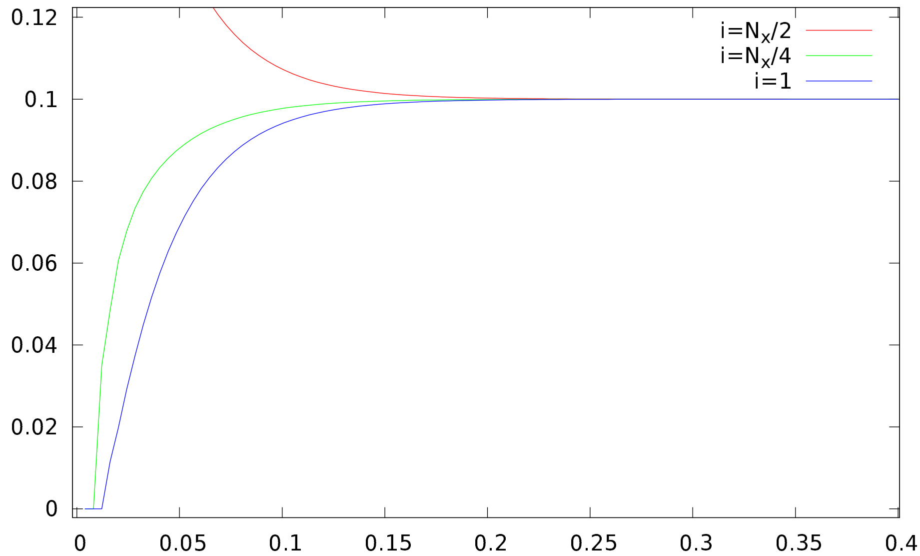

Figure 8.4: The functions $u_{N_x/2,j}$, $u_{N_x/4,j}$, $u_{1,j}$ are given as a function of t_j for $N_x = 10$, $N_t = 100$, $t_f = 0.4$. We observe that for large times they are consistent with uniform diffusion.

construct the plot in figure 8.4. We observe that for large times we obtain uniform diffusion.

The relation $U_j = 1$ can be easily confirmed by inspecting the values recorded in the file `e.dat`.

The asymptotic relation $\langle r^2 \rangle_j \sim 2t_j$ can be confirmed with the commands

```
gnuplot> plot [:][:0.11] "e.dat" u 2:4,2*x
```

which construct the plot in figure 8.5.

Finally we make a plot of the function $u(x,t)$ with the commands

```
gnuplot> ! echo "10 100 0.16" | ./d
gnuplot> set pm3d
gnuplot> splot [0:0.16][0:1][0: 1] "d.dat" w l
gnuplot> splot [0:0.16][0:1][0:.2] "d.dat" w l
```

and the result is shown in figure 8.6.

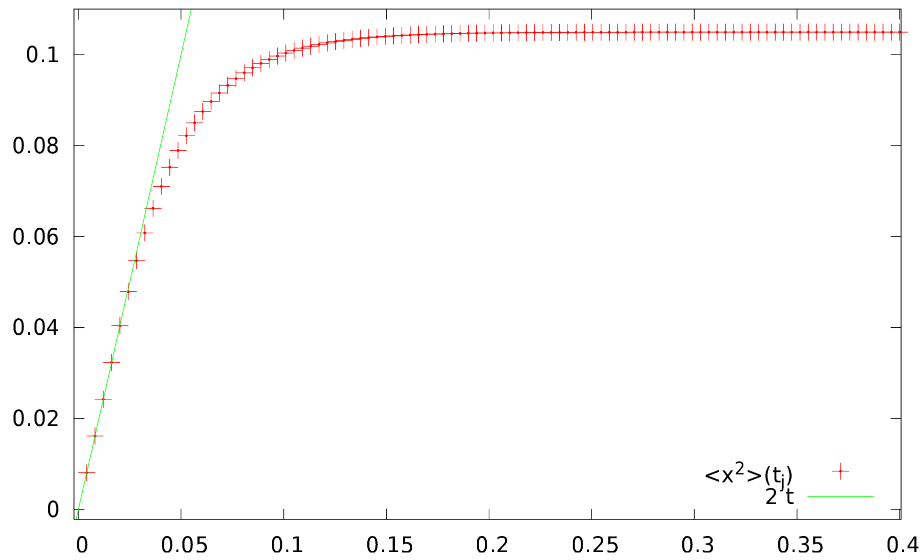

Figure 8.5: The expectation value $\langle r^2 \rangle_j$ as a function of t_j for $N_x = 10$, $N_t = 100$, $t_f = 0.4$. For small values of t_j we obtain $\langle r^2 \rangle_j \approx 2t_j$. The solid line is the straight line $2t$.

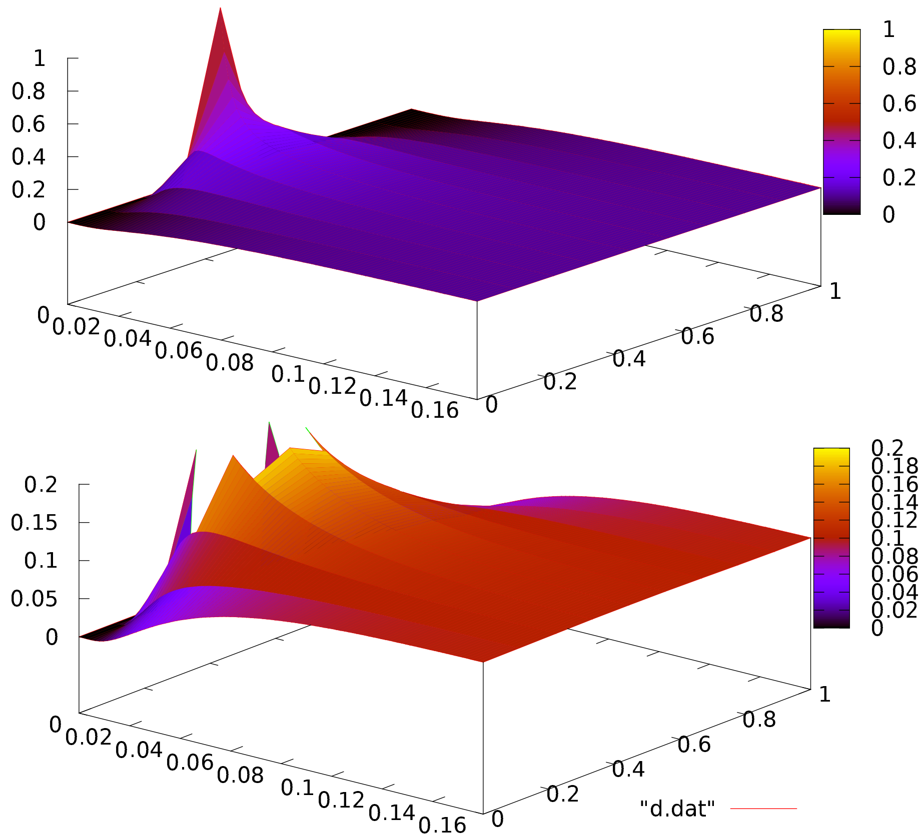

Figure 8.6: The function $u(x,t)$ for $N_x = 10$, $N_t = 100$, $t_f = 0.16$. The second plot differs only in the scale of the z axis so that we can easily see the details of the diffusion away from the point $x_0 \equiv x_{N_x/2} = x_5$.

8.8 Problems

8.1 Reproduce the results in figure 8.3.

8.2 The temperature distribution $u(x,t)$ in a thin rod satisfies equation (8.14) together with the boundary conditions (8.13) at the ends $x = 0, 1$. The initial temperature distribution at $t = 0$ is given by the function

$$u(x,0) = \begin{cases} 0.5 & x \in [x_1, x_2] \\ 0.3 & x \notin [x_1, x_2] \end{cases} ,$$

where $x_1 = 0.25$ and $x_2 = 0.75$.

(a) Calculate the temperature distribution $u(x,t_f)$ for $t_f = 0.0001$, 0.001, 0.01, 0.05. Take $N_x = 100$ and $N_t = 1000$. Do the same for $t_f = 0.1$ by choosing appropriate N_x and keeping $N_t = 1000$. Plot the functions $u(x,t_f)$ in the same plot.

(b) Calculate the maximum value of the temperature graphically for $t_f = 0.0001$, 0.001, 0.01, 0.05, 0.1, 0.15, 0.25. Take $N_x = 100$ and choose an appropriate value for the corresponding N_t.

(c) Calculate the time at which the temperature of the rod becomes everywhere less than 0.1.

Hint: Make your program print only the final temperature distribution $u(x,t_f)$.

8.3 The temperature distribution $u(x,t)$ in a thin rod satisfies the equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} .$$

The temperature at the ends of the rod is $u(0,t) = u(1,t) = 0$, and when $t = 0$

$$u(x,0) = \begin{cases} 0.5 \left[1 - \cos\left(\frac{2\pi x}{b}\right)\right] & 0 \leq x < b \\ 0 & b \leq x \leq 1 \end{cases} .$$

(a) Calculate the temperature distribution $u(x,t_f)$ for $\alpha = 0.5$, $b = 0.09$ and for $t_f = 0.0001$, 0.001, 0.01, by taking $N_x = 300$, $N_t = 1000$. Do the same for $t_f = 0.05$ by choosing appropriate N_x. Plot the functions $u(x,t_f)$ in the same plot.

(b) Using the same parameters, calculate the time evolution of the values of the temperature distribution at the points $x_1 = 0.05$, $x_2 = 0.50$ and $x_3 = 0.95$ for $0 \leq t \leq 0.05$. Plot the functions $u(x_{1,2,3}, t)$ in the same plot.

(c) Calculate the temperature distribution $u(x, t_f)$ for $b = 0.09$ and $\alpha = 5, 2, 1$ for $t_f = 0.001$. Plot the functions $u(x, t_f)$ in the same plot. Comment on the effect of the parameter α on your results.

8.4 The temperature distribution $u(x, t)$ in a thin rod of length L satisfies equation

$$\frac{\partial u}{\partial t} = D(x)\frac{\partial^2 u}{\partial x^2} - \frac{4}{L}D(x)\frac{\partial u}{\partial x}\,,$$

where $D(x) = a\mathrm{e}^{-4x/L}$ is the x-dependent thermal diffusivity. The temperature of the rod at its ends is such that $u(0, t) = u(L, t) = 0$, and at time $t = 0$, the temperature distribution is

$$u(x, 0) = C\mathrm{e}^{-(x-L/2)^2/\sigma^2}\,.$$

(a) Write a program where the user enters the parameters L, a, C, σ, N_x, N_t and t_f interactively. On exit, the program calculates $u(x, t_f)$ and writes the points $(x_i, u(x_i, t_f))$ in two columns to a file `d.dat`.

(b) Run the program for $L = 4$, $a = 0.2$, $C = 1$, $\sigma = 1/2$, $N_x = 400$, $N_t = 20000$ and calculate $u(x, t_f)$ for $t_f = 0.05, 1.0, 5.0$. Plot the functions $u(x, t_f)$ in the same plot.

(c) Using the same parameters, calculate the time evolution of the temperature distribution at the points $x_1 = 1$ and $x_2 = 2$ for $0 \leq t \leq 5$. Plot the functions $u(x_{1,2}, t)$ in the same plot.

8.5 Reproduce the results shown in figures 8.4 and 8.5.

Chapter 9

The Anharmonic Oscillator

In this chapter we will use matrix methods in order to compute the quantum mechanical energy spectrum of the anharmonic oscillator. This problem cannot be solved exactly and one has to resort to perturbative or other approximation methods. We will approach this problem numerically by representing the Hamiltonian H as a real symmetric matrix in an appropriately chosen basis of the Hilbert space $\mathcal{H}$ of quantum mechanical states. The energy spectrum is obtained from the eigenvalues of this matrix and the numerical problem reduces to that of the diagonalization of a real symmetric matrix. Since the Hamiltonian is represented in $\mathcal{H}$ by an infinite size matrix, we have to restrict ourselves to a finite dimensional subspace $\mathcal{H}_N$ of dimension N. In this space the Hamiltonian is represented by an $N \times N$ real symmetric matrix. The eigenvalues of this matrix will be calculated numerically using standard methods and the energy eigenvalues will be obtained in the $N \to \infty$ limit.

For the calculation of the eigenvalues we will use software that is found in the well known library Lapack which contains high quality, freely available, linear algebra software. Part of the goals of this chapter is to show to the reader how to link her programs with software libraries. In order to solve the same problem using Mathematica or Matlab see [42] and [43] respectively.

9.1 Introduction

The Hamiltonian of the harmonic oscillator is given by

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \,. \tag{9.1}$$

Define the position and momentum scales $x_0 = \sqrt{\hbar/(m\omega)}$, $p_0 = \sqrt{\hbar m\omega}$ so that we can express the above equation using dimensionless terms:

$$\frac{H_0}{\hbar\omega} = \frac{1}{2}\left(\frac{p}{p_0}\right)^2 + \frac{1}{2}\left(\frac{x}{x_0}\right)^2 \,. \tag{9.2}$$

If we take the units of energy, distance and momentum to be $\hbar\omega$, x_0 and p_0, then we obtain

$$H_0 = \frac{1}{2}p^2 + \frac{1}{2}x^2 \,, \tag{9.3}$$

where H_0, p and x are now dimensionless. The operator H_0 can be diagonalized with the help of the creation and annihilation operators a and $a^\dagger$, defined by the relations:

$$x = \frac{1}{\sqrt{2}}(a^\dagger + a) \qquad p = \frac{i}{\sqrt{2}}(a^\dagger - a) \,, \tag{9.4}$$

or

$$a = \frac{1}{\sqrt{2}}(x + ip) \qquad a^\dagger = \frac{1}{\sqrt{2}}(x - ip) \,, \tag{9.5}$$

which obey the commutation relation

$$[a, a^\dagger] = 1 \,, \tag{9.6}$$

which leads to

$$H_0 = a^\dagger a + \frac{1}{2} \,. \tag{9.7}$$

The eigenstates $|n\rangle$, $n = 0, 1, 2, \ldots$ of H_0 span the Hilbert space of states $\mathcal{H}$ and satisfy the relations

$$a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad a\,|n\rangle = \sqrt{n}\,|n-1\rangle \qquad a\,|0\rangle = 0 \,, \tag{9.8}$$

therefore

$$a^\dagger\,a\,|n\rangle = n\,|n\rangle \,, \tag{9.9}$$

and

$$H_0\,|n\rangle = E_n\,|n\rangle\,, \qquad E_n = n+\frac{1}{2}\,. \tag{9.10}$$

The position representation of the eigenstates $|n\rangle$ is given by the wave-functions:

$$\psi_n(x) = \langle x|n\rangle = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}\mathrm{e}^{-x^2/2}H_n(x)\,, \tag{9.11}$$

where $H_n(x)$ are the Hermite polynomials.

From equations (9.4) and (9.8) we obtain

$$x_{nm} = \langle n|\,x\,|m\rangle = \frac{1}{\sqrt{2}}\sqrt{m+1}\,\delta_{n,m+1} + \frac{1}{\sqrt{2}}\sqrt{m}\,\delta_{n,m-1} \tag{9.12}$$

$$= \frac{1}{2}\sqrt{n+m+1}\,\delta_{|n-m|,1} \tag{9.13}$$

$$p_{nm} = \langle n|\,p\,|m\rangle = \frac{i}{\sqrt{2}}\sqrt{m+1}\,\delta_{n,m+1} - \frac{i}{\sqrt{2}}\sqrt{m}\,\delta_{n,m-1}\,. \tag{9.14}$$

From the above equations we can easily calculate the Hamiltonian of the *anharmonic* oscillator

$$H(\lambda) = H_0 + \lambda x^4\,. \tag{9.15}$$

The matrix elements of H in this representation are:

$$H_{nm}(\lambda) \equiv \langle n|\,H(\lambda)\,|m\rangle = \langle n|\,H_0\,|m\rangle + \lambda\langle n|\,x^4\,|m\rangle \tag{9.16}$$

$$= (n+\frac{1}{2})\delta_{n,m} + \lambda(x^4)_{nm} \tag{9.17}$$

where $(x^4)_{nm}$ can be calculated from equation (9.12):

$$(x^4)_{nm} = \sum_{i_1,i_2,i_3=0}^{\infty} x_{ni_1}x_{i_1i_2}x_{i_2i_3}x_{i_3m}\,. \tag{9.18}$$

This relation computes the matrix elements of the matrix x^4 from the matrix product of x with itself.

The problem of the calculation of the energy spectrum has now been reduced to the problem of calculating the eigenvalues of the matrix H_{nm}.

9.2 Calculation of the Eigenvalues of $H_{nm}(\lambda)$

We start by choosing the dimension N of the subspace $\mathcal{H}_N$ of the Hilbert space of states $\mathcal{H}$. We will restrict ourselves to states within this subspace and we will use the N dimensional representation matrices of x, H_0 and $H(\lambda)$ in $\mathcal{H}_N$. For example, when $N = 4$ we obtain

$$x = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 1 & 0 \\ 0 & 1 & 0 & \sqrt{\frac{3}{2}} \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 \end{pmatrix} \tag{9.19}$$

$$H_0 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{5}{2} & 0 \\ 0 & 0 & 0 & \frac{7}{2} \end{pmatrix} \tag{9.20}$$

$$H(\lambda) = \begin{pmatrix} \frac{1}{2} + \frac{3\lambda}{4} & 0 & \frac{3\lambda}{\sqrt{2}} & 0 \\ 0 & \frac{3}{2} + \frac{15\lambda}{4} & 0 & 3\sqrt{\frac{3}{2}}\lambda \\ \frac{3\lambda}{\sqrt{2}} & 0 & \frac{5}{2} + \frac{27\lambda}{4} & 0 \\ 0 & 3\sqrt{\frac{3}{2}}\lambda & 0 & \frac{7}{2} + \frac{15\lambda}{4} \end{pmatrix} \tag{9.21}$$

Our goal is to write a program that calculates the eigenvalues $E_n(N, \lambda)$ of the $N \times N$ matrix $H_{nm}(\lambda)$. Instead of reinventing the wheel, we will use ready made routines that calculate eigenvalues and eigenvectors of matrices found in the Lapack library. This library can be found in the high quality numerical software repository `Netlib` and more specifically at `http://www.netlib.org/lapack/`. Documentation can be found at `http://www.netlib.org/lapack/lug/`, but it is also easily accessible online by a Google search or by using the man pages[1]. The programs have been written in the Fortran programming language, therefore the reader should review the discussion in Section 6.1.2.

As inexperienced users we will first look for driver routines that perform a diagonalization process. Since our task is to diagonalize a real

[1]The library can be easily installed in many Linux distributions. For example in Ubuntu or other Debian like systems you may use the command `apt-get install liblapack3 liblapack-doc liblapack-dev`.

symmetric matrix, we pick the subroutine[2] DSYEV (D = double precision, SY = symmetric, EV = eigenvalues with optional eigenvectors). If the documentation of the library is installed in our system, we may use the Linux man pages for accessing it:[3]

```
> man dsyev
```

From this page we learn how to use this subroutine:

```
SUBROUTINE DSYEV( JOBZ, UPLO, N, A, LDA, W, WORK, LWORK, INFO )
 CHARACTER      JOBZ, UPLO
 INTEGER        INFO, LDA, LWORK, N
 DOUBLE         PRECISION A( LDA, * ),W( * ), WORK( * )

ARGUMENTS
 JOBZ  (input) CHARACTER*1
       = 'N':  Compute eigenvalues only;
       = 'V':  Compute eigenvalues and eigenvectors.

 UPLO  (input) CHARACTER*1
       = 'U':  Upper triangle of A is stored;
       = 'L':  Lower triangle of A is stored.

 N     (input) INTEGER
       The order of the matrix A.  N >= 0.

 A     (input/output) DOUBLE PRECISION array, dimension (LDA, N↩
    )
       On  entry,  the symmetric matrix A.  If UPLO = 'U', the
       leading N-by-N upper triangular part of A contains the
       upper triangular part  of the matrix A.  If UPLO = 'L',
       the leading N-by-N lower triangular part of A contains
       the lower triangular part of  the matrix A.  On exit, if
       JOBZ = 'V', then if INFO = 0, A contains
       the orthonormal eigenvectors of the matrix A.  If
       JOBZ = 'N', then on exit the lower triangle (if UPLO='L↩
          ')
       or the upper triangle (if UPLO='U') of A, including the
       diagonal, is destroyed.
 LDA   (input) INTEGER
```

[2]A function of type void in Fortran, is called a subroutine.
[3]A Google search "dsyev" will easily take you to the same page.

```
        The leading dimension of the array A.  LDA >= max(1,N).

W       (output) DOUBLE PRECISION array, dimension (N)
        If INFO = 0, the eigenvalues in ascending order.

WORK    (workspace/output) DOUBLE PRECISION array, dimension
        (LWORK).
        On exit, if INFO = 0, WORK(1) returns the optimal LWORK.

LWORK   (input) INTEGER
        The  length  of  the  array  WORK.  LWORK >= max(1,3*N↩
            −1).
        For optimal efficiency, LWORK >= (NB+2)*N, where NB is
        the  blocksize for DSYTRD returned by ILAENV.

        If  LWORK  = −1, then a workspace query is assumed; the
        routine only calculates the optimal size of  the  WORK
        array,  returns this  value  as the first entry of the
        WORK array, and no error message related to LWORK is
        issued by XERBLA.

INFO    (output) INTEGER
        = 0:  successful exit
        < 0:  if INFO = −i, the i−th argument had an illegal ↩
            value
        > 0:  if INFO = i, the algorithm failed  to  converge;  ↩
            i
        off−diagonal  elements  of an intermediate tridiagonal
        form did not converge to zero.
```

These originally cryptic pages contain all the necessary information and the reader should familiarize herself with its format. For a quick and dirty use of the routine, all we need to know is the types and usage of its arguments. These are classified as “input”, “output” and “working space” variables (some are in more than one classes). Input is the necessary data that the routine needs in order to perform the computation. Output is where the results of the computation are stored. And working space is the memory provided by the user to the routine in order to store intermediate results.

From the information above we learn that the matrix to be diagonalized is `A` which is a rectangular matrix with the number of its rows and columns $\leq N$. The number of rows `LDA` (`LDA`= “leading dimension of A”) can be larger than `N` is which case `DSYEV` will diagonalize the upper

left N×N part of the matrix[4]. In our program we define a large matrix A[LDA][LDA] and diagonalize a smaller submatrix A[N][N]. This way we can study many values of N using the same matrix. The subroutine can be used in two ways:

- If JOBZ='N', it calculates only the eigenvalues of the matrix A[N][N] and stores them in the array W[N], sorted in ascending order. We have to be careful because, upon return, the routine destroys the *lower* (UPLO='U') or *upper* (UPLO='L') triangular part of A. Be careful: the documentation says the opposite, but I hope that you remember from the discussion in Section 6.1.2 that Fortran arrays are transposed from the point of view of C++! Since A is symmetric, only this part is needed by DSYEV. If we need to reuse the matrix A, we have to make a backup copy before the call to DSYEV.

- If JOBZ='V', it calculates both the eigenvalues and the eigenvectors of the matrix A[N][N]. The eigenvalues are stored in the array W[N] as before, whereas the corresponding eigenvectors in the columns of the matrix A[N][N]. The eigenvectors are stored in the *rows* of A[N][N], i.e. the n-th eigenvector corresponding to the eigenvalue λ_n =W[n-1] is v=A[n-1]. The eigenvectors are normalized to unity, i.e. $\sum_{m=0}^{N-1}$v[m]*v[m]$= 1$. The matrix A[N][N] is destroyed after the call to DSYEV and if we need it we have to make a backup copy before the call.

The reader should also familiarize herself with the use of the workspace array WORK. This is memory space given to the routine for all its intermediate calculations. Determining the size of this array needs some care. This is given by LWORK and if performance is an issue the reader should read the documentation carefully for its optimal determination. We will make the simple choice LWORK=3*LDA-1. The variable INFO is used as a flag which informs the user whether the calculation was successful, in which case its value is set to 0. In our case, if INFO takes a non zero value, the program will abort the calculation.

Before using the program in a complicated calculation, it is necessary to test its use in a simple, easily controlled problem. We will familiarize ourselves with the use of DSYEV by writing the following program:

---

[4]The number LDA is necessary because the matrix element A(i,j) is found after i+(LDA-1)*j memory positions from A(1,1).

```
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>

using namespace std;

//----------------------------------------------------------------
const int P     = 100; //P=LDA
const int LWORK = 3*P-1;
double A[P][P], W[P], WORK[LWORK];
//----------------------------------------------------------------
extern "C" void
dsyev_(const char&  JOBZ,const char& UPLO,
       const int &     N,
       double    A[P][P],const int &  LDA,
       double    W   [P],
       double    WORK[P],
       const int & LWORK,      int & INFO);
//----------------------------------------------------------------
int main(){
  int    N;
  int    i,j;
  int    LDA,INFO;
  char   JOBZ,UPLO;
  string buf;
  //Define the **symmetric** matrix to be diagonalized
  //The subroutine uses the upper triangular part
  //(UPLO='U') in the *FORTRAN* column-major mode,
  //therefore in C++, we need to define its *lower*
  //triangular part!
  N = 4; // an N x N matrix
  A[0][0]=-7.7;
  A[1][0]= 2.1;A[1][1]= 8.3;
  A[2][0]=-3.7;A[2][1]=-16.;A[2][2]=-12.;
  A[3][0]= 4.4;A[3][1]= 4.6;A[3][2]=-1.04;A[3][3]=-3.7;

  //We print the matrix A before calling DSYEV since
  //it is destroyed after the call.
  for(i=0;i<N;i++)
    for(j=0;j<=i;j++)
      cout << "A( "  << i+1 << " , " << j+1 << " )="
           << A[i][j] << endl;
  //We ask for eigenvalues AND eigenvectors (JOBZ='V')
```

```
  JOBZ='V'; UPLO='U';
  cout << "COMPUTING WITH DSYEV:"       << endl;
  // LDA: Leading dimension of A = number of rows of
  // full array
  LDA = P;
  dsyev_(JOBZ,UPLO,N,A,LDA,W,WORK,LWORK,INFO);
  cout << "DSYEV: DONE. CHECKING NOW:" << endl;
  if(INFO != 0){cerr << "DSYEV failed. INFO= "
                 << INFO << endl;exit(1);}
  //Print results: W(I) has the eigenvalues:
  cout << "DSYEV: DONE.:"               << endl;
  cout << "EIGENVALUES OF MATRIX:"      << endl;
  cout.precision(17);
  for(i=0;i<N;i++)
    cout << "LAMBDA("<< i+1
         << ")="     << W[i]            << endl;
  //Eigenvectors are in stored in the rows of A:
  cout << "EIGENVECTORS OF MATRIX:"     << endl;
  for(i=0;i<N;i++){
    cout << "EIGENVECTOR "     << i+1
         << " FOR EIGENVALUE " << W[i] << endl;
    for(j=0;j<N;j++)
      cout << "V_"  << i+1     << "("  << j+1
           << ")= " << A[i][j]          << endl;
  }
}// main()
```

The next step is to compile and link the program. In order to link the program to Lapack we have to instruct the *linker* `ld` where to find the libraries Lapack and BLAS[5] and link them to our program. A library contains compiled software in archives of *object* files. The convention for their names in a Unix environment is to start with the string "`lib`" followed by the name of the library and a `.a` or `.so` extension. For example, in our case the files we are interested in have the names `lib`**lapack**`.so` and `lib`**blas**`.so` which can be searched in the file system by the commands:

```
> locate libblas
> locate liblapack
```

[5]The library BLAS contains the basic linear algebra subroutines used by Lapack. In some versions of the library, one has to only link to Lapack ignoring the link BLAS but in some other version, linking to BLAS is necessary.

In order to see the files that they contain we give the commands[6]:

```
> ar -t /usr/lib/libblas.so
> ar -t /usr/lib/liblapack.so
```

In the commands shown above you may have to substitute `/usr/lib` with the path appropriate for your system. If the program is in the file `test.cpp`, the compilation/linking command is:

```
> g++ test.cpp -o test -L/usr/lib -llapack -lblas
```

The option `-L/usr/lib` instructs the linker to look for libraries in the `/usr/lib` directory[7], whereas the options `-llapack -lblas` instructs the linker to look for any *unresolved symbols* (i.e. names of external functions and subroutines not coded in our program) *first* in the library `lib`**lapack**`.a` and *then* in the library `lib`**blas**`.a`.

The command shown above produces the executable file `test` which, when run, produces the result:

```
EIGENVALUES  OF MATRIX:
LAMBDA(1)=-21.411990695806409
LAMBDA(2)=-9.9339436575643028
LAMBDA(3)=-2.5576560809720039
LAMBDA(4)= 18.803590434342716
EIGENVECTORS OF MATRIX:
EIGENVECTOR 1 FOR EIGENVALUE -21.411990695806409
V_1(1)=  -0.19784566233322534
V_1(2)=  -0.4647986784623277
V_1(3)=  -0.85469100929950759
V_1(4)=   0.1196769026094445
EIGENVECTOR 2 FOR EIGENVALUE -9.9339436575643028
V_2(1)=   0.82441241087467854
V_2(2)=  -0.13242939824916203
V_2(3)=  -0.19107651157591365
V_2(4)=  -0.51603914386327754
EIGENVECTOR 3 FOR EIGENVALUE -2.5576560809720039
V_3(1)=   0.50268419698022238
```

[6] If the `.so` files don't exist in your system, try `ar -t /usr/lib/libblas.a` etc.

[7] This is not necessary in our case, since `/usr/lib` is in the *path* that `ld` searches anyway. This option is useful for libraries located in non conventional paths.

```
V_3(2)=  -0.24778437244453691
V_3(3)=   0.13285333709059657
V_3(4)=   0.81747262565942114
EIGENVECTOR 4 FOR EIGENVALUE 18.803590434342716
V_4(1)=   0.16884865648881003
V_4(2)=   0.83965918547426488
V_4(3)=  -0.46405068276047351
V_4(4)=   0.22609632301334964
```

We are now ready to tackle the problem of computing the energy spectrum of the anharmonic oscillator. The main program contains the user interface where the basic parameters for the calculation are read from the `stdin`. The user can specify the dimension `DIM` $\equiv N$ of $\mathcal{H}_N$ and the coupling constant λ. Then the program computes the eigenvalues $E_n(N,\lambda)$ of the $N\times N$ matrix $H_{nm}(\lambda)$, which represents the action of the operator $H(\lambda)$ in the $\{\,|n\rangle\}_{n=0,1,\ldots,N-1}$ representation in $\mathcal{H}_N$. The tasks are allocated to the subroutines `calculate_X4`, `calculate_evs` and `calculate_H`. The subroutine `calculate_X4` calculates the $N\times N$ matrix $(x^4)_{nm}$. First, the matrix x_{nm} is calculated and then $(x^4)_{nm}$ is obtained by computing its fourth power. The matrix $(x^4)_{nm}$ can also be calculated analytically and this is left as an exercise to the reader. The subroutine `calculate_H` calculates the matrix $H_{nm}(\lambda)$ using the result for $(x^4)_{nm}$ given by `calculate_X4`. Finally the eigenvalues are calculated in the subroutine `calculate_evs` by a call to `DSYEV`, which are returned to the main program for printing to the `stdout`. The program is listed below and can be found in the file `anharmonic.cpp`:

```
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>

using namespace std;

//------------------------------------------------------------------
const int P     = 1000; //P=LDA
const int LWORK = 3*P-1;
int       DIM;
double    H[P][P], X[P][P], X4[P][P];
double    E[P], WORK[LWORK];
```

```
double   lambda;
//--------------------------------------------------------
extern "C" void
dsyev_(const char&  JOBZ,const char& UPLO,
       const int &     N,
       double     H[P][P],const int &  LDA,
       double     E   [P],
       double     WORK[P],
       const int & LWORK,       int & INFO);
//--------------------------------------------------------
void calculate_X4 ();
void calculate_evs();
void calculate_H  ();
//--------------------------------------------------------
int main(){
  string buf;

  cout << "# Enter Hilbert Space dimension:\n";
  cin  >> DIM;                           getline(cin,buf);
  cout << "# Enter lambda:\n";
  cin  >> lambda;                        getline(cin,buf);
  cout << "# lambda= " << lambda                  << endl;
  cout << "# ########################################\n";
  cout << "# Energy spectrum of anharmonic oscillator\n";
  cout << "# using matrix methods.\n";
  cout << "# Hilbert Space Dimension DIM = "<<DIM<< endl;
  cout << "# lambda coupling = " << lambda       << endl;
  cout << "# ########################################\n";
  cout << "# Output: DIM lambda E_0 E_1 .... E_{N-1} \n";
  cout << "# ----------------------------------------\n";

  cout.precision(15);
  //Calculate X^4 operator:
  calculate_X4();
  //Calculate eigenvalues:
  calculate_evs();
  cout.precision(17);
  cout << "EV " << DIM << " " << lambda << " ";
  for(int n=0;n<DIM;n++) cout << E[n] << " ";
  cout << endl;
}// main()
//--------------------------------------------------------
void calculate_evs(){
  int INFO;
  const char JOBZ='V',UPLO='U';
```

```
  calculate_H();
  dsyev_(JOBZ,UPLO,DIM,H,P,E,WORK,LWORK,INFO);
  if(INFO != 0){
    cerr << "dsyev failed. INFO= " << INFO << endl;
    exit(1);
  }
  cout << "# **************** EVEC ****************\n";
  for(  int n=0;n<DIM;n++){
    cout   << "# EVEC " << lambda << " ";
    for(int m=0;m<DIM;m++)
      cout << H[n][m]   <<  " ";
    cout   <<              '\n';
  }
}//calculate_evs()
//--------------------------------------------------------------
void calculate_H(){
  double X2[P][P];

  for(  int n =0;n<DIM;n++){
    for(int m =0;m<DIM;m++)
      H[n][m] = lambda*X4[n][m];
    H  [n][n]+= n+0.5;
  }

  cout << "# **************** H    ****************\n";
  for(int n=0;n<DIM;n++){
    cout   << "# HH ";
    for(int m=0;m<DIM;m++)
      cout << H[n][m] << " ";
    cout   << '\n';
  }
  cout << "# **************** H    ****************\n";
}//calculate_H()
//--------------------------------------------------------------
void calculate_X4(){
  double X2[P][P];
  const double isqrt2=1.0/sqrt(2.0);

  for(  int n=0;n<DIM;n++)
    for(int m=0;m<DIM;m++)
      X[n][m]=0.0;

  for(  int n=0;n<DIM;n++){
    int m=n-1;
```

```
    if(m>=0) X[n][m] = isqrt2*sqrt(double(m+1));
    m     =n+1;
    if(m<DIM)X[n][m] = isqrt2*sqrt(double(m   ));
  }
  // X2 = X  . X
  for(    int n=0;n<DIM;n++)
    for(  int m=0;m<DIM;m++){
      X2  [n][m]  = 0.0;
      for(int k=0;k<DIM;k++)
        X2[n][m] += X [n][k]*X [k][m];
    }
  // X4 = X2 . X2
  for(    int n=0;n<DIM;n++)
    for(  int m=0;m<DIM;m++){
      X4  [n][m]  = 0.0;
      for(int k=0;k<DIM;k++)
        X4[n][m] += X2[n][k]*X2[k][m];
    }
}//calculate_X4()
```

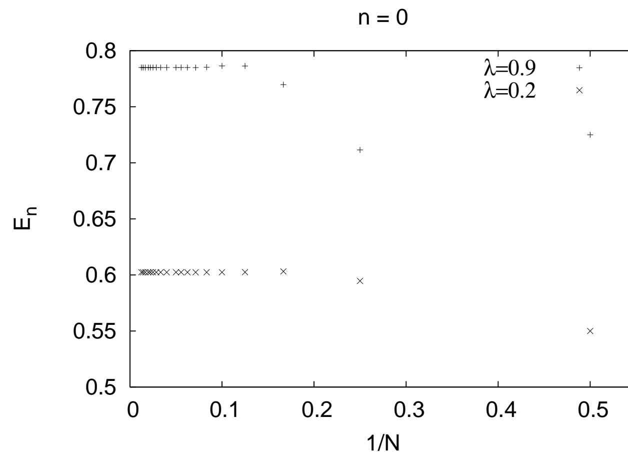

Figure 9.1: The ground state energy $E_0(\lambda)$ for $\lambda = 0.2, 0.9$ is calculated in the large N limit of the eigenvalues $E_0(N, \lambda)$. Convergence is satisfactory for relatively small values of N and it is slightly faster for $\lambda = 0.2$ than it is for $\lambda = 0.9$.

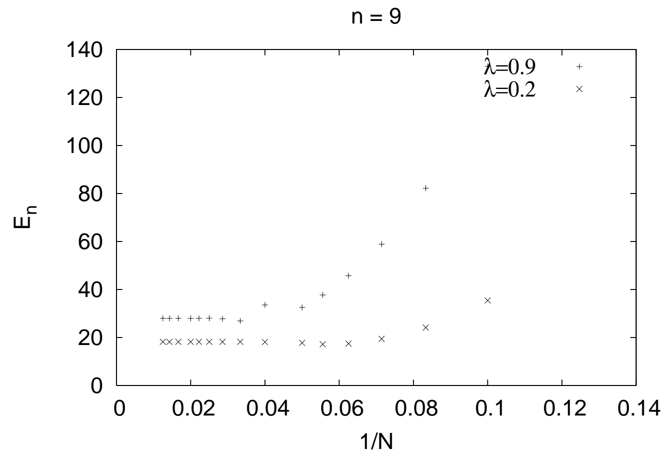

Figure 9.2: The 9th excited state $E_9(\lambda)$ for $\lambda = 0.2, 0.9$ is given by the large N limit of the eigenvalues $E_9(N, \lambda)$.

9.3 Results

Compiling and running the program can be done with the commands:

```
> g++ -O2 anharmonic.cpp -o an -llapack -lblas
> ./an
# Enter Hilbert Space dimension:
4
# Enter lambda:
0.0
.....
# ***************** H    *****************
# HH 0.5  0  0  0
# HH 0  1.5  0  0
# HH 0  0  2.5  0
# HH 0  0  0  3.5
# ***************** H    *****************
# ***************** EVEC *****************
# EVEC 0  1  0  0  0
# EVEC 0  0  1  0  0
# EVEC 0  0  0  1  0
# EVEC 0  0  0  0  1
EV 4 0 0.5 1.5 2.5 3.5
```

In the above program we used $N = 4$ and $\lambda = 0$. The $\lambda = 0$ choice leads

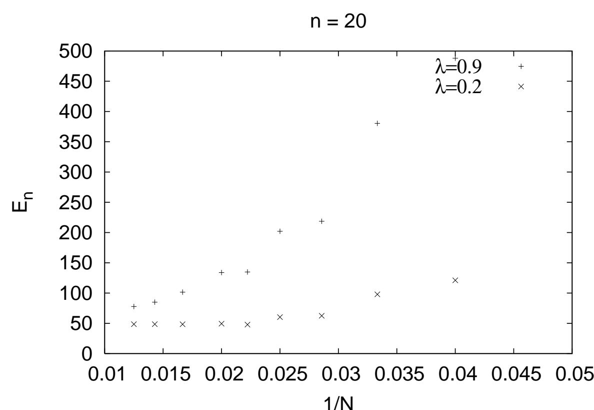

Figure 9.3: The 20th excited state $E_{20}(\lambda)$ for $\lambda = 0.2, 0.9$ is given by the large N limit of the eigenvalues $E_{20}(N, \lambda)$. Convergence has not been achieved for the displayed values of $N \leq 80$.

us to the simple harmonic oscillator and we obtain the expected solutions: $H_{nm} = (n + 1/2)\delta_{n,m}$, $E_n = (n + 1/2)$ and the eigenstates (eigenvectors of H_{nm}) $|n\rangle_{\lambda=0} = |n\rangle = \sum_{m=0}^{3} \delta_{n,m} |m\rangle$. Similar results can be obtained for larger N.

For non zero values of λ, the finite N calculation contains systematic errors from neglecting all the matrix elements $H_{nm}(\lambda)$ for $n \geq N$ or $m \geq N$. Our program calculates the eigenvalues $E_n(N, \lambda)$ of the finite matrix $H_{nm}(\lambda)$, $m, n = 0, \ldots, N-1$ and one expects that

$$E_n(\lambda) = \lim_{N\to\infty} E_n(N, \lambda) \,, \tag{9.22}$$

where

$$H(\lambda) |n\rangle_\lambda = E_n(\lambda) |n\rangle_\lambda \,, \tag{9.23}$$

is the true n-th level eigenvalue of the Hamiltonian $H(\lambda)$. In practice the limit 9.22 for given λ and n is calculated by computing $E_n(N, \lambda)$ numerically for increasing values of N. If convergence to a desired level of accuracy is achieved for the accessible values of N, then the approached limit is taken as an approximation to $E_n(\lambda)$. This process is shown graphically in figures 9.1-9.3 for $\lambda = 0.2, 0.9$. Convergence is satisfactory for quite small N for $n = 0, 9$ but larger values of N are needed for $n = 20$. Increasing the value of n for fixed λ makes the use of larger values of N necessary. Similarly for a given energy level n, increasing λ also makes

the use of larger values of N necessary. A session that computes this limit for the ground level energy $E_0(\lambda = 0.9)$ is shown below[8]:

```
> tcsh
> g++ -O2 anharmonic.cpp -llapack -lblas -o an
> foreach N (4 8 12 16 24 32)
foreach? (echo $N;echo 0.9) |./an >> data
foreach? end
> grep ^EV data | awk '{print $2,$4}'
4  0.711467845686790
8  0.786328966767866
12 0.785237674919165
16 0.784964461939594
24 0.785032515135677
32 0.785031492177730
> gnuplot
gnuplot> plot "<grep ^EV data | awk '{print 1/$2,$4}'"
```

Further automation of this process can be found in the shell script file `anharmonic.csh` in the accompanying software. We note the large N convergence of $E_0(N, 0.9)$ and that we can take $E_0(0.9) \approx 0.78503$. For higher accuracy, a computation using larger N will be necessary.

We can also compute the expectation values $\langle A\rangle_n(\lambda)$ of an operator $A = A(p, q)$ when the anharmonic oscillator is in a state $|n\rangle_\lambda$:

$$\langle A\rangle_n(\lambda) = {}_\lambda\langle n|\, A\, |n\rangle_\lambda \,. \tag{9.24}$$

In practice, the expectation value will be computed from the limit

$$\langle A\rangle_n(\lambda) = \lim_{N\to\infty} \langle A\rangle_n(N, \lambda) \equiv \lim_{N\to\infty} {}_{N,\lambda}\langle n|\, A\, |n\rangle_{N,\lambda} \,, \tag{9.25}$$

where $|n\rangle_{N,\lambda}$ are the eigenvectors of the finite $N \times N$ matrix $H_{nm}(\lambda)$ computed numerically by `DSYEV`. These are determined by their components $c_m(N, \lambda)$, where

$$|n\rangle_{N,\lambda} = \sum_{m=0}^{N-1} c_m(N, \lambda)\, |m\rangle \,, \tag{9.26}$$

which are stored in the columns of the array `H` after the call to `DSYEV`:

$$c_m(N, \lambda) = \mathtt{H}[n][m] \,. \tag{9.27}$$

[8]The `foreach` loop construct is special to the `tcsh` shell. This is why an explicit `tcsh` command is shown. For other shells use their corresponding syntax.

Substituting equation (9.26) to (9.24) we obtain

$$\langle A\rangle_n(\lambda) = \sum_{m,m'=0}^{N-1} c^*_m(N,\lambda) c_{m'}(N,\lambda) A_{mm'}\,, \tag{9.28}$$

and we can use (9.27) for the computation of the sum.

As an application, consider the expectation values of the operators x^2, x^4 and p^2. Taking into account that $\langle x\rangle_n = \langle p\rangle_n = 0$, we obtain the uncertainties $\Delta x_n \equiv \sqrt{\langle x^2\rangle_n - \langle x\rangle_n^2} = \sqrt{\langle x^2\rangle_n}$ and $\Delta p_n = \sqrt{\langle p^2\rangle_n}$. Their product should satisfy Heisenberg's uncertainty relation $\Delta x_n \cdot \Delta p_n \gtrsim 1/2$. The results are shown in table 9.1 and in figures 9.4-9.5. The calculation is left as an exercise to the reader.

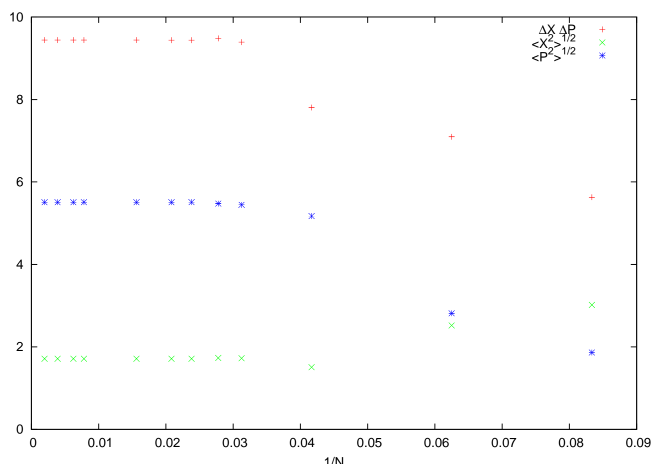

Figure 9.4: The expectation values $\langle x^2\rangle_n^{1/2}(\lambda)$, $\langle p^2\rangle_n^{1/2}(\lambda)$ and the product of uncertainties $\Delta x_n \cdot \Delta p_n$ for $n = 9$ and $\lambda = 0.5$ calculated from the large N limits of $\langle x^2\rangle_n^{1/2}(N,\lambda)$, $\langle p^2\rangle_n^{1/2}(N,\lambda)$.

The physics of the anharmonic oscillator can be better understood by studying the large λ limit. As shown in figure 9.5, the term λx^4 dominates in this limit and the expectation value $\langle x^2\rangle_n(\lambda)$ decreases. This means that states that confine the oscillator to a smaller range of x are favored. This, using the uncertainty principle, implies that the typical momentum of the oscillator also increases in magnitude. This is confirmed in figure 9.5 where we observe the expectation value $\langle p^2\rangle_n(\lambda)$ to increase with λ. In order to understand quantitatively these competing effects we will use

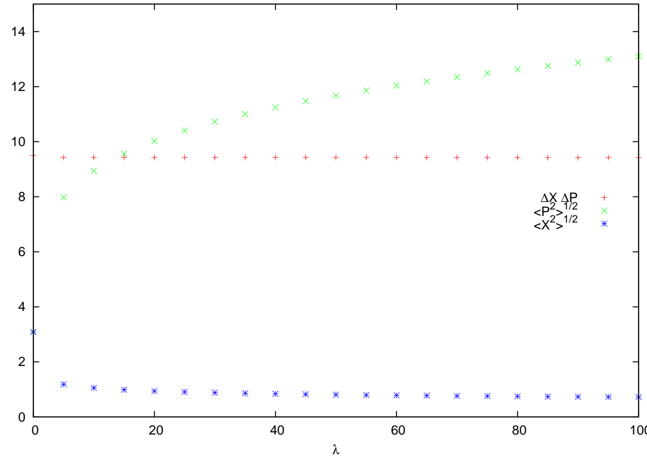

Figure 9.5: The expectation values $\langle x^2\rangle_n^{1/2}(\lambda)$, $\langle p^2\rangle_n^{1/2}(\lambda)$ and the product of uncertainties $\Delta x_n \cdot \Delta p_n$ for $n = 9$.

a scaling argument due to Symanzik. We redefine $x \to \lambda^{-1/6}x$, $p \to \lambda^{1/6}p$ in the Hamiltonian $H(\lambda) = p^2/2 + x^2/2 + \lambda x^4$ and for large enough λ we obtain[9] the asymptotic behavior

$$H(\lambda) \sim \lambda^{1/3}\, h(1)\,, \qquad \lambda \to \infty\,, \tag{9.29}$$

where $h(\lambda) = p^2/2 + \lambda x^4$ is the Hamiltonian of the anharmonic "oscillator" with $\omega = 0$. Since the operator $h(1)$ is independent of λ, the energy spectrum will have the asymptotic behavior

$$E_n(\lambda) \sim C_n \lambda^{1/3}\,, \qquad \lambda \to \infty\,. \tag{9.30}$$

In reference [44] it is shown that for $\lambda > 100$ we have that

$$E_0(\lambda) = \lambda^{1/3}\left(0.667\,986\,259\,18 + 0.143\,67\lambda^{-2/3} - 0.0088\lambda^{-4/3} + \ldots\right)\,, \tag{9.31}$$

with an accuracy better than one part in 10^6. For large values of n, the authors obtain the asymptotic behavior

$$E_n(\lambda) \sim C\lambda^{1/3}\left(n + \frac{1}{2}\right)^{4/3}\,, \qquad \lambda \to \infty\,, n \to \infty\,, \tag{9.32}$$

[9] For $x \to \lambda^{-1/6}x$, $H \to \lambda^{1/3}(p^2/2 + \lambda^{-2/3}x^2/2 + x^4)$, therefore in the limit $\lambda \to \infty$ the second term vanishes and we obtain equation (9.29).

	$\lambda = 0.5$			$\lambda = 2.0$		
n	$\langle x^2\rangle$	$\langle p^2\rangle$	$\Delta x \cdot \Delta p$	$\langle x^2\rangle$	$\langle p^2\rangle$	$\Delta x \cdot \Delta p$
0	0.305814	0.826297	0.502686	0.21223	1.19801	0.504236
1	0.801251	2.83212	1.5064	0.540792	4.21023	1.50893
2	1.15544	5.38489	2.49438	0.761156	8.15146	2.49089
3	1.46752	8.28203	3.48627	0.958233	12.6504	3.48166
4	1.75094	11.4547	4.47845	1.13698	17.596	4.47285
5	2.01407	14.8603	5.47079	1.30291	22.9179	5.46443
6	2.2617	18.4697	6.4632	1.45905	28.5683	6.45619
7	2.49696	22.2616	7.45562	1.60735	34.5124	7.44805
8	2.72198	26.2196	8.44804	1.74919	40.7234	8.43998
9	2.93836	30.3306	9.44045	1.88558	47.1801	9.43194

Table 9.1: The expectation values $\langle x^2\rangle$, $\langle p^2\rangle$, $\Delta x \cdot \Delta p$ for the anharmonic oscillator for the states $|n\rangle$, $n = 0, \ldots, 9$. We observe a decrease of $\Delta x = \sqrt{\langle x^2\rangle}$ and an increase of $\Delta p = \sqrt{\langle p^2\rangle}$ as λ is increased. The product $\Delta x \cdot \Delta p$ seems to remain very close to the values $(n + 1/2)$ of the harmonic oscillator for both values of λ.

where $C = 3^{4/3}\pi^2/\Gamma(1/4)^{8/3} \approx 1.376\,507\,40$. This relation is tested in figure 9.6 where we observe good agreement with our calculations.

9.4 The Double Well Potential

We can also use matrix methods in order to calculate the energy spectrum of a particle in a double well potential given by the Hamiltonian:

$$H = \frac{p^2}{2} - \frac{x^2}{2} + \lambda\frac{x^4}{4}\,. \tag{9.33}$$

The equilibrium points of the classical motion are located at the minima:

$$x_0 = \pm\frac{1}{\sqrt{\lambda}}, V_{min} = -\frac{1}{4\lambda}\,. \tag{9.34}$$

When the well is very deep, then for the lowest energy levels the potential can be well approximated by that of a harmonic oscillator with angular frequency $\omega^2 = V''(x_0)$, therefore

$$E_{min} \approx V_{min} + \frac{1}{2}\omega\,. \tag{9.35}$$

Table 9.2: Numerical calculation of the energy levels of the anharmonic oscillator given in reference [44].

λ	E_0	E_1	E_2	E_3	E_4
0.002	0.501 489 66	1.507 419 39	2.519 202 12	3.536 744 13	4.559 955 56
0.006	0.504 409 71	1.521 805 65	2.555 972 30	3.606 186 33	4.671 800 37
0.01	0.507 256 20	1.535 648 28	2.590 845 80	3.671 094 94	4.774 913 12
0.05	0.532 642 75	1.653 436 01	2.873 979 63	4.176 338 91	5.549 297 81
0.1	0.559 146 33	1.769 502 64	3.138 624 31	4.628 882 81	6.220 300 90
0.3	0.637 991 78	2.094 641 99	3.844 782 65	5.796 573 63	7.911 752 73
0.5	0.696 175 82	2.324 406 35	4.327 524 98	6.578 401 95	9.028 778 72
0.7	0.743 903 50	2.509 228 10	4.710 328 10	7.193 265 28	9.902 610 70
1	0.803 770 65	2.737 892 27	5.179 291 69	7.942 403 99	10.963 5831
2	0.951 568 47	3.292 867 82	6.303 880 57	9.727 323 19	13.481 2759
50	2.499 708 77	8.915 096 36	17.436 9921	27.192 6458	37.938 5022
200	3.930 931 34	14.059 2268	27.551 4347	43.005 2709	60.033 9933
1000	3.694 220 85	23.972 2061	47.017 3387	73.419 1140	102.516 157
8000	13.366 9076	47.890 7687	93.960 6046	146.745 512	204.922 711
20000	18.137 2291	64.986 6757	127.508 839	199.145 124	278.100 238
λ	E_5	E_6	E_7	E_8	
0.002	5.588 750 05	6.623 044 60	7.662 759 33	8.707 817 30	
0.006	5.752 230 87	6.846 948 47	7.955 470 29	9.077 353 66	
0.01	5.901 026 67	7.048 326 88	8.215 837 81	9.402 692 31	
0.05	6.984 963 10	8.477 397 34	10.021 9318	11.614 7761	
0.1	7.899 767 23	9.657 839 99	11.487 3156	13.378 9698	
0.3	10.166 4889	12.544 2587	15.032 7713	17.622 4482	
0.5	11.648 7207	14.417 6692	17.320 4242	20.345 1931	
0.7	12.803 9297	15.873 6836	19.094 5183	22.452 9996	
1	14.203 1394	17.634 0492	21.236 4362	24.994 9457	
2	17.514 1324	21.790 9564	26.286 1250	30.979 8830	
50	49.516 4187	61.820 3488	74.772 8290	88.314 3280	
200	78.385 6232	97.891 3315	118.427 830	139.900 400	
1000	133.876 891	167.212 258	202.311 200	239.011 580	
8000	267.628 498	334.284 478	404.468 350	477.855 700	
20000	363.201 843	453.664 875	548.916 140	648.515 330	

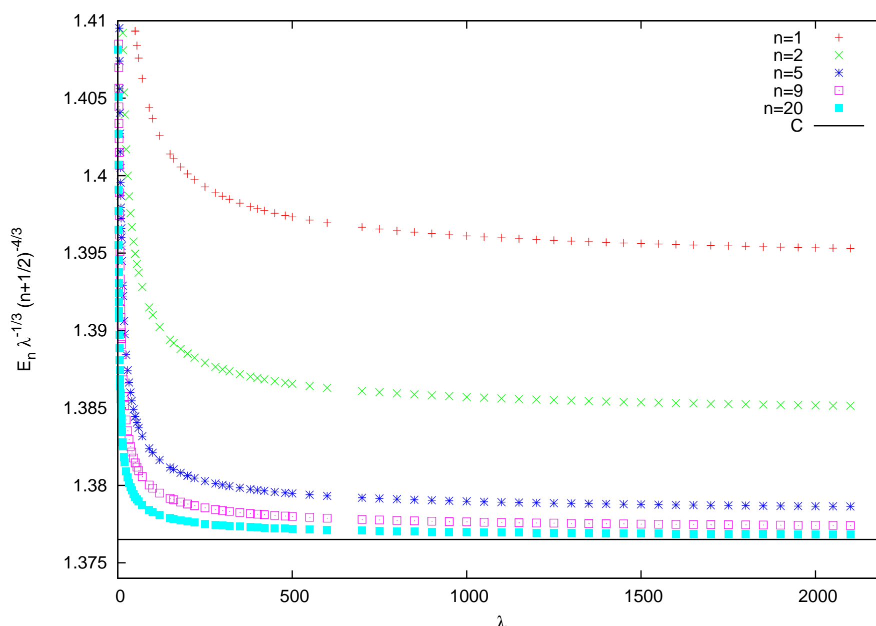

Figure 9.6: Test of the asymptotic relation (9.32). The vertical axis is $E_n\lambda^{-1/3}(n+1/2)^{-4/3}$ where for large enough n and λ should approach the value $C = 3^{4/3}\pi^2/\Gamma(1/4)^{8/3} \approx 1.376\,507\,40$ (horizontal line).

In this case the tunneling effect is very weak and the energy levels are arranged in almost degenerate pairs. The corresponding eigenstates are symmetric and antisymmetric linear combinations of states localized near the left and right minima of the potential. For example, for the two lowest energy levels we expect that

$$E_{0,1} \approx E_{min} \pm \frac{\Delta}{2}\,, \tag{9.36}$$

where $\Delta \ll |E_{min}|$ and

$$|0\rangle_\lambda \approx \frac{|+\rangle + |-\rangle}{\sqrt{2}}\,, \qquad |1\rangle_\lambda \approx \frac{|+\rangle - |-\rangle}{\sqrt{2}}\,, \tag{9.37}$$

where the states $|+\rangle$ and $|-\rangle$ are localized to the left and right well of the potential respectively (see also figure 10.4 of chapter 10).

We will use equations (9.12) in order to calculate the Hamiltonian (9.33). We need to make very small modifications to the code in the file

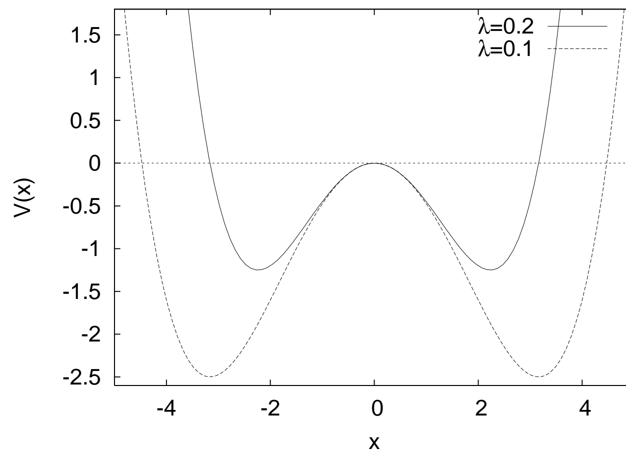

Figure 9.7: The potential energy $V(x)$ for the double well potential for $\lambda = 0.1, 0.2$.

anharmonic.cpp. We will only add a routine that calculates the matrices p_{nm}. The resulting program can be found in the file doublewell.cpp:

```
//========================================================
// H       : Hamiltonian operator H0+(lambda/4)*X^4
// H0      : Hamiltonian H0=1/2 P^2-1/2 X^2
// X,X2,X4: Position operator and its powers
// iP      : i P operator
// P2      : P^2 = -(iP)(iP) operator
// E       : Energy eigenvalues
// WORK    : Workspace for lapack routine DSYEV
//========================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>

using namespace std;

//--------------------------------------------------------
const int P     = 1000; //P=LDA
const int LWORK = 3*P-1;
int       DIM;
double    H [P][P], H0[P][P];
double    X [P][P], X2[P][P], X4[P][P];
```

```
double   iP[P][P], P2[P][P];
double   E[P], WORK[LWORK];
double   lambda;
//--------------------------------------------------------
extern "C" void
dsyev_(const char&  JOBZ,const char& UPLO,
       const int &     N,
       double    H[P][P],const int &  LDA,
       double    E   [P],
       double    WORK[P],
       const int & LWORK,      int & INFO);
//--------------------------------------------------------
void calculate_ops();
void calculate_evs();
void calculate_H  ();
//--------------------------------------------------------
int main(){
  string buf;

  cout << "# Enter Hilbert Space dimension:\n";
  cin  >> DIM;                          getline(cin,buf);
  cout << "# Enter lambda:\n";
  cin  >> lambda;                       getline(cin,buf);
  cout << "# lambda= " << lambda                  << endl;
  cout << "# ########################################\n";
  cout << "# Energy levels of double well potential  \n";
  cout << "# using matrix methods.\n";
  cout << "# Hilbert Space Dimension DIM = "<<DIM<< endl;
  cout << "# lambda coupling = " << lambda       << endl;
  cout << "# ########################################\n";
  cout << "# Output: DIM lambda E_0 E_1 .... E_{N-1} \n";
  cout << "# ----------------------------------------\n";

  cout.precision(15);
  //Calculate operators:
  calculate_ops();
  //Calculate eigenvalues:
  calculate_evs();
  cout.precision(17);
  cout << "EV " << DIM << " " << lambda << " ";
  for(int n=0;n<DIM;n++) cout << E[n]  << " ";
  cout << endl;
}// main()
//--------------------------------------------------------
void calculate_evs(){
```

```
  int INFO;
  const char JOBZ='V',UPLO='U';

  calculate_H();
  dsyev_(JOBZ,UPLO,DIM,H,P,E,WORK,LWORK,INFO);
  if(INFO != 0){
    cerr << "dsyev failed. INFO= " << INFO << endl;
    exit(1);
  }
  cout << "# ***************** EVEC *****************\n";
  for( int n=0;n<DIM;n++){
    cout  << "# EVEC " << lambda << " ";
    for(int m=0;m<DIM;m++)
      cout << H[n][m]  <<  " ";
    cout  <<              '\n';
  }
}//calculate_evs()
//----------------------------------------------------------
void calculate_H(){
  double X2[P][P];

  for( int n =0;n<DIM;n++)
    for(int m =0;m<DIM;m++)
      H[n][m] = H0[n][m] + 0.25*lambda*X4[n][m];

  cout << "# ***************** H    *****************\n";
  for(int n=0;n<DIM;n++){
    cout  << "# HH ";
    for(int m=0;m<DIM;m++)
      cout << H[n][m] << " ";
    cout  << '\n';
  }
  cout << "# ***************** H    *****************\n";
}//calculate_H()
//----------------------------------------------------------
void calculate_ops(){
  double X2[P][P];
  const double isqrt2=1.0/sqrt(2.0);

  for( int n=0;n<DIM;n++)
    for(int m=0;m<DIM;m++){
      X[n][m]=0.0;iP[n][m]=0.0;
    }

  for( int n=0;n<DIM;n++){
```

```
    int m=n-1;
    if(m>=0 )X [n][m] =  isqrt2*sqrt(double(m+1));
    if(m>=0 )iP[n][m] = -isqrt2*sqrt(double(m+1));
    m    =n+1;
    if(m<DIM)X [n][m] = isqrt2*sqrt(double(m  ));
    if(m<DIM)iP[n][m] = isqrt2*sqrt(double(m  ));
  }
  // X2 = X  . X
  for(    int n=0;n<DIM;n++)
    for(  int m=0;m<DIM;m++){
      X2  [n][m]  = 0.0;
      for(int k=0;k<DIM;k++)
        X2[n][m] += X [n][k]*X [k][m];
    }
  // X4 = X2 . X2
  for(    int n=0;n<DIM;n++)
    for(  int m=0;m<DIM;m++){
      X4  [n][m]  = 0.0;
      for(int k=0;k<DIM;k++)
        X4[n][m] += X2[n][k]*X2[k][m];
    }
  // P2 =-iP  . iP
  for(    int n=0;n<DIM;n++)
    for(  int m=0;m<DIM;m++){
      P2  [n][m]  = 0.0;
      for(int k=0;k<DIM;k++)
        P2[n][m] -= iP[n][k]*iP[k][m];
    }
  // Hamiltonian:
  for(    int n=0;n<DIM;n++)
    for(  int m=0;m<DIM;m++)
      H0[n][m] = 0.5*(P2[n][m]-X2[n][m]);

}//calculate_ops()
```

Where is the particle's favorite place when it is in the states $|+\rangle$ and $|-\rangle$? The answer to this question is obtained from the study of the expectation value of the position operator $\langle x\rangle$ in each one of them. We know that when the particle is in one of the energy eigenstates, then we have that

$$\langle x\rangle_n(\lambda) = {}_\lambda\langle n|\, x\, |n\rangle_\lambda = 0 \tag{9.38}$$

because the potential $V(x) = V(-x)$ is even. Therefore

$$\begin{aligned}
\langle x\rangle_\pm(\lambda) &= \langle\pm|\,x\,|\pm\rangle \\
&= \frac{1}{\sqrt{2}}\left({}_\lambda\langle 0|\,x\,|0\rangle_\lambda \pm {}_\lambda\langle 1|\,x\,|0\rangle_\lambda \pm {}_\lambda\langle 0|\,x\,|1\rangle_\lambda + {}_\lambda\langle 1|\,x\,|0\rangle_\lambda\right) \\
&= \pm\sqrt{2}\langle 1|\,x\,|0\rangle_\lambda\,,
\end{aligned} \tag{9.39}$$

where in the last line we used the relation (9.38) ${}_\lambda\langle 0|\,x\,|0\rangle_\lambda = {}_\lambda\langle 1|\,x\,|1\rangle_\lambda = 0$ and that the amplitudes ${}_\lambda\langle 1|\,x\,|0\rangle_\lambda = {}_\lambda\langle 0|\,x\,|1\rangle_\lambda$. Also[10] we have that ${}_\lambda\langle 1|\,x\,|0\rangle_\lambda > 0$. Therefore, if we have that $|0\rangle_\lambda = \sum_{m=0}^{\infty} c_m^{(0)}\,|m\rangle$ and $|1\rangle_\lambda = \sum_{m=0}^{\infty} c_m^{(1)}\,|m\rangle$, we obtain

$$\langle x\rangle_\pm(\lambda) = \pm\sqrt{2}\sum_{m,m'=0}^{\infty} c_m^{(0)} c_{m'}^{(1)} X_{mm'}\,. \tag{9.40}$$

Given that for finite N, the subroutine `DSYEV` returns approximations to the coefficients $c_m^{(n)}$ in the columns of the matrix `H[DIM][DIM]` so that $c_m^{(n)} \approx$ `H[n][m]`, you may compare the value of $\langle x\rangle_\pm(\lambda)$ with the classical values $x_0 = \pm 1/\sqrt{\lambda}$ as λ is increased.

[10] You may convince yourselves by looking at the wave functions in figures 10.4 of chapter 10 and by computing the relevant integrals.

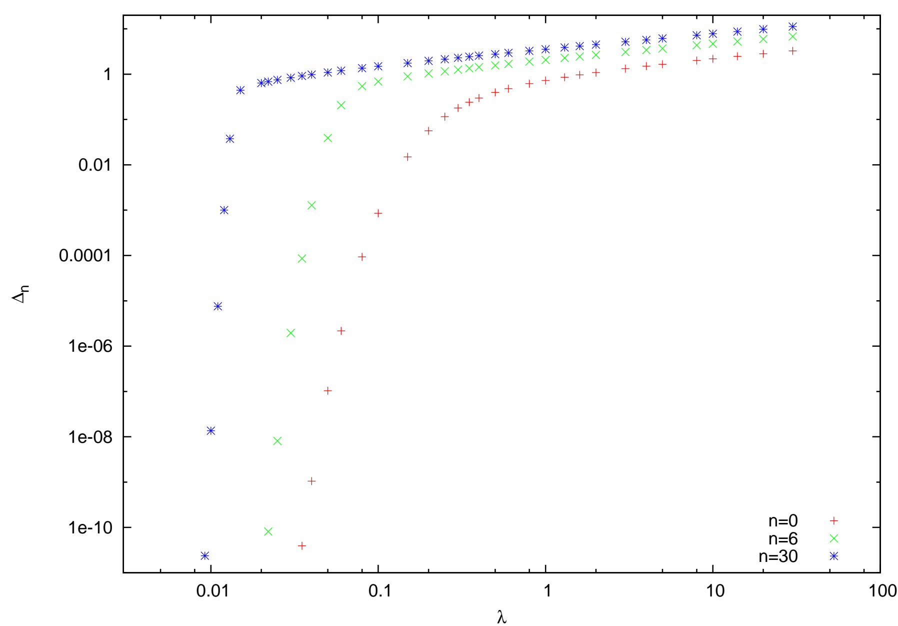

Figure 9.8: Calculation of the difference of the energy levels $\Delta_n = E_{n+1} - E_n$ for $n = 0, 6, 30$ for the double well potential from the program `doublewell.cpp`. The difference vanishes as the well becomes deeper with decreasing λ. The states $|\pm\rangle = (\,|n+1\rangle_\lambda \pm |n\rangle_\lambda)/\sqrt{2}$ are more and more localized to the right or left well respectively.

9.5 Problems

9.1 Calculate the matrix $H(\lambda)$ for $N = 2, 3$ analytically. Calculate its eigenvalues for $N = 2$. Compare your results with the numerical values that you obtain from your program.

9.2 Add the necessary code to the program in the file `test.cpp` so that it checks that the eigenvectors satisfy their defining relations $\mathbf{A}\,\mathbf{v}_i = \lambda_i \mathbf{v}_i$ and that they form an orthonormal basis $\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$.

9.3 Calculate $E_5(\lambda)$ and $E_9(\lambda)$ for $\lambda = 0.8, 1.2$ with an accuracy better than 0.01%.

9.4 For how large n can you calculate $E_n(\lambda)$ for $\lambda = 1$ with an accuracy better than 2% when $N = 64$?

9.5 Calculate $E_3(\lambda)$ and $E_{12}(\lambda)$ for $0 \leq \lambda \leq 4$ with step $\delta\lambda = 0.2$ by achieving accuracy better than 0.01%. How large should N be taken in each case?

9.6 Calculate the expression that gives the matrix elements of the operator x^4 in the $|n\rangle$ representation analytically. Modify the program in `anharmonic.cpp` in order to incorporate your calculation. Verify that the results are the same and test if it has an effect in the total computation time with and without calculating the eigenvalues and eigenvectors of the Hamiltonian. Compute in each case the dependence of the cpu time on N by computing the exponent (cpu time)$\sim N^a$ for $N = 40 - 1000$.

9.7 Modify the code in the file `anharmonic.cpp` so that the arrays `H, X, X4, E, WORK` are dynamically allocated and their dimension is determined by the variable `DIM` read by the program interactively.

9.8 Make an attempt to reproduce the results of Hioe and Montroll [44] given in table 9.2 for $n = 3$ and $n = 5$. What is the largest value of λ that you can study given your computational resources?

9.9 Make an attempt to reproduce the results of Hioe and Montroll [44] given by equation (9.31). Calculate the ground state energy E_0 for $200 < \lambda < 20000$ and then fit your results to a function of the form

$\lambda^{1/3}(a + b\lambda^{-2/3} + c\lambda^{-4/3})$. What is the accuracy in the calculation of the coefficients a, b and c and how good is the agreement with equation (9.31)?

9.10 Modify the code in the file `anharmonic.cpp` so that it calculates the expectation values $\langle x^2\rangle_n(N,\lambda)$, $\langle p^2\rangle_n(N,\lambda)$ and the corresponding products $\Delta x \cdot \Delta p$.
(Hint: See the file `anharmonicOBS.cpp`.)

9.11 Reproduce the results shown in figure 9.4. Repeat your calculation for $\lambda = 2.0, 10.0, 100.0$. Repeat your calculations for $n = 20$.

9.12 Reproduce the results shown in figure 9.5. Repeat your calculations for $n = 20$.

9.13 Reproduce the results shown in figure 9.6. Repeat your calculation for $n = 3, 7, 12, 18, 24$.

9.14 Write a program that calculates the energy levels of the anharmonic oscillator

$$H(\lambda,\mu) = \frac{1}{2}p^2 + \frac{1}{2}x^2 + \lambda x^4 + \mu x^6 \,. \tag{9.41}$$

Calculate $E_n(\lambda)$ for $n = 0, 3, 8, 20$, $\lambda = 0.2$ and $\mu = 0.2, 0.5, 1.0, 2.0, 10.0$.

9.15 Modify the program of the previous problem so that it calculates the expectation values $\langle x^2\rangle_n(N,\lambda)$, $\langle p^2\rangle_n(N,\lambda)$ and the products $\Delta x \cdot \Delta p$. Calculate the expectation values $\langle x^2\rangle_n(\lambda)$, $\langle p^2\rangle_n(\lambda)$ and $\Delta x \cdot \Delta p$ for $n = 0, 3, 8, 20$, $\lambda = 0.2$ and $\mu = 0.2, 0.5, 1.0, 2.0, 10.0$.

9.16 Use the program `doublewell.cpp` in order to calculate the energy level pairs E_n, E_{n+1} for $n = 0, 4, 20$ and $\lambda = 0.2, 0.1, 0.05, 0.02$. Calculate the difference $\Delta_n = E_{n+1} - E_n$ and comment on your results.

9.17 Define the energy values

$$\epsilon_n = -\frac{1}{4\lambda} + \left(n + \frac{1}{2}\right) \,.$$

Compare the results for E_n, E_{n+1} of the previous problem with $\epsilon_n - \Delta_n/2$ and $\epsilon_n + \Delta_n/2$ respectively. Explain your results.

9.18 Modify the program `doublewell.cpp`, so that it calculates the expectation values $\langle x\rangle_\pm(\lambda)$ given by equation (9.40). Compare $\langle x\rangle_\pm(\lambda)$ with the classical values $x_0 = \pm 1/\sqrt{\lambda}$ for $\lambda = 0.2, 0.1, 0.05, 0.02, 0.01$.

9.19 Repeat the previous problem when the states $|\pm\rangle = (1/\sqrt{2})(\,|n\rangle_\lambda \pm |n+1\rangle_\lambda)$ for $n = 6$ and $n = 30$.

9.20 For the simple harmonic oscillator, the energy levels are equidistant, i.e. $\Delta_n = E_{n+1} - E_n = 1$, $(\Delta_{n+2} - \Delta_n)/\Delta_n = 0$. Calculate these quantities for the anharmonic oscillator and the double well potential for $\lambda = 1, 10, 100, 1000$ and $n = 0, 8, 20$. What do you conclude from your results?

Chapter 10

Time Independent Schrödinger Equation

In this chapter, we will study the time independent Schrödinger equation for a non relativistic particle of mass m, without spin, moving in one dimension, in a static potential $V(x)$. We will only study bound states. The solutions in this case yield the discrete energy spectrum $\{E_n\}$ as well as the corresponding eigenstates of the Hamiltonian $\{\psi_n(x)\}$ in position representation.

From a numerical analysis point of view, the problem consists of solving for the eigensystem of a differential equation with boundary conditions. Part of the solution is the energy eigenvalue which also needs to be determined.

As an exercise, we will use two different methods, one that can be applied to a particle in an infinite well with $V(x) = V(-x)$, and one that can be applied to more general cases. The first method is introduced only for educational purposes and the reader may skip section 10.2 to go directly to section 10.3.

10.1 Introduction

The wave functions $\psi(x)$, which are the position representation of the energy eigenstates, satisfy the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x)\,, \tag{10.1}$$

with the normalization condition

$$\langle\psi|\psi\rangle = \int_{-\infty}^{+\infty} \psi^*(x)\psi(x)\, dx = 1\,. \tag{10.2}$$

The Hamiltonian operator is given in position representation by

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(\hat{x})\,, \tag{10.3}$$

and it is Hermitian, i.e. $\hat{H}^\dagger = \hat{H}$. Equation (10.1) is an eigenvalue problem

$$\hat{H}\psi(x) = E\psi(x)\,, \tag{10.4}$$

which, for bound states, has as solutions a discrete set of *real* functions $\psi_n^*(x) = \psi_n(x)$ such that $\hat{H}\psi_n(x) = E_n\psi_n(x)$. The numbers $E_0 \leq E_1 \leq E_2 \leq \ldots$ are real and they are the (bound) energy spectrum of the particle in the potential[1] $V(x)$. The minimum energy E_0 is called the ground state energy and the corresponding ground state is given by a non trivial function $\psi_0(x)$. According to the Heisenberg uncertainty principle, in this state the uncertainties in momentum $\Delta p > 0$ and position $\Delta x > 0$ so that $\Delta p \cdot \Delta x \geq \hbar/2$.

The eigenstates $\psi_n(x)$ form an orthonormal basis

$$\langle\psi_n|\psi_m\rangle = \int_{\infty}^{+\infty} \psi_n^*(x)\psi_m(x)\, dx = \delta_{n,m}\,. \tag{10.5}$$

so that any (square integrable) wave function $\phi(x)$ which represents the state $|\phi\rangle$ is given by the linear combination

$$\phi(x) = \sum_{n=0}^{\infty} c_n\psi_n(x)\,. \tag{10.6}$$

The amplitudes $c_n = \langle\psi_n|\phi\rangle = \int_{-\infty}^{+\infty} \psi_n^*(x)\phi(x)\, dx$ are complex numbers that give the probability $p_n = |c_n|^2$ to measure energy E_n in the state $|\phi\rangle$.

[1]The fact that the energy spectrum of the particle is bounded from below depends on the form of the potential. We assume that $V(x)$ is such that E_0 is finite. Also, in one dimension, the energy spectrum of a particle for reasonable potentials is non degenerate (see, however, S. Kar, R. Parwani, arXiv:0706.1135.)

For any state $|\phi\rangle$ the function

$$p_\phi(x) = |\phi(x)|^2 = \phi^*(x)\phi(x) \tag{10.7}$$

is the probability density of finding the particle at position x, i.e. the probability of detecting the particle in the interval $[x_1, x_2]$ is given by

$$\mathcal{P}_\phi(x_1 < x < x_2) = \int_{x_1}^{x_2} p_\phi(x)\, dx = \int_{x_1}^{x_2} \phi^*(x)\phi(x)\, dx\,. \tag{10.8}$$

The normalization condition (10.2) reflects the conservation of probability (independent of time, respected by the time dependent Schrödinger equation) and the completeness (in this case the certainty that the particle will be observed somewhere on the x axis).

The classical observables $\mathcal{A}(x,p)$ of this quantum mechanical system are functions of the position and the momentum and their quantum mechanical versions are given by operators $\hat{\mathcal{A}}(\hat{x},\hat{p})$. Their expectation values when the system in a state $|\phi\rangle$ are given by

$$\langle\hat{\mathcal{A}}\rangle_\phi = \langle\phi|\,\hat{\mathcal{A}}\,|\phi\rangle = \int_{-\infty}^{+\infty} \phi^*(x)\hat{\mathcal{A}}(\hat{x},\hat{p})\phi(x)\, dx\,. \tag{10.9}$$

From a numerical point of view, the eigenvalue problem (10.1) requires the solution of an ordinary second order differential equation. There are certain differences in this problem compared to the ones studied in previous sections:

- Instead of an initial value problem (i.e. the values of the function and its derivative are given at one point), we have a boundary value problem (values of the function or its derivative given at two different points).

- The eigenvalue (energy) is unknown and should be determined as part of the solution.

As an introduction to such classes of problems, we will present some simple methods which are special to one dimension.

For the numerical solution of the above equation we renormalize x, the function $\psi(x)$ and the parameters so that we deal only with dimensionless quantities. Equation (10.1) is rewritten as:

$$\frac{d^2}{dx^2}\psi(x) + \frac{2m}{\hbar^2}(E - V(x))\psi(x) = 0\,. \tag{10.10}$$

Then we choose a length scale L which is defined by the parameters of the problem[2] and we redefine $\tilde{x} = x/L$. We define $\tilde{\psi}(\tilde{x}) = \psi(x)$ $\tilde{\psi}'(\tilde{x}) = d\psi(x)/d\tilde{x} = L\, d\psi(x)/dx$ and we obtain

$$\tilde{\psi}''(\tilde{x}) + \frac{2mL^2}{\hbar^2}(E - V(\tilde{x}L))\tilde{\psi}(\tilde{x}) = 0\,. \tag{10.11}$$

We define $v(\tilde{x}) = 2mL^2V(x)/\hbar^2 = 2mL^2V(\tilde{x}L)/\hbar^2$, $\epsilon = 2mL^2E/\hbar^2$ and change notation to $\tilde{x} \to x$, $\tilde{\psi} \to \psi$. We obtain

$$\psi''(x) = -(\epsilon - v(x))\psi(x)\,. \tag{10.12}$$

The solutions of equation (10.1) can be obtained from those of equation (10.12) by using the following "dictionary"[3]:

$$x \to \frac{x}{L}\,, \quad E = \frac{\hbar^2}{2mL^2}\epsilon\,, \quad V(x) = \frac{\hbar^2}{2mL^2}v(x/L)\,. \tag{10.13}$$

The dimensionless momentum is defined as $\tilde{p} = -i\partial/\partial\tilde{x} = -iL\partial/\partial x$ and we obtain

$$\tilde{p} = \frac{L}{\hbar}p\,. \tag{10.14}$$

The commutation relation $[x,p] = i\hbar$ becomes $[\tilde{x},\tilde{p}] = i$. The kinetic energy $T = \dfrac{p^2}{2m}$ is given by

$$T = \frac{\hbar^2}{2mL^2}\tilde{p}^2 = -\frac{\hbar^2}{2mL^2}\frac{\partial^2}{\partial\tilde{x}^2}\,, \tag{10.15}$$

and the Hamiltonian $H = T + V$

$$H = \frac{\hbar^2}{2mL^2}\left(\tilde{p}^2 + v(\tilde{x})\right) = \frac{\hbar^2}{2mL^2}\left(-\frac{\partial^2}{\partial\tilde{x}^2} + v(\tilde{x})\right)\,. \tag{10.16}$$

In what follows, we will omit the tilde above the symbols and write x instead of $\tilde{x}$.

---

[2]There are m, $\hbar$ and the coupling constants in the function $V(x)$. The range of the potential will determine L in some problems and it is given explicitly in potential wells. In potentials of real physical systems, however, this is also determined by the coupling constants.

[3]If we normalize the solutions $\tilde{\psi}(\tilde{x})$ of equation (10.12) according to the relation $\int_{-\infty}^{+\infty}\tilde{\psi}^*(\tilde{x})\tilde{\psi}(\tilde{x})d\tilde{x} = 1$, we should also take $\psi(x) = (1/\sqrt{L})\tilde{\psi}(x/L)$ in order to be properly normalized $\int_{-\infty}^{+\infty}\psi^*(x)\psi(x)dx = 1$.

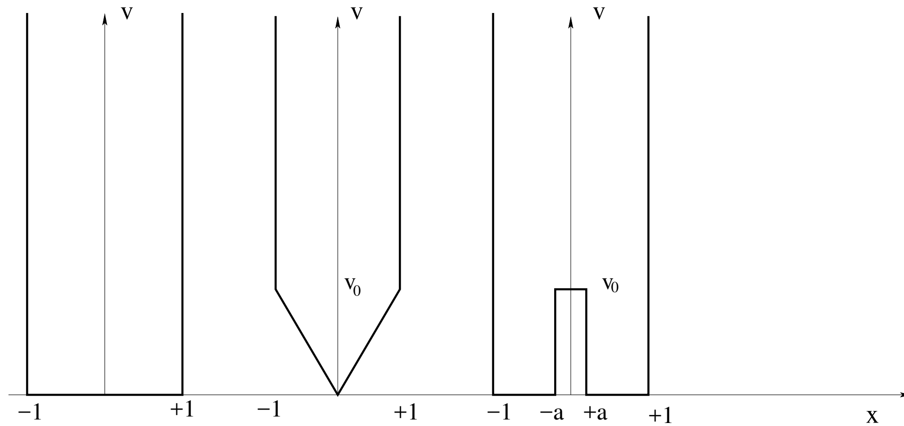

Figure 10.1: The potentials given by equations (10.17), (10.26) and (10.27).

10.2 The Infinite Potential Well

The simplest model for studying the qualitative features of bound states is the infinite potential well of width L where a particle is confined within the interval $[-L/2, L/2]$:

$$v(x) = \begin{cases} 0 & |x| < 1 \\ +\infty & |x| \geq 1 \end{cases} \tag{10.17}$$

The length scale chosen here is $L/2$ and the dimensionless variable x corresponds to $x/(L/2)$ when x is measured in length units.

The solution of (10.12) can be easily computed. Due to the symmetry

$$v(-x) = v(x)\,, \tag{10.18}$$

of the potential, the solutions have well defined *parity*. This property will be crucial to the method used below. The method discussed in the next section can also be used on non even potentials.

The solutions are divided into two categories, one with even parity $\psi_n(x) \equiv \psi_n^{(+)}(-x) = \psi_n^{(+)}(x)$ for $n = 1, 3, 5, 7, \ldots$ and one with odd parity $\psi_n(x) \equiv -\psi_n^{(-)}(-x) = \psi_n^{(-)}(x)$ for $n = 2, 4, 6, 8, \ldots$.

$$\psi_n(x) = \begin{cases} \psi_n^{(+)}(x) = \cos\left(\frac{n\pi}{2}x\right) & |x| < 1 \quad n = 1, 3, 5, 7, \ldots \\ \psi_n^{(-)}(x) = \sin\left(\frac{n\pi}{2}x\right) & |x| < 1 \quad n = 2, 4, 6, 8, \ldots \end{cases} \tag{10.19}$$

where

$$\epsilon_n = \left(\frac{n\pi}{2}\right)^2 , \tag{10.20}$$

and the normalization has been chosen so that[4] $\int_{-1}^{1} (\psi_n(x))^2 \, dx = 1$.

The solutions can be found by using the parity of the wave functions. We note that for the positive parity solutions

$$\psi_n^{(+)}(0) = A \qquad \psi_n^{(+)\prime}(0) = 0 , \tag{10.21}$$

whereas for the negative parity solutions

$$\psi_n^{(-)}(0) = 0 \qquad \psi_n^{(-)\prime}(0) = A . \tag{10.22}$$

The constant A depends on the normalization of the wave function. Therefore we can set $A = 1$ originally and then renormalize the wave function so that equation (10.2) is satisfied. If the energy is known, the relations (10.21) and (10.22) can be taken as initial conditions in relation (10.12). By using a Runge–Kutta algorithm we can evolve the solution towards $x = \pm 1$. The problem is that the energy ϵ is unknown. If the energy is not allowed by the quantum theory we will find that the boundary conditions

$$\psi_n^{(\pm)}(\pm 1) = 0 \tag{10.23}$$

are violated. As we approach the correct value of the energy, we obtain $\psi_n^{(\pm)}(\pm 1) \to 0$.

Therefore we follow the steps described below:

- We choose an initial value for the energy ϵ that is lower than the one we are looking for. We can use estimates from known solutions of similar looking potential wells or simply start from a value slightly higher than the absolute minimum of the potential.

- We choose the parity of the solution and we set initial conditions according to equations (10.21) and (10.22).

---

[4]According to the dictionary mentioned in the previous section, for a potential well where $x \in [-L/2, L/2]$ the dimensionless position variable has been chosen to be $x/(L/2) \in [-1, 1]$. Then $E_n = \frac{\hbar^2}{2m(L/2)^2}\epsilon_n = \frac{\hbar^2\pi^2}{2mL^2}n^2$ and $\psi_n^{(+)}(x) = \sqrt{2/L}\cos(n\pi x/L)$, $\psi_n^{(-)}(x) = \sqrt{2/L}\sin(n\pi x/L)$. Note that $\epsilon_n = p_n^2$ according to equations (10.13) and (10.14).

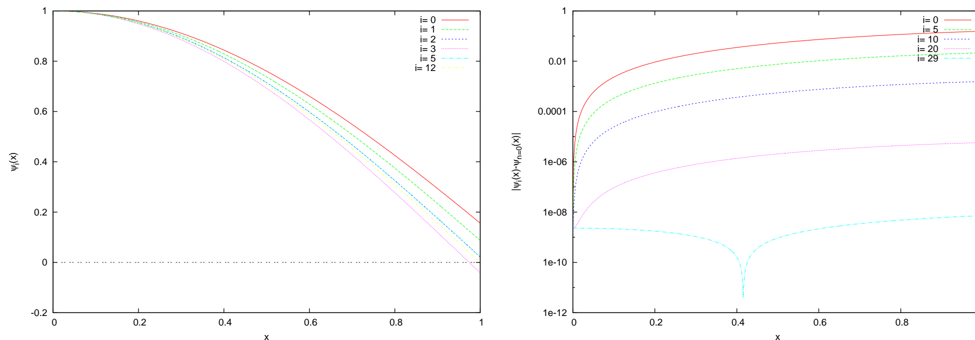

Figure 10.2: Convergence of the solution $\psi_i(x)$ of (10.12) with the potential (10.17) as a function of the number of iterations i in the program `well.cpp`. Initially `energy = 2.0` and `parity = 1`. After 29 iterations the solution converges to the ground state $\psi_1(x) = \cos(\pi x/2)$ with energy $\epsilon = (\pi/2)^2$ and with relative accuracy $\sim 10^{-9}$. The bottom plot shows the error as a function of the number of iterations in a logarithmic scale. For $i \equiv$`iter = 1,2,3,5,10,12,20` we obtain `energy = 2.4, 2.6, 2.4, 2.4625, 2.46875, 2.4673828125`.

- We evolve the solutions using a 4th order Runge-Kutta method from [5] $x = 0$ to $x = +1$.
- If equation (10.23) is not satisfied, we increase the energy by $\delta\epsilon$ and we repeat.
- We repeat until $\psi_n^{(\pm)}(1)$ changes sign. Then we lower the energy by $\delta\epsilon = -\delta\epsilon/2$.
- The process is ended when $|\psi_n^{(\pm)}(1)| < \delta$ for appropriately chosen small δ.

For the evolution of the solution from $x = 0$ to $x = 1$ we use the 4th order Runge-Kutta method programmed in the file `rk.cpp` of chapter 4. We copy the function `RKSTEP` in a local file `rk.cpp`. The integration of (10.12) can by done by using the function $\phi(x) \equiv \psi'(x)$

$$
\begin{aligned}
\psi'(x) &= \phi(x) \\
\phi'(x) &= (v(x) - \epsilon)\psi(x)\,,
\end{aligned}
\tag{10.24}
$$

[5] The function in $[-1, 0)$ is determined by the parity of the solution.

with the initial conditions

$$\begin{aligned} \psi(0) = 1 \quad , \quad \phi(0) \equiv \psi'(0) = 0 \quad &\text{even parity} \\ \psi(0) = 0 \quad , \quad \phi(0) \equiv \psi'(0) = 1 \quad &\text{odd parity}\,. \end{aligned} \tag{10.25}$$

We use the notation $\psi(x) \to$ `psi`, $\phi(x) \to$ `psip`. The functions `f1` and `f2` correspond to the right hand side of (10.24). They are the derivatives of $\psi(x)$ and $\phi(x)$ respectively and `f1=psip`, `f2=(V-energy)*psi`. The code of `f1` and `f2` is put in a different file so that we can easily reuse the code for many different potentials $v(x)$. The file `wellInfSq.cpp` contains the necessary program for the potential of equation (10.17):

```
//========================================================
//file: wellInfSq.cpp
//
//Functions used in RKSTEP function. Here:
//f1 = psip(x) = psi(x)'
//f2 = psip(x)'= psi(x)''
//
//All one has to set is V, the potential
//========================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
extern double energy;
//-------- trivial function: derivative of psi
double
f1(const double& x, const double& psi,const double& psip){
  return psip;
}
//========================================================
//-------- the second derivative of wavefunction:
//psip(x)' = psi(x)'' = -(E-V) psi(x)
double
f2(const double& x, const double& psi,const double& psip){
  double V;
  //------- potential, set here:
  V  = 0.0;
  //------- Schroedinger eq: RHS
  return (V-energy)*psi;
```

```
}
```

We stress that the energy $\epsilon =$ `energy` is put in the global scope so that it can be accessed by the main program.

The main program is in the file `well.cpp`. The user enters the parameters (`energy, parity, Nx`) and the loop

```
  while(iter < 10000){
    ....
    if(abs(psinew)   <= epsilon) break;
    if(psinew*psiold <  0.0     ) de = -0.5*de;
    energy   += de;
    ....
  }
```

exits when $\psi(1)$ =`psinew` has an absolute value which is less than `epsilon`, i.e. when the condition (10.23) is satisfied to the desired accuracy. The value of the energy increases up to the point where the sign of the wave function at $x = 1$ changes (`psinew*psiold`< 0). Then the value of the energy is overestimated and we change the sign of the step `de` and reduce its magnitude by a half. The algorithm described on page 446 is implemented inside the loop. After exiting the loop, the energy has been determined with the desired accuracy and the rest of the program stores the solution in the array `psifinal(STEPS)`. The results are written to the file `psi.dat`. Note how the variable `parity` is used so that both cases `parity`$= \pm 1$ can be studied. The full program is listed below:

```
//============================================================
//file: well.cpp
//
//Computation of energy eigenvalues and eigenfunctions
//of a particle in an infinite well with V(-x)=V(x)
//
//Input:  energy: initial guess for energy
//        parity: desired parity of solution (+/- 1)
//        Nx-1  : Number of RK4 steps from x=0 to x=1
//Output: energy: energy eigenvalue
//        psi.dat: final psi[x]
//        all.dat: all   psi[x] for trial energies
//============================================================
#include <iostream>
```

```
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
//--------------------------------------------------------
const int P = 10000;
double energy;
//--------------------------------------------------------
void
RKSTEP( double& t,      double& x1,
        double& x2,
        const           double& dt);
//--------------------------------------------------------
int main(){
  double dx,x,epsilon,de;
  double psi,psip,psinew,psiold;
  double array_psifinal[2*P+1],array_xstep[2*P+1];
  double *psifinal = array_psifinal+P;
  double *xstep    = array_xstep   +P;
  //psifinal points to (array_psifinal+P) and one can
  //use psifinal[-P]...psifinal[P]. Similarly for xstep
  int    parity,Nx,iter,i,node;
  string buf;
  //------ Input:
  cout << "Enter energy,parity,Nx:\n";
  cin  >> energy  >> parity >> Nx;     getline(cin,buf);
  if(Nx > P){cerr << "Nx>P\n";exit(1);}
  if(parity > 0) parity= 1;
  else           parity=-1;
  cout << "# #######################################\n";
  cout << "# Estart= " << energy
       << "  parity= " << parity  << endl;
  dx       =  1.0/(Nx-1);
  epsilon  = 1.0e-6;
  cout << "# Nx=  "    << Nx      << " dx = " << dx
       << " eps=  "    << epsilon << endl;
  cout << "# #######################################\n";
  //------ Calculate:
  ofstream myfile("all.dat");
  myfile.precision(17);
  cout  .precision(17);
  iter     = 0;
  psiold   = 0.0;
  psinew   = 1.0;
```

```
  de        = 0.1 * abs(energy);
  while(iter < 10000){
    // Initial conditions at x=0
    x       = 0.0;
    if(parity == 1){
      psi  = 1.0;
      psip = 0.0;
    }else{
      psi  = 0.0;
      psip = 1.0;
    }
    myfile   << iter  << " " << energy << " "
             << x     << " " << psi    << " "
             << psip  << endl;
    // Use Runge-Kutta to forward to x=1
    for(i=2;i<=Nx;i++){
      x = (i-2)*dx;
      RKSTEP(x,psi,psip,dx);
      myfile << iter  << " " << energy << " "
             << x     << " " << psi    << " "
             << psip  << endl;
    }
    psinew = psi;
    cout     << iter  << " " << energy << " "
             << de    << " " << psinew << endl;
    // Stop if value of psi close to 0
    if(abs(psinew)    <= epsilon) break;
    // Change direction of energy search:
    if(psinew*psiold  <  0.0    ) de = -0.5*de;
    energy   += de;
    psiold    = psinew;
    iter++;
  }//while(iter < 10000)
  myfile.close();
  //We found the solution:
  //calculate it once again and store it
  if(parity   == 1){
    psi       =  1.0;
    psip      =  0.0;
    node      =  0; //count number of nodes of function
  }else{
    psi       =  0.0;
    psip      =  1.0;
    node      =  1;
  }
```

```
  x            =  0.0;
  xstep   [0] =  x;
  psifinal[0] =  psi; //array that stores psi(x)
  psiold       =  0.0;
  // Use Runge-Kutta to move to x=1
  for(i=2 ; i<=Nx ; i++ ){
    x              = (i-2)*dx;
    RKSTEP(x,psi,psip,dx);
    xstep   [i-1]   = x;
    psifinal[i-1]   = psi;
    // Use parity to compute psi(-x)
    xstep   [1-i]   = -x;
    psifinal[1-i]   = psi*parity;
    // Determine zeroes of psi(x):
    // psi should not be zero within epsilon:
    if(abs(psi)     > 2.0*epsilon){
      if(psi*psiold < 0.0) node += 2;
      psiold        = psi;
    }
  }//for(i=2;i<=Nx;i++)
  node++;
  //print final solution:
  myfile.open("psi.dat");
  cout     << "Final result: E= " << energy
           << " n= "              << node
           << " parity= "         << parity       << endl;
  myfile   << "# E= "             << energy
           << " n= "              << node
           << " parity= "         << parity       << endl;
  for(i=-(Nx-1);i<=(Nx-1);i++)
    myfile << xstep   [i]         << " "
           << psifinal[i]                         << endl;
  myfile.close();
}// main()
```

The compilation and running of the program can be done with the commands

```
> g++  well.cpp wellInfSq.cpp rk.cpp -o well
> ./well
Enter energy,parity,Nx:
2.0 1 400
# ########################################
# Estart= 2  parity= 1
```

```
# Nx=  400 dx = 0.00250627 eps=  1e-06
# ########################################
0  2            0.200000000      0.155943694
1  2.2000000000  0.200000000     0.087444801
.....
28 2.4674072265  1.220703125e-05 -1.950054368e-06
29 2.4674011230 -6.103515625e-06 -7.246215909e-09
Final result: E= 2.4674011230468746 n= 1 parity= 1
```

The energy is determined to be ϵ =`2.467401123` which can be compared to the exact value $\epsilon = (\pi/2)^2 \approx$ `2.467401100`. The fractional error is $\sim 10^{-8}$. The convergence can be studied graphically in figure 10.2.

The calculation of the excited states is done by changing the parity and by choosing the initial energy slightly higher than the one determined in the previous step[6]. The results are in table 10.1. The agreement with the exact result $\epsilon_n = (n\pi/2)^2$ is excellent.

We close this section with two more examples. First, we study a potential well with triangular shape at its bottom

$$v(x) = \begin{cases} v_0|x| & |x| < 1 \\ +\infty & |x| > 1 \end{cases} \tag{10.26}$$

and then a double well potential with

$$v(x) = \begin{cases} v_0 & |x| < a \\ 0 & a < |x| < 1 \\ +\infty & 1 < |x| \end{cases} \tag{10.27}$$

where the parameters v_0, a are positive numbers. A qualitative plot of these functions is shown in figure 10.1.

For the triangular potential we take $v_0 = 10$, whereas for the double well potential $v_0 = 100$ and $a = 0.3$. The code in `wellInfSq.cpp` is appropriately modified and saved in the files `wellInfTr.cpp` and `wellInfDbl.cpp` respectively. All we have to do is to change the line computing the value of the potential in the function `f2`. For example the file `wellInfTr.cpp` contains the code

```
  //-------- potential, set here:
  V = 10.0 * abs(x);
```

[6]Careful: if the energy levels are too close, we should keep the initial energy constant and change the sign of parity.

n	$(n\pi/2)^2$	Square	Triangular	Double Well
1	2.467401100	2.467401123	5.248626709	15.294378662
2	9.869604401	9.869604492	14.760107422	15.350024414
3	22.2066099	22.2066040	27.0690216	59.1908203
4	39.47841	39.47839	44.51092	59.96887
5	61.6850275	61.6850242	66.6384315	111.3247375
6	88.82643	88.82661	93.84588	126.37628
7	120.902653	120.902664	125.878830	150.745215
8	157.91367	157.91382	162.92569	194.07578
9	199.859489	199.859490	204.845026	235.017471
10	246.74011	246.74060	251.74813	275.67383
11	298.555533	298.555554	303.545814	331.428306
12	355.3057	355.3064	360.3107	388.7444

Table 10.1: Energy eigenvalues for the square, triangular and double well potentials (equations (10.17), (10.26) with $v_0 = 10$ and equation (10.27) with $v_0 = 100$, $a = 0.3$). The agreement of the results for the square potential with the exact ones is excellent. For the other potentials, we note that as we move further from the bottom of the well we obtain energy levels very close to those of the square well: The particle does not feel the influence of the details at the bottom of the well. For the double well potential we obtain $E_1 \approx E_2$ and $E_3 \approx E_4$ according to the analysis on page 455.

whereas the file `wellInfDbl.cpp` contains the code

```
//-------- potential, set here:
if(abs(x) <= 0.3)
  V  = 100.0;
else
  V  = 0.0;
```

The analysis is performed in exactly the same way and the results are shown in table 10.1. Note that, for large enough n, the energy levels of all the potentials that we studied above tend to have identical values. This happens because, when the particle has energy much larger than v_0, the details of the potential at the bottom do not influence its dynamical properties very much. For the triangular potential, the energy levels have higher values than the corresponding ones of the square potential. This happens because, on the average, the potential energy is higher and the potential tends to confine the particle to a smaller region (Δx is decreased, therefore Δp is increased). This can be seen in figure 10.3 where the wave functions of the particle in each of the two potentials are compared.

Similar observations can be made for the double well potential. Moreover, we note the approximately degenerate energy levels, something which is expected for potentials of this form. This can be understood in terms of the localized states given by the wave functions $\psi_+(x) = (1/\sqrt{2})(\psi_1(x) + \psi_2(x))$ and $\psi_-(x) = (1/\sqrt{2})(\psi_1(x) - \psi_2(x))$. The first one represents a state where the particle is localized in the left well and the second one in the right. This is shown in figure 10.4. As $v_0 \to +\infty$ the two wells decouple and the wave functions $\psi_\pm(x)$ become equal to the energy eigenstate wave functions of two particles in separate infinite square wells of width $1-a$ with energy eigenvalues $\epsilon_{+,1} = \epsilon_{-,1} = (\pi/(1-a))^2$. The difference of ϵ_1 and ϵ_2 from these two values is due to the finite v_0 (see problem 4).

We will now discuss the limitations of this method. First, the method can be used only on potential wells that are even, i.e. $v(x) = v(-x)$. We used this assumption in equations (10.21) and (10.22) giving the initial conditions for states of well defined parity. When the potential is even, the energy eigenstates have definite parity. The other problem can be understood by solving problem 4: When $v(0) \gg \epsilon$, the wave function is almost zero around $x = 0$ and the integration from $x = 0$ to $x = 1$ will be

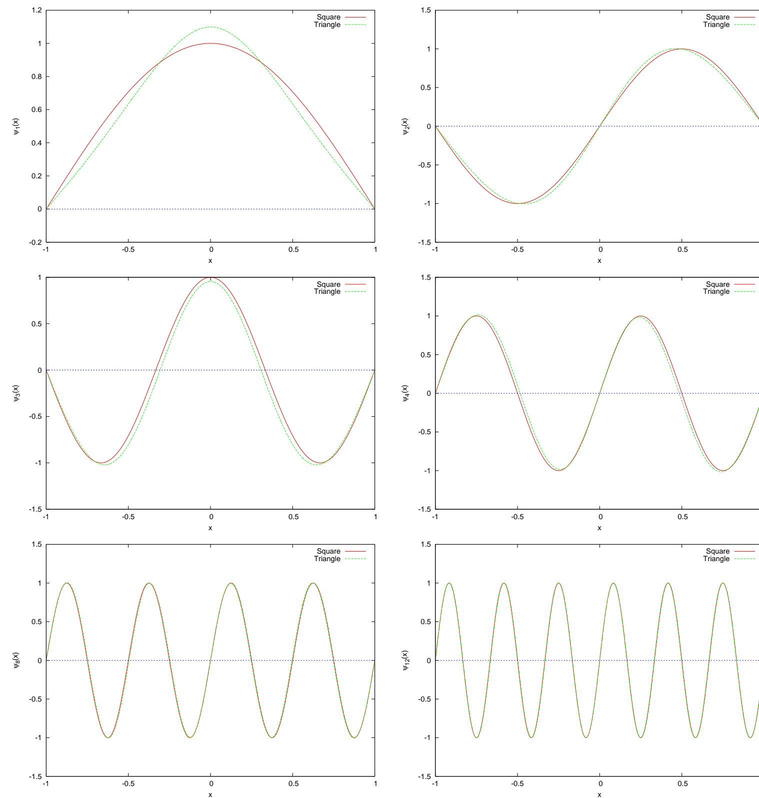

Figure 10.3: The wave functions of the energy eigenstates of the infinite square and triangular well potentials for $n = 1, 2, 3, 4, 8, 12$ given by equations (10.17) and (10.26) with $v_0 = 10$. We observe the influence of the shape of the potential on the wave functions with small n, while for $n \geq 8$ the influence becomes weaker.

dominated by numerical errors. The same is true when the particle has to go through high potential barriers.

This method can also we used on potential wells that are not infinite. In that case we can add infinite walls at points that are far enough so that the wave function is practically zero there. Then the influence of this artificial wall will be negligible (see problem 3).

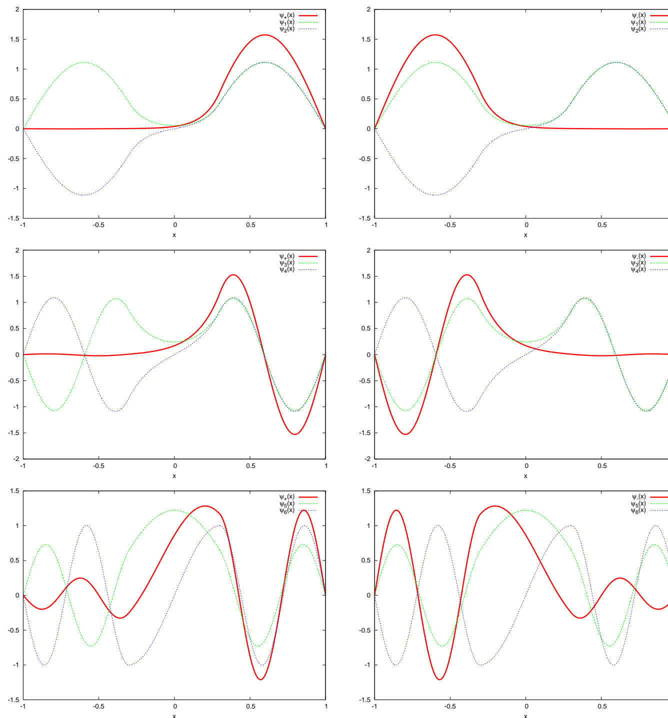

Figure 10.4: The functions $\psi_\pm(x) = (1/\sqrt{2})(\psi_n(x) \pm \psi_{n+1}(x))$ for $n = 1, 3, 5$ for the double well potential (equation (10.27) with $v_0 = 100, a = 0.3$) are plotted using bold red lines. We observe that the more degenerate the states, the stronger the localization of the particle to the left or right well. The other plots are those of the energy eigenfunctions for $n = 1, 2, 3, 4, 5, 6$.

10.3 Bound States

A serious problem with the method discussed in the previous section is that it is numerically unstable. You should have already realized that if you tried to solve problem 3. In that problem, when the walls are moved further than $|x| = 3$, the convergence of the algorithm becomes harder. You can understand this by realizing that in the integration process the solution is evolved from the classically allowed into the classically forbidden region so that an oscillating solution changes into an exponentially damped one. But as $|x| \to +\infty$ there are two solutions, one that is physically acceptable $\psi(x) \sim \mathrm{e}^{-k|x|}$ and one that is diverging $\psi(x) \sim \mathrm{e}^{+k|x|}$ which is not acceptable due to (10.2). Therefore, in order to achieve con-

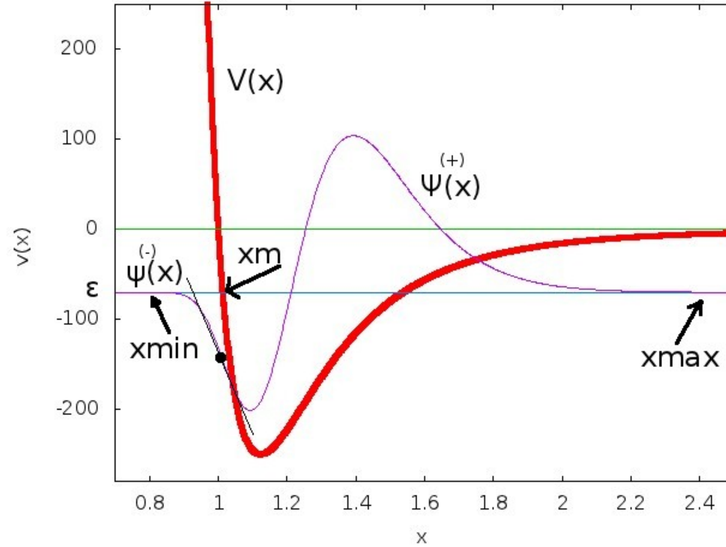

Figure 10.5: Integration of Schrödinger's equation by the use of the algorithm of section 10.3. The wave functions and their derivatives are given small trial values at `xmin` and `xmax` which are in the classically forbidden regions of x. The point x_m is calculated from the equation $v(x_m) = \epsilon$. The wave functions are evolved to x_m according to (10.24) and we obtain the solutions $\psi^{(+)}(x)$ and $\psi^{(-)}(x)$. We renormalize $\psi^{(-)}(x)$ so that $\psi^{(+)}(x_m) = \psi^{(-)}(x_m)$ and we vary the energy until the derivatives $\psi^{(+)\prime}(x_m) \approx \psi^{(-)\prime}(x_m)$.

vergence to the physically acceptable solution, the energy has to be finely tuned, especially when we integrate towards large $|x|$. For this reason it is preferable to integrate from the exponentially damped region towards the oscillating region. The idea is to start integrating from these regions and try to match the solutions and their derivatives at appropriately chosen matching points. The matching is achieved at a point x_m by trying to determine the value of the energy that sets the ratio

$$f(\epsilon) = \frac{\psi^{(+)\prime}(x_m)/\psi^{(+)}(x_m) - \psi^{(-)\prime}(x_m)/\psi^{(-)}(x_m)}{\psi^{(+)\prime}(x_m)/\psi^{(+)}(x_m) + \psi^{(-)\prime}(x_m)/\psi^{(-)}(x_m)} \tag{10.28}$$

equal to zero, within the attainable numerical accuracy. It is desirable to choose a point x_m within the classical region ($\epsilon > v(x)$) and usually we pick a turning point $\epsilon = v(x)$. By renormalizing $\psi^{(\pm)}(x)$ we can always set $\psi^{(+)}(x_m) = \psi^{(-)}(x_m)$, therefore $f(\epsilon) \ll 1$ means that $\psi^{(+)\prime}(x_m) \approx \psi^{(-)\prime}(x_m)$.

The denominator of (10.28) sets the scale of the desired accuracy[7] The idea is depicted in figure 10.5. The algorithm is the following:

- Choose the integration interval `[xmin,xmax]`.
- Choose the initial conditions $\psi^{(-)}(\texttt{xmin})$, $\psi^{(-)\prime}(\texttt{xmin})$, $\psi^{(+)}(\texttt{xmax})$, $\psi^{(+)\prime}(\texttt{xmax})$. This choice depends on the potential $v(x)$. Usually we take `xmin` and `xmax` deep enough in the classically forbidden region and choose the values $\psi^{(-)}(\texttt{xmin})$, $\psi^{(+)}(\texttt{xmax})$ to be zero or exponentially small (e.g. $\sim \mathrm{e}^{-k|x|}$, $k^2 = v(x)-\epsilon$). The corresponding values of the derivatives $\psi^{(-)\prime}(\texttt{xmin})$, $\psi^{(+)\prime}(\texttt{xmax})$ are also taken to be small. The arbitrary normalization of $\psi(x)$ allows these initial values to be chosen in a crude way. The relative sign of the derivatives at large $|x|$ (determined e.g. by the parity of the wave function for even potentials) is also taken care by the renormalization of $\psi^{(-)}(x)$ when applying the matching condition. For an infinite well, the points `xmin,xmax` are the ones where the potential becomes infinite and $\psi^{(-)}(\texttt{xmin}) = \psi^{(+)}(\texttt{xmax}) = 0$.
- Choose the initial value of the energy ϵ and of the energy variation step $\delta\epsilon$.
- Calculate `xm` from the initial value of the energy and the solution of $v(x) = \epsilon$. Choose the solution that is at the left most side[8].
- Evolve the equations (10.24) from `xmin` to `xm` and obtain the solutions $\psi^{(-)}(x)$,$\psi^{(-)\prime}(x)$.
- Evolve the equations (10.24) from `xmax` to `xm` and obtain the solutions $\psi^{(+)}(x)$,$\psi^{(+)\prime}(x)$.
- Renormalize $\psi^{(-)}(x) \to \psi^{(-)}(x)\left(\psi^{(+)}(\texttt{xm})/\psi^{(-)}(\texttt{xm})\right)$, so that $\psi^{(+)}(\texttt{xm}) = \psi^{(-)}(\texttt{xm})$.
- Compute the ratio $f(\epsilon)$ of equation (10.28).
- If $|f(\epsilon)| < \delta$ for appropriately chosen $\delta > 0$, the calculation ends. The result for the energy eigenvalue and eigenfunction is considered

[7]If we are unlucky enough to pick a point where $\psi'(x_m) = 0$, this criterion will fail.
[8]Note that this point changes when we vary ϵ

to be determined with adequate accuracy and we may proceed with the analysis of the results.

- If $f(\epsilon)$ changes sign it means that we have crossed the energy eigenvalue. Reverse the direction of search by taking $\delta\epsilon \to -\delta\epsilon/2$.

- Change the energy $\epsilon \to \epsilon + \delta\epsilon$ and repeat by going back to the fourth step.

When we exit the above loop, the current wave function is a good approximation to the eigenfunction $\psi_n(x)$ corresponding to the eigenvalue ϵ_n. We normalize the wave function according to equation (10.2) and we calculate the expectation values according to (10.9). It is also interesting to determine the number of nodes[9] n_0 of the wave function which is related to n by $n = n_0 + 1$.

Our program needs to implement the Runge–Kutta algorithm. We use the function `RKSTEP` (see page 220) which performs a 4th order Runge–Kutta step. Its code is copied to the file `rk.cpp`.

The potential $v(x)$ is coded in the function `V(x)`. The boundary conditions are programmed in the function `boundary(xmin, xmax, psixmin, psipxmin, psixmax, psipxmax)` which returns the values of `psixmin` = $\psi^{(-)}$(`xmin`), `psipxmin` = $\psi^{(-)\prime}$(`xmin`), `psixmax` = $\psi^{(+)}$(`xmax`), `psipxmax` = $\psi^{(-)\prime}$(`xmax`) to the calling program. These functions are put in a separate file for each potential that we want to study. The name of the file is related to the form of the potential, e.g. we choose `schInfSq.cpp` for the infinite potential well of (10.17). The same file contains the code for the functions `f1, f2`:

```
//========================================================
//file: schInfSq.cpp
//========================================================
//------ potential:
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
```

[9]The number of points x for which $\psi(x) = 0$ and `xmin` $< x <$ `xmax`. The relation $n = n_0 + 1$ sets ϵ_1 to be the ground state for which $n_0 = 0$.

```
extern double energy;
//--------
double V  (const double& x ){
  return 0.0;
}//V()
//-------- boundary conditions:
void
boundary  (const double& xmin,const double& xmax    ,
                 double& psixmin,   double& psipxmin,
                 double& psixmax,   double& psipxmax){
  //for infinite square well we set psi=0 at boundary
  //and psip=+/-1
  psixmin  =  0.0;
  psipxmin =  1.0;
  psixmax  =  0.0;
  psipxmax = -1.0;
  //Initial values at xmin and xmax
}
//-------- trivial function: derivative of psi
double
f1(const double& x, const double& psi,const double& psip){
  return psip;
}
//-------- the second derivative of wavefunction:
//psip(x)' = psi(x)'' = -(E-V) psi(x)
double
f2(const double& x, const double& psi,const double& psip){
  //------- Schroedinger eq: RHS
  return (V(x)-energy)*psi;
}
```

We note that if the potential becomes infinite for $x <$ `xmin` and/or $x >$`xmax`, then this will be determined by the boundary conditions at `xmin` and/or `xmax`.

The main program is in the file `sch.cpp`. The code is listed below and it includes the function `integrate(psi, dx, Nx)` used for the normalization of the wave function. It performs a numerical integration of the square of a function whose values `psi[i] i=0,...,Nx-1` are given at an odd number of `Nx` equally spaced points by a distance `dx` using Simpson's rule.

```
//==========================================================
//
```

```
// File: sch.cpp
//
// Integrate 1d Schrodinger equation from xmin to xmax.
// Determine energy eigenvalue and eigenfunction by matching
// evolving solutions from xmin and from xmax at a point xm.
// Mathing done by equating values of functions and their
// derivatives at xm. The point xm chosen at the left most
// turning point of the potential at any given value of the
// energy. The potential and boundary conditions chosen in
// different file.
// --------------------------------------------------------------
// Input:  energy: Trial value of energy
//         de: energy step, if matching fails de -> e+de, if
//             logderivative changes sign     de -> -de/2
//         xmin, xmax, Nx
// --------------------------------------------------------------
// Output: Final value of energy, number of nodes of
//    wavefunction in stdout
//    Final eigenfunction in file psi.dat
//    All trial functions and energies in file all.dat
//==============================================================
#include <iostream>
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
//-----------------------------------------------------------
const int P = 20001;
double energy;
//-----------------------------------------------------------
double V  (const double& x                          );
double integrate(double* psi ,
          const double& dx   ,const int   & Nx      );
void
boundary  (const double& xmin,const double& xmax    ,
                 double& psixmin,   double& psipxmin,
                 double& psixmax,   double& psipxmax);
void
RKSTEP(          double& t      ,   double& x1,
                 double& x2     ,
          const double& dt                          );
//-----------------------------------------------------------
int main(){
  int Nx,NxL,NxR;
```

```
  double psi[P],psip[P];
  double dx;
  double xmin,xmax,xm; //left/right/matching points
  double psixmin ,psipxmin ,psixmax ,psipxmax;
  double psileft ,psiright ,psistep ,psinorm;
  double psipleft,psipright,psipstep;
  double de,epsilon;
  double matchlogd,matchold,psiold,norm,x;
  int    iter,i,imatch,nodes;
  string buf;
  //Input:
  cout << "# Enter energy,de,xmin,xmax,Nx:\n";
  cin  >> energy>>de>>xmin>>xmax>>Nx;getline(cin,buf);
  //need even intervals for normalization integration:
  if(Nx   %2 == 0) Nx++;
  if(Nx   >P    ){cerr << "Error: Nx  > P   \n";exit(1);}
  if(xmin>=xmax){cerr << "Error: xmin>=xmax\n";exit(1);}
  dx       = (xmax-xmin)/(Nx-1);
  epsilon = 1.0e-6;
  boundary(xmin,xmax,psixmin,psipxmin,psixmax,psipxmax);
  cout <<"# #######################################\n";
  cout <<"# Estart= "<< energy <<" de= "<< de  << "\n";
  cout <<"# Nx=  " << Nx <<" eps= " << epsilon << "\n";
  cout <<"# xmin= "<< xmin << "xmax= "<< xmax
       <<" dx= "   << dx                      << "\n";
  cout <<"# psi(xmin)= "<< psixmin
       <<" psip(xmin)= "<< psipxmin           << "\n";
  cout <<"# psi(xmax)= "<< psixmax
       <<" psip(xmax)= "<< psipxmax           << "\n";
  cout <<"# #######################################\n";
  //Calculate:
  ofstream myfile("all.dat");
  myfile.precision(17);
  cout  .precision(17);
  matchold   = 0.0;
  for(iter=1;iter<=10000;iter++){
    //Determine matching point at turning point from the left:
    imatch=-1;
    for(i=0;i<Nx;i++){
      x = xmin + i * dx;
      if(imatch <  0 && (energy-V(x)) > 0.0) imatch = i;
    }
    if(imatch < 100 || imatch >= Nx-100) imatch = Nx/5-1;
    xm         = xmin + imatch*dx;
    NxL        = imatch+1;
```

```
  NxR       = Nx-imatch;
  //Evolve wavefunction from the left:
  psi  [0]  = psixmin;
  psip [0]  = psipxmin;
  psistep   = psixmin;
  psipstep  = psipxmin;
  for(i=1;i<NxL;i++){
    x       = xmin+(i-1)*dx;//this is x before the step
    RKSTEP(x,psistep,psipstep, dx);
    psi [i] = psistep;
    psip[i] = psipstep;
  }
  //use this to normalize eigenfunction to match at xm
  psinorm   = psistep;
  psipleft  = psipstep;
  //Evolve wavefunction from the right:
  psi [Nx-1]= psixmax;
  psip[Nx-1]= psipxmax;
  psistep   = psixmax;
  psipstep  = psipxmax;
  for(i=1;i<NxR;i++){
    x       = xmax-(i-1)*dx;
    RKSTEP(x,psistep,psipstep,-dx);
    psi [Nx-i-1] = psistep;
    psip[Nx-i-1] = psipstep;
  }
  psinorm   = psistep/psinorm;
  psipright = psipstep;
  //Renormalize psil so that psil(xm)=psir(xm)
  for(i=0;i<NxL-1;i++){
    psi [i] *= psinorm;
    psip[i] *= psinorm;
  }
  psipleft  *= psinorm;
  //print current solution:
  for(i=0;i<Nx;i++){
    x = xmin + i *dx;
    myfile << iter  <<" "<<energy<<" "
           << x     <<" "<<psi[i]<<" "
           <<psip[i]             <<"\n";
  }
  //matching using derivatives:
  //Careful: this can fail if psi'(xm) = 0
  //(use also |de|<1e-6 criterion)
  matchlogd =
```

```
      (psipright-psipleft)/(abs(psipright)+abs(psipleft));
    cout << "# iter ,energy ,de ,xm, logd: "
         << iter      << " "
         << energy    << " "
         << de        << " "
         << xm        << " "
         << matchlogd << "\n";
    //break condition:
    if(abs(matchlogd)<=epsilon ||
       abs(de/energy)< 1.0e-12     ) break;
    if(matchlogd * matchold < 0.0 ) de = -0.5*de;
    energy   += de;
    matchold  = matchlogd;
  }//for(iter=1;iter<=10000;iter++)
  myfile.close();
  //-----------------------------------------------------------------
  //Solution has been found and now it is stored:
  norm = integrate(psi,dx,Nx);
  norm = 1.0/sqrt(norm);
  //for(i=0;i<Nx;i++) psi[i] *= norm;
  //Count number of zeroes, add one and get energy level:
  nodes  = 1;
  psiold = psi[0];
  for(i=1;i<Nx-1;i++)
    if(abs(psi[i])      > epsilon){
      if(psiold*psi[i] < 0.0     ) nodes++;
      psiold = psi[i];
    }
  //Print final solution:
  myfile.open("psi.dat");
  cout << "Final result: E= " << energy
       << " n= "              << nodes
       << " norm= "           << norm   << endl;
  if(abs(matchlogd) > epsilon)
    cout << "Final result: SOS: logd>epsilon. logd= "
         << matchlogd                   << endl;
  myfile << "# E= "           << energy
         << " n= "            << nodes
         << " norm = "        << norm   << endl;
  for(i=0;i<Nx;i++){
    x = xmin + i * dx;
    myfile << x << " "        << psi[i]  << "\n";
  }
  myfile.close();
}//main()
```

```
//========================================================
//Simpson's rule to integrate psi(x)*psi(x) for proper
//normalization. For n intervals of width dx (n even)
//Simpson's rule is:
//int(f(x)dx) =
//(dx/3)*(f(x_0)+4 f(x_1)+2 f(x_2)+...+4 f(x_{n-1})+f(x_n))
//
//Input:   Discrete values of function psi[Nx], Nx is odd
//         Integration step dx
//Returns: Integral(psi(x)psi(x) dx)
//========================================================
double integrate(double* psi,
           const double& dx ,const int& Nx){
  double Integral;
  int    i;
  //zeroth order point:
  i         = 0;
  Integral  = psi[i]*psi[i];
  //odd  order points:
  for(i=1;i<=Nx-2;i+=2) Integral += 4.0*psi[i]*psi[i];
  //even order points:
  for(i=2;i<=Nx-3;i+=2) Integral += 2.0*psi[i]*psi[i];
  //last point:
  i         = Nx-1;
  Integral += psi[i]*psi[i];
  //measure normalization:
  Integral *=dx/3.0;

  return Integral;
}//integrate()
```

The reproduction of the results of the previous section for the infinite potential well is left as an exercise. The compilation and running of the program can be done with the commands:

```
> g++ sch.cpp schInfSq.cpp rk.cpp -o s
> ./s
# Enter energy,de,xmin,xmax,Nx:
1 0.5 -1 1 2000
# ########################################
# Estart=     1     de=    0.5
# Nx=         2001  eps=   1e-06
# xmin=      -1     xmax= 1  dx=   0.001
# psi(xmin)= 0      psip(xmin)=   1
```

```
# psi(xmax)= 0    psip(xmax)= -1
# ######################################
# iter,energy,de,xm,logd:  1 1.0000  0.500    -0.601  -0.9748
# iter,energy,de,xm,logd:  2 1.5000  0.500    -0.601  -0.6412
.....
# iter,energy,de,xm,logd: 30 2.4674 -3.815E-6 -0.601  -1.0E-6
# iter,energy,de,xm,logd: 31 2.4674  1.907E-6 -0.601   2.7E-7
Final result: E= 2.467401504516602 n= 1 norm = 1.5707965025
```

We set `xmin= -1, xmax = 1, Nx= 2000` and $\epsilon = 1$, $\delta\epsilon = 0.5$. The energy of the ground state is found to be $\epsilon_1 = 2.4674015045166016$. The wave function is stored in the file `psi.dat` and can be plotted with the `gnuplot` command

```
gnuplot> plot "psi.dat" using 1:2 with lines
```

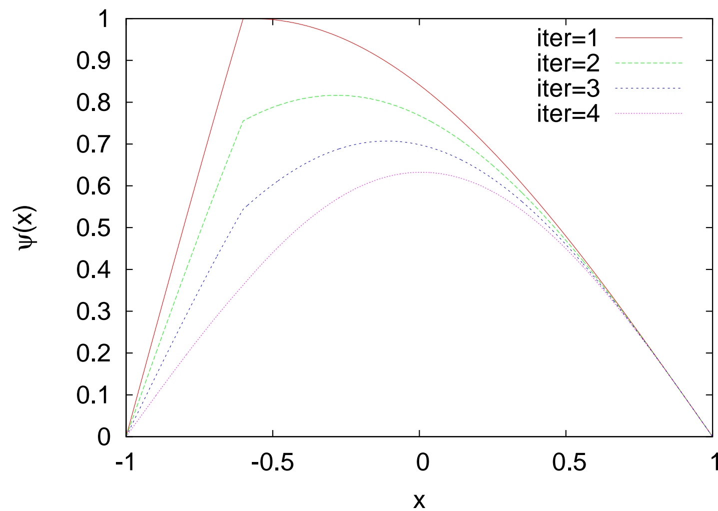

Figure 10.6: The convergence of the solutions to the solution of Schrödinger's equation for the ground state of the infinite potential well according to the discussion on page 468.

The functions computed during the iterations of the algorithm are stored in the file `all.dat`. The first column is the iteration number (here we have `iter = 0, ... 31`) and we can easily filter each one of them with the commands

```
gnuplot> plot   "<awk '$1==1' all.dat" using 3:4 w l t "iter=1"
gnuplot> replot "<awk '$1==2' all.dat" using 3:4 w l t "iter=2"
gnuplot> replot "<awk '$1==3' all.dat" using 3:4 w l t "iter=3"
gnuplot> replot "<awk '$1==4' all.dat" using 3:4 w l t "iter=4"
.....
```

which reproduce figure 10.6.

10.4 Measurements

The action of an operator $\hat{\mathcal{A}}(\hat{x},\hat{p})$ on a state $|\psi\rangle$ can be easily calculated in the position representation by its action on the corresponding wave function $\psi(x)$. The action of the operators

$$\hat{x}\psi(x) = x\psi(x) \qquad \hat{p}\psi(x) = -i\frac{\partial}{\partial x}\psi(x) \tag{10.29}$$

yield[10]

$$\hat{\mathcal{A}}(\hat{x},\hat{p})\psi(x) = \mathcal{A}(x,-i\frac{\partial}{\partial x})\psi(x)\,. \tag{10.30}$$

Using equation (10.9) we can calculate the expectation value $\langle\mathcal{A}\rangle$ of the operator $\mathcal{A}$ when the system is at the state $|\psi\rangle$. Interesting examples are the observables "position" x, "position squared" x^2, "momentum" p, "momentum squared" p^2, "kinetic energy" T, "potential energy" V, "energy" or "Hamiltonian" $H = T + V$ whose expectation values are

[10]We do not consider ordering problems of operators formed by products of non commuting operators, e.g. xp^2.

given by the relations

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{+\infty} \psi^*(x)\, x\, \psi(x)\, dx \\
\langle x^2 \rangle &= \int_{-\infty}^{+\infty} \psi^*(x)\, x^2\, \psi(x)\, dx \\
\langle p \rangle &= \int_{-\infty}^{+\infty} \psi^*(x) \left(-i\frac{\partial}{\partial x}\right) \psi(x)\, dx \\
\langle p^2 \rangle &= \int_{-\infty}^{+\infty} \psi^*(x) \left(-\frac{\partial^2}{\partial x^2}\right) \psi(x)\, dx \\
\langle T \rangle &= \frac{\hbar^2}{2mL^2} \int_{-\infty}^{+\infty} \psi^*(x) \left(-\frac{\partial^2}{\partial x^2}\right) \psi(x)\, dx \\
\langle V \rangle &= \frac{\hbar^2}{2mL^2} \int_{-\infty}^{+\infty} \psi^*(x)\, v(x)\, \psi(x)\, dx \\
\langle H \rangle &= \frac{\hbar^2}{2mL^2} \int_{-\infty}^{+\infty} \psi^*(x) \left(-\frac{\partial^2}{\partial x^2} + v(x)\right) \psi(x)\, dx\,. \qquad (10.31)
\end{aligned}$$

We remind the reader that we used the dimensionless x, p as well as equations (10.15) and (10.16). Especially interesting are the "uncertainties" $\Delta x^2 = \langle x^2 \rangle - \langle x \rangle^2$, $\Delta p^2 = \langle p^2 \rangle - \langle p \rangle^2$ that satisfy the inequality ("Heisenberg's uncertainty relation")

$$\Delta x \cdot \Delta p \geq \frac{1}{2}\,. \qquad (10.32)$$

In the previous section we described how to calculate numerically the eigenfunctions of the Hamiltonian. If $\hat{H}\psi(x) = E\psi(x)$, we obtain that $\langle H \rangle = (1/2mL^2)\epsilon$. Other operators need a numerical approximation for the calculation of their expectation values. If the values of the wave function are given at N equally spaced points $x_1, x_2, \ldots, x_N$, then we obtain

$$\frac{\partial \psi(x_i)}{\partial x} \approx \frac{\psi(x_{i+1}) - \psi(x_{i-1})}{2h} \qquad (10.33)$$

where $h = x_{i+1} - x_i$ and

$$\frac{\partial^2 \psi(x_i)}{\partial x^2} \approx \frac{\psi(x_{i+1}) - 2\psi(x_i) + \psi(x_{i-1})}{h^2}\,. \qquad (10.34)$$

Both equations entail an error of the order of $\mathcal{O}(h^2)$. Special care should be taken at the endpoints of the interval $[x_1, x_N]$. As a first approach we will use the naive approximations[11]

$$\begin{aligned}\frac{\partial\psi(x_1)}{\partial x} &\approx \frac{\psi(x_2)-\psi(x_1)}{h}\\ \frac{\partial\psi(x_N)}{\partial x} &\approx \frac{\psi(x_N)-\psi(x_{N-1})}{h}\end{aligned} \tag{10.35}$$

and

$$\begin{aligned}\frac{\partial^2\psi(x_1)}{\partial x^2} &\approx \frac{\psi(x_3)-2\psi(x_2)+\psi(x_1)}{h^2}\\ \frac{\partial^2\psi(x_N)}{\partial x^2} &\approx \frac{\psi(x_N)-2\psi(x_{N-1})+\psi(x_{N-2})}{h^2}\,.\end{aligned} \tag{10.36}$$

The relevant program that calculates $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$, Δx, Δp can be found in the file `observables.cpp` and is listed below:

```
//==========================================================
//
// File observables.cpp
// Compile: g++ observables.cpp -o o
// Usage:    ./o <psi.dat>
//
// Read in a file with a wavefunction in the format of psi.dat:
// # E= <energy> ....
// x1   psi(x1)
// x2   psi(x2)
//  ............
//
// Outputs expectation values:
// normalization Energy <x> <p> <x^2> <p^2> Dx Dp DxDp
// where Dx = sqrt(<x^2>-<x>^2) Dp = sqrt(<p^2>-<p>^2)
//  DxDp = Dx * Dp
//
//==========================================================
#include <iostream>
```

[11]See the files `observables.cpp`, `Derivatives.nb` of the accompanying software. There you can find formulas that have errors of $\mathcal{O}(h^2)$. In the examples discussed below, the influence of the $\mathcal{O}(h)$ error on the results is approximately at the fourth significant digit.

```
#include <fstream>
#include <cstdlib>
#include <string>
#include <cmath>
using namespace std;
//------------------------------------------------------------
double integrate(double* psi ,
           const double& dx  ,const int   & Nx      );
//------------------------------------------------------------

int main(int argc, char **argv){
  const int P = 50000;
  int    Nx,i;
  double xstep[P],psi[P],obs[P];
  double xav, pav, x2av, p2av, Dx, Dp, DxDp,energy,h,norm;
  string buf;
  char   *psifile;

  if(argc != 2){
    cerr << "Usage: " << argv[0] << "  <filename>\n";
    exit(1);
  }
  psifile = argv[1];
  ifstream ifile(psifile);
  if(! ifile){
    cerr << "Error reading from file " << psifile << endl;
    exit(1);
  }
  cout  << "# reading wavefunction from file: "
        << psifile << endl;
  ifile >> buf >> buf >> energy; getline(ifile,buf);
  //------------------------------------------------------------
  //Input data: psi[x]
  Nx = 0;
  while(ifile >> xstep[Nx] >> psi[Nx]){
    Nx++;
    if(Nx == P){cerr << "Too many points\n";exit(1);}
  }
  if(Nx % 2 == 0) Nx--;
  h = (xstep[Nx-1]-xstep[0])/(Nx-1);
  //------------------------------------------------------------
  //Calculate:
  //----------- norm:
  for(i=0;i<Nx;i++) obs[i] = psi[i]*psi[i];
  norm=  integrate(obs,h,Nx);
```

```
  //------------ <x> :
  for(i=0;i<Nx;i++) obs[i] = psi[i]*psi[i]*xstep[i];
  xav =  integrate(obs,h,Nx)/norm;
  //------------ <p>/i:
  obs[0] = psi[0]*(psi[1]-psi[0])/h;
  for(i=1;i<Nx-1;i++)
    obs[i] = psi[i]*(psi[i+1]-psi[i-1])/(2.0*h);
  obs[Nx-1]=psi[Nx-1]*(psi[Nx-1]-psi[Nx-2])/h;
  pav = -integrate(obs,h,Nx)/norm;
  //------------ <x^2>:
  for(i=0;i<Nx;i++) obs[i] = psi[i]*psi[i]*xstep[i]*xstep[i];
  x2av=  integrate(obs,h,Nx)/norm;
  //------------ <p^2>:
  obs[0] = psi[0]*(psi[2]-2.0*psi[1]+psi[0])/(h*h);
  for(i=1;i<Nx-1;i++)
    obs[i] = psi[i]*(psi[i+1]-2.0*psi[i]+psi[i-1])/(h*h);
  obs[Nx-1] = psi[Nx-1] *
      (psi[Nx-1]-2.0*psi[Nx-2]+psi[Nx-3])/(h*h);
  p2av= -integrate(obs,h,Nx)/norm;
  //------------ Dx
  Dx = sqrt(x2av - xav*xav);
  //------------ Dp
  Dp = sqrt(p2av - pav*pav);
  //------------ Dx.Dp
  DxDp = Dx*Dp;
  //Print results:
  cout.precision(17);
  cout << "# norm E <x> <p>/i <x^2> <p^2> Dx Dp DxDp\n";
  cout << norm   << " "
       << energy << " "
       << xav    << " "
       << pav    << " "
       << x2av   << " "
       << p2av   << " "
       << Dx     << " "
       << Dp     << " "
       << DxDp   << endl;

}//main()
//==========================================================
//Simpson's rule to integrate psi(x).
//For n intervals of width dx (n even)
//Simpson's rule is:
//int(f(x)dx) =
//(dx/3)*(f(x_0)+4 f(x_1)+2 f(x_2)+...+4 f(x_{n-1})+f(x_n))
```

```
//
//Input:   Discrete values of function psi[Nx], Nx is odd
//         Integration step dx
//Returns: Integral(psi(x) dx)
//=========================================================
double integrate(double* psi,
                 const double& dx ,const int& Nx){
  double Integral;
  int    i;
  //zeroth order point:
  i         = 0;
  Integral  = psi[i];
  //odd  order points:
  for(i=1;i<=Nx-2;i+=2) Integral += 4.0*psi[i];
  //even order points:
  for(i=2;i<=Nx-3;i+=2) Integral += 2.0*psi[i];
  //last point:
  i         = Nx-1;
  Integral += psi[i];
  //measure normalization:
  Integral *=dx/3.0;

  return Integral;
}//integrate()
```

The program needs to read in the wave function at the points $x_0, \ldots, x_{\texttt{Nx}-1}$ in the format produced by the program in `sch.cpp`. The first line should have the energy written at the 3rd column, whereas from the 2nd line and on there should be two columns with the $(x_i, \psi(x_i))$ pairs. It is not necessary to have the wave function properly normalized, the program will take care of it. If this data is stored in a file `psi.dat`, then the program can be used by running the commands

```
> g++  observables.cpp -o obs
> ./obs psi.dat
```

The program prints the normalization constant of $\psi(x)$, the value of the energy[12], $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle/i$, $\langle p^2\rangle$, Δx, Δp and $\Delta x \cdot \Delta p$ to the `stdout`.

Some details about the program: In order to read in the data from the file `psi.dat` we use the variables `argc` and `argv`. These contain the information on the number of arguments and the arguments of the

[12]The one read from the file. It is not calculated from the data.

command line. If the command line comprises of `n` *words*, then `argc=n`. These words are stored in an array of C-style strings `argv[0]`, `argv[1]`, ... , `argv[argc-1]`. The first argument `argv[0]` is the name of the program, therefore the lines

```
  if(argc != 2){
    cerr << "Usage: " << argv[0] << "  <filename>\n";
    exit(1);
  }
```

check if there are two arguments on the command line, including the path to the executable file. If not, it prints an error message containing the name of the program and exits:

```
> ./o
Usage: ./o  <filename>
```

The variables `argc` and `argv` must be declared as arguments to the `main()` function:

```
int main(int argc, char **argv){
    .....
}
```

The variable `argv` is an array of C-style strings[13], i.e. and array of an array of characters and can be declared as a pointer to a pointer to `char`.

The statements

```
  ifstream ifile("psi.dat");
  if(! ifile){ exit(1); }
```

attempt to open a file `psi.dat` for input and, if this fails, the program is terminated.

The statements

[13]A C-style string, not to be confused with variables declared as `string`, is an array of "null terminated" characters. This means that the sequence of characters ends with the character `'\0'`. Functions that treat such objects, detect the end of the string using this convention.

```
Nx = 0;
while(ifile >> xstep[Nx] >> psi[Nx]){Nx++;}
```

read in a pair of doubles and store them in the arrays `xstep` and `psi`. The loop terminates when it reaches the end of file or when it fails to read input that can be converted to two doubles. In the end, `Nx` stores the number of pairs read into the arrays.

The rest of the commands are applications of equations (10.33), (10.34), (10.35) and (10.36) to the formulas (10.31) and the reader is asked to study them carefully. The program uses the function `integrate` in order to perform the necessary integrals.

10.5 The Anharmonic Oscillator - Again...

In the previous chapter 9 we studied the quantum mechanical harmonic and anharmonic oscillator in the representation of the energy eigenstates of the harmonic oscillator $|n\rangle$. In this section we will revisit the problem by using the position representation. We will calculate the eigenfunctions $\psi_{n,\lambda}(x)$ that diagonalize the Hamiltonian (9.15), which are the solutions of the Schrödinger equation. By setting $L=\sqrt{\hbar/m\omega}$ in equation (10.13), equation (10.12) becomes

$$\psi''(x) = -(\epsilon - v(x))\psi(x)\,, \tag{10.37}$$

where $v(x)=x^2+2\lambda x^4$. For $\lambda=0$ we obtain the harmonic oscillator with

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}\mathrm{e}^{-x^2/2}H_n(x)\,,\epsilon_n = 2\left(n+\frac{1}{2}\right)\,, \tag{10.38}$$

where $H_n(x)$ are the Hermite polynomials.

We start with the simple harmonic oscillator where the exact solution is known. The potential and the initial conditions are programmed in the file `schHOC.cpp`. The changes that we need to make concern the functions `V(x)`, `boundary(xmin, xmax, psixmin, psipxmin, psixmax, psipxmax)`:

```
//========================================================
//file: schHOC.cpp
```

```
................
double V  (const double& x ){
  return x*x;
}
//-------- boundary conditions:
void
boundary  (const double& xmin,const double& xmax    ,
                 double& psixmin,   double& psipxmin,
                 double& psixmax,   double& psipxmax){

  psixmin  =  exp(-0.5*xmin*xmin);
  psipxmin = -xmin*psixmin;
  psixmax  =  exp(-0.5*xmax*xmax);
  psipxmax = -xmax*psixmax;
}
...............
```

The code omitted at the dots is identical to the one discussed in the previous section. The initial conditions are inspired by the asymptotic behavior of the solutions to Schrödinger's[14] equation $\psi_0(x) \sim \mathrm{e}^{-x^2/2}$, $\psi_n'(x) \sim -x\psi_n(x)$. You are encouraged to test the influence of other choices on the results. The results are depicted in figure 10.7 where, besides the qualitative agreement, their difference from the known values (10.38) is also shown. This difference turns out to be of the order of 10^{-11}–10^{-7}. The values of the energy ϵ_n for $n \leq 14$ are in agreement with (10.38) with relative accuracy better than 10^{-9}.

Then we calculate the expectation values $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$, Δx and Δp. These are easily calculated using equations (9.4) and (9.8). We see that $\langle x\rangle = \langle n|\,(a^\dagger + a)/\sqrt{2}\,|n\rangle = 0$, $\langle p\rangle = \langle n|\,i(a^\dagger - a)/\sqrt{2}\,|n\rangle = 0$, whereas

$$\langle x^2\rangle = \langle p^2\rangle = \langle n|\,\frac{1}{2}(a^\dagger a + aa^\dagger)\,|n\rangle = \left(n + \frac{1}{2}\right)\,. \tag{10.39}$$

The program `observables.cpp` calculates $\langle x\rangle = 0$ with accuracy $\sim 10^{-6}$ and $\langle p\rangle = 0$ with accuracy $\sim 10^{-11}$. The expectation values $\langle x^2\rangle$, $\langle p^2\rangle$ are shown in table 10.2.

Next, the calculation is repeated for the anharmonic oscillator for $\lambda = 0.5, 2.0$. We copy the file `schHOC.cpp` to `schUOC.cpp` and change the potential in the function `V(x)`:

[14] In fact $\psi_n(x) \sim x^n \mathrm{e}^{-x^2/2}$ which we neglect. This does not influence the results for the values of n studied here. Examine if this is necessary for larger values of n.

n	$\langle x^2 \rangle$	$\langle p^2 \rangle$	$\Delta x \cdot \Delta p$
0	0.500000000	0.4999977	0.4999989
1	1.500000284	1.4999883	1.4999943
2	2.499999747	2.4999711	2.4999854
3	3.499999676	3.4999441	3.4999719
4	4.499999607	4.4999082	4.4999539
5	5.499999520	5.4998633	5.4999314
6	6.499999060	6.4998098	6.4999044
7	7.499999642	7.4995484	7.4997740
8	8.499999715	8.4994203	8.4997100
9	9.499999837	9.4992762	9.4996380
10	10.500000012	10.4991160	10.4995580
11	11.499999542	11.4994042	11.4997019
12	12.499999610	12.4992961	12.4996479
13	13.499999705	13.4991791	13.4995894
14	14.499999835	14.4990529	14.4995264

Table 10.2: The expectation values $\langle x^2 \rangle$, $\langle p^2 \rangle$ and the product $\Delta x \cdot \Delta p$ for the simple harmonic oscillator for the states $|n\rangle$, $n = 0, \ldots, 14$.

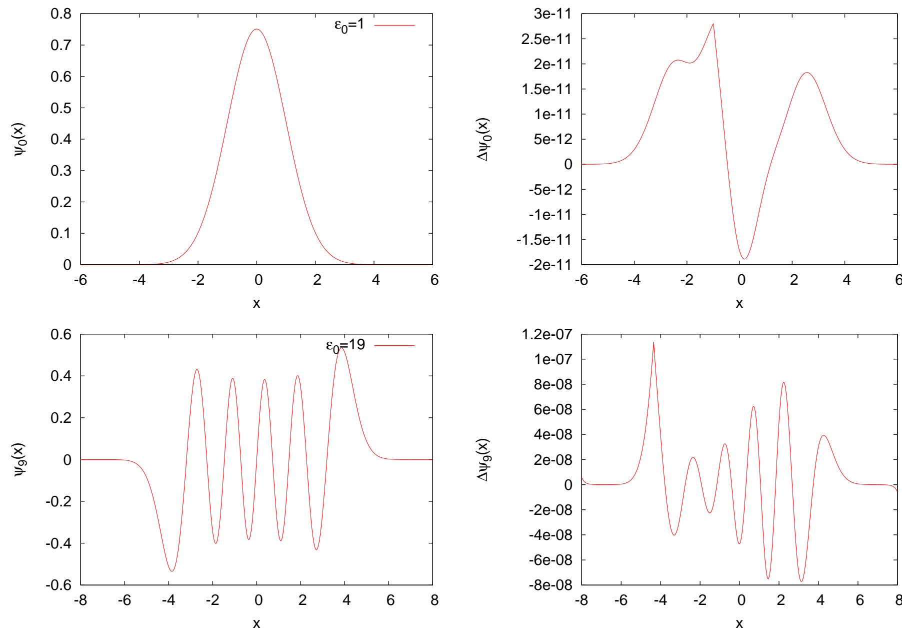

Figure 10.7: The eigenfunctions $\psi_0(x)$, $\psi_9(x)$ calculated by the program in `sch.cpp`, `schHOC.cpp`. The plot to the right shows the difference of the results from the known values (10.38).

```
//==========================================================
// file: schUOC.cpp
.................
double V  (const double& x ){
  double lambda = 2.0;
  return x*x+2.0*lambda*x*x*x*x;
}
.................
```

The wave functions are plotted in figure 10.8. We see that by increasing λ the particle becomes more confined in space as expected. In table 10.3 we list the values of the energy ϵ_n for $n = 0, ..., 9$. By increasing λ, $\epsilon_n(\lambda)$ is increased. Table 10.4 lists the expectation values $\langle x^2\rangle$, $\langle p^2\rangle$ and $\Delta x \cdot \Delta p$ for the anharmonic oscillator for the states $|n\rangle$, $n = 0, \ldots, 9$. By increasing λ, $\Delta x = \sqrt{\langle x^2\rangle}$ is decreased and $\Delta p = \sqrt{\langle p^2\rangle}$ is increased. The product of the uncertainties $\Delta x \cdot \Delta p$ seems to be quite close to the corresponding values for the harmonic oscillator. The results should be

n	ϵ_n	$\epsilon_{n,\lambda=0.5}$	$\epsilon_{n,\lambda=2.0}$
0	1.0000	1.3924	1.9031
1	3.0000	4.6488	6.5857
2	5.0000	8.6550	12.6078
3	7.0000	13.1568	19.4546
4	9.0000	18.0576	26.9626
5	11.0000	23.2974	35.0283
6	13.0000	28.8353	43.5819
7	15.0000	34.6408	52.5723
8	17.0000	40.6904	61.9598
9	19.0000	46.9650	71.7129

Table 10.3: The values of the energy ϵ_n for the harmonic and anharmonic oscillator for $\lambda = 0.5, 2.0$. The values of the corresponding energy levels are increased with increasing λ.

	$\lambda = 0.5$			$\lambda = 2.0$		
n	$\langle x^2\rangle$	$\langle p^2\rangle$	$\Delta x \cdot \Delta p$	$\langle x^2\rangle$	$\langle p^2\rangle$	$\Delta x \cdot \Delta p$
0	0.3058	0.8263	0.5027	0.2122	1.1980	0.5042
1	0.8013	2.8321	1.5064	0.5408	4.2102	1.5089
2	1.1554	5.3848	2.4944	0.7612	8.1513	2.4909
3	1.4675	8.2819	3.4862	0.9582	12.6501	3.4816
4	1.7509	11.4545	4.4784	1.1370	17.5955	4.4728
5	2.0141	14.8599	5.4707	1.3029	22.9169	5.4643
6	2.2617	18.4691	6.4631	1.4590	28.5668	6.4560
7	2.4970	22.2607	7.4555	1.6074	34.5103	7.4478
8	2.7220	26.2184	8.4478	1.7492	40.7206	8.4397
9	2.9384	30.3289	9.4402	1.8856	47.1762	9.4316

Table 10.4: The expectation values $\langle x^2\rangle$, $\langle p^2\rangle$ and the product $\Delta x \cdot \Delta p$ for the anharmonic oscillator for the states $|n\rangle$, $n = 0, \ldots, 9$. Note the decrease of $\Delta x = \sqrt{\langle x^2\rangle}$ and the increase of $\Delta p = \sqrt{\langle p^2\rangle}$ with increasing λ. The uncertainty product $\Delta x \cdot \Delta p$ seems to take values close to the corresponding ones of the harmonic oscillator for both values of λ. Compare the results in this table with the ones in table 9.1.

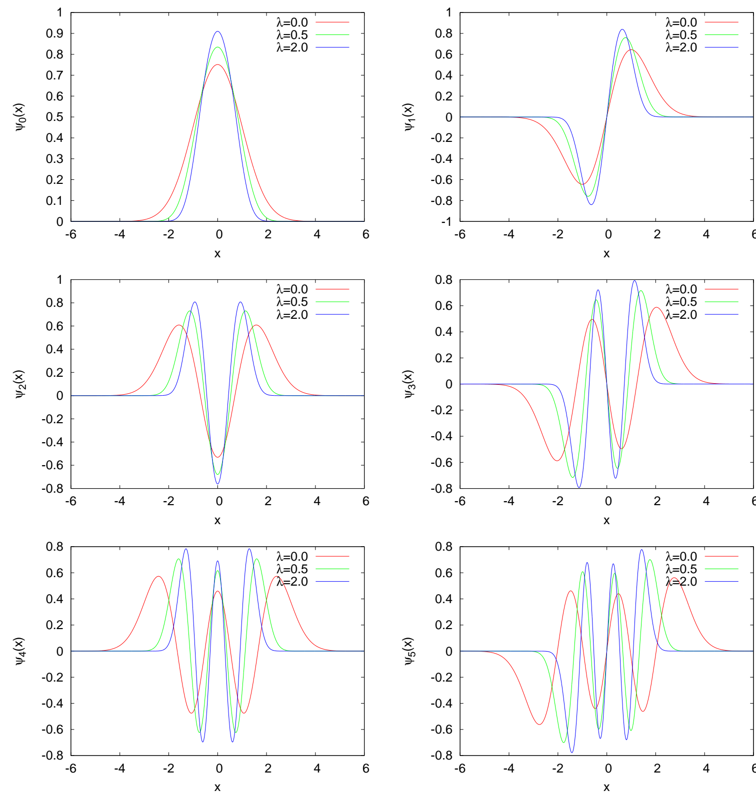

Figure 10.8: The wave functions of the anharmonic oscillator $\psi_{n,\lambda}(x)$ for $n = 0, 1, 2, 3, 4, 5$ and $\lambda = 0.5, 2.0$ compared to the respective ones of the simple harmonic oscillator. Increasing λ yields stronger confinement of the particle in space.

compared with the ones obtained in table 9.1 of chapter 9.

10.6 The Lennard–Jones Potential

The Lennard–Jones potential is a simple phenomenological model of the interaction between two neutral atoms in a diatomic molecule. This is given by

$$V(x) = 4V_0 \left\{ \left(\frac{\sigma}{x}\right)^{12} - \left(\frac{\sigma}{x}\right)^{6} \right\} . \tag{10.40}$$

The repulsive term describes the Pauli interaction due to the overlapping of the electron orbitals, whereas the attractive term describes the Van der Waals force. The first one dominates at short distances and the latter at long distances. We choose $L = \sigma$ in (10.13) and define $v_0 = 2m\sigma^2 V_0/\hbar^2$. Equation (10.40) becomes

$$v(x) = 4v_0 \left\{ \left(\frac{1}{x}\right)^{12} - \left(\frac{1}{x}\right)^{6} \right\}, \tag{10.41}$$

whereas the eigenvalues ϵ_n are related to the energy values E_n by

$$\epsilon_n = 4v_0 \left(\frac{E_n}{V_0}\right). \tag{10.42}$$

The plot of the potential is shown in figure 10.5 for $v_0 = 250$. The minimum is located at $x_m = 2^{1/6} \approx 1.12246$ and its value is $-v_0$. The code for this potential is in the file `schLJ.cpp`. The necessary changes to the code discussed in the previous sections are listed below:

```
//==========================================================
//file: schLJ.cpp
.................
//--------
double V  (const double& x ){
  double V0 = 250.0;
  return  4.0*V0*(pow(x,-12.0)-pow(x,-6.0));
}
//-------- boundary conditions:
void
boundary  (const double& xmin,const double& xmax    ,
                 double& psixmin,   double& psipxmin,
                 double& psixmax,   double& psipxmax){

  psixmin    =  exp(-xmin*sqrt(abs(energy-V(xmin))));
  psipxmin   =  sqrt(abs(energy-V(xmin)))*psixmin;
  psixmax    =  exp(-xmax*sqrt(abs(energy-V(xmax))));
  psipxmax   = -sqrt(abs(energy-V(xmax)))*psixmax;
}
..................
```

For the integration we choose $v_0 = 250$ and `xmin = 0.7`, $4 <$`xmax` < 10. The results are plotted in figure 10.9. There are four bound states.

n	ϵ_n	$\langle x \rangle$	$\langle p \rangle$	$\langle x^2 \rangle$	$\langle p^2 \rangle$	Δx	Δp	$\Delta x \cdot \Delta p$
0	-173.637	1.186	1.0e-10	1.415	34.193	0.091	5.847	0.534
1	-70.069	1.364	6.0e-11	1.893	56.832	0.178	7.539	1.338
2	-18.191	1.699	-4.5e-08	2.971	39.480	0.291	6.283	1.826
3	-1.317	2.679	-2.6e-08	7.586	9.985	0.638	3.160	2.016

Table 10.5: The results for the Lennard-Jones potential with $v_0 = 250$. We find 4 bound states.

The first two ones are quite confined within the potential well whereas the last ones begin to "spill" out of it. Table 10.5 lists the results. We observe that $\langle p \rangle = 0$ within the attained accuracy as expected for real, bound states[15].

10.7 Problems

10.1 Add the necessary code to the program in the file `well.cpp` so that the final wave function printed in the file `psi.dat` is properly normalized. The integral $\int_{-1}^{1} \psi(x)\psi(x)\,dx$ can be computed using the Simpson rule

$$\int_a^b f(x)\,dx = (h/3)\,(f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\,.)$$

The interval $[a, b]$ is discretized by n points $x_0 = a, x_1, x_2, \ldots, x_n = b$ where n is even. Each interval $[x_i, x_{i+1}]$ has width h.

10.2 Add the necessary code to the program in the file `well.cpp` in order to calculate the number of nodes (zeroes) of the wave function. Using this result, the program should print the level n of the calculated wave function $\psi_n(x)$.

10.3 Calculate the wave functions of the energy eigenstates for the potential (10.27) with $v_0 < 0$. This is the problem of the (finite)

[15] For $\psi(+\infty) = \psi(0) = 0$ and $\psi^*(x) = \psi(x)$ we have that $i\langle p \rangle/\hbar = \int_0^{+\infty} \psi(x)(d/dx)\psi(x)\,dx = -\int_0^{+\infty}(d/dx)\psi(x)\psi(x)\,dx = 0$.

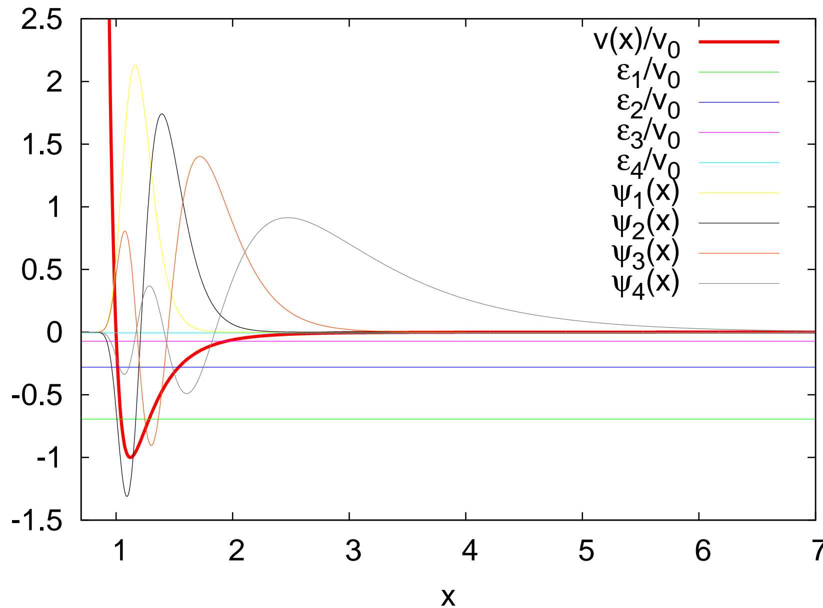

Figure 10.9: The four bound states for the Lennard-Jones potential with $v_0 = 250$. The bold red line is the potential $v(x)/v_0$. We plot the energy levels ϵ_n/v_0 and the corresponding wave functions.

potential well. Solve the problem for $v_0 = -100$ and $a = 0.3$. How many bound states do you find? Next study the influence of the wall on the solutions. Introduce a parameter b so that $v(x \geq b) = +\infty$ and study the dependence of the solutions on b. Take $b = 0.35, 0.4, 0.5, 0.6, 0.8, 1.0, 1.5, 2.0, 2.5, 3.0$ and compute the difference of the first two energy eigenvalues. Estimate the accuracy of the method. Next lower the value of $|v_0|$ until there is no bound state. What is the relation between a and v_0 when this happens? Compare with the analytic result which you know from your quantum mechanics course.
Hint: For the largest values of b, take `Nx > 1000`. When convergence is not achieved decrease `epsilon`.

10.4 Set $v_0 = 1000, 5000$ to the double well potential. Observe the (almost) degenerate states and plot the wave functions $\psi_{\pm,n} = (1/\sqrt{2})(\psi_n(x) \pm \psi_{n+1}(x))$, where n is odd. Compare the results with the correspond-

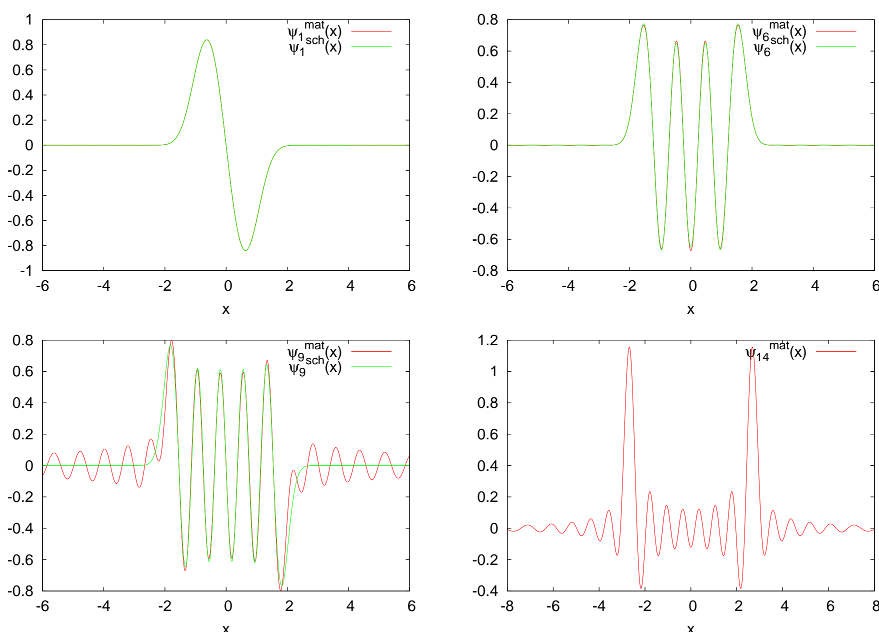

Figure 10.10: Comparison of the results of the calculation of the wave functions $\psi_{n,\lambda}(x)$ of the anharmonic oscillator for $\lambda = 2.0$ using the methods described in problem 12. The wave functions $\psi^{\text{sch}}(x)$ are the wave functions $\psi_{n,\lambda}(x)$ calculated using the methods described in this chapter. The wave functions $\psi^{\text{mat}}(x)$ are the wave functions $\psi_{n,\lambda}(x)$ calculated using the methods described in chapter 9 for Hilbert space dimension $N = 40$. Note the difference at large x. This is because the amplitudes $\psi_{n,\lambda}(x) = \langle x|n\rangle_\lambda$ for large x receive contributions from states $|m\rangle$ with large m (why?).

ing energy levels and eigenfunctions of the infinite square well. Increase v_0 to the point where you cannot solve the problem numerically.
Hint: For large v_0 the numerical effort is increased. For $|x| < a$ the wave function is almost zero and it is hard to obtain the non trivial wave function for $a < |x| < 1$. As the accuracy deteriorates, you should increase `epsilon` in the program so that convergence is achieved relatively fast.

10.5 Repeat problems 3 and 4 using the program `sch.cpp`. Compare the results.

10.6 Study the bound states in the potentials

$$v(x) = \begin{cases} 0 & a < |x| \\ -V_0 & b < |x| < a \\ -V_1 & |x| < b \end{cases}$$

for $a = 1, b = 0.2, V_0 = 100, V_1 = 0, 50$ and

$$v(x) = \begin{cases} V_1 & x < 0 \\ -V_0 & 0 < x < a \\ 0 & a < x \end{cases}$$

for $a = 1, V_0 = 100, V_1 = +\infty, 10, 100$ and

$$v(x) = \begin{cases} V_1 & a < |x| \\ -V_0 & b < |x| < a \\ 0 & c < |x| < b \\ -V_0 & |x| < c \end{cases}$$

for $a = 1, b = 0.7, c = 0.6, 0.3, V_0 = 100, V_1 = +\infty, 10, 0$. In each case calculate $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$, Δx, Δp, $\Delta x \cdot \Delta p$.

10.7 Write a program that calculates the probability that a particle is found in an interval $[x_1, x_2]$ given the wave function calculated by the program in the file `sch.cpp`. Apply your program on the results of the previous problem and calculate the intervals $[-x_1, x_1]$ where the probability to find the particle inside them is equal to $1/3$.

10.8 Fill the tables 10.3 and 10.4 with the results for $\lambda = 0.2$, 0.7, 1.0, 1.3, 1.6, 2.5, 3.0 and plot each expectation value as a function of λ.

10.9 A particle is under the influence of a potential

$$V(x) = \frac{\hbar^2}{2m}\alpha^2\lambda(\lambda-1)\left\{\frac{1}{2} - \frac{1}{\cosh^2(\alpha x)}\right\} .$$

The energy spectrum is given by

$$E_n = \frac{\hbar^2}{2m}\alpha^2\left\{\frac{\lambda(\lambda-1)}{2} - (\lambda-1-n)^2\right\}$$

for the values of $n = 0, 1, 2, \ldots$ for which $E_n > V_{\min}$. Calculate the energy levels ϵ_n of the bound states numerically by setting $L = 1/\alpha$ in equation (10.13) and $\lambda = 4$. Plot the potential $v(x)$ and the corresponding eigenfunctions. Calculate the expectation values of the position and momentum, the uncertainties in position and momentum and their product. Repeat for $\lambda = 2, 6, 8, 10$.

10.10 Write a program that reads in a wavefunction and calculates the expectation value of the Hamiltonian

$$\langle \hat{H} \rangle = \int_{-\infty}^{+\infty} \psi(x) \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) \psi(x) \, dx \, ,$$

by assuming that $\psi(x)$ is real. Calculate $\psi_n(x)$ for the harmonic oscillator for $n = 1, \ldots, 10$ and show (numerically) that $\langle \hat{H} \rangle_n = E_n$.

10.11 Consider a particle in the Morse potential

$$V(x) = D_e \left\{ \left(1 - \mathrm{e}^{-a(r-r_e)}\right)^2 - 1 \right\} \, .$$

Calculate the energy spectrum of the bound states. Choose $L = 1/a$, $x = ar$, $x_e = ar_e$, $\lambda^2 = 2mD_e/a^2\hbar^2$ and obtain

$$v(x) = \lambda^2 \left(\mathrm{e}^{-2(x-x_e)} - 2\mathrm{e}^{-(x-x_e)} \right) \, .$$

Compare your results with the known analytic solutions

$$\epsilon_n = \left(\lambda - n - \frac{1}{2} \right)^2$$

$$\psi_n(z) = N_n z^{\lambda-n-1/2} \mathrm{e}^{-z/2} L_n^{2\lambda-2n-1}(z)$$

where $z = 2\lambda \mathrm{e}^{-(x-x_e)}$, $N_n = n! \sqrt{(2\lambda - 2n - 1)/(\Gamma(n+1)\Gamma(2\lambda - n))}$, and $L_n^{\alpha}(z)$ is a Laguerre polynomial given by $L_n^{\alpha}(z) = (z^{-\alpha}\mathrm{e}^{z}/n!)$ $(d^n/dz^n)(z^{n+\alpha}\mathrm{e}^{-z}) = (\Gamma(\alpha{+}2)/\left(\Gamma(n{+}2)\,\Gamma(\alpha{-}n{+}2)\right)\, {}_1F_1(-n, \alpha{+}1, z)$. You can take $\lambda = 4$, $x_e = 1$ and calculate $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$, $\langle p^2 \rangle$, Δx, Δp, $\Delta x \cdot \Delta p$.

10.12 Calculate the wave functions of the eigenstates of the Hamiltonian for the anharmonic oscillator for $\lambda = 2.0$ and $n = 0, \ldots, 15$. Calculate

the wavefunctions using the program `anharmonic.cpp` of chapter 9 for $N = 15, 40, 100$ and compare the two results.
Hint: Write a program that calculates the energy eigenfunctions of the simple harmonic oscillator

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \mathrm{e}^{-x^2/2} H_n(x)$$

where the Hermite polynomials satisfy the relations

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad H_0(x) = 1, \quad H_1(x) = 2x\,.$$

The program `anharmonic.cpp` calculates the eigenstates of the anharmonic oscillator

$$|n\rangle_\lambda = \sum_{m=0}^{N-1} \mathtt{H[n][m]}\, |m\rangle$$

by storing the coefficients of the linear expansion in the elements of the array `H[N][N]`. The same relation holds for the corresponding wave functions $\psi_{n,\lambda}(x)$, $\psi_n(x)$. From $\psi_n(x)$ and `H[n][m]` calculate $\psi_{n,\lambda}(x)$ for $-8 < x < 8$ and determine the accuracy achieved by the calculation for each N. For which values of x do you obtain large discrepancies between your results? Remember that for large x, the states of high energy contribute more than for small x. Figure 10.10 can help you understanding this statement.

Chapter 11

The Random Walker

In this chapter we will study the typical path followed by a ... drunk when he decides to start walking from a given position. Because of his drunkenness, his steps are in random directions and uncorrelated. These are the basic properties of the models that we are going to study. These models are related to specific physical problems like the Brownian motion, the diffusion, the motion of impurities in a lattice, the large distance properties of macromolecules etc. In the physics of elementary particles random walks describe the propagation of free scalar particles and they most clearly arise in the Feynman path integral formulation of the euclidean quantum field theory. Random walks are precursors to the theory of random surfaces which is related to the theory of two dimensional "soft matter" membranes, two dimensional quantum gravity and string theory [46].

The geometry of a typical path of a simple random walk is not classical and this can be seen from two of its non classical properties. First, the average distance traveled by the random walker is proportional to the *square root* of the time traveled, i.e. the classical relation $r = vt$ does not apply. Second, the geometry of the path of the random walker has fractal dimension which is larger than one[1]. Similar structures arise in the study of quantum field theories and random surfaces, where the non classical properties of a typical configuration can be described by appropriate generalizations of these concepts. For further study we refer to [7, 45, 46, 47].

[1] More precisely, the Hausdorff dimension of the simple random walk is $d_H = 2$.

In order to simulate a stochastic system on the computer, it is necessary to use random number generators. In most of the cases, these are deterministic algorithms that generate a sequence of *pseudorandom* numbers distributed according to a desired distribution. The heart of these algorithms generate numbers distributed uniformly from which we can generate any other complex distribution. In this chapter we will study simple random number generators and learn how to use high quality, research grade, portable, random number generators.

11.1 (Pseudo)Random Numbers

The production of pseudorandom[2] numbers is at the heart of a Monte Carlo simulation. The algorithm used in their production is deterministic: The generator is put in an initial state and the sequence of pseudorandom numbers is produced during its "time evolution". The next number in the sequence is determined from the current state of the generator and it is in this sense that the generator is deterministic. Same initial conditions result in exactly the same sequence of pseudorandom numbers. But the "time evolution" is chaotic and "neighboring" initial states result in very different, uncorrelated, sequences. The chaotic properties of the generators is the key to the pseudorandomness of the numbers in the sequence: the numbers in the sequence decorrelate exponentially fast with "time". But this is also the weak point of the pseudorandom number generators. Bad generators introduce subtle correlations which produce systematic errors. Truly random numbers (useful in cryptography) can be generated by using special devices based on e.g. radioactive decay or atmospheric noise[3]. Almost random numbers are produced by the special files `/dev/random` and `/dev/urandom` available on unix systems, which read bits from an entropy pool made up from several external sources (computer temperature, device noise etc).

---

[2]We can't define what a random process is, only what it isn't. Outcomes which lack discernable patterns are assumed to be random. If there is no way to predict an event, we say it is random...Thus, there is no definition of what randomness is, only definitions of what it isn't. See Chris Wetzel, "Can you behave randomly?", `http://faculty.rhodes.edu/wetzel/random/level23intro.html`.

[3]There are online services which provide such sequences like `www.random.org`, `www.fourmilab.ch/hotbits/` and others.

Pseudorandom number generators, however, are the source of random numbers of choice when *efficiency* is important. The most popular generators are the *modulo* generators (D.H. Lehmer, 1951) because of their simplicity. Their state is determined by only one integer x_{i-1} from which the next one x_i is generated by the relation

$$x_i = a\, x_{i-1} + c \pmod m \tag{11.1}$$

for appropriately chosen values of a, c and m. In the bibliography, there is a lot of discussion on the good and bad choices of a, c and m, which depend on the programming language and whether we are on a 32–bit or 64–bit systems. For details see the chapter on random numbers in [8].

The value of the integer m determines the maximum period of the sequence. It is obvious that if the sequence encounters the same number after k steps, then the exact same sequence will be produced and k will be the period of the sequence. Since there are at most m different numbers, the period is at most equal to m. For a bad choice of a, c and m the period will be much smaller. But m cannot be arbitrarily large since there is a maximum number of bits that computers use for the storage of integers. For 4-byte (32 bit) unsigned integers the maximum number is $2^{32}-1$, whereas for signed integers $2^{31}-1$. One can prove[4] that a good choice of a, c and m results in a sequence which is a permutation $\{\pi_1, \pi_2, \ldots, \pi_m\}$ of the numbers $1, 2, \ldots, m$. This is good enough for simple applications that require fast random number generation but for serious calculations one has to carefully balance efficiency with quality. Good quality random generators are more complicated algorithms and their states are determined by more than one integer. If you need the source code for such generators you may look in the bibliography, like in e.g. [4], [5], [8], [49], [50]. If portability is an issue, we recommend the MIXMAX [49], RANLUX [50] or the Marsaglia, Zaman and Tsang generator. The code for MIXMAX can also be found in the accompanying software, whereas the MZT generator can be found in Berg's book/site [5]. RANLUX is part of the C++ Standard Library.

In order to understand the use of random number generators, but also in order to get a feeling of the problems that may arise, we list the code of the two functions `naiveran()` and `drandom()`. The first one is obviously problematic and we will use it in order to study certain

[4]See Knuth [48].

type of correlations that may exist in the generated sequences of random numbers. The second one is much better and can be used in non–trivial applications, like in the random walk generation or in the Ising model simulations studied in the following chapters.

The function `naiveran()` is a simple application of equation (11.1) with $a = 1277$, $c = 0$ and $m = 2^{17}$:

```
//============================================
//File: naiveran.cpp
//Program to demonstrate the usage of a modulo
//generator with a bad choice of constants
//resulting in strong pair correlations between
//generated numbers
//============================================
static  int    ibm     = 13337;
double naiveran(){
  const int    mult    = 1277;
  const int    modulo  = 131072 ; // equal to 2^17
  const double rmodulo = modulo;

  ibm    *= mult;
  ibm     = ibm % modulo;
  return ibm/rmodulo;
}
```

The function `drandom()` is also an application of the same equation, but now we set $a = 7^5$, $c = 0$ and $m = 2^{31} - 1$. This is the choice of Lewis, Goodman and Miller (1969) and provides a generator that passes many tests and, more importantly, it has been used countless of times successfully. One technical problem is that, when we multiply x_{i-1} by a, we may obtain a number which is outside the range of 4-byte integers and this will result in an "integer overflow". In order to have a fast and portable code, it is desirable to stay within the range of the $2^{31} - 1$ positive, 32-bit (4 byte), signed integers. Schrage has proposed to use the relation

$$(ax_{i-1}) \mod m = \begin{cases} a\,(x_{i-1} \mod q) - r\left[\frac{x_{i-1}}{q}\right] & \text{if it is} \geq 0 \\ a\,(x_{i-1} \mod q) - r\left[\frac{x_{i-1}}{q}\right] + m & \text{if it is} < 0 \end{cases} \tag{11.2}$$

where $m = aq + r$, $q = [m/a]$ and $r = m \mod a$. One can show that

if $r < q$ and if $0 < x_{i-1} < m-1$, then $0 \leq a(x_{i-1} \mod q) \leq m-1$, $0 \leq r[x_{i-1}/q] \leq m-1$ and that (11.2) is valid. The period of the generator is $2^{31}-2 \approx 2\times 10^9$. The proof of the above statements is left as an exercise to the reader.

```
//=====================================================
//File: drandom.cpp
//Implementation of the Schrage algorithm for a
//portable modulo generator for 32 bit signed integers
//(from numerical recipes)
//
//returns uniformly distributed pseudorandom numbers
// 0.0 < x < 1.0 (0 and 1 excluded)
//period: 2**31-2 = 2 147 483 646
//=====================================================
#include <iostream>
#include <fstream>
#include <iomanip>
using namespace std;
static  int seed = 323412;
double drandom(){
  const int    a = 16807;      //a = 7**5
  const int    m = 2147483647;//m = a*q+r = 2**31-1
  const int    q = 127773;     //q = [m/a]
  const int    r = 2836;       //r = m % a
  const double f = 1.0/m;
  int      p;
  double dr;

 compute:
  p    = seed/q;                 //p = [seed/q]
  //seed = a*(seed % q)-r*[seed/q] = (a*seed) % m
  seed = a*(seed-q*p) - r*p;
  if(seed < 0) seed +=m;
  dr=f*(seed-1);
  if(dr <= 0.0 || dr >= 1.0) goto compute;
  return dr;
}
```

The line that checks the result produced by the generator is necessary in order to check for the number 0 which appears once in the sequence. This adds a $10-20\%$ overhead, depending on the compiler. If you don't care about that, you may remove the line. Note that the number `seed`

is in the global scope only for functions in this file (due to the `static` keyword).

Now we will write a program in order to test the problem of correlations in the sequence of numbers produced by `naiveran()`. The program will produce pairs of integers (i,j), where $0 \leq i,j < 10000$, which are subsequently mapped on the plane. This is done by taking the integer part of the numbers $L\,u$ with $L = 10000$ and $0 \leq u < 1$ is the random number produced by the generator:

```
//============================================================
//Program that produces N random points (i,j) with
//0<= i,j < 10000. Simple qualitative test of  serial
//correlations of random number generators on the plane.
//
//compile:
//g++ correlations2ran.f90 naiveran.f90 drandom.f90
//============================================================
#include <iostream>
#include <fstream>
#include <iomanip>
#include <cstdlib>
using namespace std;

double naiveran(),drandom();

int main(int argc,char **argv){
  const int L = 10000;
  int i,N;
  N = 1000;
  //read the number of points from the command line:
  if(argc > 1) N = atoi(argv[1]);
  for(i=1;i<=N;i++){
    cout << int(L*naiveran()) << " "
         << int(L*naiveran()) << '\n';
  //cout << int(L*drandom ()) << " "
  //     << int(L*drandom ()) << '\n';
  }
}//main()
```

The program[5] can be found in the file `correlations2ran.cpp`. In order

[5]The variables `argc` and `argv` can be used in order to access the command line arguments, see page 473. The elements in the array `argv[]` are strings (the "words" in

to test `naiveran()` we compile with the command

```
> g++ correlations2ran.cpp naiveran.cpp -o naiveran
```

whereas in order to test `drandom()` we uncomment the relevant lines as follows

```
  //cout << int(L*naiveran()) << " "
  //     << int(L*naiveran()) << '\n';
    cout << int(L*drandom ()) << " "
         << int(L*drandom ()) << '\n';
```

and recompile:

```
> g++ correlations2ran.cpp drandom.cpp -o drandom
```

These commands result in two executable files `naiveran` and `drandom`. In order to see the results we run the commands

```
> ./naiveran 100000 > naiveran.out
> ./drandom  100000 > drandom.out
> gnuplot
gnuplot> plot "naiveran.out" using 1:2 with dots
gnuplot> plot "drandom.out"  using 1:2 with dots
```

which produce 10^5 points used in the plots in figures 11.1 and 11.2. In the plot of figure 11.1, we see the pair correlations between the numbers produced by `naiveran()`. Figure 11.2 shows the points produced by `drandom()`, and we can see that the correlations shown in figure 11.1 have vanished. The plot in figure 11.2 is qualitative, and a detailed, quantitative, study of `drandom()` shows that the pairs (u_i, u_{i+1}) that it produces, do not pass the χ^2 test when we have more than 10^7 points, which is much less than the period of the generator. In order to avoid such problems, there are many solutions that have been proposed and the simplest among them "shuffle" the results so that the low order serial correlations vanish. Such generators will be discussed in the next section. The uniform distribution of the random numbers produced

---

the command line), and if they are to be used as numbers they have to be converted. The function `atoi()` converts a string to an integer (`cstdlib` needs to be included).

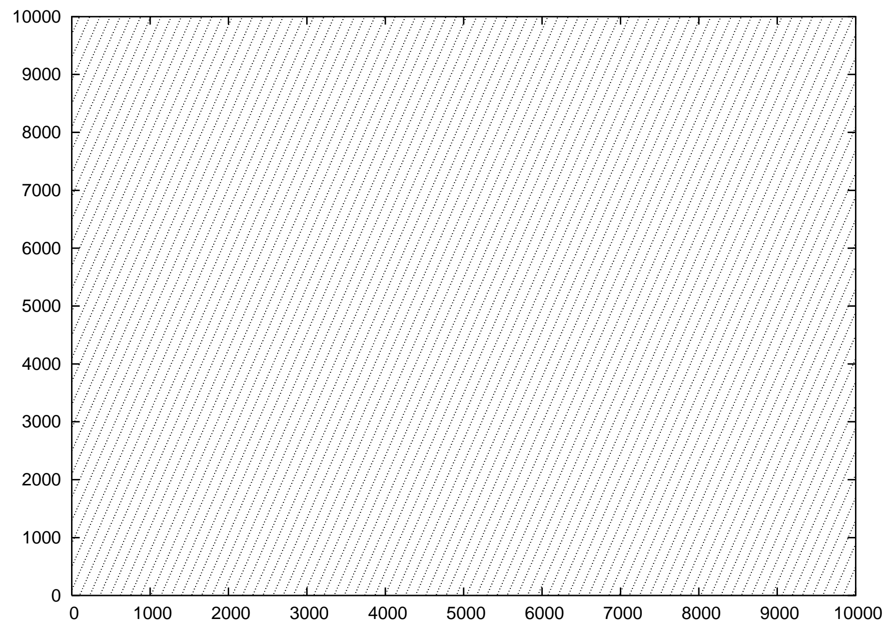

Figure 11.1: Pairs of pseudorandom numbers produced by the function `naiveran()`. The correlations among pairs of such numbers show in the distribution of such pairs on a clearly seen lattice.

can be examined graphically by constructing a histogram of the relative frequency of their appearance. In order to construct the histograms we use the script `histogram` which is written in the awk language[6] as shown below:

```
> histogram -v f=0.01 drandom.out > drandom.hst
> gnuplot
gnuplot> plot "drandom.hst" using 1:3 with histeps
gnuplot> plot [:][0:] "drandom.hst" using 1:3 with histeps
```

The command `histogram -v f=0.01` constructs a histogram of the data so that the bin width is $1/0.01 = 100$. The reciprocal of the number following the option `-v f=0.01` defines the bin width. The histogram is

[6] See the accompanying software in the `Tools` directory. Give the command `histogram -- -h` which prints short usage instructions. I hope you remember how to make the file `histogram` executable and put it in your path...

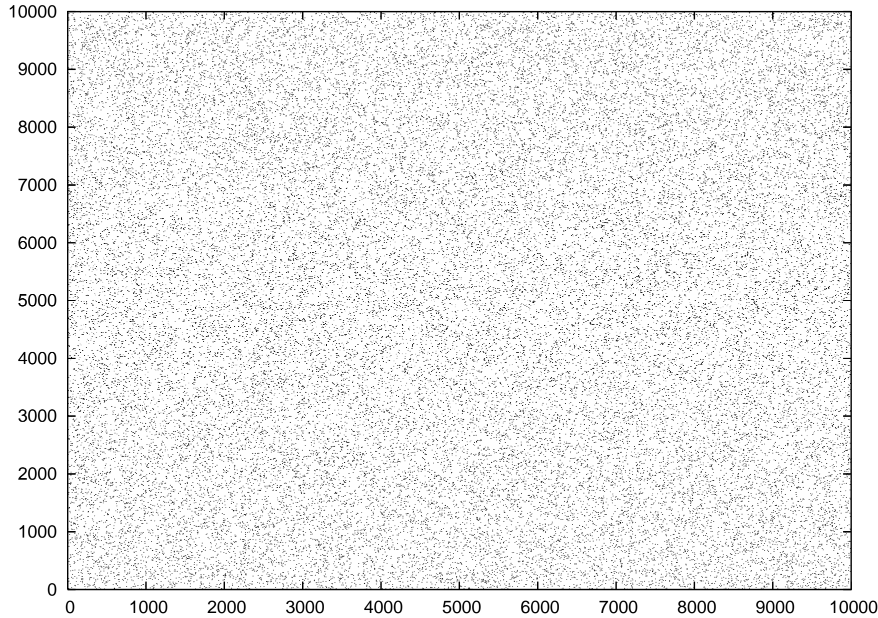

Figure 11.2: Pairs of pseudorandom numbers produced by the function `drandom()`. These points have a random distribution on the plane compared to those generated by `naiveran()`.

saved in the file `drandom.out`.

The results are shown in figures 11.3 and 11.4. Next, we study the variance of the measurements, shown in figure 11.3. The variance is decreased with the size of the sample of the collected random numbers. This is seen in the histogram of figure 11.5. For a quantitative study of the dependence of the variance on the size n of the sample, we calculate the standard deviation

$$\sigma = \sqrt{\frac{1}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2\right)}\,, \qquad (11.3)$$

where $\{x_i\}$ is the sequence of random numbers. Figure 11.6 plots this relation. By fitting

$$\ln\sigma \sim \frac{1}{2}\ln(n)\,, \qquad (11.4)$$

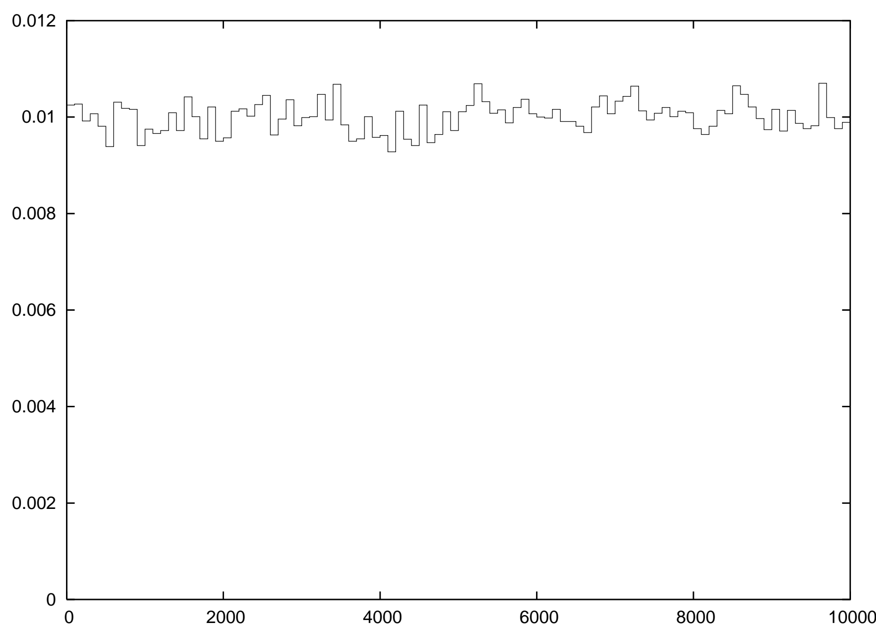

Figure 11.3: The relative frequency distribution of the pseudorandom numbers generated by `drandom()`. The distribution is uniform within $(0,1)$ and we see the deviations from the average value.

to a straight line, we see that

$$\sigma \sim \frac{1}{\sqrt{n}}\,. \tag{11.5}$$

If we need to generate random numbers which are distributed according to the probability density $f(x)$ we can use a sequence of uniformly distributed random numbers in the interval $(0,1)$ as follows: Consider the cumulative distribution function

$$0 \leq u \equiv F(x) = \int_{-\infty}^{x} f(x')\,dx' \leq 1\,, \tag{11.6}$$

which is equal to the area under the curve $f(x)$ in the interval $(-\infty, x]$ and it is equal to the probability $P(x' < x)$. If u is uniformly distributed in the interval $(0,1)$ then we have that $P(u' < u) = u$. Therefore $x = F^{-1}(u)$ is such that $P(x' < x) = u = F(x)$ and follows the $f(x)$ distribution. Therefore, if u_i form a sequence of uniformly distributed random numbers,

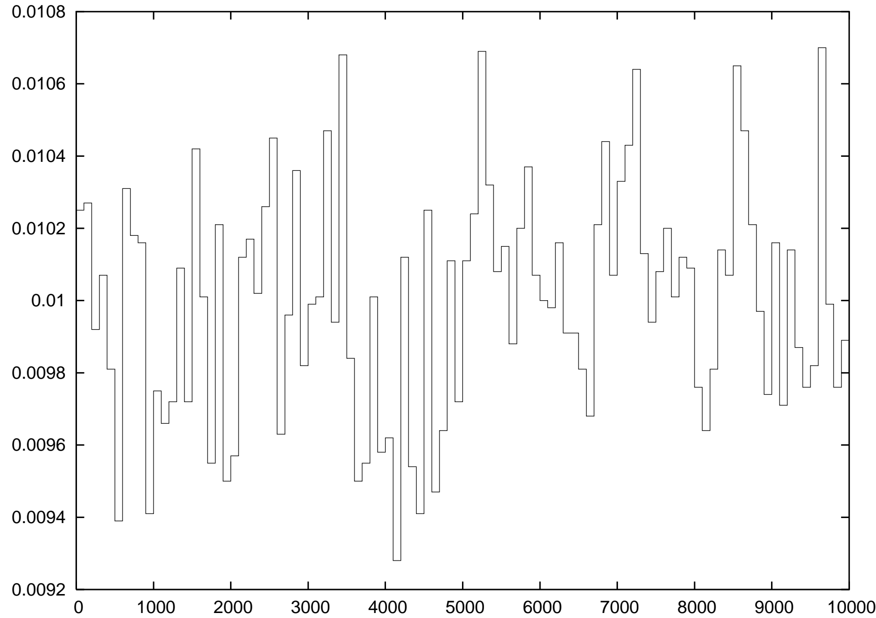

Figure 11.4: Same as in figure 11.3, but with the scale enlarged, so that the dispersion of the histogram values is clearly seen.

then the numbers

$$x_i = F^{-1}(u_i) \tag{11.7}$$

form a sequence of random numbers distributed according to $f(x)$.

Consider for example the Cauchy distribution

$$f(x) = \frac{1}{\pi}\frac{c}{c^2 + x^2} \quad c > 0\,. \tag{11.8}$$

Then

$$F(x) = \int_{-\infty}^{x} f(x')\,dx' = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x}{c}\right)\,. \tag{11.9}$$

According to the previous discussion, the random number generator is given by the equation

$$x_i = c\tan\left(\pi u_i - \pi/2\right) \tag{11.10}$$

or equivalently (for a more efficient generation)

$$x_i = c\tan\left(2\pi u_i\right)\,. \tag{11.11}$$

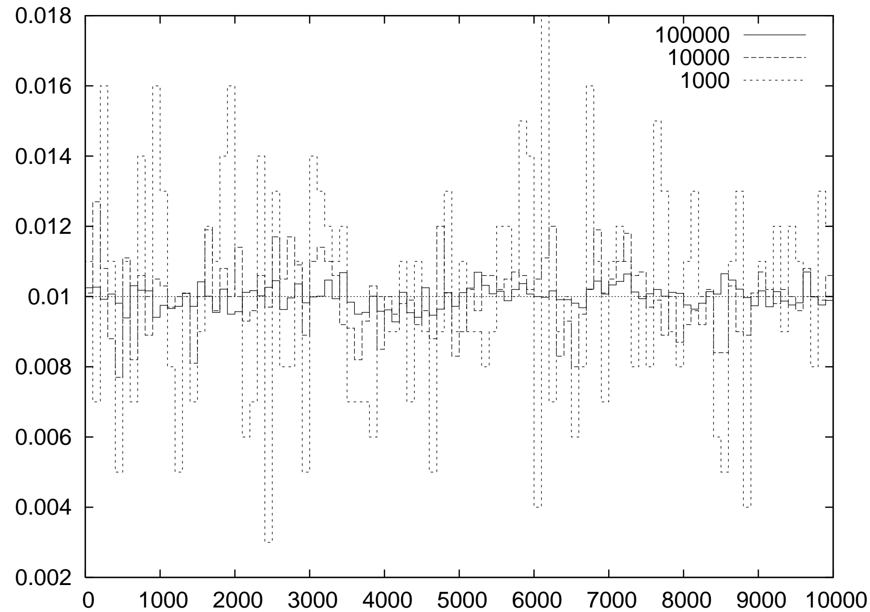

Figure 11.5: The relative frequency distribution of the pseudorandom numbers generated by `drandom()` as a function of the sample size n for $n = 1000, 10000, 100000$.

The generator of Gaussian random numbers is found in many applications. The Gaussian distribution is given by the probability density

$$g(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \mathrm{e}^{-x^2/(2\sigma^2)} \tag{11.12}$$

The cumulative distribution function is

$$G(x) = \int_{-\infty}^{x} g(x')\,dx' = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2}\,\sigma}\right) \tag{11.13}$$

where $\mathrm{erf}(x) = \int_{-\infty}^{x} \exp\{-(x')^2\}\,dx'$ is the error function. The error function, as well as its inverse, can be calculated numerically, but this would result in a slow computation. A trick to make a more efficient calculation is to consider the probability density $\rho(x,y)$ of *two* independent Gaussian random variables x and y

$$\rho(x,y)\,dx\,dy = \frac{1}{\sqrt{2\pi}\,\sigma}\mathrm{e}^{-x^2/(2\sigma^2)}\ \frac{1}{\sqrt{2\pi}\,\sigma}\mathrm{e}^{-y^2/(2\sigma^2)}\,dx\,dy = \frac{1}{2\pi\sigma^2}\mathrm{e}^{-r^2/(2\sigma^2)}\,r\,dr\,d\phi \tag{11.14}$$

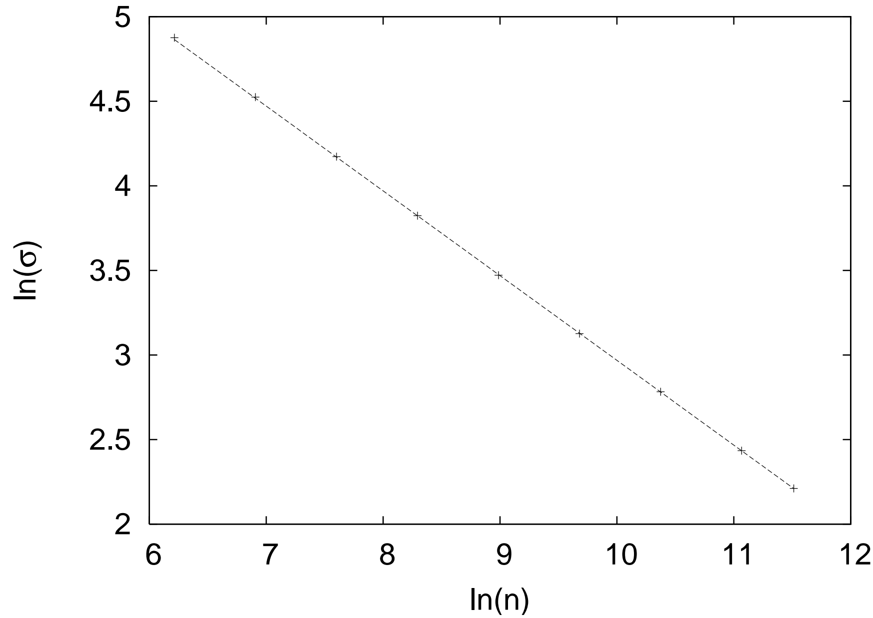

Figure 11.6: The dependence of the variance (11.3) on n for the distribution of random numbers generated by `drandom()`.

where $x = r\cos\phi$, $y = r\sin\phi$. Then we have that

$$u = G(r) = \int_0^r \int_0^{2\pi} dr\, d\phi\, \rho(r,\phi) = 1 - \mathrm{e}^{-r^2/(2\sigma^2)}\,, \tag{11.15}$$

which, upon inversion, it gives

$$r = \sigma\sqrt{-2\,\ln(1-u)}\,. \tag{11.16}$$

Therefore it is sufficient to generate a sequence $\{u_i\}$ of uniformly distributed random numbers and take

$$\begin{aligned}
r_i &= \sigma\sqrt{-2\,\ln(u_i)} & (11.17)\\
\phi_i &= 2\pi u_{i+1} & (11.18)\\
x_i &= r_i\cos\phi_i & (11.19)\\
x_{i+1} &= r_i\sin\phi_i\,. & (11.20)
\end{aligned}$$

The algorithm shown above gives a sequence of pseudorandom numbers $\{x_i\}$, which follow the Gaussian distribution[7]. The program for $\sigma = 1$ is listed below:

```
//=====================================================
//Function to produce random numbers distributed
//according to the gaussian distribution
//g(x) = 1/(sigma*sqrt(2*pi))*exp(-x^2/(2*sigma^2))
//=====================================================
#include <cmath>
double drandom ();
double gaussran(){
  const  double sigma = 1.0;
  const  double PI2   = 6.28318530717958648;
  static bool   newx  = true;
  static double x;
  double        r,phi;

  if(newx){
    newx =  false;
    r    =  drandom();
    phi  =  drandom()*PI2;
    r    =  sigma*sqrt(-2.0*log(r));
    x    =  r*cos(phi);
    return  r*sin(phi);
  }else{
    newx =  true;
    return x;
  }
}//gaussran();
```

The result is shown in figure 11.7. Notice the `static` attribute for the variables `newx` and `x`. This means that their values are saved between calls of `drandom`. We do this because each time we calculate according to (11.17), we generate two random numbers, whereas the function returns only one. The function needs to know whether it is necessary to generate a new pair (x_i, x_{i+1}) (this is what the "flag" `newx` marks) and, if not, to return the previously generated number, saved in the variable `x`. The analysis of the results is left as an exercise to the reader.

[7] It can be shown that x_i, x_{i+1} are statistically independent.

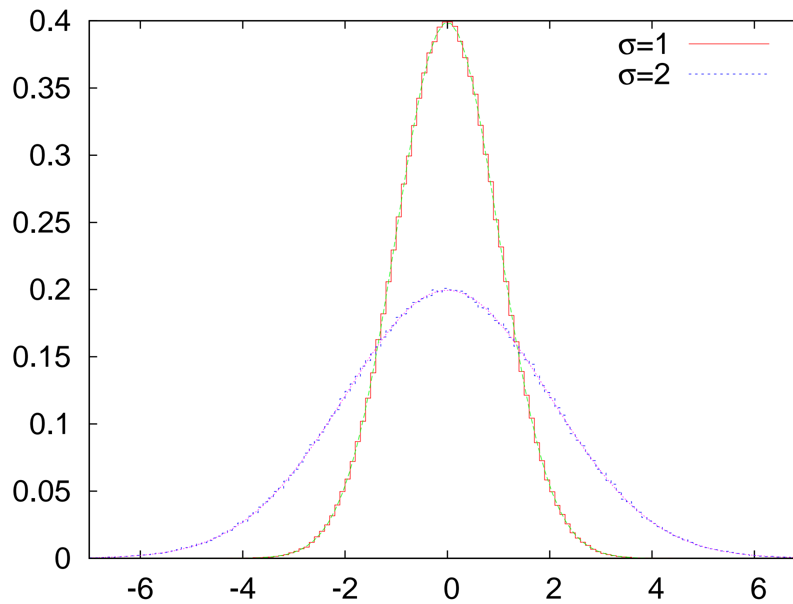

Figure 11.7: The distribution of pseudorandom numbers generated by `gaussran()` for $\sigma = 1$ and $\sigma = 2$. The histogram is superimposed to the plot of (11.12).

11.2 Using Pseudorandom Number Generators

The function `drandom()` is good enough for the problems studied in this book. However, in many demanding and high accuracy calculations, it is necessary to use higher quality random numbers and/or have the need of much longer periods. In this and the following section we will discuss how to use two high quality, efficient and portable generators which are popular among many researchers.

The first one is part of the C++ Standard Library. It is a very high quality, portable random number generator that was proposed by Martin Lüscher [50] and it is called *RANLUX*. The original code has been written by Fred James and you can download it in its original form from the links given in the bibliography [50]. The generator uses a subtract-with-borrow algorithm by Marsaglia and Zaman [52], which has a very large period but fails some of the statistical tests. Based on the chaotic properties of the algorithm, Lüscher attributed the problems to short time

autocorrelations and proposed a solution in order to eliminate them.

In order to use it, you have to learn a little bit about the interface to random number generator *engines* in the C++ Standard Library [11]. Engines are *the sources* of random number generators and they are classes which implement well known algorithms for pseudo random number generation. They generate integer values uniformly distributed between a minimum and maximum value. These numbers are processed by *distributions*, which are classes that transform them into real or integer random values distributed according to a desired random distribution. Distributions can use *any* available engine in order to produce random numbers.

There are several engines available in the standard library, like the linear congruential, Mersenne twister, subtract with carry and RANLUX. Similarly, there are several distributions available, like the uniform, gaussian, exponential, gamma, χ^2 and others.

In order to produce a sequence of random numbers, the procedure is always the same: One has to instantiate a random number engine object and a distribution object. In the following, minimal, program, `rlx` is the random number engine object and `drandom` the distribution.

```
#include <random>
using namespace std;
int main(){

  ranlux48 rlx;
  uniform_real_distribution<double> drandom;
  double x = drandom(rlx);

}
```

A random number is returned by the call to `drandom(rlx)`, where the distribution `drandom` uses the engine `rlx` in order to calculate it. The engine `rlx` uses the RANLUX algorithm, something that is determined by the `ranlux48` declaration. The distribution `drandom` produces uniformly distributed random numbers in the interval[8] $[0,1)$ (0

[8] Distributions can be parametrized by defining appropriate *parameters*. For example, `uniform_real_distribution<double> drandom(`a`,`b`)` produces numbers in the $[a,b)$ interval, `normal_distribution <double> gaussran(`μ`,`σ`)` produces numbers distributed according to $\exp\left[-(x-\mu)^2/(2\sigma^2)\right]/(\sigma\sqrt{2\pi})$ etc. If the parameters are omitted, then they take default values (here a=0, b=1, μ=0, σ=1)

included, 1 excluded) of the type `double`. This is determined by the `uniform_real_distribution <double>` declaration.

Contrary to the simple modulo generator, the state of a high quality random number generator engine is determined by more than one numbers. We should learn how to

- start from a new state
- save the current state
- restart from a previously saved state
- obtain the random numbers.

Saving and reading the state of a generator is very important when we execute a job that is split in several parts (checkpointing). This is done very often on computer systems that set time limits for jobs or when our jobs are so long (more than 8-10 hours) that it will be painful to loose the resources (time and money) spent for the calculation in the case of a computer crash. If we want to restart the job from exactly the same state as it was before we stopped, we also need to restart the random number generator from the same state.

Starting from a new, fresh state is called seeding. The simplest form of seeding is done by providing a single integer value to the `seed()` operation of the engine. For example the first of the statements below

```
rlx.seed(1234);
double x=  drandom(rlx);
```

sets the state of the engine `rlx` in a state determined by the *seed* 1234. Different seeds are guaranteed to put the engine into different states and the same seed will start the engine from the same state. That means that by repeating the above statements in different parts of the program, we will store the same value into x. A call to `rlx.seed()` without an argument puts the engine into its default state.

Sometimes we need to initialize the random number generator from as a random initial state as possible, so that each time that we run our program, a different sequence of random numbers is generated. For Unix like systems, like the GNU/Linux system, we can use the two special files `/dev/random` and `/dev/urandom` in order to generate unpredictable

random numbers. These generate random bits from the current state of the computer. A function that returns a positive `int` as a seed by using `/dev/urandom` is shown below:

```
#include <unistd.h>
#include <fcntl.h>
int seedby_urandom(){
  int ur,fd;
  fd = open("/dev/urandom", O_RDONLY);
  read (fd,&ur,sizeof(int));
  close(fd);
  return (ur>0)?ur:-ur;
}
```

It uses the low level C functions `open` and `read` for reading binary data. This is because `/dev/urandom` just provides a random set of bits. We need to determine the amount of data read into the `int` variable `ur` and for that, `sizeof(int)` is used in order to return the number of bytes in the representation of an `int`.

If we need to work in an environment where the special file `/dev/urandom` is not available, it is possible to seed using the current time and the process number ID. The latter is necessary in case we start several processes in parallel and we need different seeds. Check the file `urandom.cpp` in order to see how to do it[9].

In order to save the current state of the random number generator engine, we can simply write to a stream like in

```
  ofstream oseed("seeds");
  oseed << rlx << endl;
```

which writes the state of the engine in the file `seeds`. In order to restart the generator from a saved state, we can read from a stream like in

```
  ifstream iseed("seeds");
  iseed >> rlx;
```

[9]You can also use the operating system in order to pass random seeds to your program. Try the commands `` set x = `< /dev/urandom tr -dc "[:digit:]" | head -c9 | awk 'printf "%d",$1'` ; echo $x `` and `` set x = `perl -e 'srand();print int(100000000*rand());'` ; echo $x ``. Use the value of the variable `x` for a seed.

which reads the state of the engine stored in the file `seeds`. After this statement, the sequence of random numbers will be exactly the same as the one generated after we saved the state before.

The code in the file `test_ranlux.cpp` implements all of the above tasks and we list it below:

```
#include <random>
#include <iostream>
#include <fstream>
using namespace std;

int main(){
  //The object rlx is a RANLUX random number engine:
  ranlux48 rlx;
  //drandom is a distribution that produces uniformly
  //distributed numbers in [0,1)
  uniform_real_distribution<double> drandom;
  //--------------------------------------------------
  //random numbers starting from the default state:
  cout  << "ranlux: ";
  for(int i=1;i<=5;i++) cout << drandom(rlx) << " ";
  cout  << endl;
  //--------------------------------------------------
  //Seeding by a seed:
  rlx.seed(377493872);
  cout  << "seed   : ";
  for(int i=1;i<=5;i++) cout << drandom(rlx) << " ";
  cout  << endl;
  //--------------------------------------------------
  //Saving the state to a file "seeds":
  ofstream oseed("seeds");
  oseed << rlx << endl;
  cout  << "more   : ";
  for(int i=1;i<=5;i++) cout << drandom(rlx) << " ";
  cout  << endl;
  //--------------------------------------------------
  //Reading an old state from the file seeds:
  ifstream iseed("seeds");
  iseed >> rlx;
  cout  << "same   : ";
  for(int i=1;i<=5;i++) cout << drandom(rlx) << " ";
  cout  << endl;
}//main
```

Use the following commands[10] in order to compile and see the results:

```
> g++ -std=c++11 test_ranlux.cpp -o ranlux
> ./ranlux
ranlux: 0.101746 0.465305  0.739500 0.861557 0.196622
seed  : 0.471439 0.379696  0.887275 0.642664 0.996368
more  : 0.980649 0.0869633 0.231573 0.695236 0.226594
same  : 0.980649 0.0869633 0.231573 0.695236 0.226594
```

11.3 The MIXMAX Random Number Generator

The MIXMAX random number generator is a very high quality and efficient random number generator proposed by K. Savvidy [49]. It is faster than RANLUX and it has a huge period which, for its default function, is larger than 10^{4855}. Moreover, it passes all known statistical tests for statistically independent sequences of random numbers and it is very cleverly and efficiently implemented. It generates sequences of random numbers by using a relation of the form

$$u_i(t+1) = \sum_{j=1}^{N} A_{ij} u_j(t) \mod 1\,, \tag{11.21}$$

where $u_i(t) \in [0,1)$. The matrix A_{ij} has integer entries and has special properties. The trajectories of the above map have very strong chaotic behavior and this is the reason for the high quality of the produced random numbers[11]. By a clever choice of the matrix A_{ij}, the computational cost per random number using (11.21) does not grow with N [49]. It is particularly useful for multi-threaded simulations where independent sequences of random numbers must be simultaneously generated. For all these properties, the MIXMAX generator is expected to be the generator of choice in many scientific applications.

[10] The flag `-std=c++11` is necessary in order to instruct the compiler that the C++11 standard is used.

[11] The generator came out of the study of special dynamical systems, namely Yang-Mills classical mechanics by G. Savvidy and N. Ter-Arutyunyan Savvidy, *J.Comput.Phys.* **97** (1991) 97.

The code of the generator is available from the site `mixmax.hepforge.org` and it has a C++ interface in the file `mixmax.cpp`. The accompanying software of this chapter contains a copy of the generator's code in the subdirectory `MIXMAX`. In order to compile code with the MIXMAX random generator engine you need most of the C++ and C files in that directory. Using the generator, seeding it and saving/reading its state can be done as in the example code below:

```
#include <iostream>
#include <fstream>
#include <iomanip>
#include <random>
using namespace std;

#include "MIXMAX/mixmax.hpp"

int main(){
  //The object mxmx is a MIXMAX random number engine:
  mixmax_engine mxmx(0,0,0,1);
  //The object drandom is a uniform distribution:
  uniform_real_distribution<double> drandom;
  //--------------------------------------------------------------
  //Random numbers after seeding with a chosen seed:
  mxmx.seed(1234);
  cout   << "mixmax: ";
  for(int i=1;i<=5;i++) cout << drandom(mxmx) << " ";
  cout   << endl;
  //--------------------------------------------------------------
  //Saving the state to a file "seeds":
  ofstream oseeds("seeds");
  oseeds << mxmx << endl;oseeds.close();
  cout   << "more   : ";
  for(int i=1;i<=5;i++) cout << drandom(mxmx) << " ";
  cout   << endl;
  //--------------------------------------------------------------
  //Reading an old state from the file seeds:
  ifstream iseeds("seeds");
  iseeds >> mxmx; iseeds.close();
  cout   << "same   : ";
  for(int i=1;i<=5;i++) cout << drandom(mxmx) << " ";
  cout   << endl;

}//main()
```

Notice that the code assumes that the file `mixmax.hpp` is in the relative path `MIXMAX/mixmax.hpp`, in the subdirectory where all the code of MIXMAX is put in the examples of this chapter. Then you can compile and run the code with the commands:

```
> g++ -std=c++11 test_mixmax.cpp MIXMAX/mixmax.cpp -o mxmx
> ./mxmx
mixmax: 0.600514 0.079735 0.80067 0.657467 0.286080
more  : 0.758103 0.107861 0.20206 0.513634 0.763578
same  : 0.758103 0.107861 0.20206 0.513634 0.763578
```

As we mentioned in the previous section, the C++ Standard library provides several probability distributions, including the Gaussian. In the following program we show how to generate random numbers using MIXMAX distributed according to

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,, \tag{11.22}$$

and test the distribution by histogramming the results:

```
#include <iostream>
#include <fstream>
#include <iomanip>
#include <random>
using namespace std;

#include "MIXMAX/mixmax.hpp"

int seedby_urandom();
int main(int argc,char **argv){
  //-------------------------------------------------------------------
  //Number of random numbers that will be generated:
  int     Nrand=2000000;
  //Mean value and standard deviation of gaussian distribution:
  const double mean=0.0,sigma=1.0;
  //-------------------------------------------------------------------
  //The object mxmx is a MIXMAX random number engine:
  mixmax_engine mxmx(0,0,0,1);
  mxmx.seed(seedby_urandom()); //seed mixmax using /dev/urandom
  //The object gaussran is a gaussian distribution:
  normal_distribution<double> gaussran(mean,sigma);
```

```
  //If provided at the command line, read Nrand:
  if(argc>1) Nrand = atoi(argv[1]);
  //------------------------------------------------------------
  //Calculate random numbers and make a histogram:
  const double xm=6.0;//histogram in [-xm,xm]
  const int    nh=200;//number of histogram bins
  int         hist[nh];
  double           x,dx;
  int                ih;
  const double PI     = 2.0*atan2(1.0,0.0);
  const double sigma2 = 2.0*sigma*sigma;
  dx = 2.0*xm/nh;
  //Calculate histogram: hist[ih] counts # occurences
  for(ih=0;ih<nh;ih++) hist[ih]=0;
  for(int i=0;i<Nrand;i++){
    x = gaussran(mxmx);
    //skip if out of range:
    if(x < -xm || x > xm) continue;
    ih=int((x+xm)/dx);
    if(ih<0||ih>=nh){cerr<<"ih out of range\n";exit(1);}
    hist[ih]++;
  }
  //print results: The normalized histogram and compare to the
  //               gaussian distribution.
  cout.precision(17);
  for(ih=0;ih<nh;ih++){
    x = ih*dx - xm + 0.5*dx;
    cout << x                                     << " "
         <<        hist[ih]                       << " "
         << double(hist[ih])/dx/Nrand             << " "
         << 1.0/sqrt(PI*sigma2)*
                  exp(-(x-mean)*(x-mean)/sigma2)  << '\n';
  }

}//main()
// Use /dev/urandom for more randomness.
#include <unistd.h>
#include <fcntl.h>
int seedby_urandom(){
  int ur,fd;
  fd = open("/dev/urandom", O_RDONLY);
  read (fd,&ur,sizeof(int));
  close(fd);
  return (ur>0)?ur:-ur;
}
```

The above program can be found in the file test_gaussran.cpp. The MIXMAX files are assumed to be in the subdirectory MIXMAX/. You can change the mean μ and standard deviation σ of the distribution by changing the values of the variables mean and sigma respectively. Then you can compile and run the program with the commands:

```
> g++  -std=c++11 test_gaussran.cpp MIXMAX/mixmax.cpp -o mxmx
> ./mxmx > gauss.dat
> gnuplot
gnuplot> plot "gauss.dat" using 1:2 with lines
gnuplot> plot "gauss.dat" using 1:3 title "histogram", \
              "gauss.dat" using 1:4 with lines title "f(x)"
```

11.4 Random Walks

Consider a particle which is located at one of the sites of a two dimensional square lattice. After equilibrating at this position, it can jump randomly to one of its nearest neighbor positions. There, it might need some time to equilibrate again before jumping to a new position. During this time, the momentum that it had at its arrival is lost, therefore the next jump is made without "memory" of the previous position where it came from. This process is repeated continuously. We are not interested in the mechanism that causes the jumping[12], and we seek a simple phenomenological description of the process.

Assume that the particle jumps in each direction with equal probability and that each jump occurs after the same time τ. The minimum distance between the lattice sites is a (lattice constant). The vector that describes the change of the position of the particle during the i-th jump is a random variable $\vec{\xi}_i$ and it always has the same magnitude $|\vec{\xi}_i| = a$. This means that, given the position $\vec{r}_k$ of the particle at time $t_k = k\tau$, its position $\vec{r}_{k+1}$ at time $t_{k+1} = (k+1)\tau = t_k + \tau$ is

$$\vec{r}_{k+1} = \vec{r}_k + \vec{\xi}_k\,, \tag{11.23}$$

[12]It could be e.g. thermally stimulated sound waves, the quantum tunneling effect etc.

where

$$\vec{\xi}_k = \begin{cases} a\hat{x} & \text{with probability} \quad \frac{1}{4} \\ -a\hat{x} & \text{with probability} \quad \frac{1}{4} \\ a\hat{y} & \text{with probability} \quad \frac{1}{4} \\ -a\hat{y} & \text{with probability} \quad \frac{1}{4} \end{cases} . \tag{11.24}$$

The vectors $\vec{\xi}_i$ and $\vec{\xi}_j$ are uncorrelated for $i \neq j$ and we have that

$$\langle \vec{\xi}_i \cdot \vec{\xi}_j \rangle = \langle \vec{\xi}_i \rangle \cdot \langle \vec{\xi}_j \rangle \,. \tag{11.25}$$

The possible values of $\vec{\xi}_i$ are equally probable, therefore we obtain

$$\langle \vec{\xi}_i \rangle = \vec{0} \,. \tag{11.26}$$

This is because the positive and negative terms in the sum performed in the calculation of $\langle \vec{\xi}_i \rangle$ occur with the same frequency and they cancel each other. Therefore $\langle \vec{\xi}_i \cdot \vec{\xi}_j \rangle = 0$ for $i \neq j$. Since the magnitude of the vectors $|\vec{\xi}_i| = a$ is constant, we obtain

$$\langle \vec{\xi}_i \cdot \vec{\xi}_j \rangle = a^2 \delta_{i,j} \,. \tag{11.27}$$

The probability for a path C_N of length N to occur is[13]

$$p(C_N) = \frac{1}{z^N} \,, \tag{11.28}$$

where $z = 4$ is the number of nearest neighbors of a lattice site. This probability depends on the length of the path and not on its geometry. This can be easily seen using the obvious relation $p(C_{N+1}) = \frac{1}{z} p(C_N)$, since there are exactly z equally probable cases. The partition function is

$$Z_N = z^N \,, \tag{11.29}$$

and it counts the number of different paths of length N.

After time $t = N\tau$ the particle is displaced from its original position by

$$\vec{R} = \sum_{i=1}^{N} \vec{\xi}_i \,. \tag{11.30}$$

[13] I.e. after time $t = N\tau$, not the physical length of the path formed by the links that the particle has crossed. We also count the jumps to sites that the particle has already visited.

The average value of the displacement vanishes

$$\langle \vec{R} \rangle = \sum_{i=1}^{N} \langle \vec{\xi}_i \rangle = \vec{0}\,. \tag{11.31}$$

The expectation value of the displacement squared is non zero

$$\langle R^2 \rangle = \langle \vec{R} \cdot \vec{R} \rangle = \sum_{i,j=1}^{N} \langle \vec{\xi}_i \cdot \vec{\xi}_j \rangle = a^2 \sum_{i,j=1}^{N} \delta_{i,j} = a^2 N\,. \tag{11.32}$$

The conclusion is that the random walker has been displaced from its original position rather slowly

$$R_{rms} = \sqrt{\langle R^2 \rangle} = a\sqrt{N} \propto \sqrt{t}\,. \tag{11.33}$$

For a particle with a non zero average velocity we expect that $R_{rms} \propto t$.

Equation (11.33) defines the *critical exponent* ν

$$\langle R^2 \rangle \sim N^{2\nu}\,, \tag{11.34}$$

where $\sim$ means asymptotic behavior in the limit $N \to \infty$. For a classical walker $\nu = 1$, whereas for the random walker $\nu = \frac{1}{2}$.

The Random Walker (RW) model has several variations, like the Non Reversal Random Walker (NRRW) and the Self Avoiding Walk (SAW) . The NRRW model is defined by excluding the vector pointing to the previous position of the walker and by selecting the remaining vectors $\vec{\xi}_i$ with equal probability. The SAW is a NRRW with the additional requirement that, when the walker ends in a previously visited position, the ... walking ends! Some models studied in the literature include, besides the infinite repulsive force, an attractive contribution to the total energy for every pair of points of the path that are nearest neighbors. In this case, each path is weighted with the corresponding Boltzmann weight according to equation (12.4).

For the NRRW, equation (11.34) is similar to that of the RW, i.e. $\nu = \frac{1}{2}$. Even though the paths differ microscopically, their long distance properties are the same. They are examples of models belonging to the same *universality class* according to the discussion in section 13.1.

This is not the case for the SAW. For this system we have that [53]

$$\langle R^2 \rangle^{\mathrm{SAW}} \sim N^{2\nu} \qquad \nu = \frac{3}{4}\,, \tag{11.35}$$

therefore the typical paths in this model are longer than those of the RW. If we introduce a nearest neighbor attraction according to the previous discussion, then there is a critical temperature β_c such that for $\beta < \beta_c$ we have similar behavior given by equation (11.35), whereas for $\beta > \beta_c$ the attractive interaction dominates, the paths collapse and we obtain $\nu = 1/3 < \nu_{RW}$. For $\beta = \beta_c$ we have that $\nu = \frac{1}{2}$. For more information we refer the reader to the book of Binder and Heermann [7].

In order to write a program that simulates the RW we apply the following algorithm:

1. Set the number of the random walks to be generated
2. Set the number of steps of each walk
3. Set the initial position of the walk
4. At each step on the walk, pick a random direction with equal probability
5. After the walk is completed, measure $\vec{R}$, R^2, etc
6. After all walks have been generated, compute the expectation values of the measured quantities and the statistical error of their measurement.

All we need to explain is how to program the choice of "random direction". The program is in the file `rw.cpp`

```
#include <iostream>
#include <fstream>
#include <iomanip>
#include <random>
#include <thread>
using namespace std;

#include "MIXMAX/mixmax.hpp"

int seedby_urandom();
int main(){
  const int Nwalk = 1000;
  const int Nstep = 100000;
  double x,y;
```

```
  //------------------------------------------
  mixmax_engine mxmx(0,0,0,1);
  uniform_real_distribution<double> drandom;
  mxmx.seed(seedby_urandom());
  ofstream dataR("dataR");
  ofstream data;
  dataR.precision(17);data.precision(17);
  //------------------------------------------
  //Generate random walks:
  for(int iwalk=1;iwalk<=Nwalk;iwalk++){
    x=0.0;y=0.0;
    data.open("data");
    for(int istep=1;istep<=Nstep;istep++){
      int ir = int(drandom(mxmx)*4);
      switch(ir){
      case 0:
        x += 1.0;
        break;
      case 1:
        x -= 1.0;
        break;
      case 2:
        y += 1.0;
        break;
      case 3:
        y -= 1.0;
        break;
      }//switch(ir)
      data << x << " " << y << endl;
    }//for(istep=1;istep<=Nstep;istep++)
    data.close();
    dataR << x*x+y*y << endl;
    //wait for 2 seconds until next walk:
    this_thread::sleep_for(chrono::seconds(2));
  }//for(iwalk=1;iwalk<=Nwalk;iwalk++)
  dataR.close();
}//main()
// Use /dev/urandom for more randomness.
#include <unistd.h>
#include <fcntl.h>
int seedby_urandom(){
  int ur,fd;
  fd = open("/dev/urandom", O_RDONLY);
  read (fd,&ur,sizeof(int));
  close(fd);
```

```
  return (ur>0)?ur:-ur;
}
```

The length of the paths is Nstep and the number of the generated paths is Nwalk. Their values are hard coded and a run using different values requires recompilation. The results are written to the files dataR and data. The square of the final displacement of the walker R^2 is written to dataR and the coordinates (x, y) of the points visited by the walker in each path is written to data. The file data is truncated at the beginning of each path, therefore it contains the coordinates of the current path only.

Each path is made of Nstep steps. The random vector $\vec{\xi}_{\texttt{istep}}$ is chosen and it is added in the current position $\vec{r}_{\texttt{istep}} = (x, y)$. The choice on $\vec{\xi}_{\texttt{istep}}$ is made in the line

```
      int ir = int(drandom(mxmx)*4);
```

where the variable ir $= 0, 1, 2, 3$ because the function int returns the integer part of a double. The values of ir correspond to the four possible directions of $\vec{\xi}$. We use the construct switch(ir) in order to move in the direction chosen by ir. Depending on its value, the control of the program is transferred to the command that moves the walker to the corresponding direction.

Compiling and running the program can be done with the commands

```
> g++ -std=c++11 rw.cpp MIXMAX/mixmax.cpp -o rw
> ./rw
```

Because of the command[14]

```
    this_thread::sleep_for(chrono::seconds(2));
```

the program temporarily halts execution for 2 seconds at the end of each generated path (you should remove this line at the production stage). This allows us to monitor the generated paths graphically. During the execution of the program, use gnuplot in order to plot the random walk which is currently stored in the file data:

[14]This is why we #include <thread>.

```
gnuplot> plot "data" with lines
```

Repeat for as many times as you wish to see new random walks. The automation of this process is taken care in the script `eternal-rw`:

```
> ./rw &
> ./eternal-rw &
> killall rw eternal-rw gnuplot gnuplot_x11
```

The last command ends the execution of all programs.

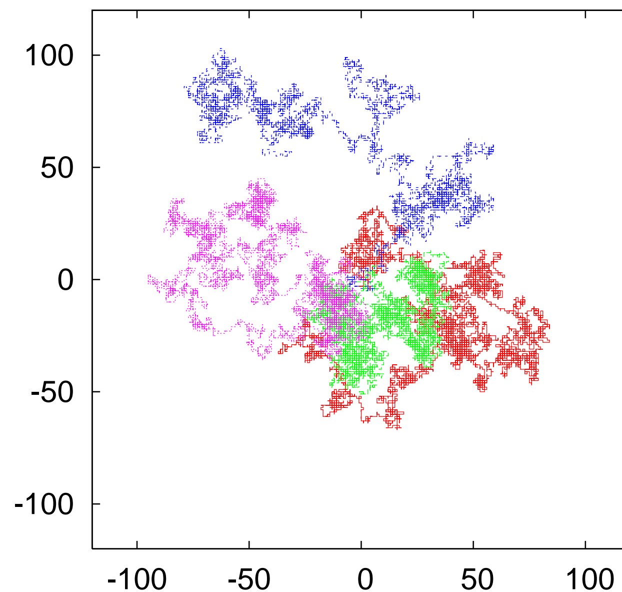

Figure 11.8: Four typical paths of the RW for $N = 10000$.

Some typical paths are shown in figure 11.8. Figure 11.9 shows the results for the expectation value $\langle R^2 \rangle$ for $N = 10, \ldots, 100000$ which confirm equation (11.32) $\langle R^2 \rangle = N$. You can reproduce this figure as follows:

1. Set the values of `Nwalk` and `Nstep` in the file `rw.cpp`. Delete the commands `sleep_for` and `data << x << " " << y` and compile the code.

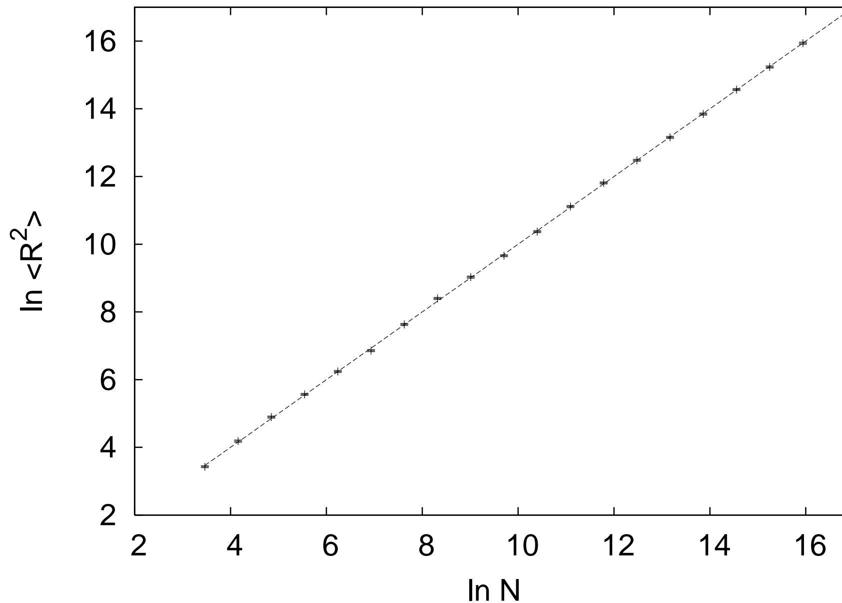

Figure 11.9: Numerical confirmation of the relation $\langle R^2 \rangle = N$ for $N = 10, \ldots, 100000$. The straight line is the fit of the data to the function $y = ax$ with $a = 0.9994(13)$.

2. Run the program and analyze the data in the file `dataR`:

```
> ./rw
> awk '{av += $1}END{print av/NR}' dataR
```

Write the results in a file `r2.dat` in two columns with the length of the paths N in the first column and with $\langle R^2 \rangle$ in the second. The command[15] `{av+=$1}` in the `awk` program adds the first column of each line of the file `dataR` to the variable `av`. After reading the whole file, the command `END{print av/NR}`, prints the variable `av` divided by the number of lines in the file (`NR` = "Number of Records"). This is a simple way for computing the mean of the first column of the file `dataR`.

3. Use a linear squares method in order to find the optimal line $y =$

[15] The command `av+=$1` is equivalent to `av=av+$1`.

$ax+b$ going through the points ($\ln N$,$\ln\langle R^2\rangle$). You can also use the `fit` command in `gnuplot` as follows:

```
gnuplot> fit  a*x+b "r2.dat" u (log($1)):(log($2)) via a,b
```

4. Construct the plot with the command:

```
gnuplot> plot a*x+b,"r2.dat" u (log($1)):(log($2)) w e
```

The obtained results are meaningless without their statistical errors. Since each measurement is statistically independent, the true expectation value is approached in the limit of infinite measurements with a speed proportional to $\sim 1/\sqrt{M}$, where M is the number of measurements. For the same reason[16], the statistical error is given by equation (11.3), e.g.

$$\delta\langle R^2\rangle = \sqrt{\frac{1}{M-1}\left(\frac{1}{M}\sum_{i=1}^{M}(R_i^2)^2 - \left(\frac{1}{M}\sum_{i=1}^{M}R_i^2\right)^2\right)}\,. \tag{11.36}$$

We can add the calculation of the error in the program in `rw.cpp` or we can leave this task to external utilities. For example we can use the `awk` script, which is written in the file `average`:

```
#!/usr/bin/awk -f
{
  av += $1;      # the sum of data
  er += $1*$1;   # the sum of squares of data
}
END{
  av /= NR;      # NR = "Number of Records" = number of lines
  er /= NR;
  # formula for error of uncorrelated measurements
  er  = sqrt( (er - av*av)/(NR-1) );
  print av, "+/-", er;
}
```

[16]If there exist statistical correlations between measurements, they should be taken into consideration. This will be discussed in detail in the following chapters.

The contents of this file is an example of a script *interpreted* by the awk program. The operating system knows which program to use for the interpretation by reading the first line #!/bin/awk -f where the first two characters of the file should be exactly #!. For the commands to be interpreted and executed, one has to make the script executable using the command chmod a+x average. Then the command

```
> ./average dataR
```

executes the script using the awk interpreter. We remind to the reader that the commands between curly brackets { ... } are executed by awk for every line of the file dataR. The commands between END{ ... } are executed after the last line of the file has been read[17]. Therefore the lines

```
av += $1;      #  the sum of data
er += $1*$1;   # the sum of squares of data
```

add the first column of the file dataR and its square to the variables av and er respectively. The commands

```
av /= NR;      # NR = "Number of Records" = number of lines
er /= NR;
```

are executed after the whole file dataR has been read and divide the variables av and er with the predefined variable NR which counts the total number of lines read so far. The last lines of the script compute the error according to equation (11.36) and print the final result. The shell script in the file rw1-anal.csh codes all of the above commands in a script. Read the comments in the file for usage instructions.

11.5 Problems

1. Reproduce the results shown in figure 11.6 and confirm the validity of equation (11.5).

[17]You can also execute a set of commands *before* the file is read by putting them between BEGIN{ ... }

2. Generate a sequence of pseudorandom numbers which follow a Gaussian distribution with standard deviation $\sigma = 1/\sqrt{2}$. Construct the plot of relative frequencies together with the plot of the probability density function.

3. Generate a sequence of pseudorandom numbers which follow the Cauchy distribution with $c = 1$. Construct the plot of relative frequencies together with the plot of the probability density function.

4. Write a program that calculates the period of the function `drandom()`. Check whether the numbers 0 and 1 belong to the sequence.

5. Compute the CPU time cost of the random number generation as follows: If you have an executable file, e.g. `random`, run the `/usr/bin/time` command with `./random` as its argument:

```
> /usr/bin/time ./random
```

   Upon exit of the command, the program `/usr/bin/time` prints the total CPU time in seconds to the `stderr`. Compute the time needed to generate 10^9 random numbers using the function `drandom()`, as well as the C++ Standard Library engine `ranlux24` and `ranlux48`. Repeat for the MIXMAX engine.

6. Show that if the expectation values of the vectors $\langle \vec{\xi}_i \rangle = \vec{v}\tau$ then $\langle \vec{R} \rangle = \vec{v}\tau N$ and we obtain a linear relation between displacement–length of path. The quantity v is the expectation value of the speed of the particle. Compute $\langle R^2 \rangle$ for large values of N.

7. Confirm the relations computed in the previous problem numerically. In your program, set the first line in (11.24) equal to $1/2$ and the rest equal to $1/6$. Compute the expectation values $\langle (\xi_i)_x \rangle$ and $\langle (\xi_i)_y \rangle$ and use them to calculate the average speed of the particle. Check the validity of the relations $\langle R^2 \rangle \sim N^\alpha$ and $\langle R_x \rangle \sim N^{2a_x}$ $\langle R_y \rangle \sim N^{2a_y}$. What is the relation between a, a_x and a_y?

8. Make the appropriate changes in the file `rw.cpp` so that the user can enter the values `Nwalk` and `Nstep` interactively using the command line arguments. For example, if she wants to generate 100 random

walks with $N = 2000$, she should run the command `./rw 100 2000`. (Hint: Look in the file `rw1.cpp`)

9. We know that for the RW we have that $\langle \vec{R} \rangle = \vec{0}$. Calculate $\langle x \rangle$ and $\langle y \rangle$ numerically for $N = 100, 100000$. Are they really equal to zero? Why? How does this depend on the number of measurements?

10. Compute the expectation value of the number of returns of the RW to his initial position as a function of N. What happens as $N \to \infty$? Why?

11. Reproduce figure 11.9 for the RW.

12. Write a program that implements the NRRW and reproduce the results in figure 11.9 for the NRRW.

13. In the program `rw.cpp` the RW's position is determined by two `double` variables `x,y`. The next position is calculated by the statements `x+=1.0, y+=1.0`. What are the limitations on the size of random walks that can be studied with this choice? What happens if one uses `float` variables `x,y` instead? Take into account the fact that $\langle R^2 \rangle = N$.

14. Repeat the previous problem by using `long` variables `x,y`. The next position is calculated by the statements `x++, y++`. Discuss the pros and the cons of each choice.

15. Repeat the previous problems by using `int` variables `x,y`. Discuss the pros and the cons of each choice by considering also the running time of the program. Use the command `/usr/bin/time`.

16. Write a straightforward code that implements the SAW. How big N can you simulate? Check whether the CPU time for computing a given number of random walks increases exponentially with N. Search the internet for the most efficient algorithm that simulates the SAW for large N.

Chapter 12

Monte Carlo Simulations

In this chapter we review the basic principles of Monte Carlo simulations in statistical mechanics. In the introduction, we review some of the fundamental concepts of statistical physics. The reader should have a basic understanding of concepts like the canonical ensemble, the partition function, the entropy, the density of states and the quantitative description of fluctuations of thermodynamic quantities. For a more in depth discussion of these concepts, see [4, 45, 54, 55, 56, 57].

For most of the interesting systems, the partition function cannot be calculated analytically, and in such a case we may resort to a numerical computation. This is what is done most effectively using Monte Carlo simulations, which consist of collecting a statistical sample of states of the system with an appropriately chosen probability distribution. It is remarkable that, by collecting a sample which is a tiny fraction of the total number of states, we can perform an accurate calculation of its thermodynamic quantities[1]. But this is no surprise: it happens in our labs all the time[2]!

---

[1]For example, for the $d=2$, $L=100$ Ising model, we have $2^{100\times 100}=2^{10000}\approx 10^{3010}$ states. A typical sample yielding a very accurate measurement consists of $\approx 10^7$ states, i.e. a fraction of $\approx 10^{-3003}$! This fraction becomes many orders of magnitude smaller for realistic complex systems studied in today's supercomputers.

[2]For a gas formed by 10^{22} molecules which has volume equal to 1 lt in room temperature and atmospheric pressure, the average velocity of its molecules is $\approx 100ms^{-1}$. This means that the typical de Broglie wavelength of the molecules is $\lambda\approx 10^{-10}m$. If we estimate that the volume occupied by each molecule is of order λ^3, then the number of states that each molecule can be is $\approx 10^{27}$. Therefore the system can be in $\approx (10^{27})^{10^{22}}$ different states. If we assume that on the average the molecules collide 10^9 times per

12.1 Statistical Physics

Statistical physics describes systems with a very large number of degrees of freedom N. For simple macroscopic systems $N \approx 10^{23} - 10^{44}$. For such systems, it is practically impossible to solve the microscopic equations that govern their dynamics. Even if we could, the solution would have had much more information than we need (and capable of analyzing!). It is enough, however, to know a small number of bulk properties of the system in order to have a useful description of it. E.g. it is enough to know the internal energy and magnetization of a magnet or the energy and density of a fluid instead of the detailed knowledge of the position, momentum, energy and angular momentum of each particle they are made of. These quantities provide a thermodynamic description of a system. Statistical physics makes an attempt to derive these quantities from the microscopic degrees of freedom and their dynamics given by the Hamiltonian of the system.

Consider a system which can be in a set of *discrete* states which belong to a countable set $\{\mu\}$. The energy spectrum of those states is assumed to consist of discrete values[3] $E_0 < E_1 < \ldots < E_n < \ldots$. This system is in contact and interacts with a large heat reservoir which has temperature $\beta = 1/kT$. The contact with the reservoir results in *random* transitions which change the energy of the system[4]. The system is described by the weights $w_\mu(t)$ which give the probability to find the system in a state μ at time t. These weights are the connection between the microscopic and statistical description of the system. When this system is in thermal equilibrium with the reservoir, its statistical properties are described by the, so called, *canonical ensemble*.

Let $R(\mu \to \nu)$ be the transition rates from the state $\mu \to \nu$, i.e.

$$R(\mu \to \nu)dt = \text{Transition probability } \mu \to \nu \text{ in time } dt\,, \tag{12.1}$$

which depend on the interaction between the system and the thermal

second, then we have $\approx 10^{31}$ changes of states per second. In order that the system visits all possible states, the time needed is $10^{10^{23}}$ times the age of the universe [4].

[3] E_0 is the *ground state* energy of the system.

[4] An isolated system always has constant energy. Such a system is studied in the microcanonical ensemble.

reservoir. The *master equation* for the weights $w_\mu(t)$ is

$$\begin{aligned}\frac{dw_\mu(t)}{dt} &= \sum_\nu \{w_\nu(t)R(\nu\to\mu) - w_\mu(t)R(\mu\to\nu)\}\\ \sum_\mu w_\mu(t) &= 1\,.\end{aligned} \tag{12.2}$$

The first of the above equations tells us that the change in $w_\mu(t)$ is equal to the rate that the system comes into the state μ from *any* other state ν, minus the rate of leaving the state μ. The second equation is a result of the probability interpretation of the weights $w_\mu(t)$ and states that the probability of finding the system in any state is equal to 1 at all times.

The transition rates $R(\mu\to\nu)$ are assumed to be time independent and then the above system of equations for $w_\mu(t)$ is linear with real parameters. This, together with the constraint $0\leq w_\mu(t)\leq 1$, implies that[5], in the large time limit, $\frac{dw_\mu(t)}{dt}=0$ and the system reaches equilibrium. Then, the $w_\mu(t)$ converge to finite numbers $p_\mu\geq 0$. These are the equilibrium occupation probabilities

$$p_\mu = \lim_{t\to\infty} w_\mu(t)\,, \qquad\qquad \sum_\mu p_\mu = 1\,. \tag{12.3}$$

For a system in thermodynamic equilibrium with a reservoir in temperature T, with $\beta = 1/kT$, the probabilities p_μ follow the Boltzmann distribution (Gibbs 1902)

$$p_\mu = \frac{1}{Z}\,\mathrm{e}^{-\beta E_\mu}\,, \tag{12.4}$$

and define the, so called, *canonical ensemble*. The parameter β will be frequently referred to as simply "the temperature" of the system, although, strictly speaking, it is the inverse of it. Its appearance in the exponential in equation (12.4), defines a characteristic energy scale of the system. The Boltzmann constant $k\approx 1.38\times 10^{-23} JK^{-1}$ is simply a conversion constant between units of energy[6].

---

[5]Note that equation (12.2) can be written in the form $\frac{dw_\mu(t)}{dt} = \sum_\nu \mathcal{R}_{\mu\nu} w_\nu(t)$, where the matrix $\mathcal{R}_{\mu\nu}$ has real, constant elements.

[6]It is *not* a fundamental constant of nature like c, $\hbar$, G, ... Temperature *is* an energy scale and the fact that it is customary to measure it in degrees Kelvin or other, is a historical accident due to the ignorance of the microscopic origin of heat exchange at the times of the original formulation of thermodynamics.

The normalization Z in equation (12.4) is the so called partition function of the system. The condition $\sum_\mu p_\mu = 1$ implies

$$Z(\beta) = \sum_\mu \mathrm{e}^{-\beta E_\mu} \tag{12.5}$$

The measurement of a physical quantity, or *observable*, of a thermodynamic system has a stochastic character. For systems with very large number of degrees of freedom N, one is interested only in the average value of such a quantity. This is because the probability of measuring the quantity to take a value significantly different from its average is ridiculously small. The average, or *expectation value*, $\langle \mathcal{O} \rangle$ of a physical observable $\mathcal{O}$ whose value in a state μ is $\mathcal{O}_\mu$ is equal to

$$\langle \mathcal{O} \rangle = \sum_\mu p_\mu \mathcal{O}_\mu = \frac{1}{Z} \sum \mathcal{O}_\mu \mathrm{e}^{-\beta E_\mu} \,. \tag{12.6}$$

As we will see later, the standard deviation $\Delta\mathcal{O}$ for a typical thermodynamic system is such that

$$\frac{\Delta\mathcal{O}}{\mathcal{O}} \sim \frac{1}{\sqrt{N}} \,, \tag{12.7}$$

which is quite small for macroscopic systems[7]. In such cases, the fluctuations of the values of $\mathcal{O}$ from its expectation value $\langle \mathcal{O} \rangle$ can be neglected. The limit $N \to \infty$ is the so called *thermodynamic limit*, and it is in this limit in which we are studying systems in statistical mechanics. Most systems in the lab are practically in this limit, but in the systems simulated on a computer we may be far from it. The state of the art is to invent methods which can be used to extrapolate the results from the study of the finite system to the thermodynamic limit efficiently.

Because of (12.5), the partition function encodes all the statistical information about the system. It is not just a simple function of one or more variables, but it counts all the states of the system with the correct weight. Its knowledge is equivalent to being able to compute any

[7] E.g. for $N \sim 10^{23}$ we have that $\Delta\mathcal{O}/\mathcal{O} \sim 10^{-11}$ and the measurements of $\mathcal{O}$ fluctuate at the 11th significant digit of their value. This is usually much smaller than other experimental errors.

thermodynamic quantity like, for example, the expectation value of the energy $\langle E \rangle$ of the system[8]:

$$\begin{aligned} U &\equiv \langle E \rangle = \frac{1}{Z}\sum_\mu E_\mu \, \mathrm{e}^{-\beta E_\mu} = -\frac{1}{Z}\sum_\mu \frac{\partial}{\partial\beta}\mathrm{e}^{-\beta E_\mu} = -\frac{1}{Z}\frac{\partial}{\partial\beta}\sum_\mu \mathrm{e}^{-\beta E_\mu} \\ &= -\frac{1}{Z}\frac{\partial Z}{\partial\beta} = -\frac{\partial \ln Z}{\partial\beta} \,. \end{aligned} \tag{12.8}$$

Similarly, one can calculate the specific heat from

$$C = \frac{\partial U}{\partial T} = \frac{\partial\beta}{\partial T}\frac{\partial U}{\partial\beta} = (-k\beta^2)(-\frac{\partial^2 \ln Z}{\partial\beta^2}) = k\beta^2\frac{\partial^2 \ln Z}{\partial\beta^2} \,. \tag{12.9}$$

12.2 Entropy

The entropy S of a thermodynamic system is defined by

$$S = -\frac{\partial F}{\partial T}\,, \quad F = U - TS \,, \tag{12.10}$$

where F is the *free energy* if the system. We will attempt to provide microscopic definitions that are consistent with the above equations.

We define the free energy from the relation

$$\mathrm{e}^{-\beta F} = Z \equiv \sum_\mu \mathrm{e}^{-\beta E_\mu} \,, \tag{12.11}$$

or equivalently

$$F = -\frac{1}{\beta}\ln Z \,. \tag{12.12}$$

Note that for $T \to 0$ the free energy becomes the ground state energy[9]. Indeed, as $\beta \to \infty$ only the lowest energy term in equation (12.11) survives. For this reason, equation (12.10) gives $\lim_{T\to 0} S = 0$, which is the third law of thermodynamics.

[8]In thermodynamics, $\langle E \rangle$ corresponds to the internal energy U of the system.

[9]For strict equality it is necessary that the ground state is not degenerate as it happens in the case of spontaneous symmetry breaking.

The definition (12.11) is consistent with (12.10) since

$$U = -\frac{\partial \ln Z}{\partial \beta} = -\frac{\partial}{\partial \beta}(-\beta F) = F + \beta\frac{\partial F}{\partial \beta} = F - T\frac{\partial F}{\partial T} = F + TS\,. \quad (12.13)$$

The relation of the entropy S to the microscopic degrees of freedom can be derived from equations (12.11) and (12.10):

$$\frac{S}{k} = \frac{U-F}{kT} = \beta(U-F) = \beta(\sum_\mu p_\mu E_\mu + \frac{1}{\beta}\ln Z)\,. \quad (12.14)$$

But

$$p_\mu = \frac{\mathrm{e}^{-\beta E_\mu}}{Z} \Rightarrow E_\mu = -\frac{1}{\beta}(\ln p_\mu + \ln Z)\,, \quad (12.15)$$

therefore

$$\begin{aligned}\frac{S}{k} &= \beta\sum_\mu\left(-\frac{1}{\beta}(\ln p_\mu + \ln Z)p_\mu + \frac{1}{\beta}\ln Z\right)\\ &= -\sum_\mu p_\mu \ln p_\mu - \ln Z\sum_\mu p_\mu + \ln Z\\ &= -\sum_\mu p_\mu \ln p_\mu\,. \end{aligned} \quad (12.16)$$

Finally

$$S = -k\sum_\mu p_\mu \,\ln p_\mu\,. \quad (12.17)$$

Let's analyze the above relation in some special cases. Consider a system[10] where all possible states have the same energy. For such a system, using equation (12.17), we obtain that

$$p_\mu = \frac{1}{g} = \text{const.} \Rightarrow S = k\ln g\,. \quad (12.18)$$

Therefore, the entropy simply counts the number of states of the system. This is also the case in the microcanonical ensemble. Indeed, equation (12.18) is also valid for the distribution

$$p_\mu = \left\{\begin{array}{cc} \frac{1}{g(E)} & E_\mu = E\\ 0 & E_\mu \neq E\end{array}\right.\,, \quad (12.19)$$

[10] E.g. the random walker, two dimensional quantum gravity without matter.

which can be considered to be equivalent to the microcanonical ensemble since it enforces $E_\mu = E = \text{const}$. Equation (12.19) can viewed as an approximation to a distribution sharply peaked at E. In such a case, S counts, more or less, the number of states of the system with energy close to E.

In general, the function[11] $g(E)$ is defined to be equal to the number of states with energy equal to E. Then the probability $p(E)$ to measure energy E in the canonical ensemble is

$$p(E) = \langle \delta_{E,E_\mu} \rangle = \sum_\mu p_\mu \delta_{E,E_\mu} = \frac{1}{Z} \sum \mathrm{e}^{-\beta E_\mu} \delta_{E,E_\mu} = \frac{1}{Z} \mathrm{e}^{-\beta E} \sum \delta_{E,E_\mu} . \tag{12.20}$$

Since $\sum_\mu \delta_{E,E_\mu} = g(E)$, we obtain

$$p(E) = \langle \delta_{E,E_\mu} \rangle = \frac{g(E)\, \mathrm{e}^{-\beta E}}{Z} . \tag{12.21}$$

For a generic system we have that

$$g(E) \sim E^{\alpha N} , \tag{12.22}$$

where N is the number of degrees of freedom of the system and α is a constant. The qualitative behavior of the distribution (12.21) is shown in figure 12.1. For such a system the most probable values of the energy are sharply peaked around a value E^* and the deviation ΔE is a measure of the energy fluctuations. The ratio $\Delta E/E$ drops with N as $1/\sqrt{N}$. Indeed, the function[12]

$$\tilde{p}(E) = E^{\alpha N} \mathrm{e}^{-\beta E} = \mathrm{e}^{-\beta E - \alpha N \ln E} \tag{12.23}$$

has a maximum when

$$\left. \frac{\partial \ln \tilde{p}(E)}{\partial E} \right|_{E=E^*} = 0 \Rightarrow \left. \frac{\partial}{\partial E} (-\beta E + \alpha N \ln E) \right|_{E=E^*} = -\beta + \frac{\alpha N}{E^*} = 0 \tag{12.24}$$

or

$$E^* = \frac{\alpha}{\beta} N . \tag{12.25}$$

[11] The notation $\Omega(E)$ is also frequently used and it is referred to as the *density of states*.

[12] $\tilde{p}(E)$ is proportional to $p(E)$ for fixed β. It is only defined for convenience.

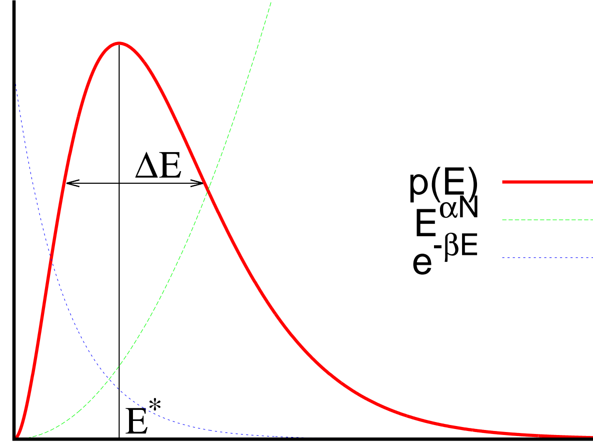

Figure 12.1: The probability $p(E)$ as a result of the competition of the Boltzmann factor $\mathrm{e}^{-\beta E}$ and the density of states $g(E) \sim E^{\alpha N}$. E^* is the most probable value of the energy and ΔE is a measure of the energy fluctuations.

As the temperature increases (β decreases), E^* shifts to larger values. E^* is proportional to the system size. By Taylor expanding around E^* we obtain

$$
\begin{aligned}
\ln \tilde{p}(E) &= \ln \tilde{p}(E^*) + (E - E^*) \left.\frac{\partial \ln \tilde{p}(E)}{\partial E}\right|_{E=E^*} \\
&\quad + \frac{1}{2}(E - E^*)^2 \left.\frac{\partial^2 \ln \tilde{p}(E)}{\partial E^2}\right|_{E=E^*} + \ldots \\
&= \ln \tilde{p}(E^*) + \frac{1}{2}(E - E^*)^2 \left(-\frac{\alpha N}{(E^*)^2}\right) + \ldots\,, \qquad (12.26)
\end{aligned}
$$

where we used equation (12.24) and computed $\left.\frac{\partial^2 \ln \tilde{p}(E)}{\partial E^2}\right|_{E=E^*}$. Therefore

$$
p(E) \approx p(E^*) \mathrm{e}^{-\alpha N \frac{(E-E^*)^2}{2(E^*)^2}}\,, \qquad (12.27)
$$

which is a Gaussian distribution with standard deviation

$$
\Delta E \sim \sqrt{\frac{(E^*)^2}{\alpha N}} = \sqrt{\frac{(\frac{\alpha N}{\beta})^2}{\alpha N}} \sim \frac{\sqrt{N}}{\beta}\,. \qquad (12.28)
$$

Therefore we confirm the relation (12.7)

$$\frac{\Delta E}{E^*} \sim \frac{\frac{\sqrt{N}}{\beta}}{\frac{N}{\beta}} = \frac{1}{\sqrt{N}}\,. \tag{12.29}$$

In the analysis above we assumed analyticity (Taylor expansion, equation (12.26)), which is not valid at a critical point of a phase transition in the thermodynamic limit.

Another important case where the above analysis becomes slightly more complicated is when the distribution $p(E)$ has more than one equally probable maxima[13] separated by a large probability barrier as shown in figure 12.2 like when the system undergoes a first order phase transition. Such a transition occurs when ice turns into water or when a ferromagnet looses its permanent magnetization due to temperature increase past its Curie point. In such a case the two states, ice – water / ferromagnet – paramagnet, are equally probable and coexist. This is qualitatively depicted in figure 12.2.

12.3 Fluctuations

The stochastic behavior of every observable $\mathcal{O}$ is given by a distribution function $p(\mathcal{O})$ which can be derived from the Boltzmann distribution (12.4). Such a distribution is completely determined by its expectation value $\langle\mathcal{O}\rangle$ and all its higher order moments, i.e. the expectation values $\langle(\mathcal{O}-\langle\mathcal{O}\rangle)^n\rangle$, $n=1,2,3.\ldots$. The most commonly studied moment is the second moment ($n=2$)

$$(\Delta\mathcal{O})^2 \equiv \langle(\mathcal{O}-\langle\mathcal{O}\rangle)^2\rangle = \langle\mathcal{O}^2\rangle - \langle\mathcal{O}\rangle^2\,. \tag{12.30}$$

For a distribution with a single maximum, $\Delta\mathcal{O}$ is a measure of the fluctuations of $\mathcal{O}$ away from its expectation value $\langle\mathcal{O}\rangle$. When $\mathcal{O}=E$ we obtain

$$(\Delta E)^2 \equiv \langle(E-\langle E\rangle)^2\rangle = \langle E^2\rangle - \langle E\rangle^2\,, \tag{12.31}$$

and using the relations

$$\langle E^2\rangle = \frac{1}{Z}\sum_\mu E_\mu^2 \mathrm{e}^{-\beta E_\mu} = \frac{1}{Z}\frac{\partial^2}{\partial\beta^2}\sum_\mu \mathrm{e}^{-\beta E_\mu} = \frac{1}{Z}\frac{\partial^2 Z}{\partial\beta^2}\,, \tag{12.32}$$

[13] When there are many local maxima, the absolute maximum dominates in the thermodynamic limit $N\to\infty$.

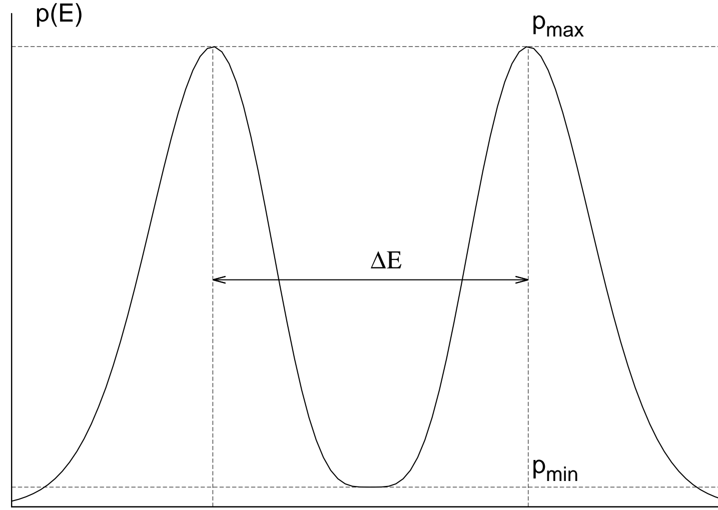

Figure 12.2: Two peak structure in the distribution $p(E)$ of the energy E for a system undergoing a first order phase transition. The two maxima correspond to two coexisting states ("ice"–"water") and $\Delta E/N$ is the latent heat. In the thermodynamic limit $N \to \infty$, $R = p_{min}/p_{max}$ decreases like $R \sim \mathrm{e}^{-fA}$, where A is the minimal surface separating the two phases and f is the interface tension.

and

$$\langle E\rangle = \frac{1}{Z}\sum_\mu E_\mu \mathrm{e}^{-\beta E_\mu} = -\frac{1}{Z}\frac{\partial}{\partial\beta}\sum_\mu \mathrm{e}^{-\beta E_\mu} = -\frac{1}{Z}\frac{\partial Z}{\partial\beta}\,, \tag{12.33}$$

we obtain that

$$(\Delta E)^2 = \langle E^2\rangle - \langle E\rangle^2 = \frac{1}{Z}\frac{\partial^2 Z}{\partial\beta^2} - \left(-\frac{1}{Z}\frac{\partial Z}{\partial\beta}\right)^2 = \frac{\partial^2 \ln Z}{\partial\beta^2}\,, \tag{12.34}$$

which, according to (12.9), is the specific heat

$$C = \frac{\partial\langle E\rangle}{\partial T} = k\beta^2(\Delta E)^2\,. \tag{12.35}$$

This way we relate the specific heat of a system (a thermodynamic quantity) with the microscopic fluctuations of the energy.

This is true for every physical quantity which is linearly coupled to an external field (in the case of E, this role is played by β). For a magnetic

system in a constant magnetic field B, such a quantity is the magnetization M. If M_μ is the magnetization of the system in the sate μ and we assume that its direction is parallel to the direction of the magnetic field $\vec{B}$, then the Hamiltonian of the system is

$$H = E - BM\,, \tag{12.36}$$

and the partition function is

$$Z = \sum_\mu \mathrm{e}^{-\beta E_\mu + \beta B M_\mu}\,. \tag{12.37}$$

"Linear coupling" signifies the presence of the linear term BM in the Hamiltonian. The quantities B and M are called conjugate to each other. Other well known conjugate quantities are the pressure/volume (P/V) in a gas or the chemical potential/number of particles (μ/N) in the grand canonical ensemble.

Because of this linear coupling we obtain

$$\langle M\rangle = \frac{1}{Z}\sum_\mu M_\mu\,\mathrm{e}^{-\beta E_\mu + \beta B M_\mu} = \frac{1}{\beta Z}\frac{\partial Z}{\partial B} = -\frac{\partial F}{\partial B}\,, \tag{12.38}$$

which is analogous to (12.8). The equation corresponding to (12.34) is obtained from (12.30) for $\mathcal{O} = M$

$$(\Delta M)^2 \equiv \langle (M - \langle M\rangle)^2\rangle = \langle M^2\rangle - \langle M\rangle^2\,. \tag{12.39}$$

From (12.37) we obtain

$$\langle M^2\rangle = \frac{1}{Z}\sum_\mu M_\mu^2\,\mathrm{e}^{-\beta E_\mu + \beta B M_\mu} = \frac{1}{\beta^2 Z}\frac{\partial^2 Z}{\partial B^2}\,, \tag{12.40}$$

therefore

$$(\Delta M)^2 = \frac{1}{\beta^2}\left\{\frac{1}{Z}\frac{\partial^2 Z}{\partial B^2} - \frac{1}{Z^2}\left(\frac{\partial Z}{\partial B}\right)^2\right\} = \frac{1}{\beta^2}\frac{\partial^2 \ln Z}{\partial B^2} = \frac{1}{\beta}\frac{\partial\langle M\rangle}{\partial B}\,. \tag{12.41}$$

The magnetic susceptibility χ is defined by the equation

$$\chi = \frac{1}{N}\frac{\partial\langle M\rangle}{\partial B} = \frac{\beta}{N}\langle (M - \langle M\rangle)^2\rangle\,, \tag{12.42}$$

where we see its relation to the fluctuations of the magnetization. This analysis can be repeated in a similar way for every pair of conjugate quantities.

12.4 Correlation Functions

The correlation functions can be obtained is a similar manner when we consider external fields which are space dependent. For simplicity, consider a system defined on a discrete lattice, whose sites are mapped to natural numbers $i = 1, \ldots, N$. Then the magnetic field B_i is a function of the position i and interacts with the spin s_i so that

$$H = E - \sum_i B_i s_i \,. \tag{12.43}$$

Then the magnetization per site $m_i \equiv s_i$[14] at position i is

$$\langle s_i \rangle = \frac{1}{\beta} \frac{\partial \ln Z}{\partial B_i} \,. \tag{12.44}$$

The connected two point correlation function is defined by

$$G_c^{(2)}(i,j) = \langle (s_i - \langle s_i \rangle)(s_j - \langle s_j \rangle) \rangle = \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle = \frac{1}{\beta^2} \frac{\partial^2 \ln Z}{\partial B_i \partial B_j} \,. \tag{12.45}$$

When the values of s_i and s_j are strongly correlated, i.e. they "vary together" in the random samples that we take, the function (12.45) takes on large positive values. When the values of s_i and s_j are not at all correlated with each other, the terms $(s_i - \langle s_i \rangle)(s_j - \langle s_j \rangle)$ in the sum over μ in the expectation value $\langle (s_i - \langle s_i \rangle)(s_j - \langle s_j \rangle) \rangle$ cancel each other and $G_c^{(2)}(i,j)$ is zero[15].

The function $G_c^{(2)}(i,j)$ takes its maximum value $\langle (s_i - \langle s_i \rangle)^2 \rangle$ for $i = j$. Then it falls off quite fast. For a generic system

$$G_c^{(2)}(i,j) \sim \mathrm{e}^{-|x_{ij}|/\xi} \,, \tag{12.46}$$

where $|x_{ij}|$ is the distance between the points i and j. The *correlation length* ξ, is a characteristic length scale of the system which is a measure of the distance where there is a measurable correlation between the magnetic

[14] Actually the two quantities are proportional to each other, but for simplicity we set the proportionality constant equal to 1.

[15] There is also the possibility (not occurring in our discussion) that s_i and s_j are strongly anti-correlated in which case $G_c^{(2)}(i,j)$ is negative.

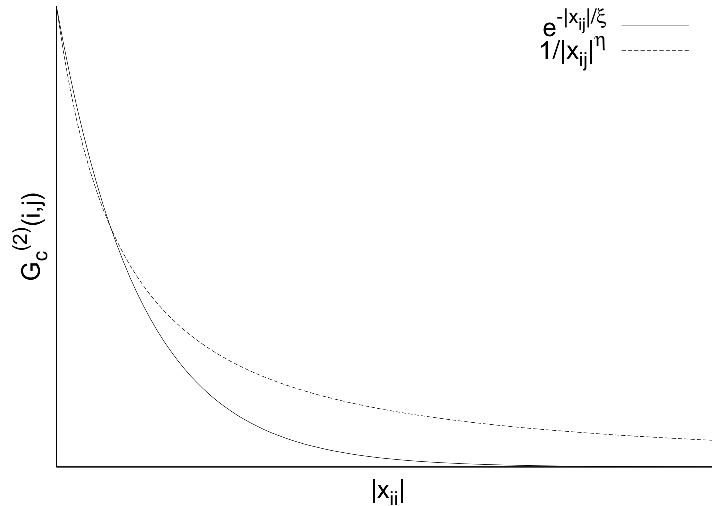

Figure 12.3: The connected two point correlation function $G_c^{(2)}(i,j)$ for $\xi < \infty$ and $\xi \to \infty$.

moments of two lattice sites. It depends on the parameters that define the system $\xi = \xi(\beta, B, N, \ldots)$. It is important to stress that it is a length scale that arises *dynamically*. In contrast, length scales like the size of the system L or the lattice constant a are parameters of the system which don't depend on the dynamics. In most of the cases, ξ is of the order of a few lattice constants a and such a system does not exhibit correlations at macroscopic scales (i.e. of the order of L).

Interesting physics arises when $\xi \to \infty$. This can happen by fine tuning the parameters on which ξ depends on to their *critical values*. For example, in the neighborhood of a continuous[16] phase transition, the exponential falloff in (12.46) vanishes and $G_c^{(2)}(i,j)$ falls off like a power (see figure 12.3)

$$G_c^{(2)}(i,j) \sim \frac{1}{|x_{ij}|^{d-2+\eta}}\,, \tag{12.47}$$

where d is the number of dimensions of space and η a *critical exponent*. As we approach the critical point[17], correlations extend to distances $|x_{ij}| \gg$

[16] I.e. not of first order.

[17] If we tune many parameters, this is a *critical surface* in the parameter space.

a. Then the system is not sensitive to the short distance details of the lattice and its dynamics are very well approximated by continuum space dynamics. Then we say that we obtain the *continuum limit* of a theory which is microscopically defined on a lattice. Since the microscopic details become irrelevant, a whole class of theories with different microscopic definitions[18] have the same continuum limit. This phenomenon is called *universality* and plays a central role in statistical physics and quantum field theories.

12.5 Sampling

Our main goal is to calculate the expectation value $\langle \mathcal{O} \rangle$,

$$\langle \mathcal{O} \rangle = \sum_\mu p_\mu \, \mathcal{O}_\mu = \frac{\sum_\mu \mathcal{O}_\mu \mathrm{e}^{-\beta E_\mu}}{\sum_\mu \mathrm{e}^{-\beta E_\mu}} \,, \qquad (12.48)$$

of a physical quantity, or *observable*, $\mathcal{O}$ of a statistical system in the canonical ensemble approximately. For this reason we construct a sample of M states $\{\mu_1, \mu_2, \ldots, \mu_M\}$ which are distributed according to a chosen probability distribution P_μ. We define the *estimator* $\mathcal{O}_M$ of $\langle \mathcal{O} \rangle$ to be

$$\mathcal{O}_M = \frac{\sum_{i=1}^M \mathcal{O}_{\mu_i} P_{\mu_i}^{-1} \mathrm{e}^{-\beta E_{\mu_i}}}{\sum_{i=1}^M P_{\mu_i}^{-1} \mathrm{e}^{-\beta E_{\mu_i}}} \,. \qquad (12.49)$$

The above equation is easily understood since, for a large enough sample, $P_{\mu_i} \approx$ "*Frequency of finding* μ_i *in the sample*", and we expect that

$$\langle \mathcal{O} \rangle = \lim_{M\to\infty} \mathcal{O}_M \,. \qquad (12.50)$$

Our goal is to find an appropriate P_μ so that the convergence of (12.50) is as fast as possible. Consider the following cases:

12.5.1 Simple Sampling

We choose $P_\mu = \text{const.}$, and equation (12.49) becomes

$$\mathcal{O}_M = \frac{\sum_{i=1}^M \mathcal{O}_{\mu_i} \mathrm{e}^{-\beta E_{\mu_i}}}{\sum_{i=1}^M \mathrm{e}^{-\beta E_{\mu_i}}} \,. \qquad (12.51)$$

[18] E.g. defined on square or triangular lattices, with nearest neighbor or next to nearest neighbor interactions.

The problem with this choice is the small *overlap* of the sample with the states that make the most important contributions to the sum in (12.48). As we have already mentioned in the introduction, the size of the sample in a Monte Carlo simulation is a minuscule fraction of the total number of states. Therefore, the probability of picking the ones that make important contributions to the sum in (12.48) is very small. Consider for example the case $\mathcal{O} = E$ in a generic model. According to equation (12.21) we have that

$$\langle E \rangle = \sum_E E\, p(E)\,, \tag{12.52}$$

where $p(E)$ is the probability of measuring energy E in the system. A qualitative plot of $p(E)$ is shown in figure 12.1. From (12.25) and (12.28) we have that $E^* \sim 1/\beta$ and $\Delta E \sim 1/\beta$, therefore for $\beta = 0$ and $\beta > 0$ the qualitative behavior of the respective $p(E)$ distributions is shown in figure 12.4. The distribution of the simple sampling corresponds to the case $\beta =$

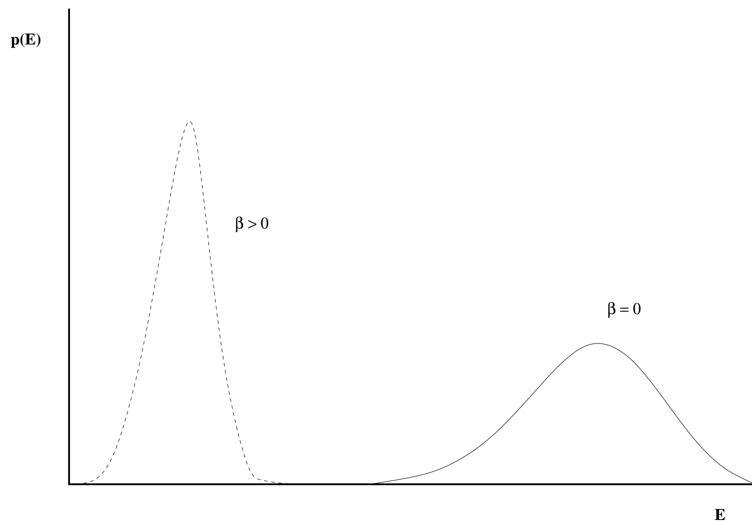

Figure 12.4: The distributions $p(E)$ for a generic model for temperatures $\beta = 0$ and $\beta > 0$. The two distributions have negligible overlap. In order that the $\beta = 0$ distribution is as shown, we assume that the energy of all states is bounded and that the system has a finite number of degrees of freedom.

0 in equation (12.4), since p_μ =const. in this case[19]. In order to calculate

[19]For these statements to be well defined, we assume that the energy of all states is bounded and that the system has a finite number of degrees of freedom. Otherwise

the sum (12.52) with acceptable accuracy for $\beta > 0$ we have to obtain a good sample in the region where the product $E\, p_{\beta>0}(E)$ is relatively important. The probability of obtaining a state such that $E\, p_{\beta>0}(E)$ is non negligible is very small when we use the $p_{\beta=0}(E)$ distribution. This can be seen pictorially in figure 12.4.

Even though this method has this serious shortcoming, it could still be useful in some cases. We have already applied it in the study of random walks. Note that, by applying equation (12.51), we can use the same sample for calculating expectation values for all values of β.

12.5.2 Importance Sampling

From the previous discussion it has become clear that, for a large system, a very small fraction of the space of states makes a significant contribution to the calculation of $\langle \mathcal{O} \rangle$. If we choose a sample with probability

$$P_\mu = p_\mu = \frac{\mathrm{e}^{-\beta E_\mu}}{Z}\,, \tag{12.53}$$

then we expect to sample exactly within this region. Indeed, the estimator, given by equation (12.49), is calculated from

$$\mathcal{O}_M = \frac{\sum_{i=1}^M \mathcal{O}_{\mu_i} \left(\mathrm{e}^{-\beta E_{\mu_i}}\right)^{-1} \mathrm{e}^{-\beta E_{\mu_i}}}{\sum_{i=1}^M \left(\mathrm{e}^{-\beta E_{\mu_i}}\right)^{-1} \mathrm{e}^{-\beta E_{\mu_i}}} = \frac{1}{M} \sum_{i=1}^M \mathcal{O}_{\mu_i}\,. \tag{12.54}$$

Sampling this way is called *importance sampling*, and it is the method of choice in most Monte Carlo simulations. The sample depends on the temperature β and the calculation of the expectation values (12.54) requires a new sample for each[20] β. This extra effort, however, is much smaller than the one required in order to overcome the overlap problem discussed in the previous subsection.

12.6 Markov Processes

Sampling according to a desired probability distribution P_μ is not possible in a direct way. For example, if we attempt to construct a sample accord-

consider the overlap for two temperatures $\beta_1 \gg \beta_2$.

[20]We can use the same sample for a range of temperatures by using the *histogram method*, see [4].

ing to $P_\mu = \frac{e^{-\beta E_\mu}}{Z}$ by picking a state μ by chance and add it to the sample with probability P_μ, then we have a very small probability to accept that state in the sample. Therefore, the difficulty of constructing the sample runs into the same overlap problem as in the case of simple sampling. For this reason we construct a *Markov chain* instead. The members of the sequence of the chain will be our sample. A Markov process, or a Markov chain, is a stochastic process which, given the system in a state μ, puts the system in a new state ν in such a way that it has the *Markov property*, i.e. that it is memoryless. This means that a chain of states

$$\mu_1 \to \mu_2 \to \ldots \to \mu_M \,, \tag{12.55}$$

is constructed in such a way that the *transition probabilities* $P(\mu \to \nu)$ from the state μ to a new state ν satisfy the following requirements:

1. They are independent of "time"

2. They depend only on the states μ and ν and not on the path that the system has followed on order to get to the sate μ (memorylessness)

3. The relation
$$\sum_\nu P(\mu \to \nu) = 1 \tag{12.56}$$
holds. Beware, in most of the cases $P(\mu \to \mu) > 0$, i.e. the system has a nonzero probability to remain in the same state

4. For $M \to \infty$ the sample $\{\mu_i\}$ follows the P_μ distribution.

Then our sample will be $\{\mu_i\} \equiv \{\mu_1, \mu_2, \ldots, \mu_M\}$. We may imagine that this construction happens in "time" $i = 1, 2, \ldots, M$. In a Monte Carlo simulation we construct a sample from a Markov chain by appropriately choosing the transition probabilities $P(\mu \to \nu)$ so that the convergence 4. is fast.

Choosing the initial state μ_1 can become a non trivial task. If it turns out not to be a typical state of the sample, then it could take a long "time" for the system to "equilibrate", i.e. for the Markov process to start sampling states typical of the simulated temperature. The required time for this to happen is called the *thermalization time* which can become a serious part of our computational effort if we make a wrong choice of μ_1 and/or $P(\mu \to \nu)$.

A necessary condition for the sample to converge to the desired distribution is for the process to be *ergodic*. This means that for every state μ it is possible to reach any other state ν in a finite number of steps. If this criterion is not satisfied and a significant part of phase space is not sampled, then sampling will fail. Usually, given a state μ, the reachable states ν at the next step (i.e. the states for which $P(\mu \to \nu) > 0$) are very few. Therefore the ergodicity of the algorithm considered must be checked carefully[21].

12.7 Detailed Balance Condition

Equation (12.2) tells us that, in order to find the system in equilibrium in the p_μ distribution, the transition probabilities should be such that

$$\sum_\nu p_\mu P(\mu \to \nu) = \sum_\mu p_\nu P(\nu \to \mu)\,. \tag{12.57}$$

This means that the rate that the system comes into the state μ is equal to the rate in which it leaves μ. From equation (12.56) we obtain

$$p_\mu = \sum_\mu p_\nu P(\nu \to \mu)\,. \tag{12.58}$$

This condition is necessary but it is not sufficient (see section 2.2.3 in [4]). A sufficient, but not necessary, condition is the *detailed balance condition*. When the transition probabilities satisfy

$$p_\mu P(\mu \to \nu) = p_\nu P(\nu \to \mu)\,, \tag{12.59}$$

then the system will equilibrate to p_μ after sufficiently long thermalization time. By summing both sides of (12.59), we obtain the equilibrium condition (12.57). For the canonical ensemble (12.4) the condition becomes

[21] There exist algorithms which are non-ergodic but the non reachable states are of "measure zero" in the space of states. These algorithms are formally non ergodic, but they are ergodic from a practical point of view. On the contrary, there exist algorithms that are formally ergodic but there are large regions of phase space where the probability of getting there is very small. This puts "ergodic barriers" in the sampling which will lead to wrong results. A common example is sampling a system in the neighborhood of a first order phase transition where, for large systems, it is very hard to sample states in both phases.

$$\frac{P(\mu \to \nu)}{P(\nu \to \mu)} = \frac{p_\nu}{p_\mu} = \mathrm{e}^{-\beta(E_\nu - E_\mu)}\,. \tag{12.60}$$

One can show that if the transition probabilities satisfy the above conditions then the equilibrium distribution of the system will be the Boltzmann distribution (12.4). A program implementing a Monte Carlo simulation of a statistical system in the canonical ensemble consists of the following main steps:

1. Write a program that codes appropriately chosen transition probabilities $P(\mu \to \nu)$ that satisfy condition (12.60)
2. Choose an initial state μ_1
3. Let the system evolve until it thermalizes to the Boltzmann distribution (12.4) (thermalization)
4. Collect data for the observables $\mathcal{O}$ and calculate the estimators $\mathcal{O}_M$ from equation (12.54)
5. Stop when the desired accuracy in the calculation of $\langle \mathcal{O} \rangle$ has been achieved.

Equation (12.60) has many solutions. For a given problem, we are looking for the most efficient one. Below we list some possible choices:

$$P(\mu \to \nu) = A \cdot \mathrm{e}^{-\frac{1}{2}\beta(E_\nu - E_\mu)}\,, \tag{12.61}$$

$$P(\mu \to \nu) = A \cdot \frac{\mathrm{e}^{-\beta(E_\nu - E_\mu)}}{1 + \mathrm{e}^{-\beta(E_\nu - E_\mu)}}\,, \tag{12.62}$$

$$P(\mu \to \nu) = A \cdot \left\{ \begin{array}{ll} \mathrm{e}^{-\beta(E_\nu - E_\mu)} & E_\nu - E_\mu > 0 \\ 1 & E_\nu - E_\mu \le 0 \end{array} \right. , \tag{12.63}$$

for appropriately chosen states $\nu \neq \mu$ and

$$P(\mu \to \mu) = 1 - \sum_\nu P(\mu \to \nu)\,. \tag{12.64}$$

$P(\mu \to \nu') = 0$ for any other state ν'. In order for (12.64) to be meaningful, the constant A has to be chosen so that

$$\sum_{\nu \neq \mu} P(\mu \to \nu) < 1 \,. \tag{12.65}$$

Equation (12.65) gives much freedom in the choice of transition probabilities. In most cases, we split $P(\mu \to \nu)$ in two independent parts

$$P(\mu \to \nu) = g(\mu \to \nu) \, A(\mu \to \nu) \,. \tag{12.66}$$

The probability $g(\mu \to \nu)$ is the *selection probability* of the state ν when the system is in the state μ. Therefore the first step in the algorithm is to select a state $\nu \neq \mu$ with probability $g(\mu \to \nu)$.

The second step is to accept the change with probability $A(\mu \to \nu)$. If the answer is no, then the system remains in the state μ. This way equation (12.64) is satisfied. The probabilities $A(\mu \to \nu)$ are called the *acceptance ratios*.

The art in the field is to device algorithms that give the maximum possible acceptance ratios for the new states ν and that the states ν are as much as possible statistically independent from the original state μ. An ideal situation is to have $A(\mu \to \nu) = 1$ for all ν for which $g(\mu \to \nu) > 0$. As we will see in a following chapter, this is what happens in the case of the Wolff cluster algorithm.

12.8 Problems

1. Prove equation (12.18).
2. Prove equation (12.19).
3. Prove equation (12.45).
4. Show that equations (12.61)–(12.63) satisfy (12.60).

Chapter 13

Simulation of the $d = 2$ Ising Model

This chapter is an introduction to the basic Monte Carlo methods used in the simulations of the Ising model on a two dimensional rectangular lattice, but also in a wide spectrum of scientific applications. We will introduce the Metropolis algorithm, which is the most common algorithm used in Monte Carlo simulations. We will discuss the thermalization of the system and the effect of correlations between successive spin configurations generated during the simulation. The *autocorrelation function* and the time scale defined by it, the *autocorrelation time*, are measures of these autocorrelations and play a central role in the study of the statistical independence of our measurements. Beating autocorrelations is crucial in Monte Carlo simulations since they are the main obstacle for studying large systems, which in turn is essential for taking the thermodynamic limit without the systematic errors introduced by finite size effects. We will also introduce methods for the computation of statistical errors that take into account autocorrelations. The determination of statistical errors is of central importance in order to assess the quality of a measurement and predict the amount of resources needed for reaching a specific accuracy goal.

13.1 The Ising Model

The Ising model (1925) [58] has played an important role in the evolution of ideas in statistical physics and quantum field theory. In particular, the two dimensional model is complicated enough in order to possess nontrivial properties but simple enough in order to be able to obtain an exact analytic solution. The zero magnetic field model has a 2nd order phase transition for a finite value of the temperature and we are able to compute critical exponents and study its continuum limit in detail. This gives us valuable information on the non analytic properties of a system undergoing a second order phase transition, the appearance of scaling, the renormalization group and universality. Using the exact solution[1] of Onsager (1948) [59] and others, we obtain exact results and compare them with those obtained via approximate methods, like Monte Carlo simulations, high and low temperature expansions, mean field theory etc. The result is also interesting from a physics point of view, since it is the simplest, phenomenologically interesting, model of a ferromagnetic material. Due to universality, the model describes also the liquid/vapor phase transition at the triple point. A well known textbook for a discussion of statistical mechanical models that can be solved exactly is the book by Baxter [57].

In order to define the model, consider a two dimensional square lattice like the one shown in figure 13.1. On each *site* or *node* of the lattice we have an "atom" or a "magnet" of spin s_i. The geometry is determined by the distance of the nearest neighbors, the *lattice constant* a, and the number of sites N. Each side consists of L sites so that $N = L \times L = L^d$, where $d = 2$ is the dimension of space. The topology is determined by the way sites are connected with each other via links. Special care is given to the sites located on the sides of the lattice. We usually take *periodic boundary conditions* which is equivalent to identifying the opposite sides of the square by connecting their sites with a link. This is depicted in figure (13.1). Periodic boundary conditions endow the plane on which the lattice is defined with a toroidal topology. The system's dynamics are determined by the spin–spin interaction. We take it to be *short range* and the simplest case considered here takes into account only nearest

[1] For a very nice proof of Onsager's solution look at the book by T. Huang [60] and the paper by C.N. Yang [61].

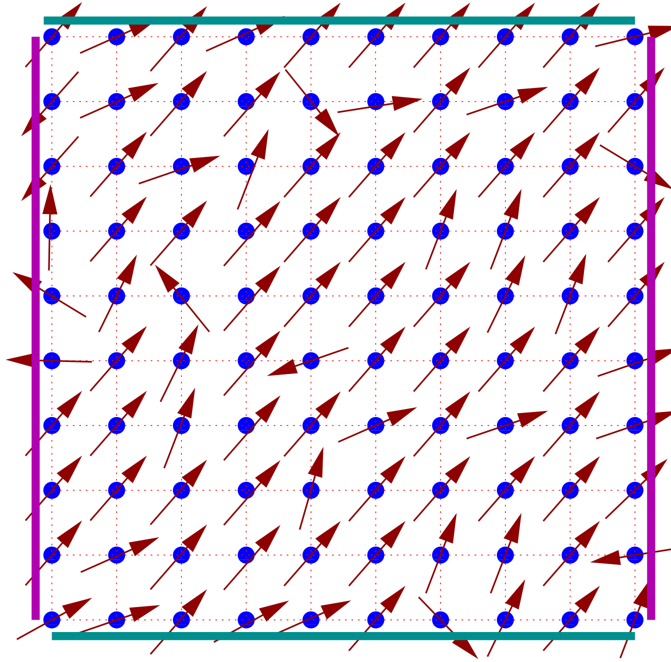

Figure 13.1: The two dimensional square lattice whose sites $i = 1, \ldots, N$ are occupied by "atoms" or "magnets" with spin s_i. In this figure spins may have any orientation on the plane (XY model). The simplest models take into account only the nearest neighbor interactions $-J\vec{s}_i \cdot \vec{s}_j$, where $\langle ij \rangle$ is a *link* of the lattice. We take periodic boundary conditions which result in a toroidal topology on the lattice where the horizontal and vertical sides of the lattice are identified. In the figure, identified sides have the same color and their respective sites are connected by a link..

neighbor interactions. In the Ising model, spins have two possible values, "up" or "down" which we map[2] to the numerical values $+1$ or -1. For the ferromagnetic model, each link is a "bond" whose energy is higher when the spins on each side of the link are pointing in the same direction and lower when they point in the opposite[3] direction. This is depicted in figure 13.1. The system could also be immersed in a constant magnetic field B whose direction is parallel to the direction of the spins.

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[2]This is only a convention. We could have picked 0 and 1 or any other pair of labels. The choice of labels affects only the expression of the Hamiltonian and related observables.

[3]The opposite is true for the antiferromagnetic model.

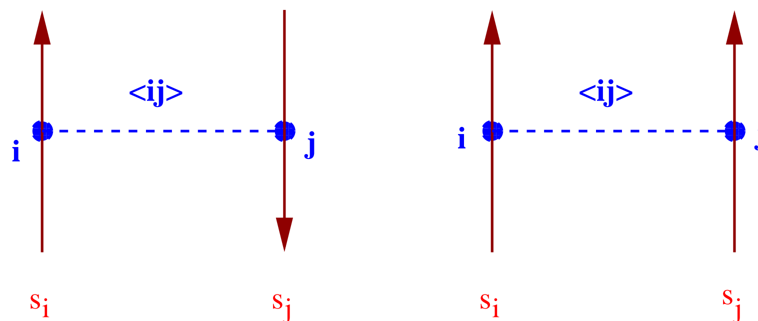


Figure 13.2: The Ising model spins take two possible values: "up" or "down" and the Hamiltonian of the system is the sum of contributions of the energy of all links ("bonds") $\langle ij \rangle$. The energy of each bond takes two values, $+J$ for opposite or $-J$ for same spins, where $J > 0$ for a ferromagnetic system. The system possesses a discrete Z_2 symmetry: The Hamiltonian is invariant when all $s_i \to -s_i$.

We are now ready to write the Hamiltonian and the partition function of the system. Consider a square lattice of N lattice sites (or *vertices*) labeled by a number $i = 1, 2, \ldots, N$. The lattice has N_l links (or *bonds*) among nearest neighbors. These are labeled by $\langle ij \rangle$, where (i, j) is the pair of vertices on each side of the link. We identify the sides of the square like in figure 13.1. Then, since two vertices are connected by one link and four links intersect at one vertex, we have that[4]

$$2N_l = 4N \Rightarrow N_l = 2N \,. \tag{13.1}$$

At each vertex we place a spin $s_i = \pm 1$. The Hamiltonian of the system is given by

$$H = -J \sum_{\langle ij \rangle} s_i s_j - B \sum_i s_i \,. \tag{13.2}$$

The first term is the spin–spin interaction and for $J > 0$ the system is ferromagnetic. In this book, we consider only the $J > 0$ case. A link connecting same spins has energy $-J$, whereas a link connecting opposite spins has energy $+J$. The difference of the energy between the two states is $2J$ and the spin-spin dynamics favor links connecting same spins. The

[4]It is easy to see that each vertex is in a one to one correspondence with a pair of links, say the east and north bound ones.

minimum energy E_0 is obtained for the *ground state*, which is the *unique*[5] state in which all spins point in the direction[6] of B. This is equal to

$$E_0 = -JN_l - BN = -(2J + B)N\,. \tag{13.3}$$

The partition function is

$$Z = \sum_{s_1=\pm 1} \sum_{s_2=\pm 1} \dots \sum_{s_N=\pm 1} \mathrm{e}^{-\beta H[\{s_i\}]} \equiv \sum_{\{s_i\}} \mathrm{e}^{\beta J \sum_{\langle ij\rangle} s_i s_j + \beta B \sum_i s_i}\,, \tag{13.4}$$

where $\{s_i\} \equiv \{s_1, s_2, \dots, s_N\}$ is a spin configuration of the system. The number of terms is equal to the number of configurations $\{s_i\}$, which is equal to 2^N, i.e. it increases exponentially with N. For a humble 5×5 lattice we have $2^{25} \approx 3.4 \times 10^6$ terms.

The two dimensional Ising model for $B = 0$ has the interesting property that, for $\beta = \beta_c$, where

$$\beta_c = \frac{1}{2}\ln(1 + \sqrt{2}) \approx 0.4406867935\dots\,, \tag{13.5}$$

it undergoes a phase transition between an *ordered* or *low temperature* phase where the system is magnetized ($\langle|M|\rangle > 0$) and a *disordered* or *high temperature* phase where the magnetization vanishes ($\langle|M|\rangle = 0$). The magnetization $\langle|M|\rangle$ distinguishes between the two phases and it is called the *order parameter*. The critical temperature β_c is the Curie temperature. The phase transition is of second order, which is a special case of a *continuous phase transition*. For a continuous phase transition the order parameter is continuous at $\beta = \beta_c$, but it is non analytic[7]. For a second order phase transition, its derivative is not continuous. This is qualitatively depicted in figure 13.1.

For $\beta \neq \beta_c$ the correlation function (12.45) behaves like in equation (12.46) resulting in a finite correlation length $\xi(\beta)$. The correlation

---

[5]When $B = 0$ the system has an "up–down" Z_2 symmetry. This means that states connected by the transformation $s_i \to -s_i$ for all i result in the same Hamiltonian. In this case we have two ground states and the system chooses one of them by *spontaneously breaking* the Z_2 symmetry.

[6]The vacuum structure of the antiferromagnetic system $J < 0$ for $B = 0$ is much richer.

[7]In contrast, a first order phase transition is a transition where the order parameter itself is discontinuous.

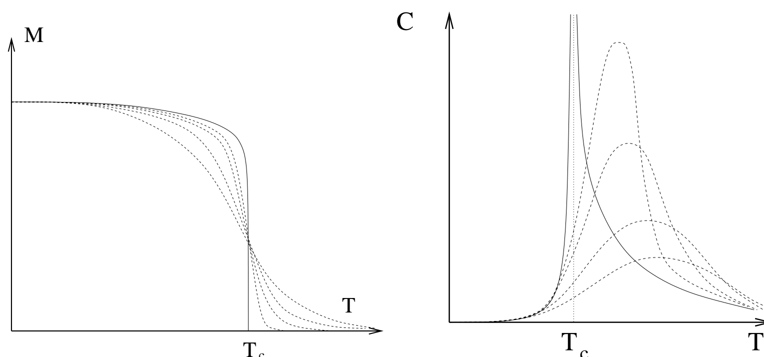


Figure 13.3: The qualitative behavior of the magnetization (left) and the specific heat (right) near the Ising model phase transition. The continuous line is the non analytic behavior in the thermodynamic limit, whereas the dashed lines show the behavior of the analytic, finite N behavior. The latter converge to the former in the large N limit (thermodynamic limit).

length[8] diverges as we approach the critical temperature, and its asymptotic behavior in this limit is given by the scaling relation

$$\xi(\beta) \equiv \xi(t) \sim |t|^{-\nu}, \qquad t = \frac{\beta_c - \beta}{\beta_c}. \tag{13.6}$$

Then the correlation function behaves according to (12.47)

$$G_c^{(2)}(i,j) \sim \frac{1}{|x_{ij}|^{\eta}}. \tag{13.7}$$

Scaling behavior is also found for the specific heat C, the magnetization $M \equiv \langle |M| \rangle$ and the magnetic susceptibility χ according to the relations

$$C \sim |t|^{-\alpha} \tag{13.8}$$
$$M \sim |t|^{\beta} \tag{13.9}$$
$$\chi \sim |t|^{-\gamma}, \tag{13.10}$$

whereas the magnetization for $t = 0$ and nonzero magnetic field $B \neq 0$ behaves like

$$M \sim B^{-1/\delta}. \tag{13.11}$$

[8]We mean the correlation length in the thermodynamic limit, i.e. we take the large N limit *first*.

The exponents in the above scaling relations are called *critical exponents* or *scaling exponents*. They take universal values, i.e. they don't depend on the details of the lattice construction or of the interaction. A whole class of such models with different microscopic definitions have the exact same long distance behavior[9]! The systems in the same *universality class* need to share the same symmetries and dimensionality of space and the fact that the interaction is of short range. In the particular model that we study, these exponents take the so called *Onsager exponent* values

$$\begin{aligned} &\alpha = 0\,, \quad \beta = \tfrac{1}{8}\,, \quad \gamma = \tfrac{7}{4} \\ &\delta = 15\,, \quad \nu = 1\,, \quad \eta = \tfrac{1}{4}\,. \end{aligned} \tag{13.12}$$

Theses exponents determine the non analytic behavior of the corresponding functions in the thermodynamic limit. Non analyticity cannot arise in the finite N model. The partition function (13.4) is a sum of a finite number of analytic terms, which of course result in an analytic function. The non analytic behavior manifests in the $N \to \infty$ limit, where the finite N analytic functions converge to a non analytic one. The loss of analyticity is related to the appearance of long distance correlations between the spins and the scaling of the correlation length according to equation (13.6).

The two phases, separated by the phase transition, are identified by the different values of an order parameter. Each phase is characterized by the appearance or the breaking of a symmetry. In the Ising model, the order parameter is the magnetization and the symmetry is the Z_2 symmetry represented by the transformation $s_i \to -s_i$. The magnetization is zero in the disordered, high temperature phase and non zero in the ordered, low temperature phase. This implies that the magnetization is a non analytic function of the temperature[10].

Universality and *scale invariance* appear in the $\xi \to \infty$ limit. In our case, this occurs by tuning only one parameter, the temperature, to its critical value. A unique, dynamical, length scale emerges from the correlation function, the correlation length ξ. Scale invariance manifests when the correlation length becomes much larger than the microscopic length

[9] i.e. at distances larger than the (diverging) correlation length.

[10] An analytic function which is zero in an arbitrarily small interval, it is - by Taylor expanding around a point in this interval - everywhere zero.

scale a when $\beta \to \beta_c$. In the critical region, all quantities which are functions of the distance become functions only of the ratio r/ξ. Everything depends on the long wavelength fluctuations required by the symmetry of the order parameter and all models in the same universality class have the same long distance behavior. This way one can study only the simplest model within a universality class in order to deduce the large distance/long wavelength properties of all systems in the class.

13.2 Metropolis

Consider a square lattice with L sites on each side so that $N = L \times L = L^2$ is the number of lattice sites (vertices) and $N_l = 2N$ is the number of links (bonds) between the sites. The relation $N_l = 2N$ holds because we choose helical boundary conditions as shown in figure 13.6. The choice of boundary conditions will be discussed later. On each site i we have one degree of freedom, the "spin" s_i which takes on two values ± 1. We consider the case of zero magnetic field $B = 0$, therefore the Hamiltonian is given by[11]

$$H = -\sum_{\langle ij \rangle} s_i s_j \,. \tag{13.13}$$

The sum $\sum_{\langle ij \rangle}$ is a sum over the *links* $\langle ij \rangle$, corresponding to the pairs of sites i, j. Then $\sum_{\langle i,j \rangle} = (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N}$ since each bond is counted twice in the second sum. The partition function is

$$Z = \sum_{s_1=\pm 1} \sum_{s_2=\pm 1} \ldots \sum_{s_N=\pm 1} \mathrm{e}^{-\beta H[\{s_i\}]} \equiv \sum_{\{s_i\}} \mathrm{e}^{\beta \sum_{\langle ij \rangle} s_i s_j} \,. \tag{13.14}$$

Our goal is to collect a sample of states that is distributed according to the Boltzmann distribution (12.4). This will be constructed via a Markov process according to the discussion in section 12.6. Sampling is made according to (12.53) and the expectation values are estimated from the sample using (12.54). At each step the next state is chosen according to (12.60), and for large enough sample, or "time steps", the sample is approximately in the desired distribution.

[11]The constant $J = 1$ by choosing appropriate units for the s_i.

Suppose that the system is in a state[12] μ. According to (12.66), the probability that in the next step the system goes into the state ν is

$$P(\mu \to \nu) = g(\mu \to \nu)\, A(\mu \to \nu)\,, \tag{13.15}$$

where $g(\mu \to \nu)$ is the *selection probability* of the state ν when the system is in the state μ and $A(\mu \to \nu)$ is the *acceptance ratio*, i.e. the probability that the system jumps into the new state. If the detailed balance condition (12.60)

$$\frac{P(\mu \to \nu)}{P(\nu \to \mu)} = \frac{g(\mu \to \nu)\, A(\mu \to \nu)}{g(\nu \to \mu)\, A(\nu \to \mu)} = \mathrm{e}^{-\beta(E_\nu - E_\mu)} \tag{13.16}$$

is satisfied, then the distribution of the sample will converge to (12.4) $p_\mu = \mathrm{e}^{-\beta E_\mu}/Z$. In order that the system changes states often enough, the probabilities $P(\mu \to \nu)$ should be of order one and the differences in the energy $E_\nu - E_\mu$ should not be too large. This means that the product of the temperature with the energy difference should be a number of order one or less. One way to accomplish this is to consider states that differ by the value of the spin on only one site $s_i = \pm 1 \to s_i' \mp 1$. Since the energy (13.13) is a local quantity, the change in energy will be small. More specifically, if each site has $z = 4$ nearest neighbors, the change of the spin on site i results in a change of sign for z terms $s_i s_j$ in (13.13). The change in the energy for each bond is ± 2. If the state μ is given by $\{s_1, \ldots, s_i, \ldots, s_N\}$ and the state ν by $\{s_1, \ldots, s_i', \ldots, s_N\}$ (i.e. all the spins are the same except the spin s_i which changes sign), the energy difference will be

$$|\Delta E| \le 2z \Leftrightarrow E_\mu - 2z \le E_\nu \le E_\mu + 2z\,. \tag{13.17}$$

If the site i is randomly chosen then

$$g(\mu \to \nu) = g(\nu \to \mu) = \left\{ \begin{array}{ll} \frac{1}{N} & (\mu,\nu) \text{ differ by one spin} \\ 0 & \text{otherwise} \end{array} \right. \,, \tag{13.18}$$

and the algorithm is ergodic. Then we have that

$$\frac{A(\mu \to \nu)}{A(\nu \to \mu)} = \mathrm{e}^{-\beta(E_\nu - E_\mu)}\,. \tag{13.19}$$

[12] The state μ is determined by a spin configuration $\{s_i\}_{i=1\ldots N}$.

A simple choice for satisfying this condition is (12.61)

$$A(\mu \to \nu) = A_0 \cdot e^{-\frac{1}{2}\beta(E_\nu - E_\mu)} . \tag{13.20}$$

In order to maximize the acceptance ratios we have to take $A_0 = e^{-\beta z}$. Remember that we should have $A(\mu \to \nu) \leq 1$ and $|\Delta E| \leq 2z$. Therefore

$$A(\mu \to \nu) = e^{-\frac{1}{2}\beta(E_\nu - E_\mu + 2z)} . \tag{13.21}$$

Figure 13.4 depicts the dependence of $A(\mu \to \nu)$ on the change in energy

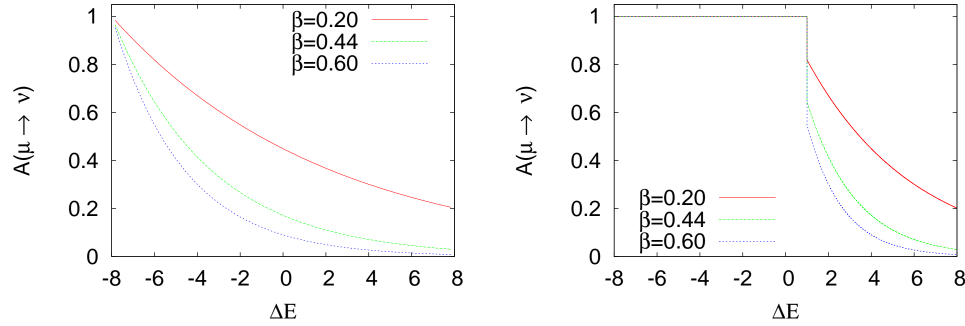


Figure 13.4: The acceptance ratio $A(\mu \to \nu)$ for the two dimensional Ising model on a square lattice given by equation (13.21) (left) and the Metropolis algorithm (right) as a function of the change in energy $\Delta E = E_\nu - E_\mu$. For the Metropolis algorithm the acceptance ratios are larger and the algorithm is expected to perform better.

for different values of β. We observe that this probability is small even for zero energy change and we expect this method not to perform very well.

It is much more efficient to use the algorithm proposed by Nicolas Metropolis *et. al.* 1953 [62] which is given by (12.63)

$$A(\mu \to \nu) = \begin{cases} e^{-\beta(E_\nu - E_\mu)} & E_\nu - E_\mu > 0 \\ 1 & E_\nu - E_\mu \leq 0 \end{cases} . \tag{13.22}$$

According to this relation, when a change in the states lowers the energy, the change is always accepted. When it increases the energy, the change is accepted with a probability less than one. As we can see in figure 13.4, this process accepts new states much more frequently than the previous algorithm.

The Metropolis algorithm is very widely used. It is applicable to any system, it is simple and efficient. We note that the choice to change the spin only locally is not a restriction put by the metropolis algorithm. There exist efficient algorithms that make non local changes to the system's configuration that (almost) conserve the Hamiltonian[13] and, consequently, the acceptance ratios are satisfactorily large.

13.3 Implementation

The first step in designing a code is to define the data structure. The degrees of freedom are the spins $s_i = \pm 1$ which are defined on N lattice sites. The most important part in designing the data structure in a lattice simulation is to define the neighboring relations among the lattice sites in the computer memory and this includes the implementation of the boundary conditions. A bad choice of boundary conditions will make the effect of the boundary on the results to be large and increase the *finite size effects*. This will affect the speed of convergence of the results to the thermodynamic limit, which is our final goal. The most popular choice is the toroidal or periodic boundary conditions. A small variation of these lead to the so called *helical* boundary conditions, which will be our choice because of their simplicity. Both choices share the fact that each site has the same number of nearest neighbors, which give the same local geometry everywhere on the lattice and minimize finite size effects due to the boundary. In contrast, if we choose fixed or free boundary conditions on the sides of the square lattice, the boundary sites have a smaller number of nearest neighbors than the ones inside the lattice.

One choice for mapping the lattice sites into the computer memory is to use their coordinates (i, j), $i, j = 0, \ldots, L-1$. Each spin is stored in an array `s[L][L]`. For a site `s[i][j]` the four nearest neighbors are `s[i±1][j]`, `s[i][j±1]`. The periodic boundary conditions are easily implemented by adding $\pm L$ to `i,j` each time they become less than one or greater than L. This is shown in figures 13.5 and 13.30.

The elements of the array `s[L][L]` are stored linearly into the computer memory. The element `s[i][j]` is at a "distance" `i*L+j` array positions from `s[0][0]` and accessing its value involves an, invisible to the

[13] An example is the Hybrid Monte Carlo used in lattice QCD simulations.

0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
5	6	7	8	9	5	6	7	8	9	5	6	7	8	9
10	11	12	13	14	10	11	12	13	14	10	11	12	13	14
15	16	17	18	19	15	16	17	18	19	15	16	17	18	19
20	21	22	23	24	20	21	22	23	24	20	21	22	23	24
0	1	2	3	4	**0**	**1**	**2**	**3**	**4**	0	1	2	3	4
5	6	7	8	9	**5**	**6**	**7**	**8**	**9**	5	6	7	8	9
10	11	12	13	14	**10**	**11**	**12**	**13**	**14**	10	11	12	13	14
15	16	17	18	19	**15**	**16**	**17**	**18**	**19**	15	16	17	18	19
20	21	22	23	24	**20**	**21**	**22**	**23**	**24**	20	21	22	23	24
0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
5	6	7	8	9	5	6	7	8	9	5	6	7	8	9
10	11	12	13	14	10	11	12	13	14	10	11	12	13	14
15	16	17	18	19	15	16	17	18	19	15	16	17	18	19
20	21	22	23	24	20	21	22	23	24	20	21	22	23	24

Figure 13.5: An $L = 5$ square lattice with periodic boundary conditions. The topology is toroidal.

programmer, multiplication. Using helical boundary conditions this multiplication can be avoided. The positions of the lattice sites are now given by one number $i = 0, \ldots, N-1$, where $N = L^2$, as shown in figures 13.6 and 13.31. The spins are stored in memory in a one dimensional array `s[N]` and the calculation of the nearest neighbors of a site `s[i]` is easily done by taking the spins `s[i±1]` and `s[i±L]`. The simplicity of the helical boundary conditions is based on the fact that, for the nearest neighbors of sites on the sides of the square, all we have to do is to make sure that the index `i` stays within the accepted range `0`≤ `i` ≤ `N-1`. This is easily done by adding or subtracting N when necessary. Therefore in a program that we want to calculate the four nearest neighbors `nn` of a site `i`, all we have to do is:

```
if((nn=i+XNN)>= N) nn -= N;
if((nn=i-XNN)<  0) nn += N;
if((nn=i+YNN)>= N) nn -= N;
if((nn=i-YNN)<  0) nn += N;
```

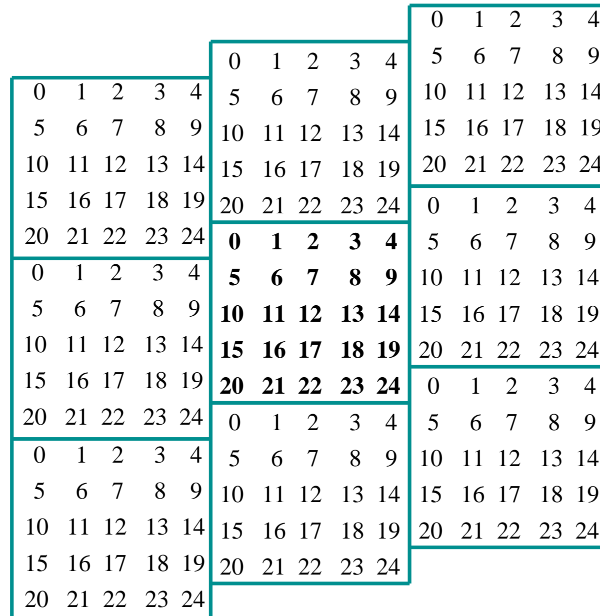


Figure 13.6: An $L = 5$ square lattice with helical boundary conditions. The topology is toroidal.

We will choose helical boundary conditions for their simplicity and efficiency in calculating nearest neighbors[14].

The dynamics of the Monte Carlo evolution is determined by the initial state and the Metropolis algorithm. A good choice of initial configuration can be important in some cases. It could lead to fast or slow thermalization, or even to no thermalization at all. In the model that we study it will not play an important role, but we will discuss it because of its importance in the study of other systems. We may choose a "cold" ($\beta = +\infty$ - all spins aligned) or a "hot" ($\beta = 0$ - all spins are equal to ± 1 with equal probability $1/2$) initial configuration. For large lattices, it is desirable to start in one of these states and then lower/increase the temperature in small steps. Each time that the temperature is changed,

[14] On the bad side, helical boundary conditions introduce a small finite size effect due to the shift of lattice positions in neighboring copies of the lattice. If one has to study small lattices, especially in higher dimensions, the best choice is to use periodic boundary conditions. We are going to study large enough lattices that this finite size effect is negligible.

the spin configuration is saved and used in the next simulation.

Ergodicity and thermalization must be checked by performing independent simulations[15] and verify that we obtain the same results. Similarly, independent simulations starting from different initial states must also be checked that yield the same results.

Consider each step in the Markov process defined by the Metropolis algorithm. Assume that the system is in the state $\mu = \{s_1^\mu, \ldots, s_k^\mu, \ldots, s_N^\mu\}$ and consider the transition to a new state $\nu = \{s_1^\nu, \ldots, s_k^\nu, \ldots, s_N^\nu\}$ which differs only by the value of the spin $s_k^\nu = -s_k^\mu$ (spin flip), whereas all the other spins are the same: $s_j^\nu = s_j^\mu \; \forall j \neq k$. The energy difference between the two states is

$$\begin{aligned}
E_\nu - E_\mu &= (-\sum_{\langle ij \rangle} s_i^\nu s_j^\nu) - (-\sum_{\langle ij \rangle} s_i^\mu s_j^\mu) \\
&= -\sum_{\langle ik \rangle} s_i^\mu (s_k^\nu - s_k^\mu) \\
&= 2\sum_{\langle ik \rangle} s_i^\mu s_k^\mu \\
&= 2s_k^\mu \left(\sum_{\langle ik \rangle} s_i^\mu \right) ,
\end{aligned} \tag{13.23}$$

where the second line is obtained after the cancellation of the common terms in the sums. In the third line we used the relation $s_k^\nu - s_k^\mu = -2s_k^\mu$, which you can prove easily by examining the cases $s_k^\mu = \pm 1$ separately. The important property of this relation is that it is local since it depends only on the nearest neighbors. The calculation of the energy difference $E_\nu - E_\mu$ is fast and is always a number of order one[16].

The Metropolis condition is easily implemented. We calculate the sum in the parenthesis of the last line of equation (13.23) and obtain the energy difference $E_\nu - E_\mu$. If the energy decreases, i.e. $E_\nu - E_\mu \leq 0$, the new state ν is accepted and "we flip the spin". If the energy increases, i.e. $E_\nu - E_\mu > 0$, then the acceptance ratio is $A(\mu \to \nu) = \mathrm{e}^{-\beta(E_\nu - E_\mu)} < 1$. In order to accept the new state with this probability we pick a random number uniformly distributed in $0 \leq x < 1$. The probability that this

[15] Different sequence of random numbers.

[16] An important fact is that it does not increase with the system size.

number is $x < A(\mu \to \nu)$ is equal to[17] $A(\mu \to \nu)$. Therefore if $x \le A(\mu \to \nu)$ the change is accepted. If $x > A(\mu \to \nu)$ the change is rejected and the system remains in the same state μ.

A small technical remark is in order: The possible values of the sum $\left(\sum_{\langle ik\rangle} s_i^\mu\right) = -4, -2, 0, 2, 4$ and these are the only values that enter in the calculation of $A(\mu \to \nu)$. Moreover, only the values that increase the energy, i.e. $2, 4$ are of interest to us. Therefore we only need two values of $A(\mu \to \nu)$, which depend only on the temperature. These can by calculated once and for all in the initialization phase of the program, stored in an array and avoid the repeated calculation of the exponential $\mathrm{e}^{-\beta(E_\nu - E_\mu)}$ which is expensive.

In our program we also need to implement the calculation of the observables that we want to measure. These are the energy (13.13)

$$E = -\sum_{\langle ij\rangle} s_i s_j \,, \tag{13.24}$$

and the magnetization

$$M = \left|\sum_i s_i\right| \,. \tag{13.25}$$

Beware of the absolute value in the last equation! The Hamiltonian H has a Z_2 symmetry because it is symmetric under reflection of all the spins. The probability of appearance of a state depends only on the value of H, therefore two configurations with opposite spin are equally probable. But such configurations have opposite magnetization, therefore the average magnetization $\langle \sum_i s_i\rangle$ will be zero due to this cancellation[18].

We can measure the energy and the magnetization in two ways. The first one is by updating their values each time a Metropolis step is accepted. This is easy and cheap since the difference in the sum in equations (13.24) and (13.25) depends only on the value of the spin s_k^μ and its nearest neighbors. The energy difference is already calculated by (13.23)

[17]For the uniform distribution $P(x < a) = a$.

[18]This does not show for very small temperatures in the simulation with the Metropolis algorithm. As we decrease the temperature $\beta \gg \beta_c$, it takes many improbable steps to move from a state with $\sum_i s_i = M_1$ to a state with $\sum_i s_i = -M_1$. The Monte Carlo simulation consists of a finite number of steps, therefore we may obtain a non zero $\langle \sum_i s_i\rangle$, an incorrect result.

whereas the difference in the magnetization in (13.25) is given by

$$\sum_i s_i^\nu - \sum_i s_i^\mu = s_k^\nu - s_k^\mu = -2s_k^\mu \tag{13.26}$$

The second way is by calculating the full sums in (13.24) and (13.25) every time that we want to take a measurement. The optimal choice depends on how often one obtains a statistically *independent* measurement[19]. If the average acceptance ratio is $\bar{A}$, then the calculation of the magnetization using the first method requires $\bar{A}N$ additions per N Monte Carlo steps, whereas the second one requires N additions per measurement.

We use the normalization

$$\langle e \rangle = \frac{1}{N_l}\langle E \rangle = \frac{1}{2N}\langle E \rangle\,, \tag{13.27}$$

which gives the energy per link. We have that $-1 \leq e \leq +1$, where $e = -1$ for the ground state in which all $2N$ links have energy equal to -1. The magnetization per site is

$$\langle m \rangle = \frac{1}{N}\langle M \rangle\,. \tag{13.28}$$

We have that $0 \leq m \leq 1$, where $m = 0$ for $\beta = 0$ (perfect disorder) and $m = 1$ for the ground state at $\beta = \infty$ (perfect order). We call m the order parameter since its value determines the phase that the system is in.

The specific heat is given by the fluctuations of the energy

$$c = \beta^2 N \langle (e - \langle e \rangle)^2 \rangle = \beta^2 N (\langle e^2 \rangle - \langle e \rangle^2)\,, \tag{13.29}$$

and the magnetic susceptibility by the fluctuations of the magnetization

$$\chi = \beta N \langle (m - \langle m \rangle)^2 \rangle = \beta N (\langle m^2 \rangle - \langle m \rangle^2)\,. \tag{13.30}$$

In order to estimate the amount of data necessary for an accurate measurement of these quantities, we consider the fact that for n independent measurements the statistical error drops as $\sim 1/\sqrt{n}$. The problem of determining how often we have independent measurements is very important and it will be discussed in detail later in this chapter.

[19] This is given by the autocorrelation time, which will be discussed in detail later.

13.3.1 The Program

In this section we discuss the program[20] that implements the Monte Carlo simulation of the Ising model. The code in this section can be found in the accompanying software of this chapter in the directory `Ising_Introduction`.

In the design of the code, we will follow the philosophy of modular programming. Different independent sections of the program will be coded in different files. This makes the development, maintenance and correction of the code by one or a team of programmers easier. A *header file* contains the definitions which are common for the code in one or more files. Then, all the parameters and common functions are in one place and they are easier to modify. In our case we have one such file only, named `include.h`, whose code will be included in the beginning of each program unit using an `#include` directive:

```
//============== include.h   ==================
#include <iostream>
#include <fstream>
#include <iomanip>
#include <random>
using namespace std;

#include "MIXMAX/mixmax.hpp"

const  int     L   = 12;
const  int     N   = L*L;
const  int     XNN = 1;
const  int     YNN = L;
extern int     s[N];
extern int     seed;
extern double  beta;
extern double  prob[5];
extern mixmax_engine mxmx;
extern uniform_real_distribution<double> drandom;

int E(),M();
void init(int ), met(), measure();
```

[20]The basic ideas in the program are taken from the book by Newmann and Barkema [4].

The lattice size L is a constant parameter, whereas the arrays and variables encoding the spins and the simulation parameters are put in the global scope. For that, they must be declared as `external` in all files used, and then be defined in only one of the files (in our case in the file `init.cpp`). The array `s[N]` stores the spin of each lattice site which takes values ± 1. The variable `beta` is the temperature β and the array `prob[5]` stores the useful values of the acceptance ratios $A(\mu \to \nu)$ according to the discussion on page 559. The distribution `drandom` generates pseudo-random numbers uniformly distributed in the interval $[0, 1)$ and `mxmx` is a MIXMAX random generator engine object. The parameters `XNN` and `YNN` are used for computing the nearest neighbors in the `X` and `Y` directions according to the discussion of section 13.3 on helical boundary conditions. For example, for an internal site `i`, `i+XNN` is the nearest neighbor in the $+x$ direction and `i-YNN` is the nearest neighbor in the $-y$ direction.

The main program is in the file `main.cpp` and drives the simulation:

```
//============== main.cpp    ==================
#include "include.h"

int main(int argc, char **argv){

  const int nsweep  = 200000;
  int start=1;//(0 cold)/(1 hot)

  beta=0.21;seed=9873;
  init(start);
  for(int isweep=0;isweep<nsweep;isweep++){
    met();
    measure();
  }
}
```

In the beginning we set the simulation parameters. The initial configuration is determined by the value of `start`. If `start=0`, then it is a cold configuration and if `start=1`, then it is a hot configuration. The temperature is set by the value of `beta` and the number of sweeps of the lattice by the value of `nsweep`. One sweep of the lattice is defined by `N` attempted spin flips. The flow of the simulation is determined by the initial call to `init`, which performs all initialization tasks, and the subsequent calls to `met` and `measure`, which perform `nsweep` Metropolis

sweeps and measurements respectively.

One level down lies the function `init`. The value of `start` is passed through its argument so that the desired initial state is set:

```
//============== init.cpp    ==================
// file init.cpp
// init(start): start = 0: cold start
//              start = 1: hot  start
//==============================================
#include "include.h"
//Global variables:
int    s[N];
int    seed;
double beta;
double prob[5];
mixmax_engine mxmx(0,0,0,1);
uniform_real_distribution<double> drandom;

void init(int start){
  int i;

  mxmx.seed(seed);
  //Initialize probabilities:
  for(i=2;i<5;i+=2) prob[i] = exp(-2.0*beta*i);
  //Initial configuration:
  switch(start){
  case 0://cold start
    for(i=0;i<N;i++) s[i]=1;
    break;
  case 1://hot start
    for(i=0;i<N;i++){
      if(drandom(mxmx) < 0.5)
    s[i] =  1;
      else
    s[i] = -1;
    }
    break;
  default:
    cout << "start= " << start
         << " not valid. Exiting....\n";
    exit(1);
    break;
  }
}
```

Notice that all variables in the global scope, declared as `external` in the header `include.h`, are defined in the beginning of this file, despite of the fact that the `external` statements are also included.

At first the array `prob[5]` is initialized to the values of the acceptance ratios $A(\mu \to \nu) = \mathrm{e}^{-\beta(E_\nu - E_\mu)} = \mathrm{e}^{-2\beta s_k^\mu \left(\sum_{\langle ik\rangle} s_i^\mu\right)}$. Those probabilities are going to be used when $s_k^\mu \left(\sum_{\langle ik\rangle} s_i^\mu\right) > 0$ and the possible values are obtained when this expression takes the values 2 and 4. These are the values stored in the array `prob[5]`, and we remember that the index of the array is the expression $s_k^\mu \left(\sum_{\langle ik\rangle} s_i^\mu\right)$, when it is positive.

The initial spin configuration is determined by the integer `start`. The use of the `switch` block allows us to add more options in the future. When `start=0` all spins are set equal to 1, whereas when `start=1` each spin's value is set to ± 1 with equal probability. The probability that `drandom(mxmx)<0.5` is[21] $1/2$, in which case we set `s[i]=1`, otherwise (probability $1 - 1/2 = 1/2$) we set `s[i]=-1`.

The heart of the program is in the function `met()` which attempts `N` Metropolis steps. It picks `N` random sites and asks the question whether to perform a spin flip. This is done using the Metropolis algorithm by calculating the change in the energy of the system before and after the change of the spin value according to (13.23):

```
//============== met.cpp      ==================
#include "include.h"

void met(){
  int i,k;
  int nn,snn,dE;

  for(k=0;k<N;k++){
    i=N*drandom(mxmx);//pick a random site i
    //Sum of neighboring spins:
    if((nn=i+XNN)>= N) nn -= N; snn  = s[nn];
    if((nn=i-XNN)<  0) nn += N; snn += s[nn];
    if((nn=i+YNN)>= N) nn -= N; snn += s[nn];
    if((nn=i-YNN)<  0) nn += N; snn += s[nn];
    //change in energy/2
    dE = snn*s[i];
```

[21]Remember that for the uniform distribution, $P(x < a) = a$

```
    //flip
    if(dE <=0 )                           {s[i] = -s[i];}  //accept
    else  if(drandom(mxmx)<prob[dE]){s[i] = -s[i];}  //accept
 }//sweep ends
}
```

The line

```
    i=N*drandom(mxmx);
```

picks a site `i=0,...,N-1` with equal probability. It is important that the value `i=N` never appears, something that could have happened if `drandom(mxmx)=1.0` were possible.

Next, we calculate the sum $\left(\sum_{\langle ik\rangle} s_i^\mu\right)$ in (13.23). The nearest neighbors of the site `i` have to be determined and this happens in the lines

```
    if((nn=i+XNN)>= N) nn -= N; snn  = s[nn];
    if((nn=i-XNN)<  0) nn += N; snn += s[nn];
    if((nn=i+YNN)>= N) nn -= N; snn += s[nn];
    if((nn=i-YNN)<  0) nn += N; snn += s[nn];
    dE = snn*s[i];
```

The variable `dE` is set equal to the product (13.23) $s_k^\mu\left(\sum_{\langle ik\rangle} s_i^\mu\right)$. If it turns out to be negative, then the change in the energy is negative and the spin flip is accepted. If it turns out to be positive, then we apply the criterion (13.22) by using the array `prob[5]`, which has been defined in the function `init`. The probability that `drandom(mxmx)<prob[dE]` is equal to `prob[dE]`, in which case the spin flip is accepted. In all other cases, the spin flip is rejected and `s[i]` remains the same.

After each Metropolis sweep we perform a measurement. The code is minimal and simply prints the value of the energy and the magnetization to the `stdout`. The analysis is assumed to be performed by external programs. This way we keep the production code simple and store the raw data for a detailed and flexible analysis. The printed values of the energy and the magnetization will be used as monitors of the progress of the simulation, check thermalization and measure autocorrelation times. Plots of the measured values of an observable as a function of the Monte Carlo "time" are the so called "time histories". Time histories of appropriately chosen observables should *always* be viewed and used in order

to check the progress and spot possible problems in the simulation.

The function `measure` calculates the total energy and magnetization (without the absolute value) by a call to the functions `E()` and `M()`, which apply the formulas (13.24) and (13.25).

```
//============== measure.cpp ==================
#include "include.h"

void measure(){
  cout << E() << " " << M() << '\n';
}

int E(){
  int e,snn,i,nn;
  e=0;
  for(i=0;i<N;i++){
    //Sum of neighboring spins:
    //only forward nn necessary in the sum
    if((nn=i+XNN)>= N) nn -= N; snn  = s[nn];
    if((nn=i+YNN)>= N) nn -= N; snn += s[nn];
    e += snn*s[i];
  }
  return -e;
}

int M(){
  int i,m;
  m=0;
  for(i=0;i<N;i++) m+=s[i];
  return m;
}
```

The compilation of the code is done with the command

```
> g++ -std=c++11 main.cpp met.cpp init.cpp measure.cpp \
                 MIXMAX/mixmax.cpp -o is
```

which results in the executable file `is`:

```
> ./is > out.dat
> less out.dat
-72 2
```

```
-80 26
-60 30
-80 12
......
```

The output of the program is in two columns containing the values of the total energy and magnetization (without the absolute value). In order to construct their time histories we give the `gnuplot` commands:

```
gnuplot> plot "out.dat" using 1 with lines
gnuplot> plot "out.dat" using 2 with lines
gnuplot> plot "out.dat" using (($2>0)?$2:-$2) with lines
```

The last line calculates the absolute values of the second column. The expression `($2>0)?$2:-$2` checks whether `($2>0)` is true. If it is, then it returns $2, otherwise it returns `-$2`.

13.3.2 Towards a Convenient User Interface

In this section we will improve the user interface of the program. This is a nice exercise on the interaction of the programming language with the shell and the operating system. The code presented can be found in the accompanying software of this chapter in the directory `Ising_Metropolis`.

An annoying feature of the program discussed in the previous section is that the simulation parameters are hard coded and the user needs to recompile the program each time she changes them. This is not very convenient if she has to do a large number of simulations. Another notable change that needs to be made in the code is that the final configuration of the simulation must be saved in a file, in order to be read as an initial configuration by another simulation.

One of the parameters that the user might want to set interactively at run time is the size of the lattice `L`. But this is the parameter that determines the required memory for the array `s[N]`. Therefore we have to use dynamic memory allocation for this array using `new`. In the file `include.h` we put all the global variables as follows:

```
//============== include.h   ==================
#include <iostream>
#include <fstream>
```

```
#include <iomanip>
#include <random>
using namespace std;

#include "MIXMAX/mixmax.hpp"

extern int    L;
extern int    N;
extern int    XNN;
extern int    YNN;
extern int    *s;
extern int    seed,nsweep,start;
extern double beta;
extern double prob[5];
extern double acceptance;
extern mixmax_engine mxmx;
extern uniform_real_distribution<double> drandom;

int E(),M();
void init(int,char** ), met(), measure();
void simmessage(ostream&), locerr(string);
void usage(char **);
void get_the_options(int,char**),endsim();
```

The array `s[]` is declared as a pointer to an `int` and its storage space will be allocated in the function `init`. The variables `L`, `N`, `XNN` and `YNN` are not parameters anymore and their values will also be set in `init`. Notice the variable `acceptance` which will store the fraction of accepted spin flips in a simulation.

The main program has very few changes:

```
//============== main.cpp    ==================
#include "include.h"

int main(int argc, char **argv){

  init(argc,argv);
  for(int isweep=0;isweep<nsweep;isweep++){
    met();
    measure();
  }
  endsim();
}
```

The function `endsim` finishes off the simulation. Its most important function is to store the final configuration to a file for later use.

The function `init` is changed quite a bit since it performs most of the functions that have to do with the user interface:

```
//============== init.cpp    ==================
// file init.cpp
// init(start): start = 0: cold start
//              start = 1: hot  start
//              start =-1: old  configuration
//==============================================
#include "include.h"
//Global variables:
int    L;
int    N;
int    XNN;
int    YNN;
int    *s;
int    seed;
int    nsweep;
int    start;
double beta;
double prob[5];
double acceptance = 0.0;
mixmax_engine mxmx(0,0,0,1);
uniform_real_distribution<double> drandom;

void init(int argc, char **argv){
  int    i;
  int    OL=-1;
  double obeta=-1.0;
  string buf;
  //define parameters from options:
  L=-1;beta=-1.;nsweep=-1;start=-1;seed=-1;
  get_the_options(argc,argv);
  if( start == 0 || start == 1){
    if(L    <  0 )locerr("L has not been set."   );
    if(seed <  0 )locerr("seed has not been set.");
    if(beta <  0 )locerr("beta has not been set.");
    //derived parameters
    N=L*L; XNN=1; YNN = L;
    //allocate memory space for spins:
    s = new int[N];
    if (!s) locerr("allocation failure for s[N]");
```

```
}//if( start == 0 || start == 1)
if(start  < 0 )locerr("start  has not been set.");
if(nsweep < 0 )locerr("nsweep has not been set.");
//Initialize probabilities:
for(i=2;i<5;i+=2) prob[i] = exp(-2.0*beta*i);
//--------------------------------------------------------------
//Initial configuration: cold(0),hot(1),old(2)
//--------------------------------------------------------------
switch(start){
//--------------------------------------------------------------
case 0://cold start
  simmessage(cout);
  mxmx.seed (seed);
  for(i=0;i<N;i++) s[i]=1;
  break;
//--------------------------------------------------------------
case 1://hot start
  simmessage(cout);
  mxmx.seed (seed);
  for(i=0;i<N;i++){
    if(drandom(mxmx) < 0.5)
     s[i] =  1;
    else
     s[i] = -1;
  }
  break;
//--------------------------------------------------------------
case 2:{ //old configuration
  ifstream conf("conf");
  if(!conf.is_open())
    locerr("Configuration file conf not found");
  getline(conf,buf);//discard comment line
  conf >> buf >> OL >> buf >> OL >> buf >> obeta;
  getline(conf,buf);
  if( L <  0  ) L=OL;
  if( L != OL )
    locerr("Given L different from the one read from conf");
  N=L*L; XNN=1;  YNN = L;
  if(beta < 0.) beta = obeta;
  //allocate memory space for spins:
  s = new int[N];
  if (!s) locerr("allocation failure in s[N]");
  for(i=0;i<N;i++)
    if((!(conf >> s[i])) ||((s[i] != -1) && (s[i] != 1)))
      locerr("conf ended before reading s[N]");
```

```
    if(seed >=0 ) mxmx.seed (seed);
    if(seed < 0 )
      if(!(conf >> mxmx))
        locerr("conf ended before reading mixmax state");
    conf.close();
    simmessage(cout);
    break;
  }//case 2
  //------------------------------------------------
  default:
    cout << "start= "             << start
         << " not valid. Exiting....\n";
    exit(1);
    break;
  }//switch(start)
  //------------------------------------------------
}//init()
```

Global variables, which are declared as external in include.h, are defined in this file. The simulation parameters that are to be determined by the user are given invalid default values. This way they are flagged as not been set. The function[22] get_the_options sets the parameters to the values that the user passes through the command line:

```
  L=-1;beta=-1.;nsweep=-1;start=-1;seed=-1;
  get_the_options(argc,argv);
```

Upon return of get_the_options, one has to check if all the parameters have been set to acceptable values. For example, if the user has forgotten to set the lattice size L, the call to the function[23] locerr stops the program and prints the error message passed through its argument:

```
    if(L     <  0 )locerr("L has not been set."   );
```

When the value of N is calculated from L, the program allocates memory for the array s[N]:

```
    N = L*L;
```

[22]The function get_the_options is defined in the file options.cpp.
[23]The function locerr is defined in the file options.cpp.

```
  s = new int[N];
  if (!s) locerr("allocation failure in s[N]");
```

If memory allocation fails, the pointer `s` is `NULL` and the program terminates abnormally.

Using the construct `switch(start)` we set the initial configuration of the simulation. A value of `start=0` sets all spins equal to 1. The function `simmmessage(cout)` prints important information about the simulation to the output stream `cout`. The random number generator `MIXMAX` is initialized with a call to `mxmx.seed (seed)` according to the discussion in section 11.2, page 509. If `start=0` the initial configuration is hot.

If `start=2` we attempt to read a configuration stored in a file named `conf`. The format of the file is strictly set by the way we print the configuration in the function `endsim`. If the file does not exist `conf.is_open()` is `false` and the program terminates abnormally. In order to read the configuration properly we need to know the format of the data in the file `conf` which is, more or less, as follows:

```
# Configuration of 2d Ising model on square lattice....
Lx= 12 Ly= 12 beta= 0.21
 -1
  1
  1
 .....
  1
 -1
 (...MIXMAX state...)
```

All comments of the first line are discarded in the string `buf` by a call to `getline`. The parameters `L` and `beta` of the stored configuration are read in temporary variables `OL, obeta`, so that they can be compared with the values set by the user.

If the user provides a seed, then her seed will be used for seeding. Otherwise `MIXMAX` is initialized to the state read from the file `conf`. Both choices are desirable in different cases: If the user wants to split a long simulation into several short runs, then each time she wants to restart the random number generator at exactly the same state. If she wants to use the same configuration in order to produce many *independent* results, then `MIXMAX` has to produce different sequences of random numbers each

time[24]. This feature is coded in the lines:

```
    if(seed > 0 ) mxmx.seed (seed);
    if(seed < 0 )
      if(!(conf >> mxmx))
        locerr("conf ended before reading mixmax state");
```

Notice that the state of MIXMAX is given by many integers and we check whether they have been read correctly. If the information in the file `conf` is not complete, the expression `(conf >> mxmx)` will be `false` and the program will be terminated abnormally.

When reading the spins, we have to make sure that they take only the legal values ± 1 and that the data is enough to fill the array `s[N]`. Reading enough data is checked by the value of the expression `(conf >> s[i])`, which becomes false when the stream `conf` fails to read an `int` value into `s[i]`. The rest of the expression `((s[i] != -1) && (s[i] != 1))` checks if the spin values are legal.

The function `endsim` saves the last configuration in the file `conf` and can be found in the file `end.cpp`:

```
//============== end.cpp      ==================
#include "include.h"
#include <cstdio>
void endsim(){
  rename("conf","conf.old");
  ofstream conf("conf");
  conf << "# Configuration of 2d Ising model...\n";
  conf << "Lx= "    <<L     << " Ly= " << L
       << " beta= " << beta             << "\n";
  for(int i=0;i<N;i++) conf << s[i]    << '\n';
  conf << mxmx << endl;
  conf.close();
```

The state of the random number generator `MIXMAX` is saved by writing to the output stream `conf << mxmx`. The call to the function `rename` renames the file `conf` (if it exists) to the backup file `conf.old`.

The function `get_the_options()` reads the parameters, passed through *options* from the command line. The choice to use options for passing

[24]Assuming that the configuration in `conf` is thermalized, the simulations become statistically independent after time 2τ, where τ is the autocorrelation time.

parameters to the program has the advantage that they can be passed optionally and in any order desired. Let's see how they work. Assume that the executable file is named `is`. The command

```
> ./is -L 10 -b 0.44 -s 1 -S 5342 -n 1000
```

will run the program after setting `L=10` (`-L 10`), `beta=0.44` (`-b 0.44`), `start=1` (`-s 1`), `seed=5342` (`-S 5342`) and `nsweep=1000` (`-n 1000`). The `-L, -b, -s, -S, -n` are *options* or *switches* and can be put in any order in the arguments of the command line. The arguments following an option are the values passed to the corresponding variables. Options can also be used without arguments, in which case a common use is to make the command function differently[25]. In our case, the option `-h` is an option without an argument which makes the program print a usage message and exit without running the simulation:

```
> ./is -h
Usage: is  [options]
       -L: Lattice length (N=L*L)
       -b: beta  (options beta overrides the one in config)
       -s: start (0 cold, 1 hot, 2 old config.)
       -S: seed  (options seed overrides the one in config)
       -n: number of sweeps and measurements of E and M
       -u: seed  from /dev/urandom
Monte Carlo simulation of 2d Ising Model. Metropolis is used by
default. Using the options, the parameters of the simulations
must be set for a new run (start=0,1). If start=2,
a configuration is read from the file conf.
```

This is a way to provide a short documentation on the usage of a program.

Let's see the code, which is found in the file `options.cpp`:

```
//============== options.cpp ==================
#include "include.h"
#include <unistd.h>
#include <libgen.h>
//----------------------------------------------------------
//Set options: Option letters are defined with this string
```

[25]Remember how the option `-l` changes the results of the command `ls` if executed as `ls -l`

```
#define OPTARGS  "hL:b:s:S:n:u"
//------------------------------------------------------------
string prog;
int seedby_urandom();
void get_the_options(int argc,char **argv){
  int c,errflg = 0;
  //prog is the name of executable file
  prog.assign((char *)basename(argv[0]));
  while (!errflg && (c = getopt(argc, argv, OPTARGS)) != -1){
    switch(c){
    case 'L':
      L      = atoi(optarg);
      break;
    case 'b':
      beta   = atof(optarg);
      break;
    case 's':
      start  = atoi(optarg);
      break;
    case 'S':
      seed   = atoi(optarg);
      break;
    case 'n':
      nsweep = atoi(optarg);
      break;
    case 'u':
      seed   = seedby_urandom();
      break;
    case 'h':
      errflg++;/*call usage*/
      break;
    default:
      errflg++;
    }/*switch*/
    if(errflg) usage(argv);
  }/*while...*/
}//get_the_options()
```

The command `prog.assign( (char*) basename(argv[0]))` stores the name of the program in the command line to the `string` variable `prog`. The function `getopt` is the one that processes the options (see `man 3 getopt` for documentation). The string `OPTARGS "hL:b:s:S:n:u"` defines the allowed options `'h'`, `'L'`, `'b'`, `'s'`, `'S'`, `'n'`, `'u'`. When a user passes one of those through the command line (e.g. `-L 100`, `-h`) the

`while` loop takes us to the corresponding `case`. If an option does not take an argument (e.g. `-h`), then a set of commands can be executed, like `usage(argv)`. If an option takes an argument, this is marked by a semicolon in the argument of `getopt` (e.g. `L:`, `b:`, ...) and the argument can be accessed through the `char*` variable `optarg`. For example, the statements

```
    case 'L':
      L      = atoi(optarg);
      break;
```

and the command line arguments `-L 10` set `optarg` to be equal to `"10"`. Be careful, `"10"` is not a number, but a string of characters! In order to convert the *characters* `"10"` to the *integer* 10 we use `atoi()`. We do the same for the other simulation parameters.

The function `locerr` takes a `string` variable in its argument which prints it to the `stderr` together with the name of the program in the command line. Then it stops the execution of the program:

```
void locerr( string errmes ){
  cerr << prog << ": " << errmes << " Exiting....\n";
  exit(1);
}
```

The function `usage` is ... used very often! It is a constant reminder of the way that the program is used and helps users with weak long and/or short term memory!

```
void usage(char **argv){
  /*Careful: New lines end with \n\ :
    No space after last backslacsh
    indicates line is broken....*/
  cerr << "\
Usage: " << prog << "  [options]                              \n\
  -L: Lattice length (N=L*L)                                  \n\
  -b: beta  (options beta overrides the one in config)        \n\
  -s: start (0 cold, 1 hot, 2 old config.)                    \n\
  -S: seed  (options seed overrides the one in config)        \n\
  -n: number of sweeps and measurements of E and M            \n\
  -u: seed  from /dev/urandom                                 \n\
```

```
Monte Carlo simulation of 2d Ising Model. Metropolis is   \n\
used by default. Using the options, the parameters of     \n\
the simulations must be set for a new run (start=0,1).    \n\
If start=2, a configuration is read from the file conf.   \n";
  exit(1);
}//usage()
```

Notice how a long string of characters is broken over several lines. For this, the last character of the line should be a backslash '\', with no extra spaces or other non-newline characters following it.

The function simmessage is also quite important. It "labels" our results by printing all the information that defines the simulation. It is very important to label all of our data with this information, otherwise it can be dangerously useless! Imagine a set of energy measurements without knowing the lattice size and/or the temperature... Other useful information may turn out to be crucial, even though we might not appreciate it at programming time: The name of the computer, the operating system, the user name, the date etc. By varying the output stream in the argument, we can print the same information in any file we want.

```
void simmessage(ostream& ostr){
  time_t t;
  time(&t);//store time in seconds, see: "man 2 time"
  char* TIME = ctime (&t         );
  char* USER = getenv("USER"     );
  char* MACH = getenv("HOSTTYPE");
  char* HOST = getenv("HOST"     );

  ostr << "\
# ##########################################################\n\
# 2d Ising Model, Metropolis algorithm, square lattice     \n\
# Run on "    <<HOST  <<" ( "<<MACH<<" ) by "
<<USER<<" on "<<TIME  <<"\
# L       = " <<L     <<" (Lattice linear dimension,N=L*L)\n\
# seed    = " <<seed  <<" (random number gener. seed)     \n\
# nsweeps = " <<nsweep<<" (No. of sweeps)                 \n\
# beta    = " <<beta  <<"                                 \n\
# start   = " <<start <<" (0 cold, 1 hot, 2 old config)"<<endl;
}//simmessage()
```

The compilation can be done with the command:

```
> g++ -std=c++11 main.cpp    init.cpp    met.cpp \
                 measure.cpp options.cpp end.cpp \
                 MIXMAX/mixmax.cpp -o is
```

In order to run the program we pass the parameters through options in the command line, like for example:

```
> /usr/bin/time ./is -L 10 -b 0.44 -s 1 -S 5342 -n 10000 \
   >& out.dat &
```

The command `time` is added in order to measure the computer resources (CPU time, memory, etc) that the program uses at run time.

A useful tool for complicated compilations is the utility `make`. Its documentation is several hundred pages which can be accessed through the `info` pages[26] and the interested reader is encouraged to browse through it. If in the current directory there is a file named `Makefile` whose contents[27] are

```
# ####################    Makefile   ############################
OBJS    = main.o init.o met.o measure.o end.o options.o mixmax.o
CXXFLAGS = -O2 -std=c++11

is: $(OBJS)
        $(CXX)  $(CXXFLAGS) $^   -o $@

mixmax.o:
        $(CXX)  $(CXXFLAGS) -c -o $@ MIXMAX/mixmax.cpp

$(OBJS): include.h

clean:
    /bin/rm -f *.o is core*
```

then this instructs the program `make` how to "make" the executable file `is`. What have we gained? In order to see that, run `make` for the first time.

[26]Use the command `info make` or visit the www address `www.gnu.org/software/make/manual/make.html`

[27]Beware: one of the quirks of the program `make` is that all executable commands in `Makefile` *must* be in a line that starts exactly with a `TAB`. In the example `Makefile` shown above, the empty space before such lines is one `TAB` and *not* 8 empty spaces.

Then try making a trivial change in the file `main.cpp` and rerun `make`. Then only the modified file is compiled and not the ones that have not been touched. This is accomplished by defining *dependencies* in `Makefile` which execute commands conditionally depending on the time stamps on the relevant files. Dependencies are defined in lines which are of the form `keyword: word1 word2 ...`. For example, the line `is: $(OBJS)` defines a dependency of the file `is` from the files `main.o ... mixmax.o`. Lines 2-3 in the above `Makefile` define variables which can be used in the commands that follow. There are *many* predefined variables[28] in `make` which makes `make` programming easier. By using `make` in a large project, we can automatically link to libraries, pass complicated compiler options, do conditional compilation (depending, e.g., on the operating system, the compiler used etc), etc. A serious programmer needs to invest some time in order to use the full potential of `make` for the needs of her project.

13.4 Thermalization

The problem of thermalization can be important for some systems studied with Monte Carlo simulations. Even though it will not be so important in the simulations performed in this book, we will discuss it because of its importance in other problems. The reader should bear in mind that the thermalization problem becomes more serious with increasing system size and when autocorrelation times are large.

In a Monte Carlo simulation, the system is first put in a properly chosen initial configuration in order to start the Markov process. In section 12.2 we saw that when a system is in thermal equilibrium with a reservoir at a given temperature, then a typical state has energy that differs very little from its average value and belongs to a quite restricted region of phase space. Therefore, if we choose an initial state that is far from this region, then the system has to perform a random walk in the space of states until it finds the region of typical states. This is the thermalization process in a Monte Carlo simulation.

There are two problems that need to be addressed: The first one is the appropriate choice of the initial configuration and the second one is to find criteria that will determine when the system is thermalized. For

[28] Try the command `make -p`

the Ising model the initial configuration is either, (a) cold, (b) hot *or* (c) old state. It is obvious that choosing a hot state in order to simulate the system at a cool temperature is not the best choice, and the system will take longer to thermalize than if we choose a cold state or an old state at a nearby temperature. This is clearly seen in figure 13.7. Thermalization

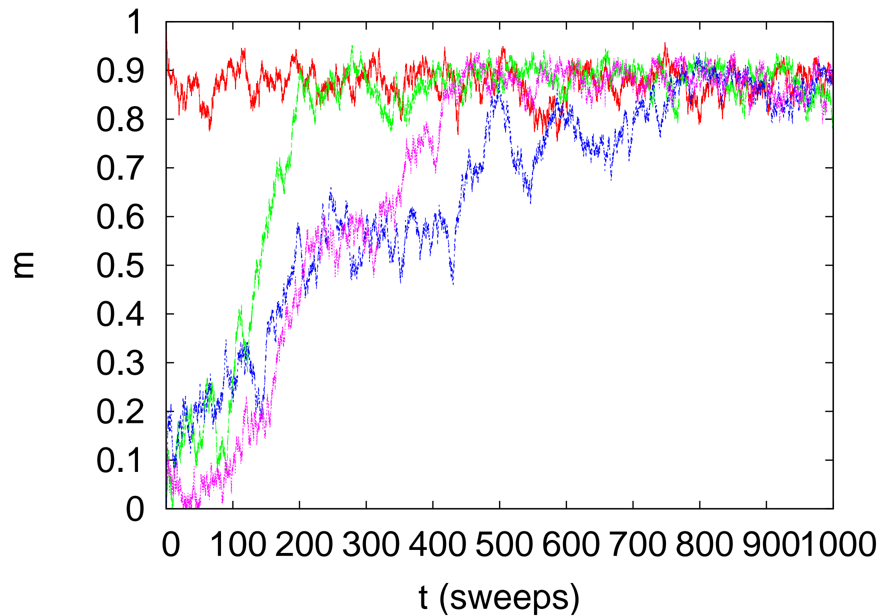


Figure 13.7: Magnetization per site for the Ising model in the ordered phase with $L = 40$, $\beta = 0.48$. We show the thermalization of the system by starting from a cold state and three hot ones. For a hot start, thermalization takes up to 1000 sweeps.

depends on the temperature and the system size, but it also depends on the physical quantity that we measure. Energy is thermalized faster than magnetization. In general, a local quantity thermalizes fast and a non local one slower. For the Ising model, thermalization is easier far from the critical temperature, provided that we choose an initial configuration in the same phase. It is easier to thermalize a small system rather than a large one.

The second problem is to determine when the system becomes thermalized and discard all measurements before that. One way is to start simulations using different initial states, or by keeping the same initial

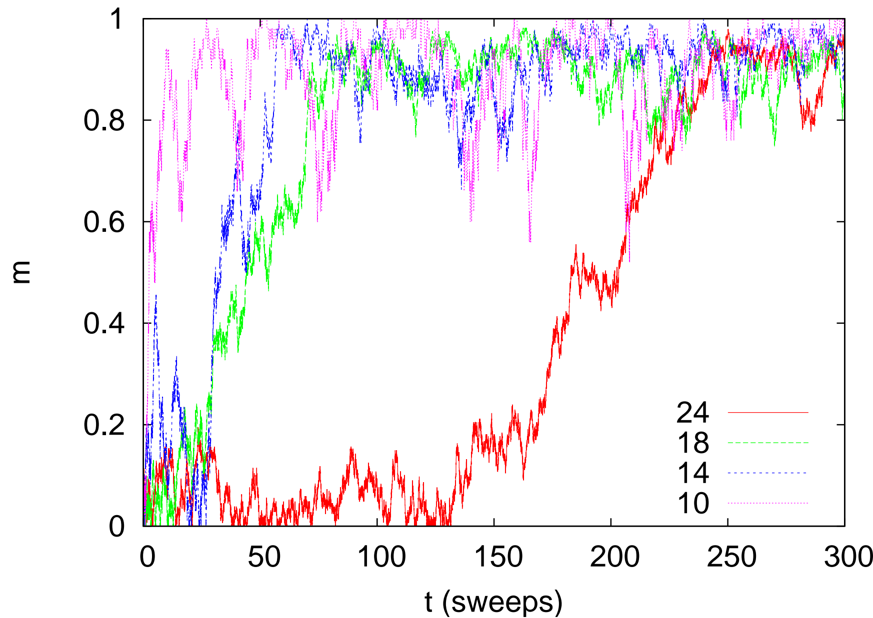


Figure 13.8: Magnetization per site for the Ising model for $L = 10, 14, 18, 24$ and $\beta = 0.50$. Thermalization from a hot start takes longer for a large system.

state and using a different sequence of random numbers. When the times histories of the monitored quantities converge, we are confident that the system has been thermalized. Figure 13.7 shows that the thermalization time can vary quite a lot.

A more systematic way is to compute an expectation value by removing an increasing number of initial measurements. When the results converge within the statistical error, then the physical quantity that we measure has thermalized. This process is shown in figures 13.10 and 13.11 where we progressively drop $0, 20, 50, 100, 200, 400, 800, 1600, 3200$ and 6400 initial measurements until the expectation value of the magnetization stabilizes within the limits of its statistical error.

13.5 Autocorrelations

In order to construct a set of independent measurements using a Markov process, the states put in the sample should be statistically uncorrelated.

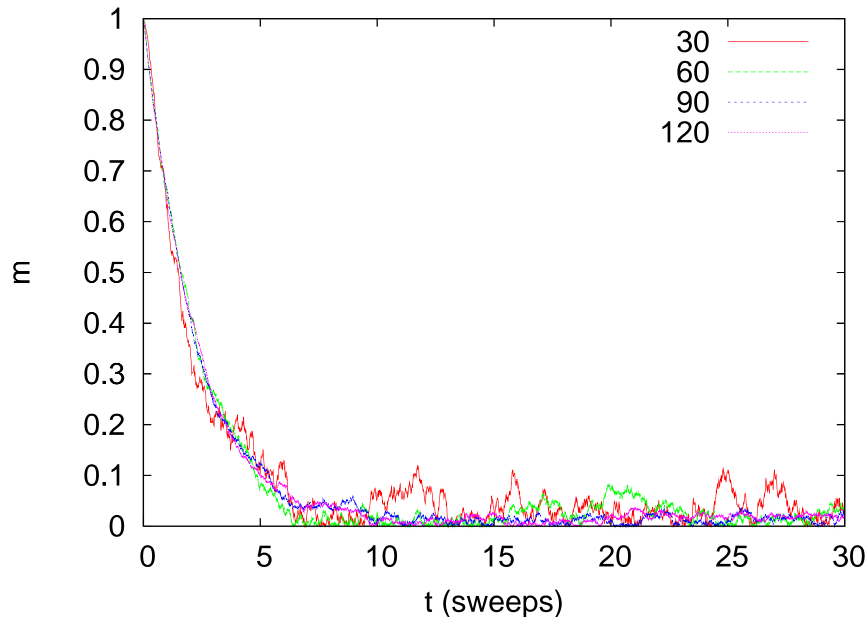


Figure 13.9: Magnetization per site for the Ising model for $L = 30, 60, 90, 120$ and $\beta = 0.20$. Thermalization starting from a cold start does not depend on the system size.

But for a process using the Metropolis algorithm this is not possible. The next state differs from the previous one by at most one value of their spins. We would expect that we could obtain an almost statistically independent configuration after one spin update per site, a so called sweep of the lattice. This is indeed the case for the Ising model for temperatures far from the critical region. But as one approaches β_c, correlations between configurations obtained after a few sweeps remain strong. It is easy to understand why this is happening. As the correlation length ξ (12.46) becomes much larger than a few lattice spacings, large clusters of same spins are formed, as can be seen in figure 13.32. For two statistically independent configurations, the size, shape and position of those clusters should be quite different. For a single flip algorithm, like the Metropolis algorithm, this process takes a lot of time[29].

[29]The Metropolis algorithm changes the clusters mostly by modifying their boundaries, since it is less probable to change the value of a spin in the cluster where all its nearest neighbors have the same spin.

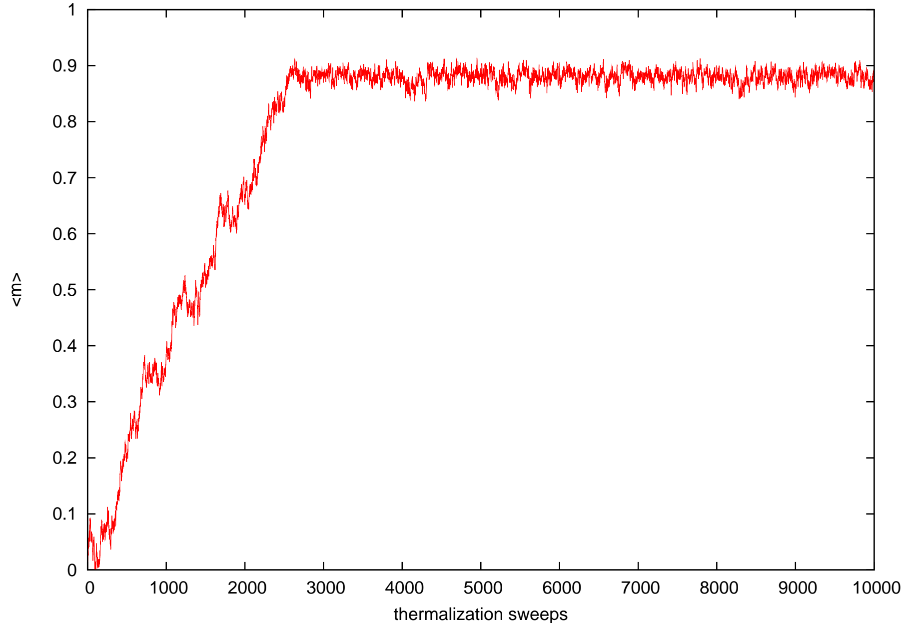


Figure 13.10: Magnetization per site for the Ising model with $L = 100$ and $\beta = 0.48$. Thermalization starts from a hot state.

For the quantitative study of autocorrelations between configurations we use the *autocorrelation function*. Consider a physical quantity $\mathcal{O}$ (e.g. energy, magnetization, etc) and let $\mathcal{O}(t)$ be its value after Monte Carlo "time" t. t can be measured in sweeps or multiples of it. The autocorrelation function $\rho_{\mathcal{O}}(t)$ of $\mathcal{O}$ is

$$\rho_{\mathcal{O}}(t) = \frac{\langle (\mathcal{O}(t') - \langle \mathcal{O} \rangle)(\mathcal{O}(t'+t) - \langle \mathcal{O} \rangle) \rangle_{t'}}{\langle (\mathcal{O} - \langle \mathcal{O} \rangle)^2 \rangle}\,, \tag{13.31}$$

where $\langle \ldots \rangle_{t'}$ is the average value over the configurations in the sample for $t' < t_{\text{max}} - t$. The normalization is such that $\rho_{\mathcal{O}}(0) = 1$.

The above definition reminds us the correlation function of spins in space (see equation (12.45)) and the discussion about its properties is similar to the one of section 12.4. In a few words, when the value of $\mathcal{O}$ after time t is strongly correlated to the one at $t = 0$, then the product in the numerator in (13.31) will be positive most of the time and the value of $\rho_{\mathcal{O}}(t)$ will be positive. When the correlation is weak, the product will

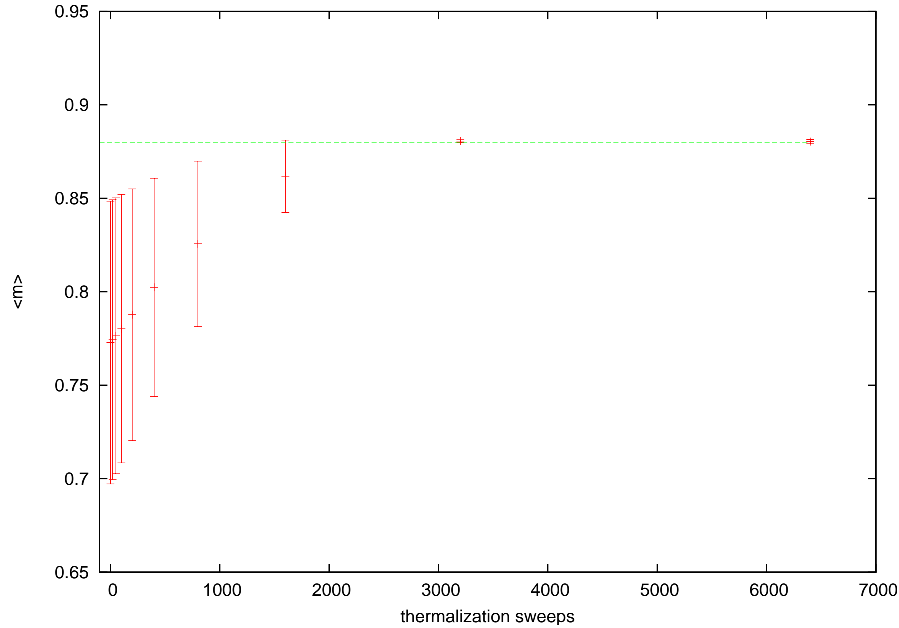


Figure 13.11: Magnetization per site for the Ising model with $L = 100$ and $\beta = 0.48$. We calculate the expectation value $\langle m \rangle$ by neglecting an increasing number of "thermalization sweeps" from the measurements in figure 13.10. When the neglected sweeps reach the thermalized state, the result converges to $\langle m \rangle = 0.880(1)$. This is an indication that the system has thermalized.

be positive and negative the same number of times and $\rho_{\mathcal{O}}(t)$ will be almost zero. In the case of anti-correlations $\rho_{\mathcal{O}}(t)$ is negative. Negative values of $\rho_{\mathcal{O}}(t)$ occur, but these are artifacts of the finite size of the sample and should be rejected.

Asymptotically $\rho_{\mathcal{O}}(t)$ drops exponentially

$$\rho_{\mathcal{O}}(t) \sim \mathrm{e}^{-t/\tau_{\mathcal{O}}} \,. \tag{13.32}$$

$\tau_{\mathcal{O}}$ is the time scale of decorrelation of the measurements of $\mathcal{O}$ and it is called the *autocorrelation time* of $\mathcal{O}$. After time $2\tau_{\mathcal{O}}$, $\rho_{\mathcal{O}}(t)$ has dropped to the $1/\mathrm{e}^2 \approx 14\%$ of its initial value and then we say that we have an independent measurement of[30] $\mathcal{O}$. Therefore, if we have $t_{\max}$ measurements,

[30]Autocorrelation times can be quite different for different $\mathcal{O}$.

the number of independent measurements of $\mathcal{O}$ is

$$n_{\mathcal{O}} = \frac{t_{\max}}{2\tau_{\mathcal{O}}}\,. \tag{13.33}$$

For expensive measurements we should measure every $\sim \tau_{\mathcal{O}}$ sweeps. If the cost of measurement is not significant, then we usually measure more often, since there is still statistical information even in slightly correlated configurations. An accurate determination of $\tau_{\mathcal{O}}$ is not easy since it requires measuring for $t \gg \tau_{\mathcal{O}}$.

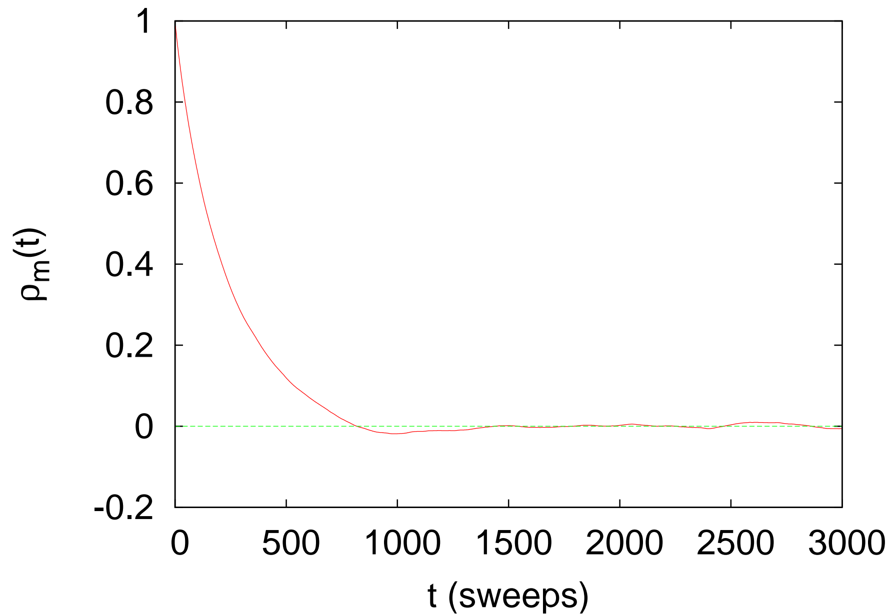


Figure 13.12: The autocorrelation function of the magnetization $\rho_m(t)$ for the Ising model for $L=100$, $\beta=0.42$. We see its exponential decay and that $\tau_m \approx 200$ sweeps. One can see the finite sample effects (the sample consists of about 1,000,000 measurements) when ρ starts fluctuating around 0.

An example is shown in figure 13.12 for the case of the magnetization ($\mathcal{O} = m$). We calculate the function $\rho_m(t)$ and we see that a fit to equation (13.32) is quite good for $\tau_m = 235 \pm 3$ sweeps. The calculation is performed on a sample of 10^6 measurements with 1 measurement/sweep. Therefore the number of independent measurements is $\approx 10^6/(2\times 235) \approx 2128$.

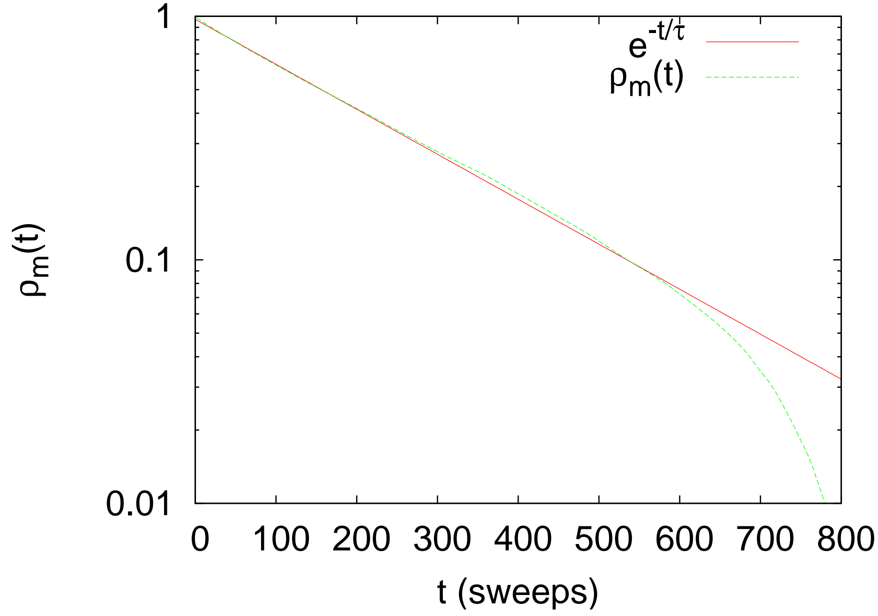


Figure 13.13: The autocorrelation function shown in figure 13.12 of the magnetization $\rho_m(t)$ for the Ising model for $L = 100$, $\beta = 0.42$ in a log plot. The plot shows a fit to $C\mathrm{e}^{-t/\tau}$ (see equation (13.32)) with $\tau = 235(3)$ sweeps.

Another estimator of the autocorrelation time is the so called *integrated autocorrelation time* $\tau_{\text{int},\mathcal{O}}$. Its definition stems from equation (13.32) where we take

$$\tau_{\text{int},\mathcal{O}} = \int_0^{+\infty} dt\, \rho_{\mathcal{O}}(t) \sim \int_0^{+\infty} dt\, \mathrm{e}^{-t/\tau_{\mathcal{O}}} = \tau_{\mathcal{O}}\,. \tag{13.34}$$

The values of $\tau_{\text{int},\mathcal{O}}$ and $\tau_{\mathcal{O}}$ differ slightly due to systematic errors that come from the corrections[31] to equation (13.32). The upper limit of the integral is cut off by a maximum value t_{max}

$$\tau_{\text{int},\mathcal{O}}(t_{\text{max}}) = \int_0^{t_{\text{max}}} dt\, \rho_{\mathcal{O}}(t)\,. \tag{13.35}$$

For large enough t_{max} we observe a plateau in the plot of the value of $\tau_{\text{int},\mathcal{O}}(t_{\text{max}})$ which indicates convergence, and we take this as the estimator

[31] In our calculations, we will see differences of the order of 10%. The actual values can be different but their scaling properties are same.

of $\tau_{\text{int},\mathcal{O}}$. For even larger t_{max}, finite sample effects enter in the sum that should be discarded.

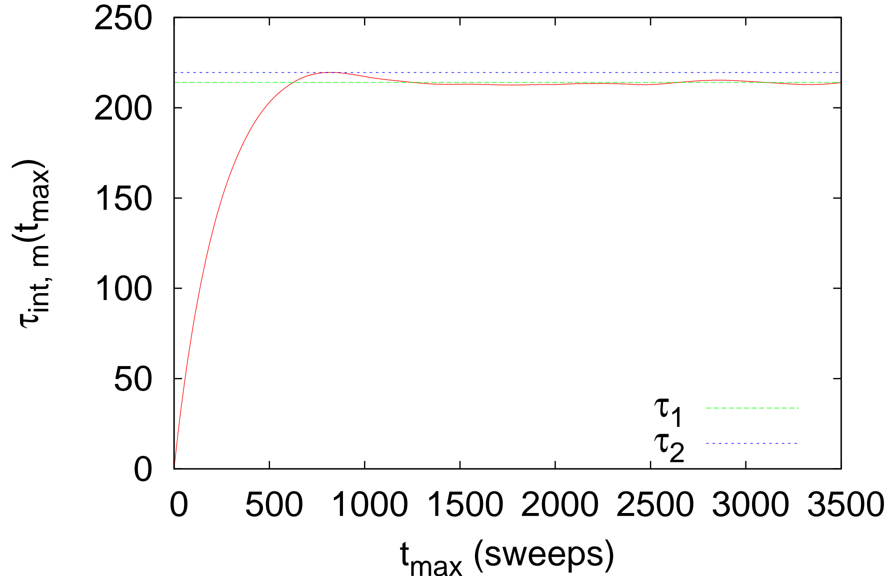


Figure 13.14: Calculation of the integrated autocorrelation time of the magnetization for the same data used in figure 13.19. There is a plateau in the values of $\tau_{\text{int},m}$ for $\tau_1 = 214(1)$sweeps and a maximum for $\tau_2 \approx 219.5$ sweeps. The fall from τ_2 to $\approx \tau_1$ is due to the negative values of $\rho_m(t)$ due to the noise coming from finite sample effects. We estimate that $\tau_{\text{int},m} = 217(3)$ sweeps.

This calculation is shown in figure 13.14 where we used the same measurements as the ones in figure 13.12. We find that $\tau_{\text{int},m} = 217(3)$sweeps, which is somewhat smaller than the autocorrelation time that we calculated using the exponential fit to the autocorrelation function. If we are interested in the scaling properties of the autocorrelation time with the size of the system L or the temperature β, then this difference is not important[32]. The calculation of $\tau_{\text{int},\mathcal{O}}$ is quicker since it involves no

[32]The actual value of $\tau_{\mathcal{O}}$ is used in computing the number of independent configurations from (13.33) and the correction of the statistical error in (13.47). In both cases, the difference in the values of $\tau_{\mathcal{O}}$ is not significant. For (13.47), this is because the concept of the error of the error is slightly fuzzy.

fitting[33].

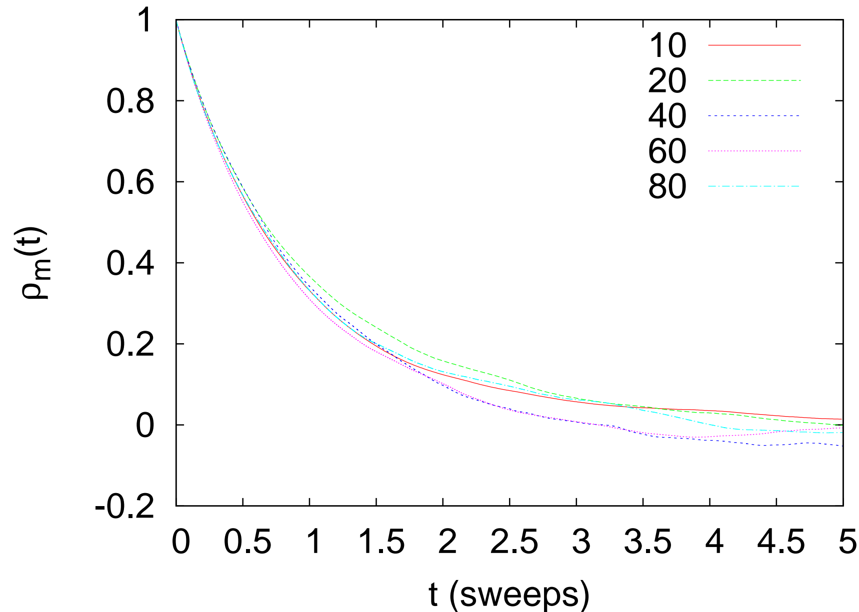


Figure 13.15: The autocorrelation time of the magnetization for the Ising model at (high) temperature $\beta = 0.20$ for $L = 10, 20, 40, 60, 80$. The autocorrelation time in sweeps is independent of L.

Autocorrelation times are not a serious problem away from the critical region. Figures 13.15 and 13.16 show that they are no longer than a few sweeps and that they are independent of the system size L. As we approach the critical region, autocorrelation times increase. At the critical region we observe scaling of their values with the system size, which means that for large L we have that

$$\tau \sim L^z \,. \tag{13.36}$$

This is the phenomenon of *critical slowing down*. For the Metropolis algorithm and the autocorrelation time of the magnetization, we have

[33]As we will see later, there are other, smaller autocorrelation times present as well. These are not taken into account in the definition of the integrated autocorrelation time and the detailed study of the autocorrelation function is necessary if more accuracy is desired.

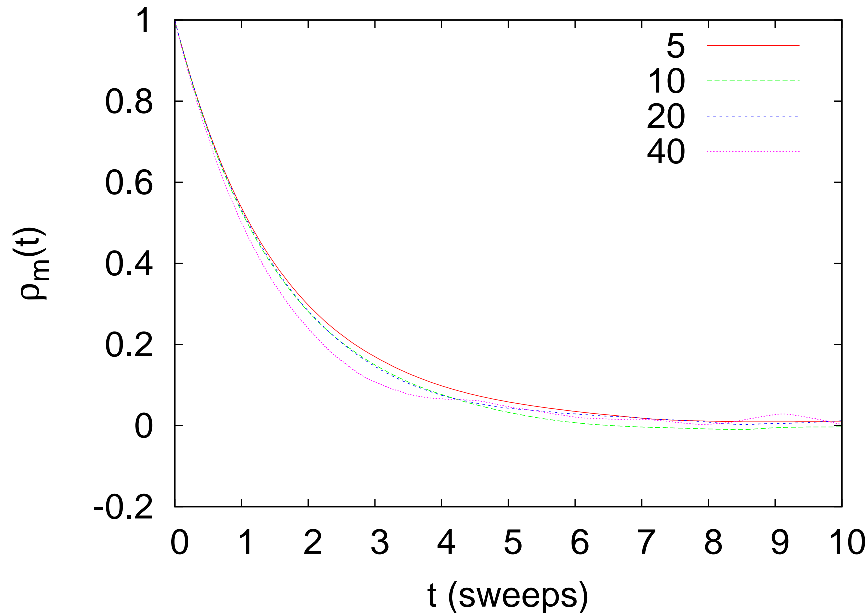


Figure 13.16: The autocorrelation time of the magnetization for the Ising model at (low) temperature $\beta = 0.65$ for $L = 5, 10, 20, 40$. The autocorrelation time in sweeps is independent of L.

that $z = 2.1665 \pm 0.0012$ [63]. This is a large value and that makes the algorithm expensive for the study of the critical properties of the Ising model. It means that the simulation time necessary for obtaining a given number of independent configurations increases as

$$t_{\text{CPU}} \sim L^{d+z} \approx L^{4.17}\,. \tag{13.37}$$

In the next chapter, we will discuss the scaling relation (13.36) in more detail and present new algorithms that reduce critical slowing down drastically.

13.6 Statistical Errors

The estimate of the expectation value of an observable from its average value in a sample gives no information about the quality of the measurement. The complete information is provided by the full distribution, but

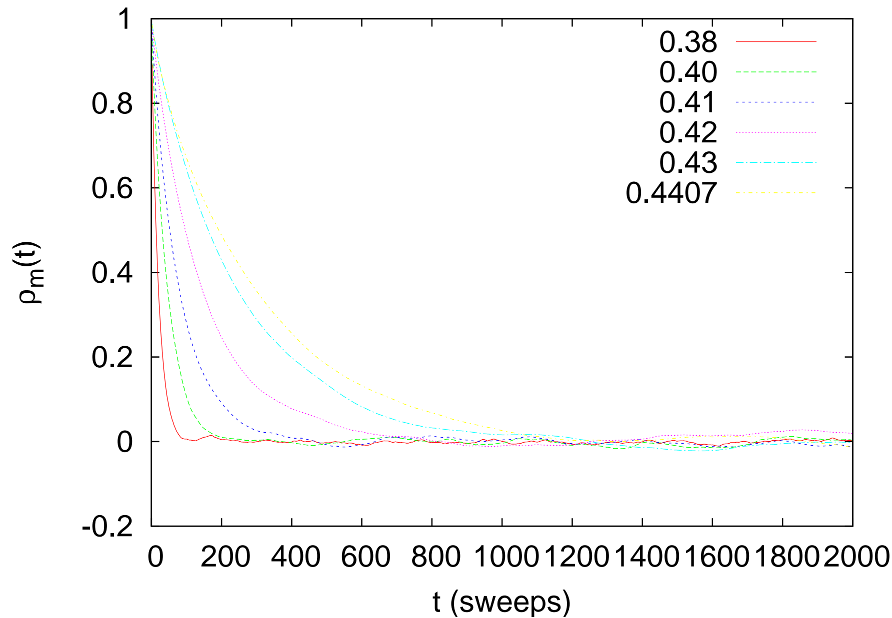


Figure 13.17: The autocorrelation function of the magnetization for the Ising model for $L = 40$. It shows how the autocorrelation time increases as we approach the critical temperature from the disordered (hot) phase.

in practice we are usually content with the determination of the "statistical error" of the measurement. This is defined using the assumption that the distribution of the measurements is Gaussian, which is a very good approximation if the measurements are *independent*. The statistical error is determined by the fluctuations of the values of the observable in the sample around its average (see discussion in section 12.2 and in particular equation (12.27)). Statistical errors can be made to vanish, because they decrease as the inverse *square root* of the size of the sample.

Besides statistical errors, one has *systematic* errors, which are harder to control. Some of them are easier to control (like e.g. poor thermalization) and others maybe hard even to realize their effect (like e.g. a subtle problem in a random number generator). In the case of a discrete, finite, lattice, approximating a continuous theory, there are systematic errors due to the discretization and the finite size of the system. These errors are reduced by simulating larger systems and by using several

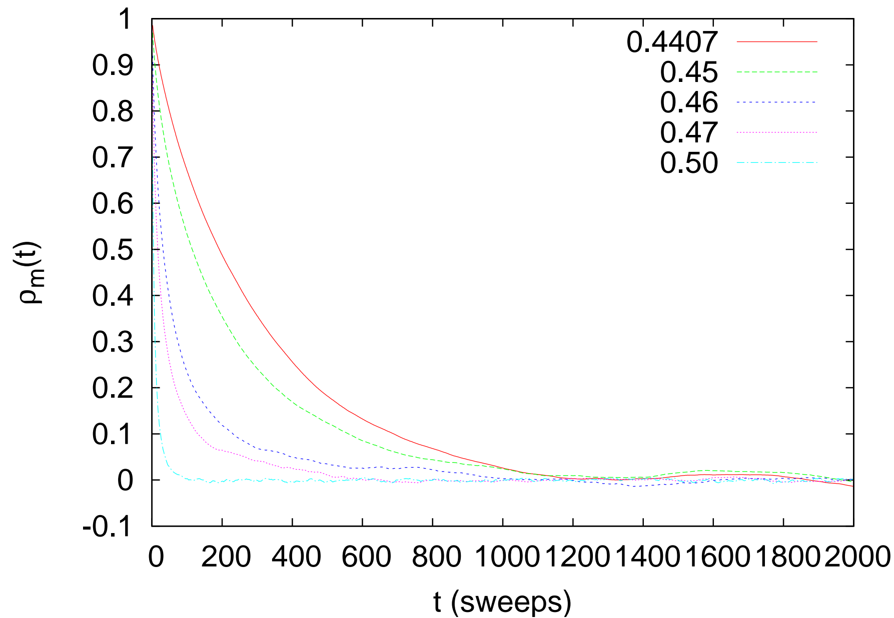


Figure 13.18: The autocorrelation function of the magnetization for the Ising model for $L = 40$. It shows how the autocorrelation time increases as we approach the critical temperature from the ordered (cold) phase.

techniques (e.g. finite size scaling) in order to extrapolate the results to the thermodynamic limit. These will be studied in detail in the following chapter.

13.6.1 Errors of Independent Measurements

Using the assumption that the source of statistical errors are the thermal fluctuations around the average value of an observable, we conclude that its expectation value can be estimated by the mean of the sample and its error by the error of the mean. Therefore if we have a sample of n measurements $\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_{n-1}$, their mean is an estimator of $\langle \mathcal{O} \rangle$

$$\langle \mathcal{O} \rangle = \frac{1}{n} \sum_{i=0}^{n-1} \mathcal{O}_i \,. \tag{13.38}$$

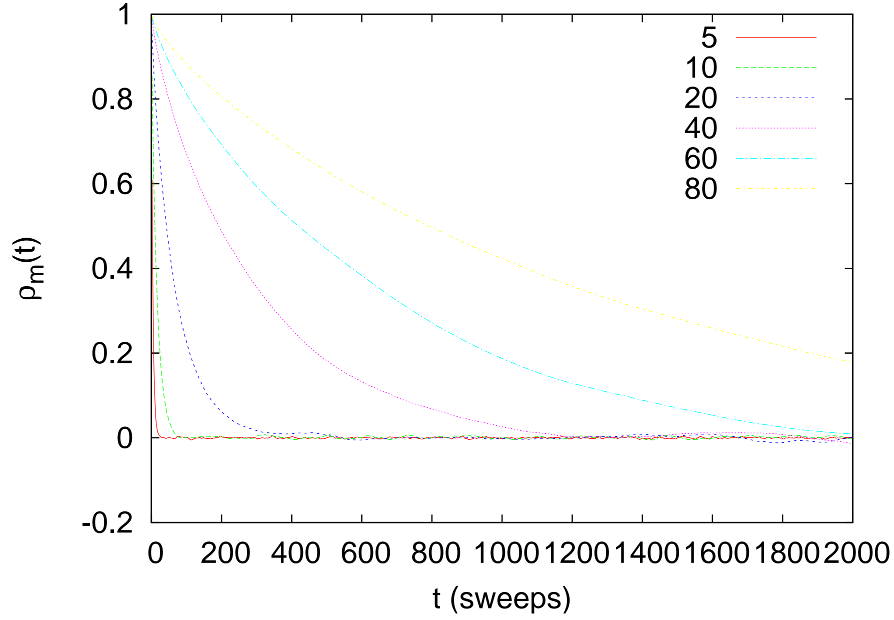


Figure 13.19: The autocorrelation function for the Ising model for $\beta = 0.4407 \approx \beta_c$ and different L. We observe the increase of the autocorrelation time with the system size in the critical region.

The error of the mean is an estimator of the statistical error $\delta\mathcal{O}$

$$(\delta\mathcal{O})^2 \equiv \sigma^2_{\mathcal{O}} = \frac{1}{n-1}\left\{\frac{1}{n}\sum_{i=0}^{n-1}(\mathcal{O}_i - \langle\mathcal{O}\rangle)^2\right\} = \frac{1}{n-1}\left(\langle\mathcal{O}^2\rangle - \langle\mathcal{O}\rangle^2\right) . \tag{13.39}$$

The above equations assume that the sample is a set of statistically independent measurements. This is not true in a Monte Carlo simulation due to the presence of autocorrelations. If the autocorrelation time, measured in number of measurements, is $\tau_{\mathcal{O}}$, then according to equation 13.33 we will have $n_{\mathcal{O}} = n/(2\tau_{\mathcal{O}})$ independent measurements. One can show that in this case, the statistical error in the measurement of $\mathcal{O}$ is[34] [64]

$$(\delta\mathcal{O})^2 = \frac{1+2\tau_{\mathcal{O}}}{n-1}\left(\langle\mathcal{O}^2\rangle - \langle\mathcal{O}\rangle^2\right) . \tag{13.40}$$

[34] See also chapter 4.1 in [5]

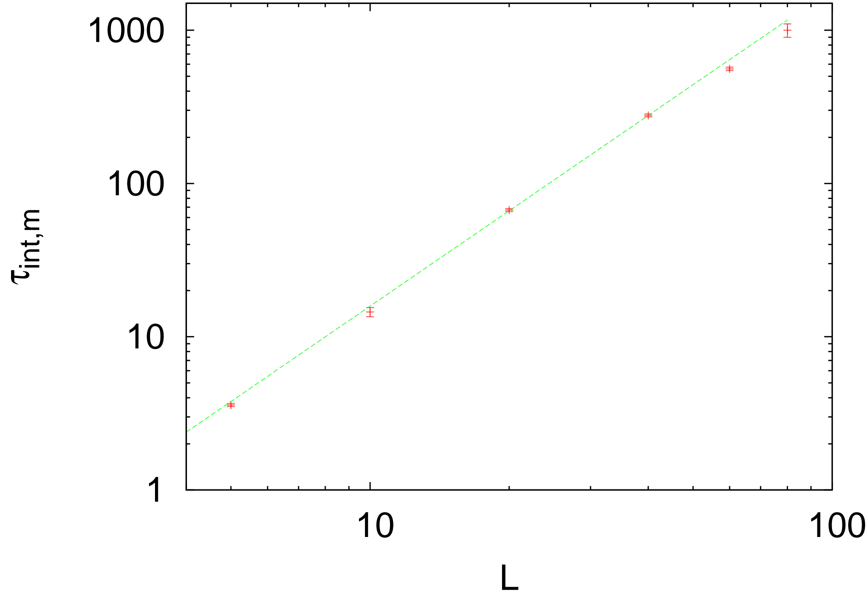


Figure 13.20: The integrated autocorrelation time $\tau_{\mathrm{int},m}$ for $\beta = \beta_c$ in a logarithmic scale. The continuous line is the fit to $0.136(10)L^{2.067(21)}$. The expected result from the bibliography is $z = 2.1665(12)$ and the difference is a finite size effect.

If $\tau_{\mathcal{O}} \ll 1$, then we obtain equation (13.39). If $\tau_{\mathcal{O}} \gg 1$

$$\begin{aligned}(\delta\mathcal{O})^2 &\approx \frac{2\tau_{\mathcal{O}}}{n-1}\left(\langle\mathcal{O}^2\rangle - \langle\mathcal{O}\rangle^2\right)\\ &\approx \frac{1}{(n/2\tau_{\mathcal{O}})}\left(\langle\mathcal{O}^2\rangle - \langle\mathcal{O}\rangle^2\right)\\ &\approx \frac{1}{n_{\mathcal{O}}-1}\left(\langle\mathcal{O}^2\rangle - \langle\mathcal{O}\rangle^2\right)\end{aligned} \tag{13.41}$$

which is nothing but equation (13.39) for $n_{\mathcal{O}}$ independent measurements (we assumed that $1 \ll n_{\mathcal{O}} \ll n$). The above relation is consistent with our assumption that measurements become independent after time $\sim 2\tau_{\mathcal{O}}$.

In some cases, the straightforward application of equations (13.41) is not convenient. This happens when, measuring the autocorrelation time according to the discussion in section 13.5, becomes laborious and time consuming. Moreover, one has to compute the errors of observables that

are functions of correlated quantities, like in the case of the magnetic susceptibility (13.30). The calculation requires the knowledge of quantities that are not defined on one spin configuration, like $\langle m\rangle$ and $\langle m^2\rangle$ (or $(m_i - \langle m\rangle)$ on each configuration i). After these are calculated on the sample, the error $\delta\chi$ is not a simple function of $\delta\langle m\rangle$ and $\delta\langle m^2\rangle$. This is because of the correlation between the two quantities and the well known formula of error propagation $(\delta(\langle m^2\rangle - \langle m\rangle^2))^2 = (\delta\langle m^2\rangle)^2 + (\delta\langle m\rangle^2)^2$ cannot be applied.

13.6.2 Jackknife

The simplest solution to the problems arising in the calculation of statistical errors discussed in the previous section is to divide a sample into blocks or *bins*. If one has n measurements, she can put them in n_b "bins" and each bin is to be taken as an independent measurement. This will be true if the number of measurements per bin $b = (n/n_b) \gg \tau_{\mathcal{O}}$. If $\mathcal{O}^{\,b}_i$ $i = 0, \ldots, n_b - 1$ is average value of $\mathcal{O}$ in the bin i, then the error is given by (13.39)

$$(\delta\mathcal{O})^2 = \frac{1}{n_b - 1}\left\{\frac{1}{n_b}\sum_{i=0}^{n_b-1}(\mathcal{O}^{\,b}_i - \langle\mathcal{O}^{\,b}\rangle)^2\right\} \tag{13.42}$$

This is the *binning* or *blocking* method and it is quite simple in its use. Note that quantities, like the magnetic susceptibilities, are calculated in each bin as if the bin were an independent sample. Then the error is easily calculated by equation (13.42). If the bin is too small and the samples are not independent, then the error is underestimated by a factor of $2\tau_{\mathcal{O}}/(n_b - 1)$ (see equation (13.40)). The bins are statistically independent if $b \sim 2\tau_{\mathcal{O}}$. If $\tau_{\mathcal{O}}$ is not a priori known we compute the error (13.42) by decreasing the number of bins n_b. When the error is not increasing anymore and takes on a constant value, then the calculation converges to the true statistical error.

But the method of choice in this book is the *jackknife method*. It is more stable and more reliable, especially if the sample is small. The basic idea is similar to the binning method. The difference is that the bins are constructed in a different way and equation (13.42) is slightly modified. The data is split in n_b bins which contain $b = n - (n/n_b)$ elements as follows: The bin j contains the part of the sample obtained after we we *erase* the contents of the j-th bin of the binning method

from the full sample $\mathcal{O}_0, \ldots, \mathcal{O}_{n-1}$. The procedure is depicted in figure 13.21. We calculate the average value of $\mathcal{O}$ in each bin and we obtain

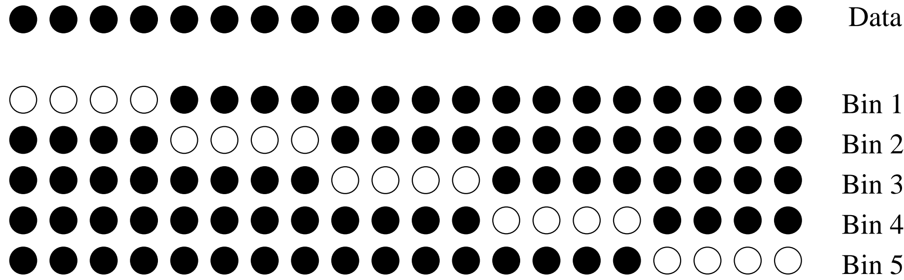


Figure 13.21: The jackknife method applied on a sample of $n = 20$ measurements. The data is split to $n_b = 5$ bins and each bin contains $b = n - (n/n_b) = 20 - 4 = 16$ measurements (the black disks). We calculate the average value $\mathcal{O}^b_i$ in each bin and, by using them, we calculate the error $\delta\mathcal{O} = \sqrt{n_b\left(\langle(\mathcal{O}^b)^2\rangle - \langle\mathcal{O}^b\rangle^2\right)}$.

$\mathcal{O}^b_0, \mathcal{O}^b_1, \ldots, \mathcal{O}^b_{n_b-1}$. Then the statistical error in the measurement of $\mathcal{O}$ is

$$(\delta\mathcal{O})^2 = \sum_{j=0}^{n_b-1}\left(\mathcal{O}^b_j - \langle\mathcal{O}^b\rangle\right)^2 = n_b\left(\langle(\mathcal{O}^b)^2\rangle - \langle\mathcal{O}^b\rangle^2\right) . \tag{13.43}$$

In order to determine the error, one has to vary the number of bins and check for the convergence of (13.43), like in the case of the binning method.

For more details and proofs of the above statements, the reader is referred to the book of Berg [5]. Appendix 13.8.1 provides examples and a program for the calculation of jackknife errors.

13.6.3 Bootstrap

Another useful method for the estimation of statistical errors is the *bootstrap method*. Suppose that we have n *independent* measurements. From these we create n_S random samples as follows: We choose one of the n measurements with equal probability. We repeat n times using the *same* set of n measurements - i.e. by putting the chosen measurements back to the sample. This means that on the average $\sim 1 - 1/e \approx 63\%$ of the sample will consist of the same measurements. In each sample

$i=0,\ldots,n_S-1$ we calculate the average values $\mathcal{O}_i^S$ and from those

$$\langle \mathcal{O}^S\rangle = \frac{1}{n_S}\sum_{i=0}^{n_S-1}\mathcal{O}_i^S\,, \tag{13.44}$$

and

$$\langle \left(\mathcal{O}^S\right)^2\rangle = \frac{1}{n_S}\sum_{i=0}^{n_S-1}\left(\mathcal{O}_i^S\right)^2\,. \tag{13.45}$$

The estimate for the error in $\langle\mathcal{O}\rangle$ is[35]

$$(\delta\mathcal{O})^2 = \langle\left(\mathcal{O}^S\right)^2\rangle - \langle\mathcal{O}^S\rangle^2\,. \tag{13.46}$$

We stress that the above formula gives the error for independent measurements. If we have non negligible autocorrelation times, then we must use the correction

$$(\delta\mathcal{O})^2 = (1+2\tau_{\mathcal{O}})\left(\langle\left(\mathcal{O}^S\right)^2\rangle - \langle\mathcal{O}^S\rangle^2\right) \tag{13.47}$$

Appendix 13.8.2 discusses how to use the bootstrap method in order to calculate the true error $\delta\mathcal{O}$ without an a priori knowledge of $\tau_{\mathcal{O}}$. For more details, the reader is referred to the articles of Bradley Efron [65]. In appendix 13.8.2 you will find examples and a program that implements the bootstrap method.

13.7 Appendix: Autocorrelation Function

This appendix discusses the technical details of the calculation of the autocorrelation function (13.31) and the autocorrelation time given by equations (13.32) and (13.34). The programs can be found in the directory `Tools` in the accompanying software.

If we have a finite sample of n measurements $\mathcal{O}(0),\mathcal{O}(1),\ldots,\mathcal{O}(n-1)$, then we can use the following estimator for the autocorrelation function, given by equation (13.31),

$$\rho_{\mathcal{O}}(t) = \frac{1}{\rho_0}\frac{1}{n-t}\sum_{t'=0}^{n-1-t}(\mathcal{O}(t')-\langle\mathcal{O}\rangle_0)(\mathcal{O}(t'+t)-\langle\mathcal{O}\rangle_t)\,, \tag{13.48}$$

[35] Notice that the right hand side of equation (13.46) is *not* divided by $1/(n_s-1)$.

where the average values are computed from the equations[36]

$$\langle \mathcal{O} \rangle_0 \equiv \frac{1}{n-t} \sum_{t'=0}^{n-1-t} \mathcal{O}(t') \qquad \langle \mathcal{O} \rangle_t \equiv \frac{1}{n-t} \sum_{t'=0}^{n-1-t} \mathcal{O}(t'+t)\,. \tag{13.49}$$

The constant ρ_0 is chosen so that $\rho_{\mathcal{O}}(0) = 1$.

The program for the calculation of (13.48) and the autocorrelation time (13.34) is listed below. It is in the file `autoc.cpp` which can be found in the `Tools` directory of the accompanying software.

```
//===========================================================
// file: autoc.cpp
//===========================================================
//Autocorrelation function: you can use this function in
//any of your programs.
#include <iostream>
#include <fstream>
#include <iomanip>
#include <cstdlib>
#include <unistd.h>
#include <libgen.h>
using namespace std;
string prog;
int    NMAX,tmax;
double rho (double *,int ,int );
void   get_the_options(int,char**);
void   usage (char**);
void   locerr(string);
int    main(int argc,char** argv){
  double *r,*tau,*x;
  double norm;
  int    i,ndat,t,tcut;
  prog.assign((char *)basename(argv[0]));//name of program
  //-----------------------------------------------------------
  //Default values for max number of data and max time for
  //rho and tau:
  NMAX=2000001;tmax=1000;//NMAX=2e6 requires ~ 2e6*8=16MB
  get_the_options(argc,argv);
```

[36] Alternatively one can take $\langle \mathcal{O} \rangle_0 = \langle \mathcal{O} \rangle_t = (1/n)\sum_{t'=0}^{n} \mathcal{O}(t')$ without noticeable difference for $t \ll n$. The choice in (13.48) results in a more accurate calculation and smaller finite sample effects. The choice (13.48), instead of $\rho_{\mathcal{O}}(t) \propto (1/(n-t))\sum_{t'=0}^{n-1-t} \mathcal{O}(t')\mathcal{O}(t'+t) - \langle \mathcal{O} \rangle_0 \langle \mathcal{O} \rangle_t$, has smaller roundoff errors.

```
  x = new double[NMAX+1];
  ndat=0;
  while((ndat<=NMAX) && (cin >> x[ndat++]));
  ndat--;
  if(ndat == NMAX)
    cerr << prog
         << ": Warning: read ndat= "   << ndat
         << " and reached the limit: " << NMAX << endl;
  //We decrease tmax if it is comparable or larger of ndat
  if(tmax > (ndat/10)) tmax = ndat/10;
  //r[t] stores the values of the autocorrelation
  //function rho(t)
  r = new double[tmax];
  for(t=0;t<tmax;t++) r[t]  = rho(x,ndat,t);      //rho(t)
  norm=1.0/r[0];
  for(t=0;t<tmax;t++)r[t] *= norm;      //normalize r[0]=1
  //tau[t] stores integrated autocorrelation times with tcut=t
  tau = new double[tmax];
  for(tcut=0;tcut<tmax;tcut++){
    tau[tcut] = 0.0;
    for(t=0;t<=tcut;t++) tau[tcut] += r[t];//sum of r[t]
  }
  //Output:
  cout << "# ============================================\n";
  cout << "# Autoc func rho and integrated autoc time tau\n";
  cout << "# ndat= "<< ndat << "  tmax= " << tmax     <<'\n';
  cout << "# t          rho(t)               tau(tcut=t)    \n";
  cout << "# ============================================\n";
  for(t=0;t<tmax;t++)
    cout << t << " " << r[t] << " " << tau[t] << '\n';
}//main()
//======================================================
double rho(double *x,int ndat,int t){
  int n,t0;
  double xav0=0.0,xavt=0.0,r=0.0;
  n=ndat-t;
  if(n<1)locerr("rho: n<1");
  //Calculate the two averages: xav0= <x>_0 and xavt= <x>_t
  for(t0=0;t0<n;t0++){
    xav0 += x[t0];
    xavt += x[t0+t];
  }
  xav0/=n;xavt/=n; //normalize the averages to number of data
  //Calculate the t-correlations:
  for(t0=0;t0<n;t0++)
```

```
    r += (x[t0]-xav0)*(x[t0+t]-xavt);
  r/=n;              //normalize the averages to number of data
  return r;
}
//==========================================================
void locerr(string errmes){
  cerr << prog <<": "<< errmes <<"Exiting....."<< endl;
  exit(1);
}
//==========================================================
#define OPTARGS  "?h1234567890.t:n:"
void get_the_options(int argc,char **argv){

  int c,errflg = 0;
  while (!errflg &&
         (c = getopt(argc, argv, OPTARGS)) != -1){
    switch(c){
    case 'n':
      NMAX = atoi(optarg);
      break;
    case 't':
      tmax = atoi(optarg);
      break;
    case 'h':
      errflg++;/*call usage*/
      break;
    default:
      errflg++;
    }/*switch*/
    if(errflg) usage(argv);
  }/*while...*/
}//get_the_options()
//==========================================================
void usage(char **argv){
  cerr << "\
Usage: " << prog << " [-t <maxtime>] [-n <ndata>]     \n\
       Reads data from stdin (one column) and computes\n\
       autocorrelation function and integrated        \n\
       autocorrelation time." << endl;
  exit(1);
}/*usage()*/
```

The compilation is done with the commands

```
> g++ -O2 autoc.cpp -o autoc
```

If our data is written in a file named `data` in one column, then the calculation of the autocorrelation function and the autocorrelation time is done with the command

```
> cat data | ./autoc > data.rho
```

The results are written to the file `data.rho` in three columns. The first one is the time t, the second one is $\rho_{\mathcal{O}}(t)$ and the third one is $\tau_{\text{int},\mathcal{O}}(t)$ (equation (13.35)). The corresponding plots are constructed by the `gnuplot` commands:

```
gnuplot> plot "data.rho" using 1:2 with lines
gnuplot> plot "data.rho" using 1:3 with lines
```

If we wish to increase the maximum number of data `NMAX` or the maximum time `tmax`, then we use the options `-n` and `-t` respectively:

```
> cat data | autoc -n 20000000 -t 20000 > data.rho
```

For doing all the work at once using `gnuplot`, we can give the command:

```
gnuplot> plot "<./is -L 20 -b 0.4407 -s 1 -S 345 -n 400000|\
            grep -v '#'|awk '{print ($2>0)?$2:-$2;}'  |\
            ./autoc -t 500" using 1:2 with lines
```

The above command is long and it is broken into 3 lines for better printing. You can type it in one line by removing the trailing \.

A script that works out many calculations together is listed below. It is in the file `autoc_L` and computes the data shown in figure 13.19.

```
#!/bin/tcsh -f

set nmeas  = 2100000
set Ls     = (5 10 20 40 60 80)
set beta   = 0.4407
set tmax   = 2000
foreach L ($Ls)
```

```
 set N      = `awk -v L=$L 'BEGIN{print L*L}'`
 set rand   = `perl -e 'srand();print int(3000000*rand())+1;'`
 set out    = outL${L}b${beta}
 echo "Running L${L}b${beta}"
 ./is -L $L -b $beta -s 1 -S $rand -n $nmeas > $out
 echo "Autocorrelations L${L}b${beta}"
 grep -v '#' $out | \
  awk -v N=$N 'NR>100000{print ($2>0)?($2/N):(-$2/N)}'|\
  autoc -t $tmax  > $out.rhom
end
```

Then we give the gnuplot commands:

```
gnuplot>   plot  "outL5b0.4407.rhom" u 1:2 w lines title "5"
gnuplot> replot "outL10b0.4407.rhom" u 1:2 w lines title "10"
gnuplot> replot "outL20b0.4407.rhom" u 1:2 w lines title "20"
gnuplot> replot "outL40b0.4407.rhom" u 1:2 w lines title "40"
gnuplot> replot "outL60b0.4407.rhom" u 1:2 w lines title "60"
gnuplot> replot "outL80b0.4407.rhom" u 1:2 w lines title "80"
```

The plots in figure 13.17 are constructed in a similar way.

For the calculation of τ_m we do the following:

```
gnuplot> f(x) = c * exp(-x/t)
gnuplot> set log y
gnuplot> plot [:1000] "outL40b0.4407.rhom" u 1:2 with lines
gnuplot> c = 1 ; t = 300
gnuplot> fit [150:650] f(x) "outL40b0.4407.rhom" u 1:2 via c,t
gnuplot> plot [:1000] "outL40b0.4407.rhom" u 1:2 w lines,f(x)
gnuplot> plot [:]     "outL40b0.4407.rhom" u 1:3 w lines
```

where in the last line we compute $\tau_{\text{int},m}$. The `fit` command is just an example and one should try different fitting ranges. The first plot command shows graphically the approximate range of the exponential falloff of the autocorrelation function. We should vary the upper and lower limits of the fitting range until the value of τ_m stabilizes and the[37] χ^2/dof

[37]For the data $\{(x_i, y_i)\}$, $i = 1, \ldots, n$ with error δy_i which are fitted to $f(x; c, t) = c\,e^{-x/t}$, the $\chi^2(c,t) = \sum_{i=1}^{n}(y_i - f(x_i; c, t))^2/\delta y_i^2$. The χ^2/dof is normalized to the number of degrees of freedom (dof = degrees of freedom= $n-2$) which is the number of data points n used in the fit minus the number of fitting parameters (here c, t which makes 2 parameters).

is minimized[38]. The χ^2/dof of the fit can be read off from the output of the command `fit`

```
.....
degrees of freedom    (FIT_NDF)                        : 449
rms of residuals      (FIT_STDFIT) = sqrt(WSSR/ndf) : ↩
   0.000939201
variance of residuals(reduced chisquare) = WSSR/ndf: 8.82099e↩
   −07

Final set of parameters              Asymptotic Standard Error
=======================              ==========================

c               = 0.925371           +/− 0.0003773      (0.04078%)
t               = 285.736            +/− 0.1141         (0.03995%)
.....
```

from the line "`variance of residuals`". From the next lines we read the values of the fitted parameters with their errors[39] and we conclude that $\tau_m = 285.7 \pm 0.1$. We stress that this is the *statistical error* of the fit for the given fitting range. But usually the largest contributions to the error come from systematic errors, which, in our case, are seen by varying the fitting range[40]. By trying different fitting ranges and using the criterion that the minimum χ^2/dof doubles its minimum value, we find that $\tau_m = 285(2)$.

In our case the largest systematic error comes from neglecting the effect of smaller autocorrelation times. These make non negligible contributions for small t.

By fitting to

$$f(t) = c\,\mathrm{e}^{-t/\tau}\,, \tag{13.50}$$

---

[38]The acceptable $\chi^2/\mathrm{dof} \sim 1$. Since we don't calculate the errors of the autocorrelation functions, the χ^2/dof is not properly normalized. The program sets $\delta y_i = 1\ \forall i$.

[39]In the parentheses we see the confidence level. This is defined as the probability that the parameters are within the range defined by the error. For a correct calculation of the confidence level, every point should be weighted by its error - in our example this is not happening and this is the reason why the confidence level is so low. A value below 5% is too low and it indicates that the model needs corrections. It is also assumed that the distribution of the measurements is Gaussian and if not, the computed numerical values are only indicative.

[40]For a careful calculation, one should also try more functions that include corrections to the asymptotic behavior.

we have taken into account only the largest autocorrelation time.

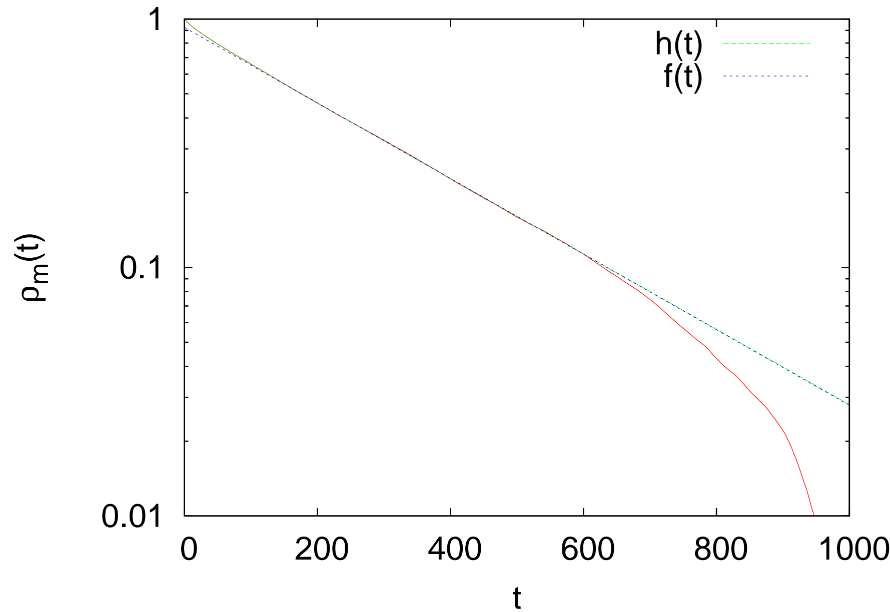


Figure 13.22: Fit of the autocorrelation function $\rho_m(t)$ to the functions $f(t) = c\,\mathrm{e}^{-t/\tau}$ and $h(t) = a_1\mathrm{e}^{-t/\tau_1} + a_2\mathrm{e}^{-t/\tau_2} + a_3\mathrm{e}^{-t/\tau_3}$. For large times $f(t) \approx h(t)$, but $h(t)$ is necessary in order to capture the small t behavior. This choice results in $\tau_m = \tau = \tau_1$. The values of the parameters are given in the text. The vertical axes are in logarithmic scale.

One should take into account also the smaller autocorrelation times. In this case we expect that $\rho_m(t) \sim a_1\mathrm{e}^{-t/\tau_1} + a_2\mathrm{e}^{-t/\tau_2} + \ldots$. We find that the data for the autocorrelation function fit perfectly to the function

$$h(x) = a_1\mathrm{e}^{-t/\tau_1} + a_2\mathrm{e}^{-t/\tau_2} + a_3\mathrm{e}^{-t/\tau_3}\,. \tag{13.51}$$

As we can see in figures 13.22 and 13.23, the small t fit is excellent and the result for the dominant autocorrelation time is $\tau_m \equiv \tau_1 = 286.3(3)$. The secondary autocorrelation times are $\tau_2 = 57(3)$, $\tau_3 = 10.5(8)$ which are considerably smaller that τ_1.

The commands for the analysis are listed below:

```
gnuplot> h(x) = a1*exp(-x/t1) + a2*exp(-x/t2) + a3*exp(-x/t3)
```

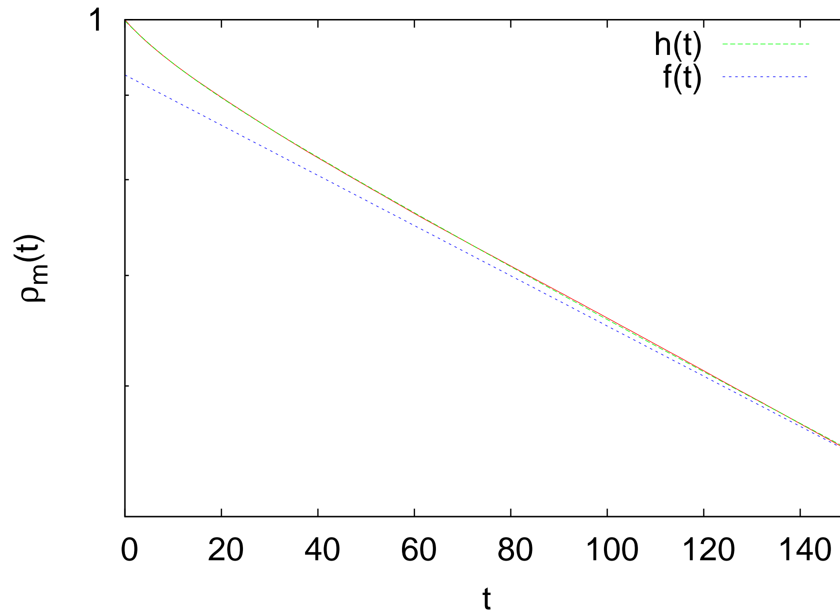


Figure 13.23: The plot of figure 13.22 for small times where the effect of smaller autocorrelation times is most clearly seen.

```
gnuplot> a1 = 1;    t1 = 285; a2 = 0.04; t2 = 56; \
         a3 = 0.03; t3 = 10
gnuplot> fit [1:600] h(x) "outL40b0.4407.rhom" \
         using 1:2 via a1,t1,a2,t2,a3,t3
...
Final set of parameters            Asymptotic Standard Error
=======================            ==========================
a1              = 0.922111         +/- 0.001046     (0.1135%)
t1              = 286.325          +/- 0.2354       (0.08221%)
a2              = 0.0462523        +/- 0.001219     (2.635%)
t2              = 56.6783          +/- 2.824        (4.982%)
a3              = 0.0300761        +/- 0.001558     (5.18%)
t3              = 10.5227          +/- 0.8382       (7.965%)
gnuplot> plot [:150][0.5:]   "outL40b0.4407.rhom" using 1:2 \
         with lines notit,h(x) ,f(x)
gnuplot> plot [:1000][0.01:] "outL40b0.4407.rhom" using 1:2 \
         with lines notit,h(x) ,f(x)
```

13.8 Appendix: Error Analysis

13.8.1 The Jackknife Method

In this section we present a program that calculates the errors using the jackknife method discussed in section 13.6.2. Figure 13.21 shows the division of the data into bins. For each bin we calculate the average value of the quantity $\mathcal{O}$ and then we use equation (13.43) in order to calculate the error. The program is in the file `jack.cpp` which you can find in the directory `Tools` in the accompanying software. The program calculates $\langle \mathcal{O} \rangle$, $\delta\mathcal{O}$, $\chi \equiv \langle(\mathcal{O} - \langle \mathcal{O} \rangle)^2\rangle$ and $\delta\chi$.

```
//========================================================
// file: jack.cpp
//--------------------------------------------------------
//jackknife function: you can use this function in
//any of your programs.
//--------------------------------------------------------
#include <iostream>
#include <fstream>
#include <iomanip>
#include <cstdlib>
#include <cmath>
#include <unistd.h>
#include <libgen.h>
using namespace std;
int maxdat,JACK;
string prog;
void jackknife(int ,int&,double *,double &,
               double &,double &,double &);
void   get_the_options(int ,char**);
void   usage (char**);
void   locerr(string);

int    main(int argc,char** argv){
  int    ndat;
  double O,dO,chi,dchi;
  double *x;
  prog.assign((char *)basename(argv[0]));//name of program
  maxdat=2000000;JACK=10;
  get_the_options(argc,argv);
  x = new double[maxdat];
  ndat=0;
```

```
  while((ndat<=maxdat) && (cin >> x[ndat++]));
  ndat--;
  if(    ndat==maxdat)
    cerr << prog
         << ": Warning: read ndat= "    << ndat
         << " and reached the limit: "  << maxdat  << endl;
  jackknife(ndat,JACK,x,O,dO,chi,dchi);
  cout << "# NDAT = "<<ndat<<" data. JACK = "<<JACK<< endl;
  cout << "# <o>, chi= (<o^2>-<o>^2)\n";
  cout << "# <o> +/- err                         chi +/- err\n";
  cout.precision(17);
  cout << O <<" "<< dO <<" "<< chi <<" "<< dchi    << endl;
}//main()
//=========================================================
void jackknife(int ndat,int& jack,double* x,double& avO,
               double& erO,double& avchi,double& erchi){
  int    i,j,binw,bin;
  double *O,*chi;
  O    = new double[jack]();//() initializes to 0
  chi  = new double[jack]();
  binw = ndat/jack;
  if(binw<1)locerr("jackknife: binw < 1");
  //------------------------------------------
  //average value:
  for(  i=0;i<ndat;i++)
    for(j=0;j<jack;j++){
      if( (i/binw) != j ){ //then we add this data
        O [j] += x[i];     //point to the bin
      }
    }                  //average in each bin:
  for(j=0;j<jack;j++) O  [j] /= (ndat-binw);
  //------------------------------------------
  //susceptibility:
  for(  i=0;i<ndat;i++)
    for(j=0;j<jack;j++){
      if( (i/binw) != j ){
        //then we add this data point to the bin:
    chi[j] += (x[i]-O[j])*(x[i]-O[j]);
      }
    }                  //average in each bin:
  for(j=0;j<jack;j++) chi[j] /= (ndat-binw);
  //------------------------------------------
  //Compute averages:
  avO = 0.0; avchi = 0.0;
  for(j=0;j<jack;j++){
```

```
    avO += O[j]; avchi += chi[j];
  }
  avO /= jack; avchi /= jack;
  //----------------------------------------------
  //Compute errors:
  erO = 0.0; erchi = 0.0;
  for(j=0;j<jack;j++){
    erO   += (O   [j]-avO   )*(O   [j]-avO   );
    erchi += (chi[j]-avchi)*(chi[j]-avchi);
  }
  erO = sqrt(erO); erchi = sqrt(erchi);
  delete [] O;
  delete [] chi;
}//jackknife()
//============================================================
void locerr(string errmes){
  cerr << prog <<": "<< errmes <<"Exiting....."<< endl;
  exit(1);
}
//============================================================
#define OPTARGS  "?hj:d:"
void get_the_options(int argc,char **argv){

  int c,errflg = 0;
  while (!errflg &&
         (c = getopt(argc, argv, OPTARGS)) != -1){
    switch(c){
    case 'j':
      JACK   = atoi(optarg);
      break;
    case 'd':
      maxdat = atoi(optarg);
      break;
    case 'h':
      errflg++;/*call usage*/
      break;
    default:
      errflg++;
    }/*switch*/
    if(errflg) usage(argv);
  }/*while...*/
}//get_the_options()
//============================================================
void usage(char **argv){
  cerr << "\
```

```
Usage: " << prog << "  [options]                        \n\
       -j : No. jack groups                             \n\
       -d : Give the maximum number of data points read\n\
Computes <o>, chi= (<o^2>-<o>^2)                        \n\
Data is in one column from stdin.\n" << endl;
  exit(1);
}/*usage()*/
```

For the compilation we use the command

```
> g++ -O2 jack.cpp -o jack
```

If we assume that our data is in one column in the file `data`, the command that calculates the jackknife errors using 50 bins is:

```
> cat data | jack -j 50
```

The default of the program is that `maxdat`=1,000,000 measurements. If we need to analyze more data, we have to use the switch `-d`. For example, for 3,000,000 measurements, use `-d 3000000`. The program reads data from the `stdin` and we can construct filters in order to do complicated analysis tasks. For example, the analysis of the magnetization produced by the output of the Ising model program can be done with the command:

```
> ./is -L 20 -b 0.4407 -s 1 -S 342 -n 3000000 | grep -v #   | \
  awk  -v L=20 '{print ($2>0)?($2/(L*L)):(-$2/(L*L))}'      | \
  ./jack -j 50 -d 3000000                    | grep -v #   | \
  awk  -v b=0.4407 -v L=20 '{print $1,$2,b*L*L*$3,b*L*L*$4}'
```

The command shown above can be written in one line by removing the backslashes ('\') at the end of each line. Let's explain it in detail: The first line runs the program `is` for the Ising model with $N = L\times L = 20\times 20$ lattice sites (`-L 20`) and $\beta = 0.4407$ (-b 0.4407). It starts the simulation from a hot configuration (`-s 1`) and makes 3,000,000 measurements (`-n 3000000`). The command `grep -v` filters out the comments from the output of the program, which are lines starting with a #. The second line calls `awk` and defines the `awk` variable `L` to be equal to 20 (`-v L=20`). For each line in its input, it prints the absolute value of the second column (\$2) divided by the number of lattice sites `L*L`. The third line makes the

jackknife calculation of the average values of $\langle m \rangle$ and $\langle (m - \langle m \rangle)^2 \rangle$ with their errors using the program `jack`. The comments of the output of the command `jack` are removed with the command `grep -v`. The fourth line is needed only for the calculation of the magnetic susceptibility, using equation (13.30). There, we need to multiply the fluctuations $\langle (m - \langle m \rangle)^2 \rangle$ and their error by the factor $\beta N = \beta L^2$ in order to obtain χ.

13.8.2 The Bootstrap Method

In this subsection we present a program for the calculation of the errors using the bootstrap method according to the discussion in section 13.6.3. The program is in the file `boot.cpp`:

```
//========================================================
//file: boot.cpp
//--------------------------------------------------------
#include <iostream>
#include <fstream>
#include <iomanip>
#include <cstdlib>
#include <cmath>
#include <random>
#include <unistd.h>
#include <libgen.h>
using namespace std;
int maxdat,SAMPLES;
string prog;
void bootstrap(int,int&,
               double *,double &,
               double &,double &,double &);
void   get_the_options(int,char**);
void   usage (char**);
void   locerr(string);

int    main(int argc,char** argv){
  int    ndat;
  double O,dO,chi,dchi;
  double *x;
  prog.assign((char *)basename(argv[0]));//name of program
  maxdat=2000000;SAMPLES=1000;
  get_the_options(argc,argv);
  x = new double[maxdat];
```

```
  ndat=0;
  while((ndat<=maxdat) && (cin >> x[ndat++]));
  ndat--;
  if(    ndat==maxdat)
    cerr << prog
         << ": Warning: read ndat= "    << ndat
         << " and reached the limit: "  << maxdat  << endl;
  bootstrap(ndat,SAMPLES,x,O,dO,chi,dchi);
  cout << "# NDAT = "                   << ndat
       << " data. SAMPLES = "           << SAMPLES << endl;
  cout << "# <o>, chi= (<o^2>-<o>^2)\n";
  cout << "# <o> +/- err                        chi +/- err\n";
  cout.precision(17);
  cout << O <<" "<< dO <<" "<< chi <<" "<< dchi    << endl;
}//main()
//==========================================================
#include "MIXMAX/mixmax.hpp"
int  seedby_urandom();
void bootstrap(int    ndat,int   & samples,
               double* x   ,double& avO    ,
               double& erO,double& avchi,double& erchi){
  int    i,j,k;
  double *O,*O2,*chi;
  mixmax_engine mxmx(0,0,0,1);
  uniform_real_distribution<double> drandom;
  O    = new double[samples]();//() initializes to 0
  O2   = new double[samples]();
  chi  = new double[samples]();
  mxmx.seed(seedby_urandom());
  for(j=0;j<samples;j++){
    for(i=0;i<ndat;i++){
      k = int(ndat*drandom(mxmx));
      O [j] += x[k];
      O2[j] += x[k]*x[k];
    }
    O  [j]  /=ndat;
    O2 [j]  /=ndat;
    chi[j]   = O2[j] - O[j]*O[j];
  }
  //------------------------------------------
  //Compute averages:
  avO = 0.0; avchi = 0.0;
  for(j=0;j<samples;j++){
    avO += O[j]; avchi += chi[j];
  }
```

```
  av0 /= samples; avchi /= samples;
  //-----------------------------------------
  //Compute errors:
  er0 = 0.0; erchi = 0.0;
  for(j=0;j<samples;j++){
    er0   += (O   [j]-av0   )*(O   [j]-av0   );
    erchi += (chi[j]-avchi)*(chi[j]-avchi);
  }
  er0 /= samples   ; erchi /= samples;
  er0  = sqrt(er0) ; erchi  = sqrt(erchi);
  //Compute the real av0:
  av0 = 0.0;
  for(i=0;i<ndat;i++) av0 += x[i];
  av0 /= ndat;
  delete [] O;
  delete [] O2;
  delete [] chi;
}//bootstrap()
//=============================================
// Use /dev/urandom for more randomness.
#include <fcntl.h>
int seedby_urandom(){
  int ur,fd,ir;
  fd = open("/dev/urandom", O_RDONLY);
  ir = read (fd,&ur,sizeof(int));
  close(fd);
  return (ur>0)?ur:-ur;
}
//======================================================
void locerr(string errmes){
  cerr << prog <<": "<< errmes <<"Exiting....."<< endl;
  exit(1);
}
//======================================================
#define OPTARGS  "?hs:d:"
void get_the_options(int argc,char **argv){

  int c,errflg = 0;
  while (!errflg &&
         (c = getopt(argc, argv, OPTARGS)) != -1){
    switch(c){
    case 's':
      SAMPLES = atoi(optarg);
      break;
    case 'd':
```

```
      maxdat  = atoi(optarg);
      break;
    case 'h':
      errflg++;/*call usage*/
      break;
    default:
      errflg++;
    }/*switch*/
    if(errflg) usage(argv);
  }/*while...*/
}//get_the_options()
//========================================================
void usage(char **argv){
  cerr << "\
Usage: " << prog << "  [options] <file>                    \n\
       -s  : No. samples                                   \n\
       -d  : Give the maximum number of data points read.\n\
Computes <o>, chi= (<o^2>-<o>^2)                           \n\
Data is in one column from stdin." << endl;
  exit(1);
}/*usage()*/
```

For the compilation we use the command

```
> g++ -O2 -std=c++11 MIXMAX/mixmax.cpp boot.cpp -o boot
```

If our data is in one column in the file `data`, then the command that calculates the errors using 500 samples is:

```
> cat data | boot -s 500
```

The maximum number of measurements is set to 2,000,000 as in the `jack` program. For more measurements we should use the `-d` switch, e.g. for 3,000,000 measurements use `-d 3000000`. For the analysis of the magnetization from the output of the program is we can use the following command:

```
>  is -L 20 -b 0.4407 -s 1 -S 342 -n 3000000 | grep -v #  | \
  awk  -v L=20 '{print ($2>0)?($2/(L*L)):(-$2/(L*L))}'   | \
  boot -s 1000  -d 3000000                        | grep -v #  | \
  awk  -v b=0.4407 -v L=20 '{print $1,$2,b*L*L*$3,b*L*L*$4}'
```

13.8.3 Comparing the Methods

In this subsection we will compute errors using equation (13.40), the jackknife method (13.43) and the bootstrap method (13.47). In order to appreciate the differences, we will use data with large autocorrelation times. We use the Metropolis algorithm on the Ising model with $L = 40$, $\beta = 0.4407 \approx \beta_c$ and measure the magnetization per site (13.28). We take $1,000,000$ measurements using the commands:

```
> ./is -L 40 -b 0.4407 -s 1 -S 5434365 -n 1000000 \
                            > outL40b0.4407.dat &
> grep -v # outL40b0.4407.dat | \
  awk -v L=40 '{if($2<0){$2=-$2};print $2/(L*L)}' \
                            > outL40b0.4407.m
> cat outL40b0.4407.m | autoc -t 10000 -n 1000000 \
                            > outL40b0.4407.rhom
```

The file `outL40b0.4407.m` has the measurements of the magnetization in one column and the file `outL40b0.4407.rhom` has the autocorrelation function and the integrated autocorrelation time as described after page 600. We obtain $\tau_m = 286.3(3)$. The integrated autocorrelation time is found to be $\tau_{\text{int,m}} = 254(1)$.

The expectation value is $\langle m \rangle = 0.638682$. The application of equation (13.39), valid for independent measurements, gives the (underestimated) error $\delta_c m = 0.00017$. Using equation (13.40) we obtain $\delta m = \sqrt{1+2\tau}\delta_c m \approx 0.004$. The error of the magnetic susceptibility cannot be calculated this way.

$$\langle m \rangle = 0.639 \pm 0.004 \equiv 0.639(4) \tag{13.52}$$

For the calculation of the error of the magnetic susceptibility we have to resort to the jackknife or to the bootstrap method. The latter is applied initially using a variable number of samples n_S so that the optimal number of samples is be determined. Figure 13.24 shows the results for the magnetization. We observe a very fast convergence to $\delta_c m = 0.00017$ for quite small number of samples. The analysis could have safely used $n_S = 100$. In the case of the magnetic susceptibility, convergence is slower, but we can still use $n_S = 500$. We obtain $\chi = 20.39$ and $\delta_c \chi = 0.0435$. The error assumes independent measurements, something that is not true in our case. We should use the correction factor $\sqrt{1+2\tau_m}$ which gives

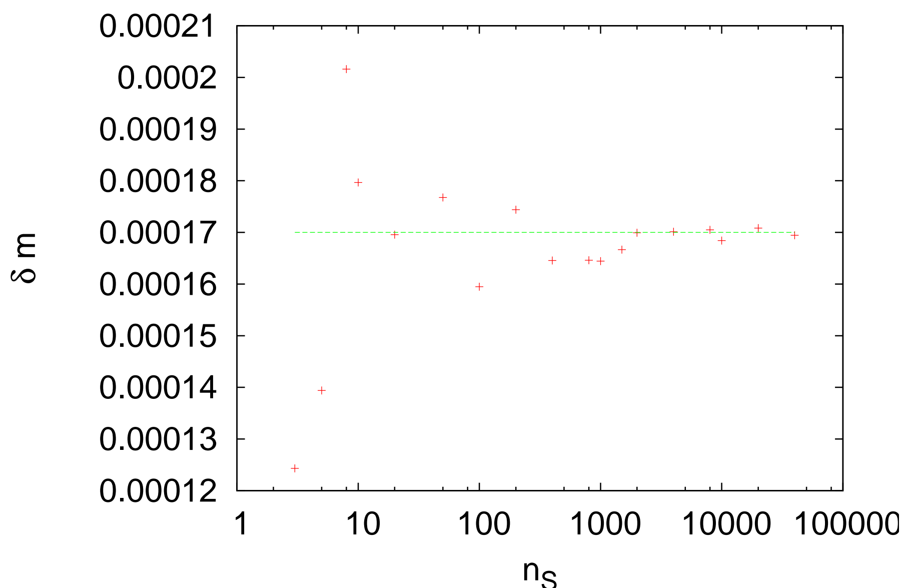


Figure 13.24: The error δm calculated using the bootstrap method as a function of the number of samples n_S. We observe a very fast convergence to the value obtained by equation (13.39) $\delta_c m = 0.00017$.

$\delta\chi = 1$. Therefore

$$\chi = 20 \pm 1 \equiv 20(1) \tag{13.53}$$

We note that the error is quite large, which is because we have few independent measurements: $n/(2\tau_m) \approx 1,000,000/(2 \times 286) \approx 1750$. The a priori knowledge of τ_m is necessary in this calculation. In the case of the jackknife method, the calculation can proceed without an priori knowledge of τ_m. The errors are calculated for a variable number of bins n_b. Figure 13.26 shows the results for the magnetization. When $n_b = n$ the samples consist of all the measurements except one. Then the error is equal to the error calculated using the standard deviation formula and it is underestimated by the factor $\sqrt{1 + 2\tau_m}$. This is shown in figure 13.26, where we observe a slow convergence to the value $\delta_c m = 0.00017$. The effect of the autocorrelations vanishes when we delete (bin width) $\approx 2\tau_m$ measurements from each bin. This happens when $n_b \approx n/(\text{bin width}) = n/(2\tau_m) = 1,000,000/572 \approx 1750$. Of course this an order of magnitude

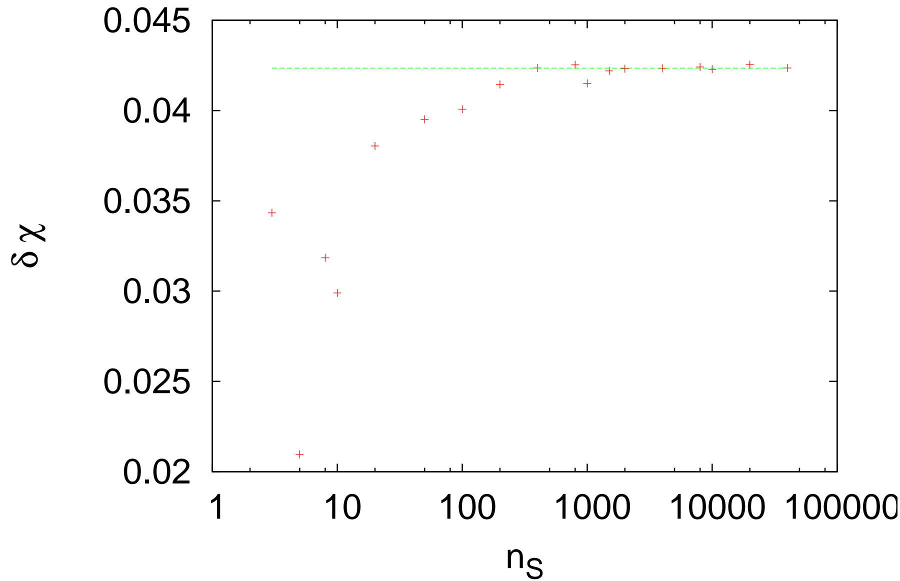


Figure 13.25: The error $\delta\chi$ of the magnetic susceptibility calculated using the bootstrap method as a function of the number of samples n_S. We observe convergence for $n_S > 1000$ to the value $\delta_c\chi = 0.0435$.

estimate and a careful study is necessary in order to determine the correct value for n_b. Figure 13.26 shows that the error converges for $100 < n_b < 800$ to the value $\delta m = 0.0036$, which is quite close to the value $\sqrt{1+2\tau_m}\delta_c m \approx 0.004$. We note that, by using a small number $n_b \approx 20-40$, we obtain an acceptable estimate, a rule of the thumb that can be used for quick calculations.

Similar results are obtained for the magnetic susceptibility χ, where the error converges to the value $\delta\chi = 0.86$, in accordance with the previous estimates. For $n_b \to n$ the error converges to the underestimated error $\delta_c\chi = 0.0421$.

We can use the bootstrap method, in a similar way to the jackknife method, in order to determine the real error δm, $\delta\chi$ without calculating τ_m directly. The data is split into n_b bins, whose bin width is (bin width) $= n/n_b$. Each jackknife bin contains $n - n/n_b$ data elements and we apply the bootstrap method on this data, by taking n_S samples of $n - n/n_b$

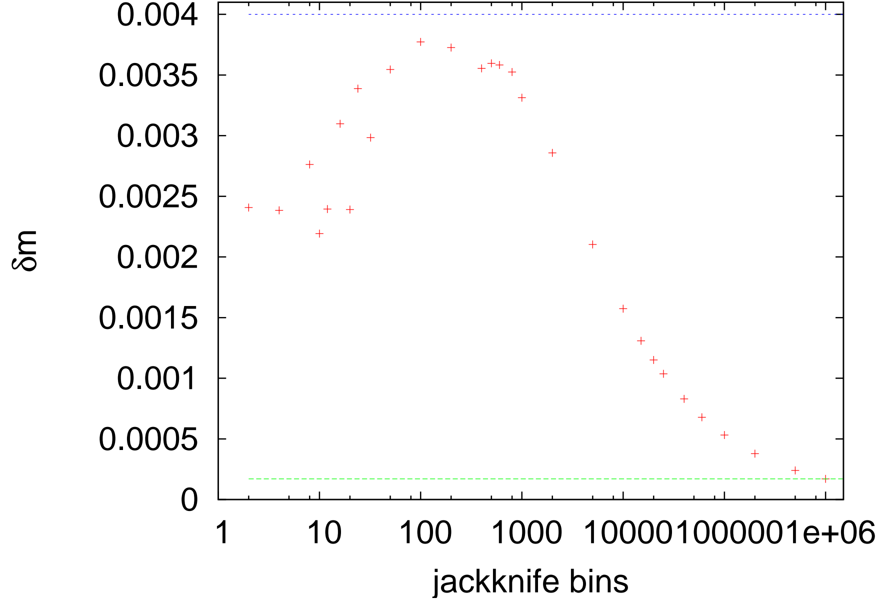


Figure 13.26: The error δm calculated using the jackknife method as a function of the number of bins n_b. Convergence is observed for $100 < n_b < 800$ to $\delta m = 0.0036$. The plot shows that as we approach the limit $n_b = n$, the error approaches the value calculated by equation (13.39) $\delta_c m = 0.00017$. The horizontal lines correspond to the values $\delta_c m$ and $\sqrt{1+2\tau_m}\delta_c m \approx 0.004$ where $\tau_m = 286.3$. The ratio $\delta m/\delta_c m \approx \sqrt{1+2\tau_m}$.

random data. Then each jackknife bin gives a measurement on which we apply equation (13.43) in order to calculate errors.

The above calculations can be reversed and used for the calculation of the autocorrelation time. By computing the underestimated error $\delta_c\mathcal{O}$ and the true error $\delta\mathcal{O}$ using one of the methods described above, we can calculate τ_m using the relation $\delta\mathcal{O}/\delta_c\mathcal{O} = \sqrt{1+2\tau_{\mathcal{O}}}$. Therefore

$$\tau_m = \frac{1}{2}\left(\left(\frac{\delta m}{\delta_c m}\right)^2 - 1\right) = \frac{1}{2}\left(\left(\frac{\delta \chi}{\delta_c \chi}\right)^2 - 1\right) = \ldots \tag{13.54}$$

By calculating τ_m using all the methods described here, these relations can also be used in order to check the analysis for self-consistency and see if they agree. This is not always a trivial work since a system may

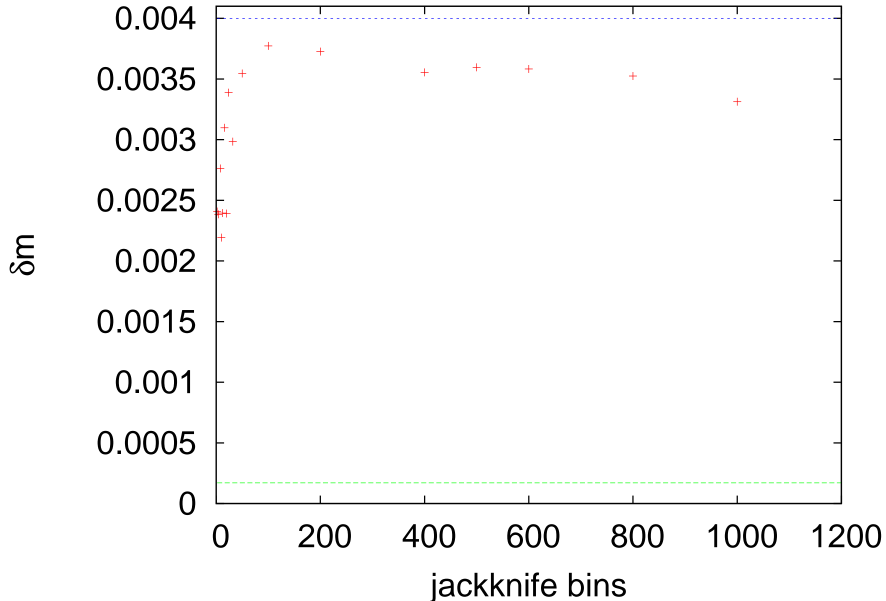


Figure 13.27: Figure 13.26 magnified in the region of the plateau in the values of δm. The horizontal lines correspond to the values $\delta_c m$ and $\sqrt{1+2\tau_m}\delta_c m \approx 0.004$ where $\tau_m = 286.3$. The ratio $\delta m/\delta_c m \approx \sqrt{1+2\tau_m}$.

have many autocorrelation times which influence each observable in a different way (fast modes, slow modes).

13.9 Problems

1. Prove that equation (13.22) satisfies the detailed balance condition.

2. Make the appropriate changes in the Ising model program so that it measures the average acceptance ratio $\bar{A}$ of the Metropolis steps. I.e. compute the ratio of accepted spin flips to the number of attempted spin flips. Compute the dependence of $\bar{A}$ on the temperature and the size of the system. Take $L = 20$ and $\beta = 0.20,\ 0.30,\ 0.40,\ 0.42,\ 0.44,\ 0.46,\ 0.48,\ 0.50$. Then take $\beta = 0.20$, $L = 10,\ 20,\ 40,\ 80,\ 100$. Repeat for the same values of L for $\beta = 0.44$ and $\beta = 0.48$.

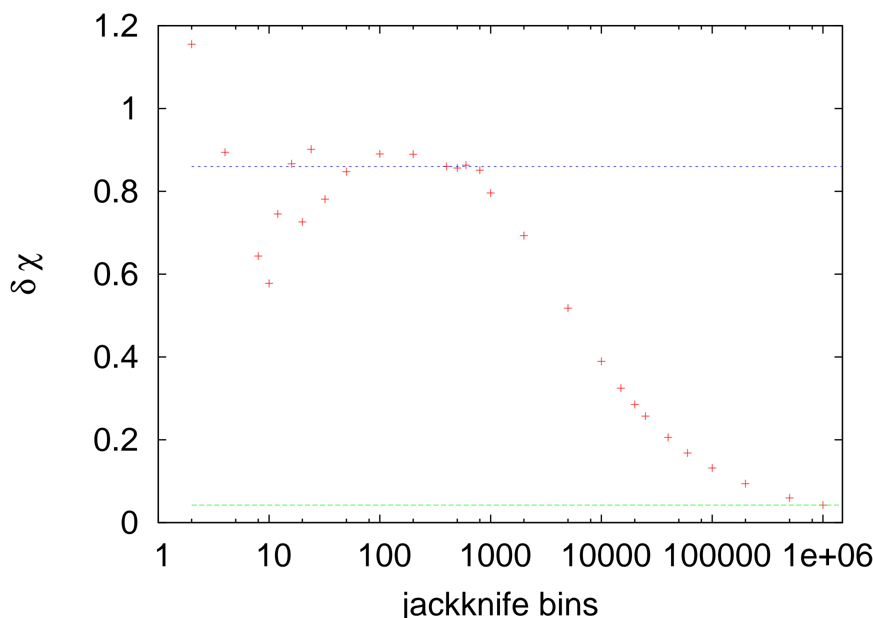


Figure 13.28: The error $\delta\chi$ calculated using the jackknife method as a function of the number of bins n_b. Convergence is observed for $100 < n_b < 800$ to $\delta\chi = 0.86$. The plot shows that as we approach the limit $n_b = n$, the error approaches the same value $\delta_c\chi = 0.0421$ that would have been obtained if we had falsely considered the measurements to be independent. These values are very close to the ones obtained using the bootstrap method. The values $\delta\chi$ and $\delta_c\chi$ are shown in the plots by the two horizontal lines. The ratio $\delta\chi/\delta_c\chi \approx \sqrt{1 + 2\tau_m}$.

3. Reproduce the plots in figure 13.12 and compute τ_m. Repeat for τ_e. Compare your results with $\tau_{\text{int,m}}$ and $\tau_{\text{int,e}}$.

4. Reproduce the plots in figure 13.15 and repeat your calculation for the energy.

5. Reproduce the plots in figure 13.17. Repeat your calculation for the energy. Then, construct similar plots for $\tau_{\text{int,m}}$ and $\tau_{\text{int,e}}$ as a function of t_{max} (see figure 13.14).

6. Reproduce the plots in figures 13.19 and 13.20. Repeat your calculation for the energy. Then, construct similar plots for $\tau_{\text{int,m}}$ and $\tau_{\text{int,e}}$ as a function of t_{max} (see figure 13.14).

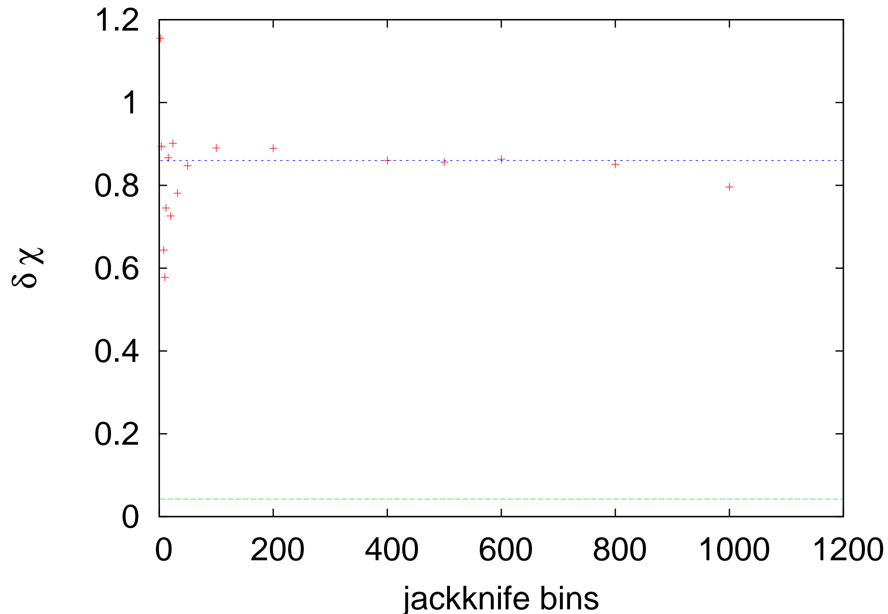


Figure 13.29: Figure 13.28 magnified in the region of the plateau in the values of $\delta\chi$. Convergence is observed for $100 < n_b < 800$ to $\delta\chi = 0.86$. The values $\delta\chi$ and $\delta_c\chi$ are shown in the plots by the two horizontal lines. The ratio $\delta\chi/\delta_c\chi \approx \sqrt{1+2\tau_m}$.

7. Modify the Ising model program presented in the text so that it can simulate the Ising model in the presence of an external magnetic field B (see equation (13.2)). Calculate the magnetization per site $m(\beta, B)$ for $L = 32$ and $B = 0.2, 0.4, 0.6, 0.8, 1.0$ at an interesting range of temperatures. Use different initial configuration in order to study the thermalization of the system as B increases: Cold state with spins parallel to B, cold state with spins antiparallel to B and hot state. Study the dependence of the critical temperature separating the ordered from the disordered state on the value of B.

8. Hysteresis: In the previous problem, the Ising model with $B \neq 0$ has a first order phase transition, i.e. a discontinuity in the value of the order parameter which in our case is the magnetization as a function of B. Near a first order transition we observe the phenomenon of hysteresis. In order to see it, set $L = 32$ and $\beta = 0.55$ and

(a) thermalize the system for $B = 0$

(b) simulate the system for $B = 0.2$ using as an initial state the last one coming from the previous step. Do 100 sweeps and calculate $\langle m \rangle$.

(c) continue by increasing each time the magnetic field by $\delta B = 0.2$. Stop when $\langle m \rangle \approx 0.95$.

(d) using the last configuration from the previous step, repeat by decreasing the magnetic field by $\delta B = -0.2$ until $\langle m \rangle \approx -0.95$.

(e) using the last configuration from the previous step, repeat by increasing the magnetic field by $\delta B = 0.2$ until $\langle m \rangle \approx 0.95$.

Make the plot (B, m). What do you observe?

For systems near a first order phase transition, the order parameter can take two different values with almost equal probability. This means that the free energy has two local minima. Only one of them is the true, global minimum. This is depicted in figure 12.2 where two equally probable values of the order parameter are shown. This happens exactly at the critical point. When we move away from the critical point, one of the peaks grows and it is favored corresponding to the global minimum of the free energy. The local minimum is called a *metastable state* and when the system is in such a state, it takes a long time until a thermal fluctuation makes it overcome the free energy barrier and find the global minimum. In a Monte Carlo simulation such a case presents a great difficulty in sampling states correctly near the two local minima. Repeat the above simulations, this time making $100,000$ sweeps per point. Plot the time series of the magnetization and observe the transitions from the metastable state to the stable one and backwards. Compute the histogram of the values of the magnetization and determine which state is the metastable in each case. How is the histogram changing as B is increased?

9. Write a program that simulates the 2 dimensional Ising model on a *triangular* lattice using the Metropolis algorithm. The main difference is that the number of nearest neighbors is $z = 6$ instead of $z = 4$. Look into chapter 13.1.2 of Newman and Barkema (esp. figure 13.4). Compute the change in energy for each spin flip for the

Metropolis step. Calculate the maxima of the magnetic susceptibility and of the specific heat and see if they are close to the expected critical temperature $\beta_c \approx 0.274653072$. Note that even though β_c is different than the corresponding value on the square lattice, the critical exponents are the same due to universality.

10. Write a program that simulates the *three* dimensional Ising model on a cubic lattice using the Metropolis algorithm. Use helical boundary conditions (all you need in this case is to add a parameter `ZNN=L*L` together with the `XNN=1` and `YNN=L`).

11. Write a program that simulates the three dimensional Ising model on a cubic lattice using the Metropolis algorithm. Use *periodic* boundary conditions.

12. Simulate the *antiferromagnetic* two dimensional Ising model on a square lattice using the Metropolis algorithm. You may use the same code that you have and enter negative temperatures. Find the ground state(s) of the system.

    Define the *staggered magnetization* m_s to be the magnetization per site of the sublattice consisting of sites with odd x *and* y coordinate. Set $L = 32$ and compute the energy, the m_s, the specific heat, the magnetic susceptibility χ and the *staggered* magnetic susceptibility $\chi_s = \beta N/4\langle(m_s - \langle m_s\rangle)^2\rangle$.

    χ has a maximum in the region $\beta \approx 0.4407$. Compute its value at this temperature for $L = 32 - 120$. Show that χ does not diverge as $L \to \infty$, therefore χ does not show a phase transition.

    Repeat the calculation for χ_s. What do you conclude? Compare the behavior of $\langle m_s\rangle$ for the antiferromagnetic Ising model with $\langle m\rangle$ of the ferromagnetic.

0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
5	6	7	8	9	5	6	7	8	9	5	6	7	8	9
10	11	12	13	14	10	11	12	13	14	10	11	12	13	14
15	16	17	18	19	15	16	17	18	19	15	16	17	18	19
20	21	22	23	24	20	21	22	23	24	20	21	22	23	24
0	1	2	3	4	**0**	**1**	**2**	**3**	**4**	0	1	2	3	4
5	6	7	8	9	**5**	**6**	**7**	**8**	**9**	5	6	7	8	9
10	11	12	13	14	**10**	**11**	**12**	**13**	**14**	10	11	12	13	14
15	16	17	18	19	**15**	**16**	**17**	**18**	**19**	15	16	17	18	19
20	21	22	23	24	**20**	**21**	**22**	**23**	**24**	20	21	22	23	24
0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
5	6	7	8	9	5	6	7	8	9	5	6	7	8	9
10	11	12	13	14	10	11	12	13	14	10	11	12	13	14
15	16	17	18	19	15	16	17	18	19	15	16	17	18	19
20	21	22	23	24	20	21	22	23	24	20	21	22	23	24

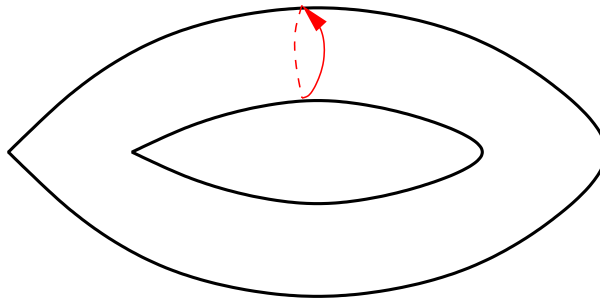
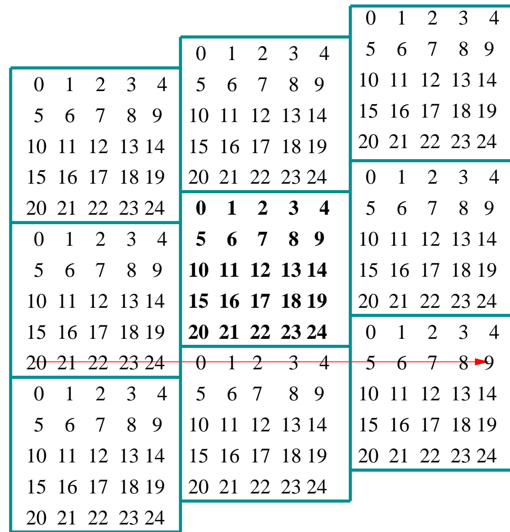


Figure 13.30: Horizontal motion on the $L = 5$ square lattice with periodic boundary conditions. The trajectory is a circle.



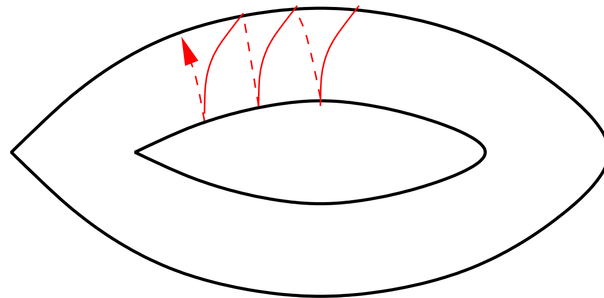


Figure 13.31: Horizontal motion on the $L = 5$ square lattice with helical boundary conditions. The trajectory is a spiral.

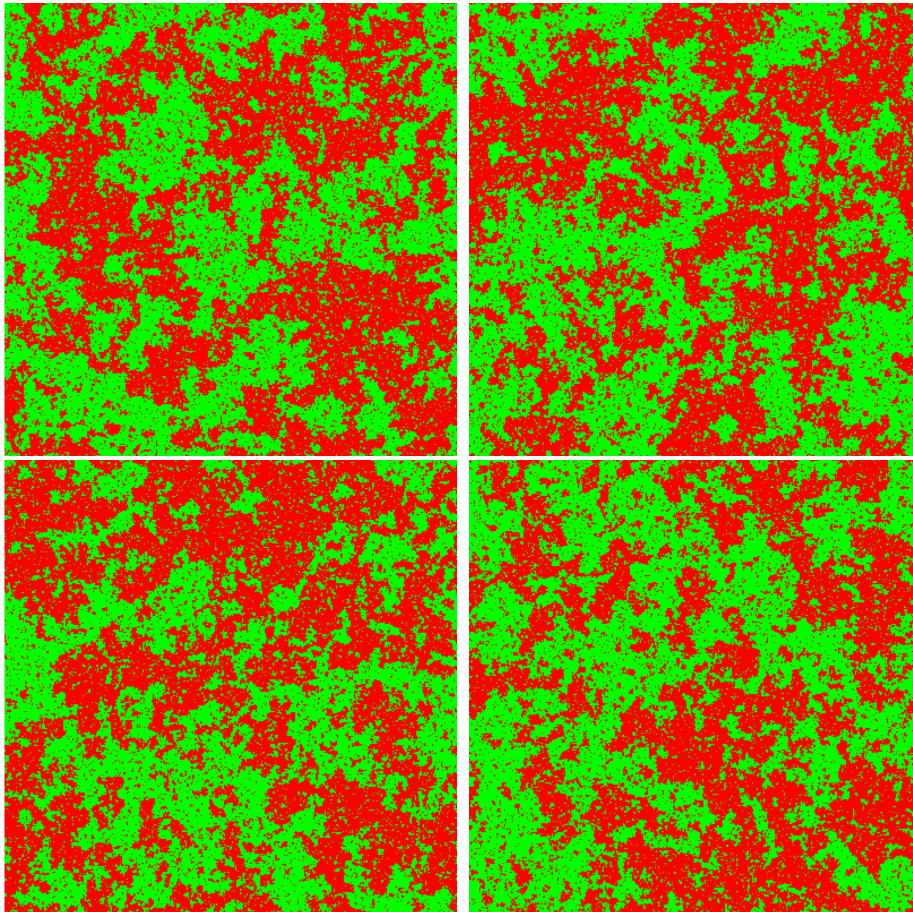


Figure 13.32: Spin configurations for the Ising model with $L = 400$, $\beta = 0.4292$ after 4000, 9000, 12000 and 45000 sweeps respectively. We observe the formation of large clusters of same spin. This makes hard to form a new independent configuration with the Metropolis algorithm and results in large autocorrelation times.

Chapter 14

Critical Exponents

In the previous chapters, we saw that when a system undergoes a continuous phase transition as $\beta \to \beta_c$, or equivalently as the reduced temperature[1]

$$t \equiv \frac{\beta_c - \beta}{\beta_c} \to 0\,, \tag{14.1}$$

the correlation length $\xi \equiv \xi(\beta, L = \infty)$, calculated in *the thermodynamic limit* diverges according to the relation

$$\xi \sim |t|^{-\nu} \qquad (\nu = 1 \text{ for 2d-Ising})\,. \tag{14.2}$$

The behavior of such systems near the phase transition is characterized by *critical exponents*, such as the exponent ν, which are the same for all systems in the same *universality class*. The critical exponents describe the leading non analytic behavior of the observables in the thermodynamic limit[2] $L \to \infty$, when $t \to 0$. Systems with the same long distance behavior, but which could possibly differ microscopically, belong to the same *universality class*. For example, if we add a next to nearest neighbor interaction in the Hamiltonian of the Ising model or if we consider the system on a triangular instead of a square lattice, the system will still belong to the same universality class. As $\xi \to \infty$ these details become irrelevant and all these systems have the same long distance behavior. Microscopic degrees of freedom of systems in the same universality class can be quite

[1]You may also find the definition $t = (T - T_c)/T_c$ but as $t \ll 1$ the two definitions are almost equivalent (they differ by a term of order $\sim t^2$).

[2]Beware: We *first* take $L \to \infty$ and *then* $t \to 0$.

different, as is the case of the liquid/vapor phase transition at the triple point and the Ising model.

The critical exponents of the 2d Ising model universality class are the Onsager exponents:

$$\chi \sim |t|^{-\gamma}\,, \qquad \gamma = 7/4\,, \tag{14.3}$$

$$c \sim |t|^{-\alpha}\,, \qquad \alpha = 0 \qquad \text{and} \tag{14.4}$$

$$\langle m \rangle \sim |t|^{\beta} \qquad t < 0\,, \qquad \beta = 1/8\,. \tag{14.5}$$

This behavior is seen only in the thermodynamic limit $L \to \infty$. For a finite lattice, all observables are analytic since they are calculated from the analytic[3] partition function $Z(\beta)$ given by equation (13.4). When $1 \ll \xi \ll L$ the model behaves approximately as the infinite system. As $\beta \approx \beta_c$ and $\xi \sim L$ *finite size effects* dominate. Then the fluctuations, e.g. χ and c, on the finite lattice have a maximum for a *pseudocritical temperature* $\beta_c(L)$ for which we have that[4]

$$\lim_{L\to\infty} \beta_c(L) = \beta_c\,. \tag{14.6}$$

For the Ising model on the square lattice, defined by (13.14), we have that $\beta_c = \log(1+\sqrt{2})/2$.

Because of (14.2), when on the finite lattice we take $\beta = \beta_c(L)$, we have that $\xi(t, L) \sim L \Rightarrow |t| = |(\beta_c - \beta_c(L))/\beta_c| \sim L^{-1/\nu}$, therefore equations (14.3)–(14.5) become

$$\chi \sim L^{\gamma/\nu}\,, \tag{14.7}$$

$$c \sim L^{\alpha/\nu}\,, \tag{14.8}$$

$$m \sim L^{-\beta/\nu}\,. \tag{14.9}$$

The left hand sides of the above relations are normally evaluated at $\beta = \beta_c(L)$, but they can also be evaluated at any temperature in the

[3]It is a finite sum of analytic functions, therefore an analytic function.

[4]Each observable may have a slightly different pseudocritical temperature, so we may write $\beta_c^{\chi}(L)$, $\beta_c^{c}(L)$ etc.

pseudocritical region. Most of the times, one calculates the observables for $\beta = \beta_c(L)$, but one can also use e.g. $\beta = \beta_c$[5]. In the next sections we will show how to calculate the critical exponents by using the scaling relations (14.3)–(14.5) and (14.7)–(14.9).

14.1 Critical Slowing Down

The computation of critical exponents is quite involved and requires accurate measurements, as well as simulations of large systems in order to reduce finite size effects. The Metropolis algorithm suffers from severe *critical slowing down*, i.e. diverging autocorrelation times with large dynamic exponent z according to (13.36), near the critical region, which makes it impossible to study large systems. In this section we will discuss the cause of this effect whose understanding will lead us to new algorithms that beat critical slowing down. These are the *cluster algorithms* and, in particular, the *Wolff algorithm*. The success of these algorithms is based on the dynamics of the system and, therefore, they have a more specialized range of applications. In contrast, the Metropolis algorithm can, in principle, be applied on any system studied with the Monte Carlo method.

According to the discussion in section 13.5, the Ising model simulation using the Metropolis algorithm near the critical region exhibits an increase in autocorrelation times given by the scaling relation (13.36)

$$\tau \sim \xi^z \,. \tag{14.10}$$

The correlation length of the finite system becomes $\xi \sim L$ in this region, and we obtain equation (13.36), $\tau \sim L^z$. When $z > 0$ we have the effect of critical slowing down.

Critical slowing down is the main reason that prohibits the simulation of very large systems, at least as far as CPU time t_{CPU} is concerned[6]. The generation of a given number of configuration requires an effort $t_{\text{CPU}} \sim L^d$. But the measurement of a local quantity, like $\langle m \rangle$, for a given

[5]In the limit $L \to \infty$ the difference is not important, it comes from analytic terms which don't contribute to the non analytic behavior. In practice, the speed of convergence to the asymptotic behavior may differ.

[6]Of course the amount of available memory can be another inhibiting factor.

number of times requires no extra cost, since each configuration yields L^d measurements[7]. In this case, measuring for the largest possible L is preferable, since it reduces finite size effects. We see that, in the absence of critical slowing down, the cost of measurement of $\langle m \rangle$ is $t_{\rm CPU}^{\langle m \rangle} \sim L^0$.

Critical slowing down, however, adds to the cost of production of independent configurations and we obtain $t_{\rm CPU}^{\langle m \rangle} \sim L^z$, making the large L simulations prohibitively expensive. For the Metropolis algorithm on the two dimensional Ising model we have that $z \approx 2.17$; and the problem is severe. Therefore, it is important to invent new algorithms that beat critical slowing down. In the case of the Ising model and similar spin systems, the solution is relatively easy. It is special to the specific dynamics of spin systems and does not have a universal application.

The reason for the appearance of critical slowing down is the divergence of the correlation length ξ. As we approach the critical temperature $\beta \to \beta_c$ from the disordered phase, the typical configurations are dominated by large clusters of same spins. The Metropolis algorithm makes at most one spin flip per step and the acceptance ratios for spins inside a cluster are small. For example, a spin with four same neighboring spins can flip with probability $e^{-8\beta_c} \approx 0.029$, which is quite small. The spins that change more often are the ones with more neighbors having opposite spins, therefore the largest activity is observed at the boundaries of the large clusters. In order to obtain a statistically independent configuration, we need to destroy and create many clusters, something that happens very slowly using the Metropolis algorithm who realizes this process mostly by moving the boundaries of the clusters.

14.2 Wolff Cluster Algorithm

Beating critical slowing down requires new algorithms so that at each step a spin configuration is changed at the scale of a spin cluster[8]. The cluster algorithms construct such regions of same spins in a way that the proposed new configuration has all the spins of the clusters flipped. For such an algorithm to be successful, the acceptance ratios should be large.

[7] Each site contributes one measurement!

[8] A spin cluster is a subset of the lattice composed of connected lattice sites of same spins.

The most famous ones are the Swendsen-Wang [66] and the Wolff [67] cluster algorithms.

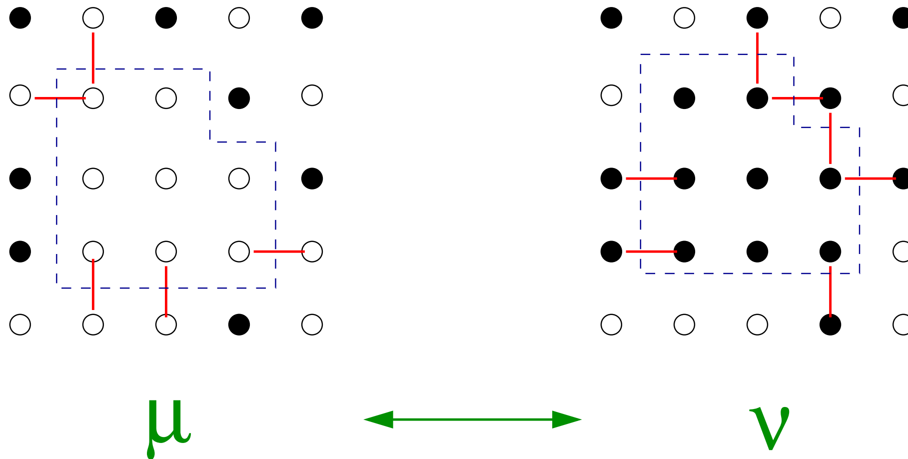


Figure 14.1: Two spin configurations that differ by the flip of a *Wolff cluster*. The bonds that are destroyed/created during the transition belong to the boundary of the cluster.

The process of constructing the clusters is stochastic and depends on the temperature. Small clusters should be favored for $\beta \ll \beta_c$, whereas large clusters of size $\sim L$ should dominate for $\beta \gg \beta_c$.

The basic idea of the Wolff algorithm is to choose a site randomly, a so called *seed* of the cluster, and construct a spin cluster around it. At each step, we add new members to the cluster with probability $P_{\text{add}} = P_{\text{add}}(\beta)$. If $P_{\text{add}}(\beta)$ is properly chosen, the detailed balance condition (12.59) is satisfied and the new configuration is *always* accepted. This process is depicted in figure 14.1. In the state μ, the cluster is enclosed by the dashed line. The new state ν is obtained by flipping all the spins in the cluster, leaving the rest of the spins to be the same.

The correct choice of P_{add} will yield equation (12.60)

$$\frac{P(\mu \to \nu)}{P(\nu \to \mu)} = \mathrm{e}^{-\beta(E_\nu - E_\mu)} \,. \tag{14.11}$$

The discussion that follows proves (14.11) and can be found in the book by Newman and Barkema [4]. The crucial observation is that the change in energy in the exponent of the right hand side of (14.11) is due to

the creation/destruction of bonds on the *boundary* of the cluster. The structure of the bonds in the interior of the cluster is identical in the two configurations μ and ν. This can be seen in the simple example of figure 14.1. By properly choosing the selection probability $g(\mu \to \nu)$ of the new state ν and the acceptance ratio $A(\mu \to \nu)$, so that

$$P(\mu \to \nu) = g(\mu \to \nu)\, A(\mu \to \nu)\,, \tag{14.12}$$

we will succeed in satisfying (14.11) and maximize the acceptance ratio. In fact in our case we will find that $A(\mu \to \nu) = 1$!

The selection probability $g(\mu \to \nu)$ is the probability of constructing a particular cluster and can be split in three factors:

$$g(\mu \to \nu) = p_{\text{seed}} \times p_{\text{yes}}^{\text{int}} \times p_{\text{no}}^{\text{border}}\,. \tag{14.13}$$

The first term is the probability to start the cluster from the particular seed. By choosing a lattice site with equal probability we obtain

$$p_{\text{seed}} = \frac{1}{N}\,. \tag{14.14}$$

Then the cluster starts growing around its seed. The second term $p_{\text{yes}}^{\text{int}}$ is the probability to include all cluster members found in the *interior* of the cluster. This probability is complicated and depends on the size and shape of the cluster. Fortunately, it is not important to calculate it. The reason is that in the opposite transition $\nu \to \mu$, the corresponding term is exactly the same since the two clusters are exactly the same (the only differ by the value of the spin)!

$$p_{\text{yes}}^{\text{int}}(\mu \to \nu) = p_{\text{yes}}^{\text{int}}(\nu \to \mu) \equiv C_{\mu\nu}\,. \tag{14.15}$$

The third term is the most interesting one. The cluster stops growing when we are on the boundary and say “no” to including all nearest neighbors with same spins, which are not already in the cluster (obviously, the opposite spins are not included). If P_{add} is the probability to include a nearest neighbor of same spin to the cluster, the probability of saying “no” is $1 - P_{\text{add}}$. Assume that we have m “bonds”[9] of same spins on the boundary of the cluster in the state μ, and that we have n such

[9] A link with same spins on both sides.

bonds in the state ν. In figure 14.1, for example, we have that $m = 5$ and $n = 7$. Therefore, the probability to stop the cluster in the state μ is to say "no" m times, which happens with probability $(1 - P_{\text{add}})^m$:

$$p_{\text{no}}^{\text{border}}(\mu \to \nu) = (1 - P_{\text{add}})^m \,. \tag{14.16}$$

Similarly, the cluster in the state ν stops at the same boundary with probability

$$p_{\text{no}}^{\text{border}}(\nu \to \mu) = (1 - P_{\text{add}})^n \,. \tag{14.17}$$

Therefore

$$\frac{P(\mu \to \nu)}{P(\nu \to \mu)} = \frac{\frac{1}{N}\, C_{\mu\nu}\, (1 - P_{\text{add}})^m\, A(\mu \to \nu)}{\frac{1}{N}\, C_{\mu\nu}\, (1 - P_{\text{add}})^n\, A(\nu \to \mu)} = \mathrm{e}^{-\beta(E_\nu - E_\mu)} \,. \tag{14.18}$$

The right hand side of the above equation depends only on the number of bonds on the boundary of the cluster. The energy difference depends only on the creation/destruction of bonds on the boundary of the cluster and the internal bonds don't make any contribution to it. Each bond created during the transition $\mu \to \nu$ decreases the energy by 2 and each bond destroyed increases the energy by 2:

$$E_\nu - E_\mu = (-2n) - (-2m) = 2(m - n) \,, \tag{14.19}$$

which yields

$$(1 - P_{\text{add}})^{m-n} \frac{A(\mu \to \nu)}{A(\nu \to \mu)} = \mathrm{e}^{-2\beta(m-n)} \Rightarrow \frac{A(\mu \to \nu)}{A(\nu \to \mu)} = \left[\mathrm{e}^{2\beta}(1 - P_{\text{add}})\right]^{n-m} \,. \tag{14.20}$$

From the above relation we see that if we choose

$$1 - P_{\text{add}} = \mathrm{e}^{-2\beta} \Rightarrow P_{\text{add}} = 1 - \mathrm{e}^{-2\beta} \,, \tag{14.21}$$

then we can also choose

$$A(\mu \to \nu) = A(\nu \to \mu) = 1\,! \tag{14.22}$$

Therefore, we can make the condition (14.11) to hold by constructing a cluster using the P_{add} given by (14.21), flipping its spins, and *always* accepting the resulting configuration as the new state.

Summarizing, the algorithm for the construction of a Wolff cluster consists of the following steps:

1. Choose a seed by picking a lattice site with probability $p_{\text{seed}} = \frac{1}{N}$. This is the first new member of the cluster

2. Repeat: For each new member of the cluster, visit its nearest neighbors that do not already belong to the cluster. If they have the same spin, add them to the "new members" of the cluster with probability P_{add}. The original spin is not a "new member" anymore

3. When there are no more "new members", the construction of the cluster ends

4. Flip the spin of all the members of the cluster.

The algorithm is ergodic, since every state can be obtained from any other state by constructing a series of clusters of size 1 (equivalent to single flips).

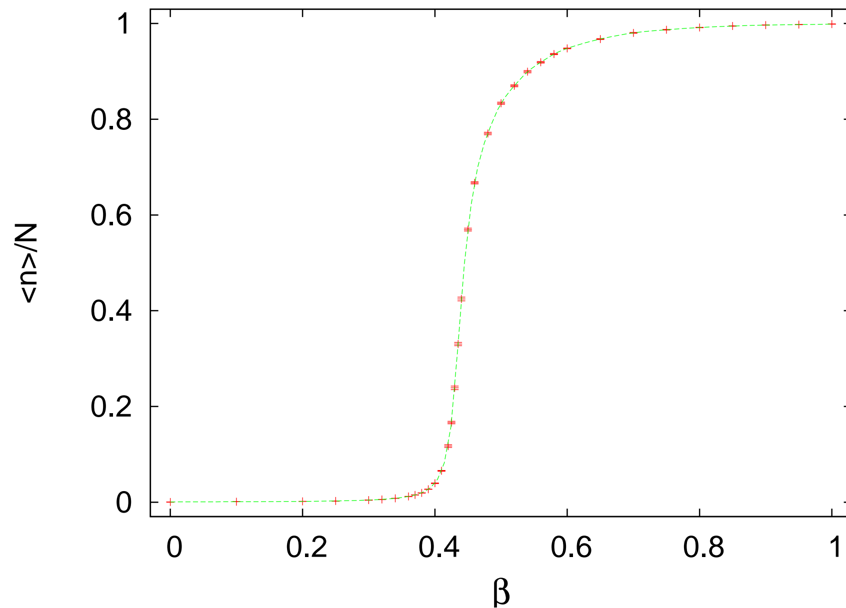


Figure 14.2: The Wolf cluster size as a function of the temperature. The plot shows the average cluster size as a fraction of the lattice size N. In the high temperature regime, $\beta \ll \beta_c$, this is $\sim 1/N$, and in the low temperature regime, $\beta \gg \beta_c$, it becomes ~ 1. The data is for the Ising model on the square lattice for $L = 40$.

The probability P_{add} depends on the temperature β. It is quite small for $\beta \ll \beta_c$ and almost 1 for $\beta \gg \beta_c$. Therefore, in the first case the algorithm favors very small clusters (they are of size 1 for $\beta = 0$) and in the second case it favors large clusters. In the high temperature regime, we have almost random spin flips, like in the Metropolis algorithm. In the low temperature regime, we have large probability of flipping the dominant cluster of the lattice. This is clearly seen in figure 14.2, where the fraction of the average cluster size to the lattice size $\langle n\rangle/N$ is plotted as a function of the temperature. For small β, $\langle n\rangle/N \to 1/N$ whereas for large β, $\langle n\rangle/N \to 1$.

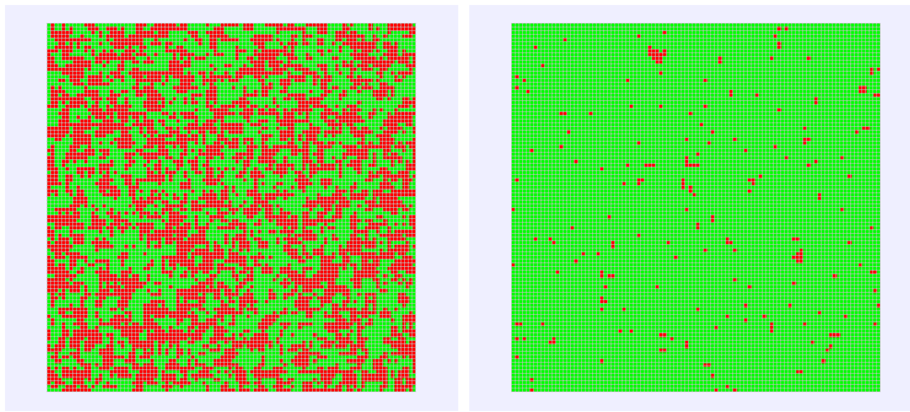


Figure 14.3: A typical spin configuration in the disordered phase (left, $\beta = 0.25$) and in the ordered phase (right, $\beta = 0.5556$) for the Ising model on the square lattice for $L = 100$.

Figure 14.3 shows typical spin configurations in the high and low temperature regimes. For small β, most of the time the algorithm chooses a lattice site randomly and constructs a small cluster around it and flips its spins. The Metropolis algorithm picks a lattice site randomly and flips it most of the times. In both cases, the two algorithms function almost the same way and construct the high temperature disordered spin configurations. For large β, a typical spin configuration is a “frozen” one: A large cluster of same spins with a few isolated thermal fluctuations of different spins. Most of the times, the Wolff algorithm picks a seed in the dominant cluster and the new cluster is almost the same as the dominant cluster: Most of its sites are included with few ones ex-

cluded, which upon flipping of the spins, they will form the new thermal fluctuations. After the flips, the old thermal fluctuations have the same spin as the dominant cluster and they become part of the new dominant cluster. The Metropolis algorithm picks lattice sites randomly: When they belong to the dominant cluster they are seldomly flipped, whereas the thermal fluctuations are flipped most of the time. Both algorithms function similarly and have the same efficiency.

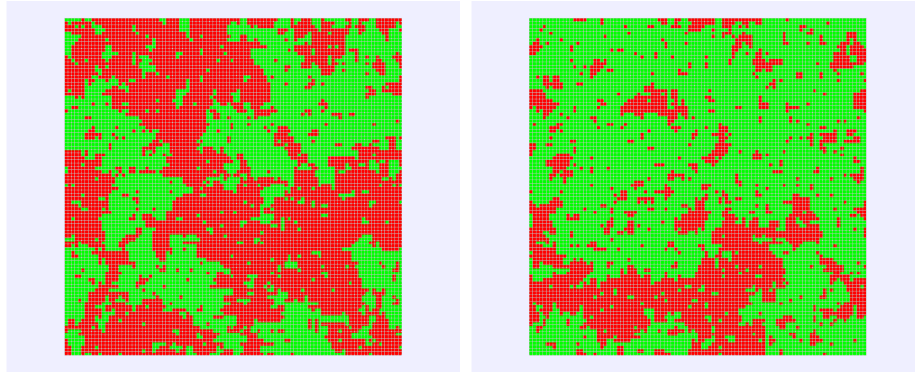


Figure 14.4: Two typical spin configurations in the (pseudo)critical region ($\beta = 0.4348$) for the Ising model on the square lattice for $L = 100$. The two configurations differ by 5000 Metropolis steps.

Figure 14.4 shows typical spin configurations in the critical region. These are dominated by large clusters whose size, shape and position are random. The Wolff algorithm constructs large clusters easily, therefore, large clusters are easily created and destroyed in a few steps (figure 14.2 shows that $\langle n\rangle/N \approx 0.5$). In contrast, the Metropolis algorithm modifies clusters by slowly moving their boundaries and large clusters are destroyed/created very slowly. Autocorrelation times are expected to reduce drastically when using the Wolff algorithm in the critical region.

The expectation value of the size of the Wolff clusters is a dynamical quantity. In order to see this, we will show that in the disordered phase ($\beta < \beta_c$) we have that

$$\chi = \beta\langle n\rangle\,. \tag{14.23}$$

We take the discussion from Newmann and Barkema [4]: Create a bond on each link of the lattice connecting two same spins with proba-

bility $P_{\text{add}} = 1 - \mathrm{e}^{-2\beta}$. In the end, the lattice will be divided in N_c Wolff[10] clusters. Each one will consist of n_i sites, whose spin is S_i. Choose a lattice site randomly and flip the spins of the cluster it belongs to. Destroy the bonds and repeat the process[11]. The total magnetization is:

$$M = \sum_{i=1}^{N_c} S_i n_i \,, \tag{14.24}$$

and

$$\langle M^2 \rangle = \langle \left(\sum_{i=1}^{N_c} S_i n_i \right) \left(\sum_{j=1}^{N_c} S_j n_j \right) \rangle = \langle \sum_{i \neq j} S_i S_j n_i n_j \rangle + \langle \sum_i S_i^2 n_i^2 \rangle \,. \tag{14.25}$$

The values $S_i = \pm 1$ are equally probable due to the symmetry of the model, therefore the first term vanishes. Since $S_i^2 = 1$, we obtain

$$\langle m^2 \rangle = \frac{1}{N^2} \langle M^2 \rangle = \frac{1}{N^2} \langle \sum_i n_i^2 \rangle \,. \tag{14.26}$$

In the Wolff algorithm, the creation of a cluster is equivalent to the choice of one of the clusters we created by following the procedure described above. The probability of selecting the cluster i is

$$p_i = \frac{n_i}{N} \,, \tag{14.27}$$

therefore the average value of the size of the Wolff clusters will be

$$\langle n \rangle = \langle \sum_i p_i n_i \rangle = \langle \sum_i \frac{n_i}{N} n_i \rangle = N \langle m^2 \rangle \,. \tag{14.28}$$

By using equation (14.26) and the fact that for $\beta < \beta_c$ we have that $\langle m \rangle = 0$[12], therefore

$$\chi = \beta N (\langle m^2 \rangle - \langle m \rangle^2) = \beta \langle n \rangle \,. \tag{14.29}$$

[10]These are true Wolff clusters since the bonds have been activated with the correct probability. There are of course isolated sites with no activated bonds around them which are Wolff clusters of size 1.

[11]The result is equivalent to performing one step of the Wolff algorithm, not a very efficient one of course...

[12]This is exactly true only in the thermodynamics limit. For a finite lattice of size N, the two quantities differ by a factor of $\beta N \langle m \rangle^2 > 0$, which vanishes in the large N limit.

14.3 Implementation

In order to create a cluster around a seed, we need a memory buffer for storing the new members of the cluster. We draw cluster sites from this buffer, and examine whether to add their nearest neighbors to the cluster.

There are two data structures that can be used in this job. The first one is the *stack* (or *LIFO*: last in – first out) and the second one is the *queue* (or *FIFO*: first in – first out). They are both one dimensional arrays, the only difference is how we draw data from them. In the case of a stack, we draw the last element that we stored in it. In the case of the queue, we draw the first element that we stored in it.

The stack is implemented as a one dimensional array `stack[N]` in which we "push" a new value that we want to store and we "pop" one that we want to retrieve. We use an integer `m` as a pointer to the last value that we stored in the position `stack[m-1]`. `m` is also the number of active elements in the stack. In order to push a value `e` into the stack we:

1. check if there exist available positions in the stack (i.e. if `m<N`)
2. set `stack[m] = e`
3. increase `m` by 1.

In order to pop a value and store it in the variable `e` we:

1. check if the stack is non empty (i.e. if `m>0`)
2. reduce `m` by 1
3. set `e = stack[m]`

The queue implementation is different. The data topology is cyclic, as shown in figure 14.5. We use an array `queue[N]` and *two* integers `m, n` which point at the beginning and at the end of the buffer. The beginning of the data is the element `queue[m-1]` and the end of the data is the element `queue[n]`. When the queue is empty, we have that `m=n` and the same is true when it is full. Therefore we need a flag that flags whether the queue is empty or full. In the beginning we set `flag=0` (queue is

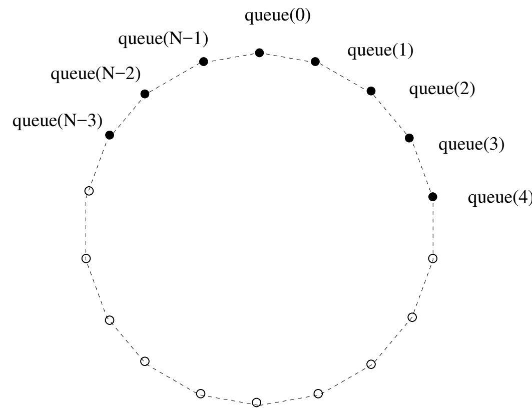


Figure 14.5: Data topology in a queue. In the array depicted here, we have 8 elements stored in `queue(N-3) ... queue(4)`. We have that `m=5, n=N-3, m-n = 8 mod N`. An element is added to the `queue(m)=queue(5)` and an element is popped by calling `queue(n)=queue(N-3)`.

empty). The number `(m-n) mod N` is the number of stored elements[13]. When the queue has data, we set `flag=1`. In order to store a value `e` into the queue we:

1. check whether the queue is full (`m=n` and `flag=1`)
2. set `flag=1`
3. set `queue[m] = e`
4. increase `m` by $1 \mod N$.

In order to pop a value and store it in the variable `e` we:

1. check whether the queue is empty (`m=n` and `flag=0`)
2. set `e = queue[n]`
3. increase `n` by $1 \mod N$
4. if `m=n` set `flag=0`.

[13]Except if `m=n` in which case the number of stored elements is 0 or `N` according to the value of `flag`.

Summarizing, the algorithm for constructing a Wolff cluster for the Ising model is the following:

1. choose a seed by randomly picking a site with probability $1/N$
2. check its nearest neighbors. If they have the same spin, add them to the cluster with probability $P_{\text{add}} = 1 - \mathrm{e}^{-2\beta}$. The new members of the cluster are pushed into the stack `stack[N]` according to the previous discussion
3. pop a site from the stack `stack[N]`. If the stack is empty we stop the construction and move on to the next step. If not, we check the site's nearest neighbors. If they are not already in the cluster and they have the same spin, we add them to the cluster with probability P_{add}
4. record the size of the cluster and flip the spin of its members.

The choice between stack or queue is not important. The results are the same and the performance similar. The only difference is the way that the clusters are constructed (for the stack, the cluster increases around the seed whereas for the queue it increases first in one direction and then in another). The careful programmer will try both during the debugging phase of the development. Bad random number generators can be revealed in such a test, since the Wolff algorithm turns out to be sensitive to their shortcomings.

14.3.1 The Program

The heart of the algorithm is coded in the function[14] `wolff()` in the file `wolff.cpp`. Each call to `wolff()` constructs a Wolff cluster, flips its spin and records its size.

The buffer `stack[N]` is used in order to store the new members of the cluster. We use the operator `new []` for dynamically allocating the necessary memory and `delete []` before returning to the calling program in order to return this memory back to the system - and avoid memory leaks.

[14] It is essentially the program in the book by Newman and Barkema [4].

```
#include <new>
  ...
  try{stack = new int[N];}
  catch(bad_alloc& stack_failed){
    string err(stack_failed.what());
    err = "allocation failure for stack in wolff(). " + err;
    locerr(err);
  }
  ...
  delete [] stack;
```

If the requested memory is not available, then `new []` throws a `bad_alloc` exception and we use `catch`. Using the `what` member of the `bad_alloc` class we obtain information identifying the exception.

The seed is chosen randomly using the distribution `drandom`:

```
  cseed    =  N*drandom(mxmx);
  stack[0] =    cseed;
  nstack   =  1;          //the stack has 1 member, the seed
  sold     =  s[cseed];
  snew     = -s[cseed];   //the new spin value of the  cluster
  s[cseed] =  snew;       //we flip all new members of cluster
  ncluster =  1;          //size of cluster=1
```

The seed is stored in `cseed` which is immediately added to the cluster (`stack[0]=cseed`). The variable `nstack` records the number of elements in the stack and it is originally set equal to 1. The variable `ncluster` counts the number of sites in the cluster and it is originally set equal to 1. `sold=s[cseed]` is the old value of the spin of the cluster and `snew=-sold` is the new one. The value of the spin of a new member of the cluster is *immediately* changed (`s[cseed]=snew`)! This increases the efficiency of the algorithm. By checking whether the spin of a nearest neighbor is equal to `sold`, we check whether the spin is the same as that of the cluster *and* if it has already been included in the cluster during a previous check.

The loop over the new members of the cluster is summarized below:

```
  while(nstack>0){
    //Pull a site off the stack:
    scluster  =  stack[--nstack];
    //check its four neighbours:
```

```
    if((nn    =  scluster+XNN)>=N) nn -= N;
    if(s[nn]  == sold)
      if(drandom(mxmx)<padd){
        //value of nstack++ is **before** increment
        stack[nstack++] = nn;
        s[nn] = snew; //flip the spin of cluster
        ncluster++;
      }
    //... check other 3 nearest neighbors ...
  }
```

The loop `while(nstack > 0)` is executed while `nstack > 0`, i.e. as long as the stack is not empty and there exist new members in the cluster. The variable `scluster` is the current site drawn from the stack in order to check its nearest neighbors. The line `if( (nn = scluster + XNN) >= N ) nn -= N;` chooses the nearest neighbor to the right and stores it in the variable `nn`. If the spin `s[nn]` of `nn` is equal to `sold`, then this neighbor has the same spin as that of the cluster *and* it has not already been included to the cluster (otherwise its spin would have been flipped). The variable `padd` is equal to P_{add} (it has been set in `init`) and if `drandom (mxmx) < padd` (which happens with probability P_{add}), then we add `nn` to the cluster: We add `nn` to the `stack`, we flip its spin (`s[nn] = snew`) and increase the cluster size by 1. We repeat for the rest of the nearest neighbors. The full code is listed below:

```
#include "include.h"
#include <new>  // bad_alloc
void wolff(){

  int  cseed,nstack,sold,snew,scluster,nn;
  int *stack;
  int  ncluster;

  //ask for the stack memory
  try{stack = new int[N];}
  catch(bad_alloc& stack_failed){
    string err(stack_failed.what());
    err = "allocation failure for stack in wolff(). " + err;
    locerr(err);
  }

  //choose the seed  spin for the cluster,
```

```
  //put it on the stack and flip it
  cseed    =  N*drandom(mxmx);
  stack[0] =    cseed;
  nstack   =  1;         //the stack has 1 member, the seed
  sold     =  s[cseed];
  snew     = -s[cseed]; //the new spin value of the  cluster
  s[cseed] =  snew;      //we flip all new members of cluster
  ncluster =  1;         //size of cluster=1
  //start loop on spins on the stack:
  while(nstack>0){
    //Pull a site off the stack:
    //value of --nstack is **after** decrement
    scluster  =  stack[--nstack];
    //check its four neighbours:
    if((nn    =  scluster+XNN)>=N) nn -= N;
    if(s[nn]  == sold)
      if(drandom(mxmx)<padd){
        //value of nstack++ is **before** increment
        stack[nstack++] = nn;
        s[nn] = snew; //flip the spin of cluster
        ncluster++;
      }
    if((nn    =  scluster-XNN)<0) nn += N;
    if(s[nn]  == sold)
      if(drandom(mxmx)<padd){
        //value of nstack++ is **before** increment
        stack[nstack++] = nn;
        s[nn] = snew; //flip the spin of cluster
        ncluster++;
      }
    if((nn    =  scluster+YNN)>=N) nn -= N;
    if(s[nn]  == sold)
      if(drandom(mxmx)<padd){
        //value of nstack++ is **before** increment
        stack[nstack++] = nn;
        s[nn] = snew; //flip the spin of cluster
        ncluster++;
      }
    if((nn    =  scluster-YNN)<0) nn += N;
    if(s[nn]  == sold)
      if(drandom(mxmx)<padd){
        //value of nstack++ is **before** increment
        stack[nstack++] = nn;
        s[nn] = snew; //flip the spin of cluster
        ncluster++;
```

```
    }
  }/*while(nstack>0)*/
  cout << "#clu " << ncluster << '\n';
  delete [] stack;
}/*wolff()*/
```

In order to link the function with the rest of the program so that we construct one cluster per "sweep"[15], we modify `main()` accordingly:

```
//============== main.cpp    ===================
#include "include.h"

int main(int argc, char **argv){

  init(argc,argv);
  for(int isweep=0;isweep<nsweep;isweep++){
    if(algorithm == 1){wolff();}
    else{met();}
    measure();
  }
  endsim();
}
```

The (global) variable `algorithm` controls whether the Wolff[16] or the Metropolis algorithm will be used for the spin updates. The (global) variable padd$\equiv P_{\mathrm{add}} = 1 - \mathrm{e}^{-2\beta}$ is defined in `init()`. The following lines are added: into the file `include.h`

```
extern double acceptance;
extern double padd;
```

The following lines are added to the file `init.cpp`

```
double padd;
  ...
  algorithm=0;
  padd = 1.0 - exp(-2.0*beta);
```

[15]Beware: A Metropolis sweep is not the same as a cluster update. The average size of the cluster changes with β and it is quite small for large temperatures (small β).

[16]Make the appropriate changes so that `wolff()` is called until the clusters constructed will have total size at least equal to `N`.

```
...
```

The following lines are added to the file `options.cpp`

```
#define OPTARGS  "hL:b:s:S:n:uw"
    ...
    case 'w':
      algorithm = 1;
      break;
    ...
```

in order to add the option `-w` to the command line. This option sets `algorithm=1`, which makes the program run the Wolff algorithm instead of the Metropolis. Some extra info must also be added to the help message printed by `usage` and `simmessage` and ... we are ready! For the compilation we use the `Makefile`

```
OBJS     = main.o init.o wolff.o met.o measure.o end.o \
           options.o mixmax.o
CXXFLAGS = -O2 -std=c++11

is: $(OBJS)
    $(CXX)  $(CXXFLAGS) $^   -o $@

mixmax.o:
    $(CXX)  $(CXXFLAGS) -c -o $@ MIXMAX/mixmax.cpp
$(OBJS): include.h
```

The commands

```
> make
> ./is -h
Usage: is  [options]
       -L: Lattice length (N=L*L)
       -b: beta  (options beta overrides the one in config)
       -s: start (0 cold, 1 hot, 2 old config.)
       -S: seed  (options seed overrides the one in config)
       -n: number of sweeps and measurements of E and M
       -u: seed  from /dev/urandom
       -w: use wolff algorithm for the updates
> ./is -L 20 -b 0.44 -s 1 -S 34235322 -n 5000 -w > outL20b0.44
```

do the compilation, print the usage instructions of the program and perform a test run for $L = 40$, $\beta = 0.44$, by constructing 5000 clusters, starting from a hot configuration and writing the data to the file `outL20b0.44`.

14.4 Production

In order to study the Ising model on a square lattice of given size N, we have to perform simulations for many values of β. Then, we want to study the finite size properties and extrapolate the results to the thermodynamic limit, by repeating the process for several values of N. The process is long and ... boring. Moreover, a bored researcher makes mistakes and several bugs can enter into her calculations. Laziness is a virtue in this case and it is worth the trouble and the time investment in order to learn some techniques that will make our life easier, our work more efficient, and our results more reliable. Shell scripting can be used in order to code repeated tasks of the command line. In its simplest form, it is just a series of commands written into a text file. Such an example can be found in the file `run1`:

```
# ################### run1 ########################
./is -L 20 -b 0.10 -s 1 -n 5000 -w -S 3423 > outL20b0.10
./is -L 20 -b 0.20 -s 2 -n 5000 -w > outL20b0.20
./is -L 20 -b 0.30 -s 2 -n 5000 -w > outL20b0.30
./is -L 20 -b 0.40 -s 2 -n 5000 -w > outL20b0.40
./is -L 20 -b 0.42 -s 2 -n 5000 -w > outL20b0.42
./is -L 20 -b 0.44 -s 2 -n 5000 -w > outL20b0.44
./is -L 20 -b 0.46 -s 2 -n 5000 -w > outL20b0.46
./is -L 20 -b 0.48 -s 2 -n 5000 -w > outL20b0.48
./is -L 20 -b 0.50 -s 2 -n 5000 -w > outL20b0.50
./is -L 20 -b 0.60 -s 2 -n 5000 -w > outL20b0.60
./is -L 20 -b 0.70 -s 2 -n 5000 -w > outL20b0.70
```

The first line is a comment, since everything after a `#` is ignored by the shell. The second line starts a simulation from a hot configuration (`-s 1`), lattice size `L=20` (`-L 20`) and temperature $\beta = 0.10$ (`-b 0.10`). The seed for the random number generator is set equal to 3423 (`-S 3423`) and we measure on 5000 Wolff clusters (`-n 5000 -w`). The results, printed to the `stdout`, are redirected to the file `outL20b0.10` (`> outL20b0.10`).

The next ten lines continue the simulation for $\beta = 0.20 - 0.70$. Each

simulation starts from the configuration stored in the file `conf` at the end of the previous simulation.

In order to run these commands, the file `run1` should be given execute permissions (only once, the permissions ... stay after that) using the command `chmod`:

```
> chmod a+x run1
```

Then `run1` can be executed like any other command:

```
> ./run1
```

Not bad... But we can do better! Instead of adding one line for each simulation, we can use the programming capabilities of the shell. Let's see how. The file `run2` contains the commands:

```
#!/bin/tcsh -f
# ################### run2 ################################
set L       = 20
set betas   = (0.10 0.20 0.30 0.40 0.42 0.44 0.46 0.48 0.50\
               0.60 0.70)
set start   = "-s 1 -S 3423"
set nsweeps = 5000

foreach beta ($betas)
 echo "L= $L beta= $beta"
 ./is -L $L -b $beta -n $nsweeps -w $start > outL${L}b${beta}
 set start = "-s 2"
end
```

The first line[17] calls the shell `tcsh` in order to interpret the script. This was not necessary in `run1`, since every shell can interpret the commands that it contains. But in this case we use syntax which is special to the shell `tcsh`.

The second line is a comment.

The third line defines a shell variable whose name is `L`. Its value is set after the `=` character equal to the *string* `"20"`. This value is accessible by

[17]The syntax is very strict and the line has to start *exactly* with the characters `#!` The string following `#!` can be the name of any program in the filesystem, which will be used to interpret the script.

adding a `$` in front of the name of the variable. Therefore, whenever we write `$L` (or `${L}`), the shell substitutes the string of characters `20`. For example, in place of `outL${L}b` the shell constructs the string `outL20b`.

The fourth line defines an array, whose name is `betas`. The different elements of the array can be accessed by using the syntax `$betas[number]`, where "number" is the array element starting from 1. In the example shown above `$betas[1]= 0.10, $betas[2]= 0.20, ..., $betas[11]= 0.70`. The special variable `$#betas` is the number of elements in the array, which is equal to 11. When we write `$betas`, the shell expands it to *all* the values in the array[18].

The fifth line defines the variable `start` to be equal to the string of characters `"-s 1 -S 3423"`. The quotes have been put because we want it to be treated as a single string of characters. If we omit them, then the shell treats `-s`, `1`, `-S` and `3423` as separate words, and we obtain a syntax error. Everything after the character `#` is a comment.

The command `foreach` is a way to construct a loop in `tcsh`. The commands between the `foreach` and `end` repeat once for every word in the parentheses in the `foreach` line. Each time, the loop variable, whose name is put after the keyword `foreach`, is set equal to the next word in the parenthesis. In our case, these words are the values of the array `betas`, and the loop will execute 11 times, once for each value 0.10, 0.20, ... , 0.70, each time with `$beta` set equal to one of those values.

The next three lines are the commands that are repeated by the `foreach` loop. The command `echo` "echoes" its arguments to the `stdout` and informs us about the current value of the parameters used in the simulation (quite useful, especially when the simulations take a long time). The command `./is` runs the program, each time using a different value of `beta`. Notice that the name of the file in which we redirect the `stdout` changes each time that `beta` changes value. Therefore our data will be stored in the files `outL20b0.10, outL20b0.20, ..., outL20b0.70`. The third command forces the program to read the initial configuration from the file `conf`. The first time that the loop is executed, the value of `start` is `"-s 1 -S 3423"` (hot configuration, seed equal to 3423), whereas for all the next simulations, `start` is equal to `"-s 2"` (old configuration).

We can also include a loop over many values of `L` as follows:

[18] Try the command: `echo $betas[3] $#betas $betas`

```
#!/bin/tcsh -f

set Ls      = (10 20 40)
set betas   = (0.10 0.20 0.30 0.40 0.42 0.44 0.46 0.48 0.50\
               0.60 0.70)
set nsweeps = 5000

foreach  L    ($Ls    )
 set start   = "-s 1 -S 3423"
 foreach beta ($betas)
  echo "L= $L beta= $beta"
  ./is -L $L -b $beta -n $nsweeps -w $start > outL${L}b${beta}
  set start = "-s 2"
 end
end
```

The array variable `Ls` stores as many `L` as we wish. Note that the definitions of `start` are put in a special place (why?).

14.5 Data Analysis

Data production must be monitored by looking at the time histories of properly chosen observables. This will allow us to spot gross mistakes and it will serve as a qualitative check of whether the system has thermalized and how long are the autocorrelation times. It is easy to construct time histories using `gnuplot`. For example, the commands[19]:

```
gnuplot> plot "<grep -v '#'             outL40b0.44" \
                u 1          with lines title "E"
gnuplot> plot "<grep -v '#'             outL40b0.44" \
                u (abs($2)) with lines title "|M|"
gnuplot> plot "<awk '/#clu/{print $2}' outL40b0.44" \
                u 1          with lines title "n"
```

show us the time histories of the energy, of the (absolute value of the) magnetization and of the size of the clusters in a simulation with $L = 40$ and $\beta = 0.44$.

[19]The `plot` command accepts, in place of the name of a data file, the `stdout` of a command `command` using the syntax `plot "<command"`.

The expectation values of the energy per link $\langle e\rangle = \frac{1}{2N}\langle E\rangle$ and the magnetization per site $\langle m\rangle = \frac{1}{N}\langle M\rangle$ with their errors can be calculated by the jackknife program, which can be found in the file `jack.cpp` in the directory `Tools` (see appendix 13.8). We compile the program into an executable file `jack` which we copy into the current working directory. The expectation value $\langle e\rangle$ can be calculated using the command:

```
> grep -v # outL40b0.44 |\
  awk  -v L=40 'NR>500{print $1/(2*L*L)}' | ./jack
```

We pass the value `L=40` to the program `awk` by using the option `-v`, therefore making possible the calculation of the ratio of the first column `$1` by $2N = 2L^2$. The condition `NR>500` makes the printing command to be executed only after `awk` reads the first 500 lines[20]. This way we can discard a number of thermalization sweeps. The result of the above command is printed to the `stdout` as follows:

```
# NDAT = 4500 data. JACK = 10 groups
# <o>, chi= (<o^2>-<o>^2)
# <o> +/- err                      chi +/- err
-0.71091166666 0.0024162628283 0.0015719190590 7.819205433e-05
```

The first three lines are the comments printed by the program `jack`, which inform the user about the important parameters of the analysis. The last line gives $\langle e\rangle$ and its error and then the fluctuations $\langle e^2\rangle - \langle e\rangle^2$ and their error. The latter must be multiplied by $\beta^2 N$, in order to obtain the specific heat c and its error according to (13.29). By adding a few more lines to the command shown above, this multiplication can be done on the fly:

```
> set L = 40; set b = 0.44 ;                    \
  grep -v # outL${L}b${b}                      | \
  awk -v L=$L 'NR>500{print $1/(2*L*L)}'  | \
  ./jack | grep -v #                           | \
  awk -v L=$L -v b=$b                           \
   '{print "e",L,b,$1,$2,b*b*L*L*$3,b*b*L*L*$4}'
```

[20] `NR` is the number of lines (number of records) read by `awk` so far.

Well, why all this fuzz? Notice that all the commands shown above can be given in one single line in the command line (by removing the trailing \ of each line). By recalling the command, it is easy to obtain the results for a different value of L and/or β, by editing the values of the variables `L` and/or `b`. The result is

```
e 40 0.42 -0.619523333 0.00189807 0.311391 0.0228302
```

i.e. $\langle e \rangle = -0.6195(19)$ and $c = 0.311(23)$.

We can work in a similar way for computing the magnetization. We have to calculate the absolute value of the second column of the `stdout` of the command `./is`, for every line that does not start with a `#`:

```
> set L = 40 ; set b = 0.42 ;                               \
  grep -v '#' outL${L}b${b}                               | \
  awk -v L=$L 'NR>500{m=($2>0)?$2:-$2;print m/(L*L)}' | \
  ./jack | grep -v '#'                                    | \
  awk -v L=$L -v b=$b                                       \
   '{print "m",L,b,$1,$2,b*L*L*$3,b*L*L*$4}'
```

The absolute value is calculated by the expression `($2>0)?$2:-$2`, and it is stored in the variable `m`, which in turn is printed after being divided by $N = L^2$. The result is

```
m 40 0.44 0.6250527778 0.00900370 21.8345 1.39975
```

which gives $\langle m \rangle = 0.6251(90)$ and $\chi = 21.8(14)$.

Similarly we can calculate $\langle n \rangle / N$:

```
> set L = 40 ; set b = 0.44 ;              \
  grep '#clu' outL${L}b${b}              | \
  awk -v L=$L 'NR>500{print $2/(L*L)}' | \
  ./jack | grep -v #                     | \
  awk -v L=$L -v b=$b '{print "n",L,b,$1,$2}'
```

The result is

```
n 40 0.44 0.4257476389 0.01302602
```

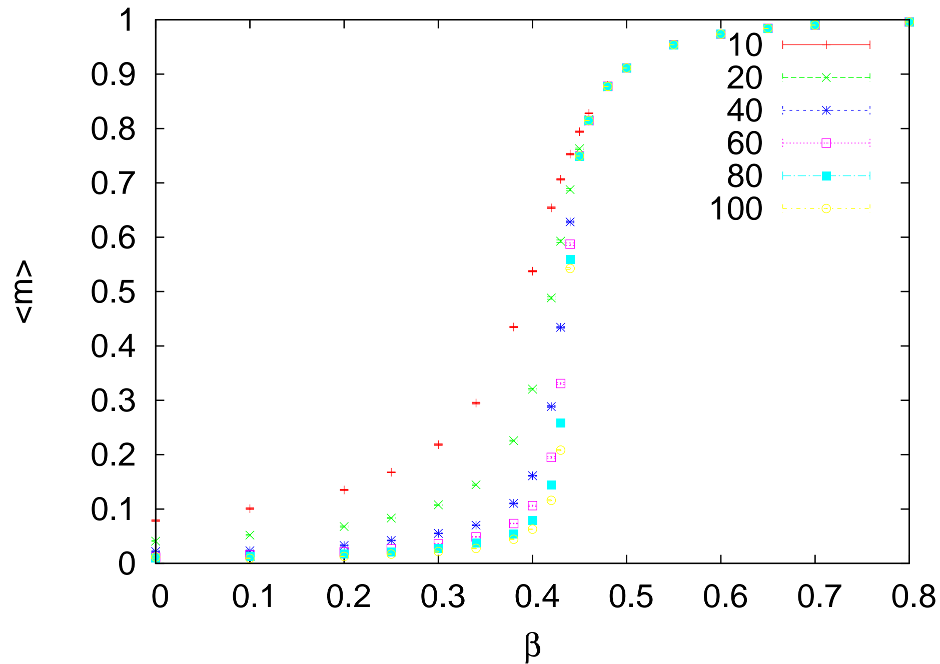


Figure 14.6: The results of the simulations performed by the shell script in the file run3. The expectation value of $\langle m \rangle$ is shown to decrease as $1/L$ at high temperatures $\beta \ll \beta_c$.

which gives $\langle n \rangle / N = 0.426(13)$.

All of the above commands can be summarized in the script in the file run3:

```
#!/bin/tcsh -f
set Ls       = (10 20 40 60 80 100)
set betas    = (0.00 0.10 0.20 0.25 0.30 0.34 0.38 \
                0.40 0.42 0.43 0.44 0.45 0.46 0.48 \
                0.48 0.50 0.55 0.60 0.65 0.70 0.80 )
set nsweeps = 100000

foreach  L    ($Ls    )
 set start    = "-s 1 -S 3423"
 foreach beta ($betas)
  ./is -L $L -b $beta -n $nsweeps -w $start > outL${L}b${beta}
  set start = "-s 2"
```

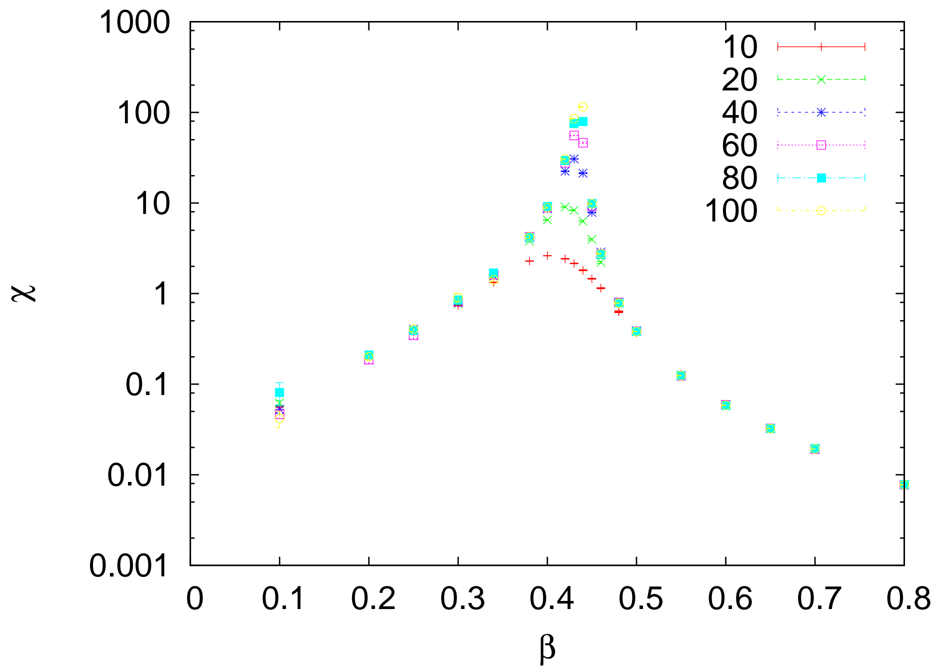


Figure 14.7: The results of the simulations performed by the shell script in the file run3. The magnetic susceptibility χ is shown to be almost independent of the lattice size when β takes values away from the critical region. In the critical region, its value increases as shown in equation (13.10).

```
  # Calculate <e> = <E>/(2N and c=beta^2*N*(<e^2>-<e>^2):
  grep -v '#' outL${L}b${beta}              | \
  awk -v L=$L 'NR>500{print $1/(2*L*L)}' | \
  ./jack | grep -v '#'                     | \
  awk -v L=$L -v b=$beta                     \
    '{print "e",L,b,$1,$2,b*b*L*L*$3,b*b*L*L*$4}'
  # Calculate <m> = <|M|>/N and chi=beta*N*(<m^2>-<m>^2)
  grep -v '#' outL${L}b${beta}                         | \
  awk -v L=$L 'NR>500{m=($2>0)?$2:-$2;print m/(L*L)}' | \
  ./jack | grep -v '#'                                | \
  awk -v L=$L -v b=$beta                                \
    '{print "m",L,b,$1,$2,b*L*L*$3,b*L*L*$4}'
 end
end
```

The script is run with the command

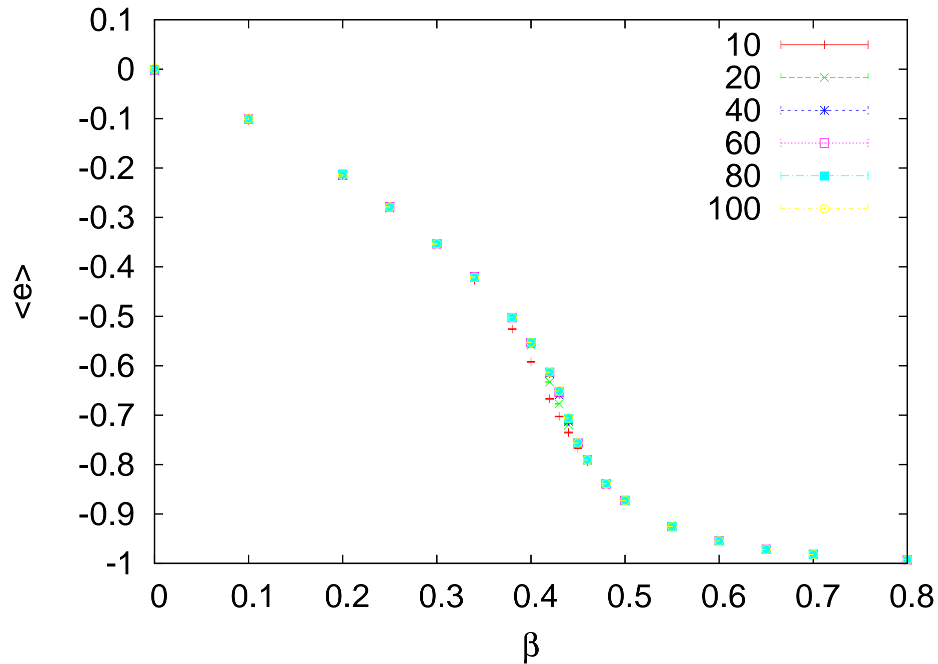


Figure 14.8: The results of the simulations performed by the shell script in the file `run3`. The plot shows the expectation value $\langle e \rangle$.

```
> ./run3 > out &
```

Then, we can plot the results using `gnuplot`[21]:

```
set xlabel "beta"
set ylabel "<m>"
plot   "<grep '^m 10 '  out" u 3:4:5 with errorbars title " 10"
replot "<grep '^m 20 '  out" u 3:4:5 with errorbars title " 20"
replot "<grep '^m 40 '  out" u 3:4:5 with errorbars title " 40"
replot "<grep '^m 60 '  out" u 3:4:5 with errorbars title " 60"
replot "<grep '^m 80 '  out" u 3:4:5 with errorbars title " 80"
replot "<grep '^m 100 ' out" u 3:4:5 with errorbars title "100"
```

The above commands plot the magnetization.

[21]These are `gnuplot` commands, even though we do not follow the usual convention to show the prompt `gnuplot>` explicitly

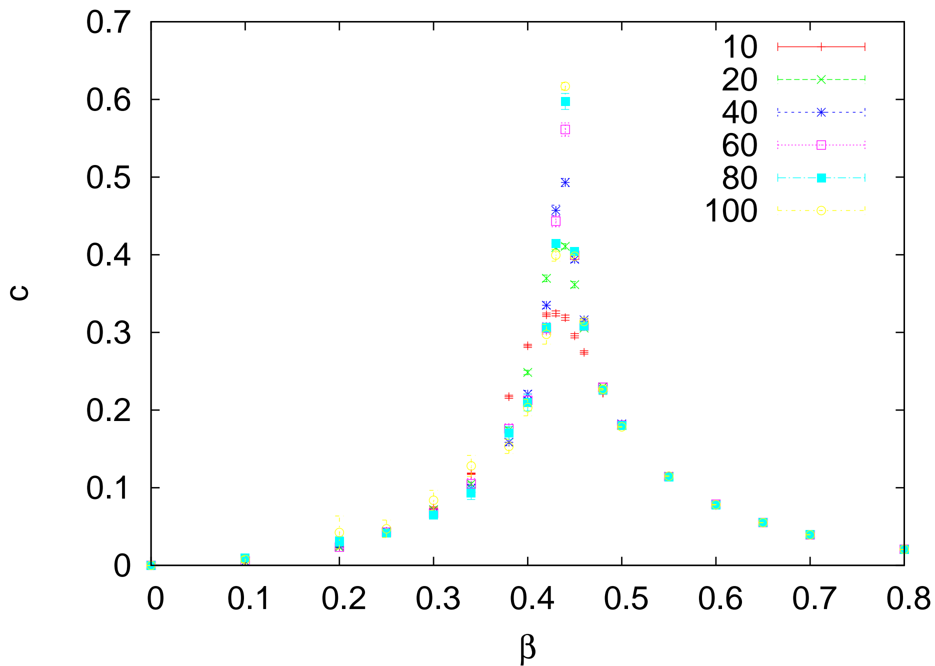


Figure 14.9: The results of the simulations performed by the shell script in the file `run3`. The plot shows the specific heat c which is shown to be almost independent of L away from the critical region, whereas in the critical region it increases according to equation (13.8).

```
set ylabel "chi"
set log y
plot   "<grep '^m 10 '  out" u 3:6:7 with errorbars title " 10"
replot "<grep '^m 20 '  out" u 3:6:7 with errorbars title " 20"
replot "<grep '^m 40 '  out" u 3:6:7 with errorbars title " 40"
replot "<grep '^m 60 '  out" u 3:6:7 with errorbars title " 60"
replot "<grep '^m 80 '  out" u 3:6:7 with errorbars title " 80"
replot "<grep '^m 100 ' out" u 3:6:7 with errorbars title "100"
```

The above commands plot the magnetic susceptibility.

```
set ylabel "<e>"
plot   "<grep '^e 10 '  out" u 3:4:5 with errorbars title " 10"
replot "<grep '^e 20 '  out" u 3:4:5 with errorbars title " 20"
```

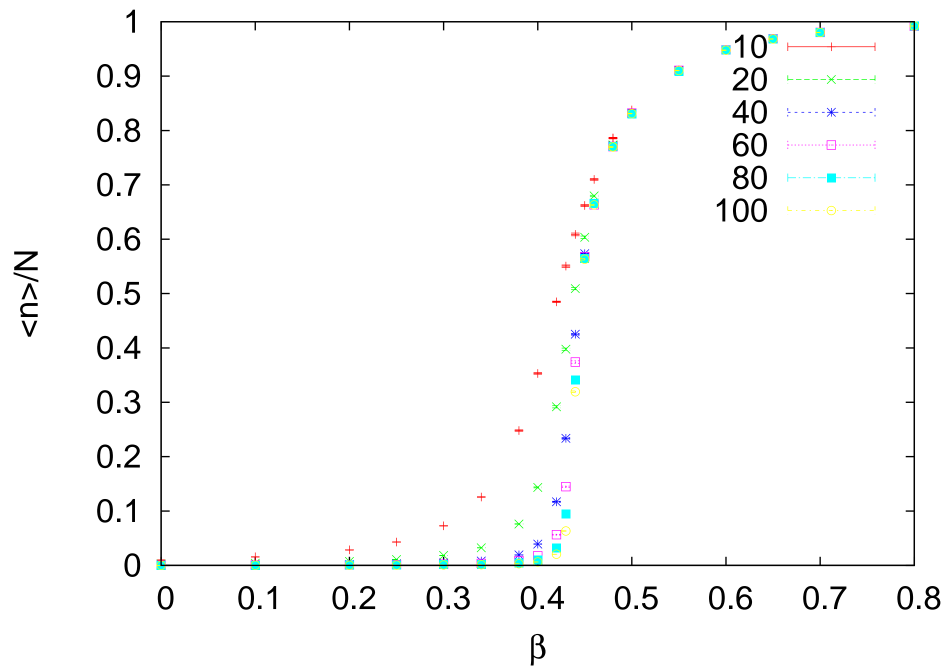


Figure 14.10: The results of the simulations performed by the shell script in the file run3. The plot shows $\langle n\rangle/N$.

```
replot "<grep '^e 40 '  out" u 3:4:5 with errorbars title " 40"
replot "<grep '^e 60 '  out" u 3:4:5 with errorbars title " 60"
replot "<grep '^e 80 '  out" u 3:4:5 with errorbars title " 80"
replot "<grep '^e 100 ' out" u 3:4:5 with errorbars title "100"
```

The above commands plot the energy.

```
set ylabel "c"
plot   "<grep '^e 10 '  out" u 3:6:7 with errorbars title " 10"
replot "<grep '^e 20 '  out" u 3:6:7 with errorbars title " 20"
replot "<grep '^e 40 '  out" u 3:6:7 with errorbars title " 40"
replot "<grep '^e 60 '  out" u 3:6:7 with errorbars title " 60"
replot "<grep '^e 80 '  out" u 3:6:7 with errorbars title " 80"
replot "<grep '^e 100 ' out" u 3:6:7 with errorbars title "100"
```

The above commands plot the specific heat.

```
set ylabel "<n>/N"
plot   "<grep '^n 10 '  out" u 3:4:5 with errorbars title " 10"
replot "<grep '^n 20 '  out" u 3:4:5 with errorbars title " 20"
replot "<grep '^n 40 '  out" u 3:4:5 with errorbars title " 40"
replot "<grep '^n 60 '  out" u 3:4:5 with errorbars title " 60"
replot "<grep '^n 80 '  out" u 3:4:5 with errorbars title " 80"
replot "<grep '^n 100 ' out" u 3:4:5 with errorbars title "100"
```

The above commands plot $\langle n\rangle/N$.

14.6 Autocorrelation Times

In the case of the Metropolis algorithm, the "unit of time" in the Monte Carlo simulation is one "sweep", which is equal to N *attempted* spin flips. In the case of the Wolff algorithm, the size of the clusters is a stochastic variable, which depends on temperature. Therefore, flipping the spins of a cluster is not a convenient unit of time, and we define:

$$(1\ \text{sweep}) = \frac{N}{\langle n\rangle}(\text{Wolff cluster updates}) \tag{14.30}$$

This definition of a sweep can be compared to a Metropolis sweep defined as N *accepted* spin flips[22]. For convenience, we also use the β–dependent unit of time equal to one Wolff cluster update. We use the notation $\tau_{\mathcal{O}}^W$ when the autocorrelation time of $\mathcal{O}$ is measured in Wolf cluster updates, and $\tau_{\mathcal{O}}$ when using the definition (14.30). Their relation is:

$$\tau_{\mathcal{O}} = \tau_{\mathcal{O}}^W \frac{\langle n\rangle}{N}\,. \tag{14.31}$$

We simulate the Ising model for $L = 10, 20, 40, 60, 80$ and 100 at $\beta = 0.4407$ using the Wolff algorithm. We construct 5×10^6 Wolff clusters. The results are written to files with names `outL${L}b0.4407`. We also perform simulations using the Metropolis algorithm with 10×10^6 sweeps. The results are written to files with names `outL${L}b0.4407met`. The following shell script makes life easier:

[22]The two definitions of a sweep in the Metropolis algorithm differ by a factor equal to the average acceptance rate. Both definitions are used in the bibliography, and the reader (as well as the author) of a scientific article should be aware of that.

```
#!/bin/tcsh -f
set Ls       = (10 20 40 60 80 100)
set beta     = 0.4407
set nsweeps = 5000000
set start    = "-s 1 -S 3423"
# Wolf cluster algorithm:
foreach  L    ($Ls)
 ./is -w -L $L -b $beta -n $nsweeps $start > outL${L}b${beta}
 # Mean cluster size <n>/N
 grep '#clu' outL${L}b${beta}                  | \
 awk -v L=$L 'NR>10000{print $2/(L*L)}'    | \
 ./jack -d $nsweeps | grep -v '#'          | \
 awk -v L=$L -v b=$beta '{print "n",L,b,$1,$2}'
end
# Metropolis algorithm
set nsweeps = 10000000
foreach  L    ($Ls)
 ./is -L $L -b $beta -n $nsweeps $start   > outL${L}b${beta}met
end
```

We compile the file `autoc.cpp` from the `Tools` directory and the executable file is named `autoc` and copied to the current working directory. Then, the following shell script calculates the autocorrelation functions $\rho_m(t)$:

```
#!/bin/tcsh -f
set Ls = (10 20 40 60 80 100)
set b  = 0.4407
# Wolff
set tmax  = 1000
set ndata = 5000000
foreach L ($Ls)
 set f = outL${L}b${b}
 grep -v '#' $f | \
 awk -v L=$L      \
  'BEGIN{N=L*L}NR>100000{print  ($2>0)?($2/N):(-$2/N)}'|\
 ./autoc -t $tmax -n $ndata> $f.rhom
end
# Metropolis
set tmax  = 8000
set ndata = 10000000
foreach L ($Ls)
 set f = outL${L}b${b}met
```

L	τ_m^W	$\langle n\rangle/N$	τ_m	$\tau_{m,\text{Metropolis}}$
10	2.18(2)	0.6124(2)	1.33(1)	16.1(1)
20	3.48(5)	0.5159(1)	1.80(3)	70.7(4)
40	5.10(6)	0.4342(2)	2.21(3)	330(6)
60	6.12(6)	0.3927(2)	2.40(2)	795(5)
80	7.33(7)	0.3653(3)	2.68(3)	1740(150)
100	8.36(6)	0.3457(1)	2.89(2)	2660(170)

Table 14.1: The autocorrelation times for the magnetization calculated as described in the text. The second column contains the autocorrelation time τ_m^W for the Wolff algorithm, using one cluster update as the unit of time. The fourth column contains τ_m in sweeps according to (14.30) and we have that $\tau_m = \tau_m^W\langle n\rangle/N$ (see (14.31)). The fifth column contains the autocorrelation times for the Metropolis algorithm in units of sweeps defined as N attempted spin flips.

```
 grep -v '#' $f | \
 awk -v L=$L        \
  'BEGIN{N=L*L}NR>100000{print  ($2>0)?($2/N):(-$2/N)}'|\
 ./autoc -t $tmax  -n $ndata> $f.rhom
end
```

We throw away 100 000 sweeps for thermalization. The results are written to files whose names have file extension `.rhom`. The function $\rho_m(t)$ is fitted to (13.51) using three autocorrelation times according to the discussion in appendix 13.7. The results are shown in table 14.1[23]

From (14.10) we expect that $\tau_m \sim L^z$ where z is the dynamic exponent. z can be calculated by the `gnuplot` commands:

```
gnuplot> tau(x) = c*x**z
gnuplot> fit tau(x) "autoc.dat" u 1:2:3 via c,z
gnuplot> plot "autoc.dat" u 1:2:3 w e t "W steps    ", tau(x)
gnuplot> fit tau(x) "autoc.dat" u 1:6:7 via c,z
gnuplot> plot "autoc.dat" u 1:6:7 w e t "W sweeps   ", tau(x)
gnuplot> fit tau(x) "autoc.dat" u 1:8:9 via c,z
gnuplot> plot "autoc.dat" u 1:8:9 w e t "Metropolis", tau(x)
```

[23]Notice the difference between the results of the Metropolis algorithm and the ones shown in appendix 13.7. The difference is due to a fivefold increase in the statistics and shows that the real error in the calculation of τ includes systematic errors that have been neglected.

The exponent z is calculated for the Wolff algorithm in Wolff steps and Wolff sweeps. The results are

$$\tau_m^W \sim L^{z^W}, \qquad z^W = 0.54 \pm 0.02 \tag{14.32}$$

$$\tau_m \sim L^z, \qquad z = 0.29 \pm 0.02 \tag{14.33}$$

$$\tau_{m,\text{Metropolis}} \sim L^z, \qquad z = 2.21 \pm 0.02 \tag{14.34}$$

The plots are shown in figures 14.11-14.13. The values of z reported in

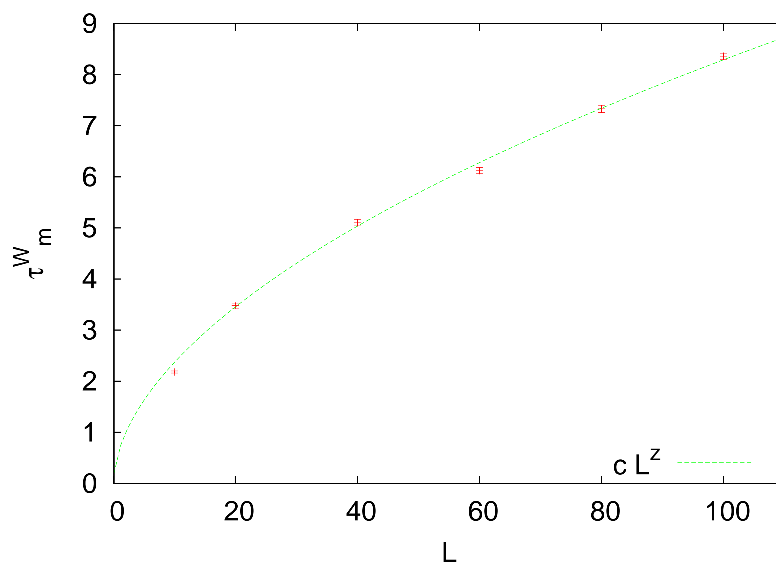


Figure 14.11: Autocorrelation times τ_m^W for the magnetization using the Wolff algorithm at $\beta = 0.4407$. The unit of time is one Wolff cluster update. The dynamic exponent is calculated from the fit to cL^{z^W} which gives $z^W = 0.54(2)$.

the bibliography are $0.50(1)$, $0.25(1)$ and $2.167(1)$ respectively [4, 63, 70]. We can obtain better results by increasing the statistics and the lattice size and this is left as an exercise for the reader.

We also mention the relation between the dynamic exponents given by equations (14.32) and (14.33). From (14.29) $\chi = \beta\langle n\rangle$, (13.10) $\chi \sim |t|^{-\gamma}$,

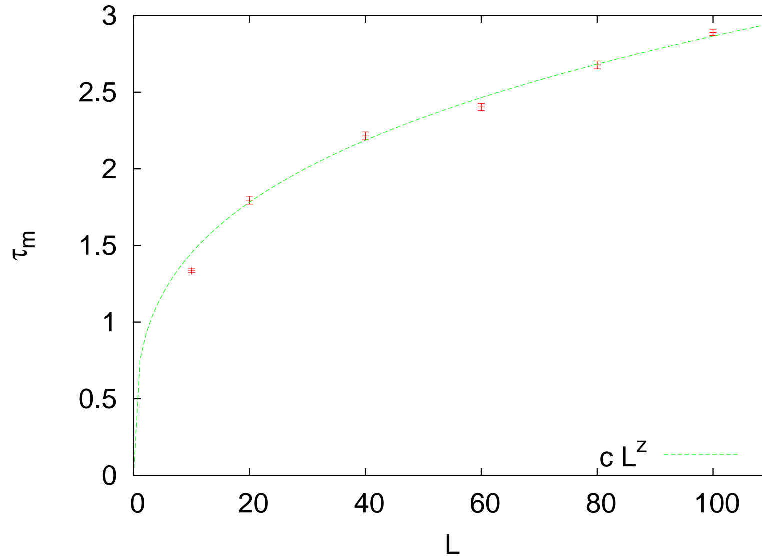


Figure 14.12: Autocorrelation times τ_m for the magnetization using the Wolff algorithm at $\beta = 0.4407$. The unit of time is one Wolff sweep. The dynamic exponent is calculated from the fit to cL^z, which gives $z = 0.29(2)$.

and (13.6) $\xi \sim |t|^{-\nu}$ and using $\xi \sim L$, valid in the critical region, we obtain

$$\tau_m = \tau_m^W \frac{\langle n \rangle}{L^2} \sim L^{z^W} \frac{L^{\gamma/\nu}}{L^2} = L^{z^W + \gamma/\nu - 2}, \qquad (14.35)$$

where we assumed that $\tau_m^W \sim L^{z^W}$, $z^W \equiv 0.54(2)$ and $\tau_m \sim L^z$. Therefore

$$z = z^W + \frac{\gamma}{\nu} - 2\,. \qquad (14.36)$$

Using the values given in (13.12), $\gamma = 7/4$, $\nu = 1$, we obtain

$$z = z^W - \frac{1}{4}\,, \qquad (14.37)$$

which is in agreement, within error, with the calculated values and the values in the bibliography.

L	$\gamma(t<0)$	$\gamma(t>0)$
40	1.7598(44)	1.730(17)
60	1.7455(24)	1.691(14)
80	1.7409(21)	1.737(12)
100	1.7420(24)	1.7226(75)
120	1.7390(15)	1.7725(69)
140	1.7390(23)	1.7354(72)
160	1.7387(10)	1.746(17)
200	1.7380(11)	1.759(15)
500	1.7335(8)	1.7485(83)

Table 14.2: Calculation of the critical exponent γ from fitting the data shown in figures 14.14 and 14.15. The second column contains the results for $\beta > \beta_c (t < 0)$ and the third one for $\beta < \beta_c (t > 0)$. The parentheses report the *statistical* errors of the fits and not the systematic. We expect that $\gamma = 7/4$.

L	$\beta(t<0)$	$\beta_+(t>0)$
40	0.1101(7)	0.1122(29)
60	0.1129(5)	0.1102(19)
80	0.1147(5)	0.1118(21)
100	0.1175(3)	0.1170(11)
120	0.1167(4)	0.1172(16)
140	0.1190(2)	0.1187(19)
160	0.1191(4)	0.1134(20)
200	0.1205(10)	0.1138(24)
500	0.1221(2)	0.1294(50)

Table 14.3: The calculation of the critical exponent β from fitting the data shown in figures 14.16. The second column contains the results for $\beta > \beta_c (t < 0)$ and the third for $\beta < \beta_c (t > 0)$. The parentheses report the *statistical* errors of the fits and not the systematic. We expect that $\beta = \beta_+ = 1/8$.

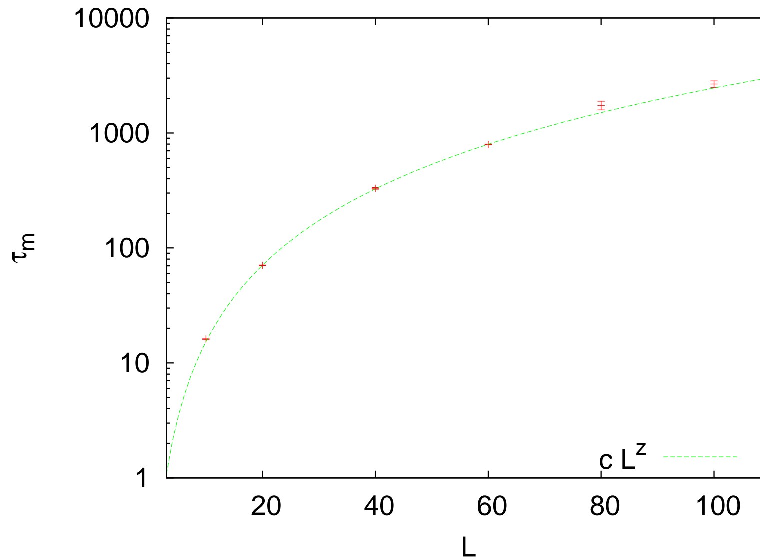


Figure 14.13: Autocorrelation times $\tau_{m,\mathrm{Metropolis}}$ for the magnetization using the Metropolis algorithm at $\beta = 0.4407$. The unit of time is a Metropolis sweep defined by N attempted spin flips. The dynamic exponent is calculated from the fit to cL^z, which gives $z = 2.21(2)$.

14.7 Temperature Scaling

In this section we will discuss the extent to which relations (14.3)–(14.5) can be used for the calculation of the critical exponents γ, α and β. The result is that, although using them it is possible to compute correct results, these relations are not the best choice[24]. In order to see clear scaling and reduce finite size effects, we have to consider $t \ll 1$ and large L. The results depend strongly on the choice of range of the data included in the fits. The systematic errors are large and the results in some cases plain wrong[25].

[24]Note that for the Ising model on the square lattice, the critical temperature is exactly known. In a model where it is not known we have larger systematic errors than the ones discussed here. The numerical calculation of the critical temperature we will discussed in a following section.

[25]In [4] it is mentioned that the random field Ising model exhibits pseudoscaling for a range of t and for even smaller t there is a crossover to a different scaling that gives the correct critical exponent. See also [71], [72].

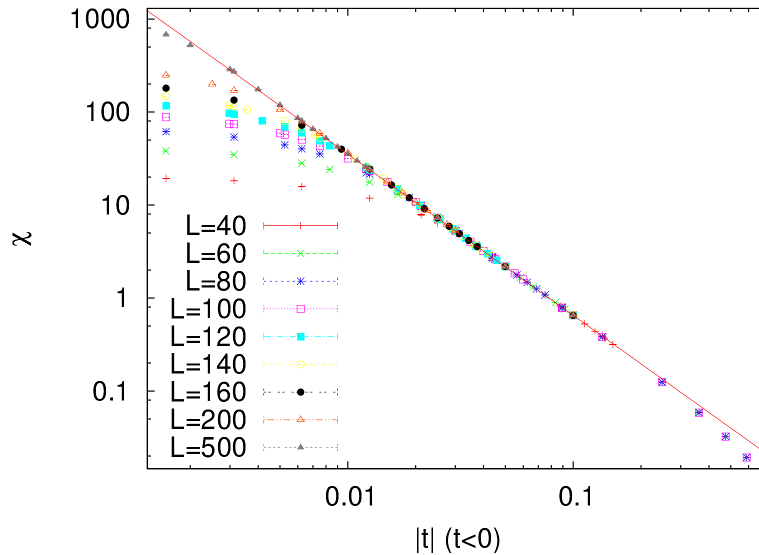


Figure 14.14: The magnetic susceptibility $\chi(t, L)$ in the scaling region according to equation (14.3). The straight line is the fit to this relation for the largest lattice. We observe that finite size effects decrease as L increases and that the range of temperatures included in the fit extends to smaller $|t|$. The data is for $\beta > \beta_c (t < 0)$ and the critical point is approached from the ordered phase.

We simulate the Ising model for $L = 40$, 60, 80, 100, 120, 140, 160, 200 and 500. The temperatures chosen correspond to small enough t in order to observe scaling. For the values of β used in the simulations, see the shell scripts in the accompanying software.

First we compute the exponent γ from the relation (14.3). For given L, we fit the data for $\chi(t)$ for an appropriate range of $|t|$ to the function $a\,|t|^{-\gamma}$, which has two fitting parameters, γ and a. We determine the range of t where $\chi(t)$ gives a linear plot in a log–log scale[26]. For large $|t|$, we observe deviations from the linear behavior and for very small $|t|$ we observe finite size effects when $\xi \approx L$. As L increases, finite size effects decrease, and the data get closer to the asymptotic behavior $|t|^{-\gamma}$ for even smaller $|t|$. The results are more clear for $\beta > \beta_c (t < 0)$, because for $t > 0$ the fluctuations near the pseudocritical temperature $\beta_c(L) < \beta_c$ are larger and the finite size effects are larger.

[26]The fit can also be done by linearly fitting the points $(\log|t_i|, \log\chi(t_i))$ to a straight line.

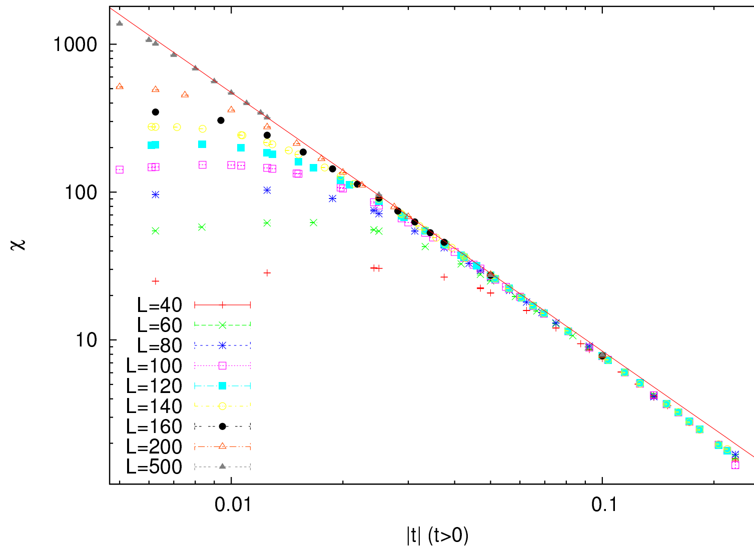


Figure 14.15: The magnetic susceptibility $\chi(t, L)$ in the scaling region according to equation (14.3). The straight line is the fit to this relation for the largest lattice. We observe that finite size effects decrease as L increases and that the range of temperatures included in the fit extends to smaller $|t|$. The data is for $\beta < \beta_c(t > 0)$ and the critical point is approached from the disordered phase. Finite size effects are larger for $t < 0$ due to the larger fluctuations at the pseudocritical point $\beta_c(L) < \beta_c$.

Table 14.2 shows the results for the exponent γ for all the measured values of L. The errors reported are the statistical errors of the fits, which are smaller than the systematic errors coming from the choice or range of t of the data included in the fits. One has to vary this range as long as the χ^2/dof of the fit remains acceptable, and the resulting variation in the values of the parameters has to be included in the estimate of the error. Sometimes, this method gives an overestimated error, and it is a matter or experience to decide which parameter values to include in the estimate. For example, figures 14.14 and 14.15 show that the acceptable range of fitting becomes more clear by studying $\chi(t)$ for increasing L. As L increases, the points approach the asymptotic curve even closer. Even though for fixed L one obtains acceptable power fits over a larger range of t, by studying the large L convergence, we can determine the scaling region with higher accuracy. Another point to consider is whether the parameters of the fits have reasonable values. For example, even though

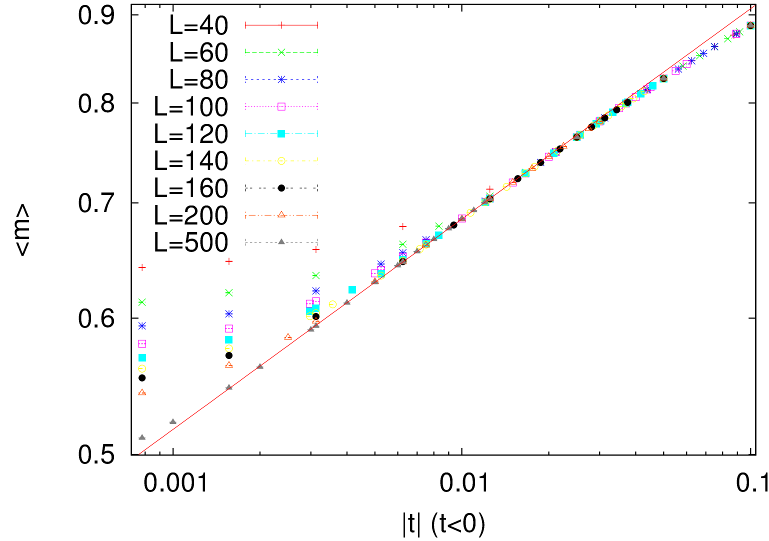


Figure 14.16: The magnetization $\langle m\rangle(t,L)$ in the scaling region according to equation (14.5). The straight line is the fit to this relation for the largest lattice. We observe that finite size effects decrease as L increases and that the range of temperatures included in the fit extends to smaller $|t|$. The data is for $\beta > \beta_c(t<0)$ and the critical point is approached from the ordered phase.

the value of a is unknown, it is reasonable to expect that its value is of order ~ 1. By taking all these remarks into consideration we obtain

$$\gamma = 1.74 \pm 0.02 \qquad (t<0)\,, \tag{14.38}$$

$$\gamma = 1.73 \pm 0.04 \qquad (t>0)\,. \tag{14.39}$$

Next, we compute the critical exponent β using relation (14.5). This relation is valid as we approach the critical point from the low temperature phase, $\beta > \beta_c$ or $t < 0$. In the thermodynamic limit, the magnetization is everywhere zero for all $\beta < \beta_c$. For a finite lattice $\langle m\rangle > 0$, and it is reasonable to expect a scaling of the form

$$\langle m\rangle \sim |t|^{\beta_+ - 1}\,, \qquad t>0\,, \tag{14.40}$$

where β_+ is defined so that

$$\beta_+ = \beta = 1/8\,. \tag{14.41}$$

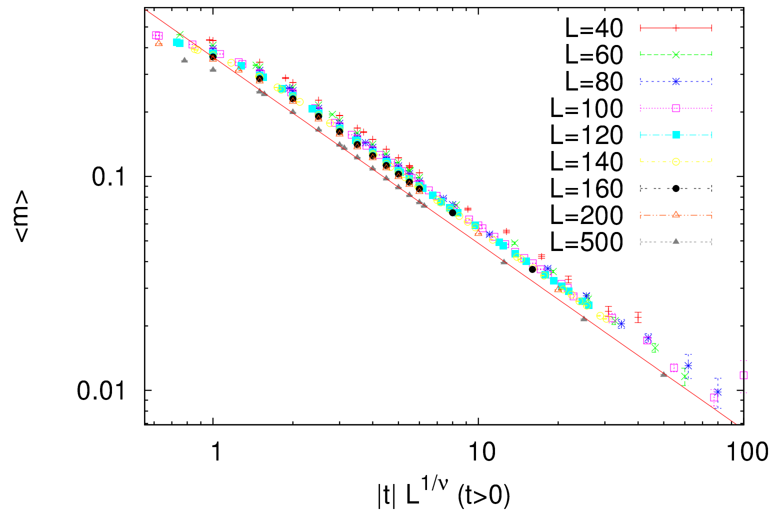


Figure 14.17: The magnetization $\langle m\rangle(t,L)$ in the scaling region fitted to equation (14.40). The straight line is the fit to this relation for the largest lattice. We observe that finite size effects decrease as L increases and that the range of temperatures included in the fit extends to smaller $|t|$. The data is for $\beta < \beta_c (t > 0)$ and the critical point is approached from the disordered phase.

By following the same procedure, we calculate the exponents β and β_+ shown in table 14.3. By taking the systematic errors described above into consideration, we find that

$$\beta = 0.121 \pm 0.003 \qquad t < 0\,, \tag{14.42}$$

$$\beta_+ = 0.120 \pm 0.007 \qquad t < 0\,, \tag{14.43}$$

which should be compared to the expected values $\beta = \beta_+ = 1/8$.

The calculation of the exponent α needs special care. The expected value is $\alpha = 0$. This does not imply that $c \sim$ const. but that [27]

$$c \sim |\log|t||\ . \tag{14.44}$$

In this case, we find that the data is better fitted to the above relation instead of being fitted to a power. This can be seen pictorially by making

[27]This does not exclude more exotic behaviors of logarithmic powers or logarithms of logarithms etc. This needs to be studied carefully when the analytic result is not known.

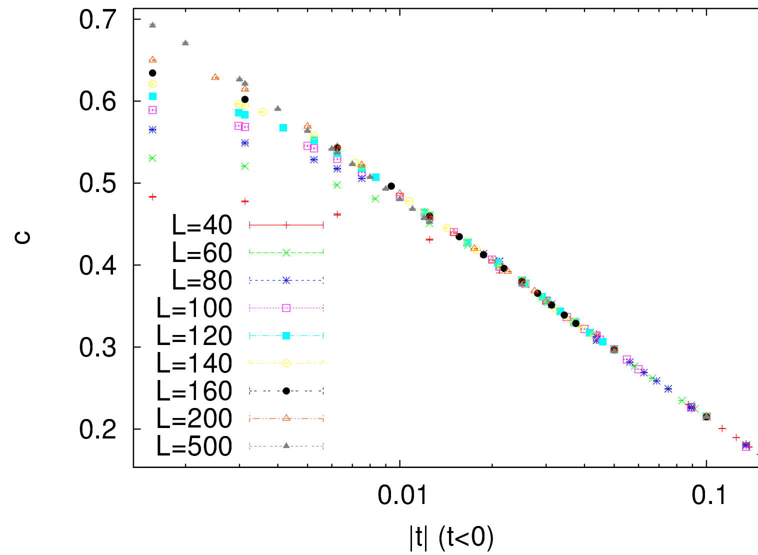


Figure 14.18: The specific heat $c(t, L)$ in the scaling region fitted to equation (14.44). Only the $|t|$ axis is in logarithmic scale. The data is for $\beta > \beta_c(t < 0)$ and the critical point is approached from the ordered phase.

a log–log plot and comparing it to a $c - |\log|t||$ plot. We see that the second choice leads to a better linear plot than the first one. A careful study will compute the quality of the fits and choose the better model this way. This is left as an exercise for the reader.

L	γ/ν	β/ν
40–100	1.754(1)	0.1253(1)
140–1000	1.740(2)	0.1239(3)
40–1000	1.749(1)	0.1246(1)

Table 14.4: The critical exponents γ/ν and β/ν given by the finite size scaling relations (14.7) and (14.9). The first column contains the range in L included in the fits of $\chi(\beta_c, L)$ and $\langle m\rangle(\beta_c, L)$ to aL^g.

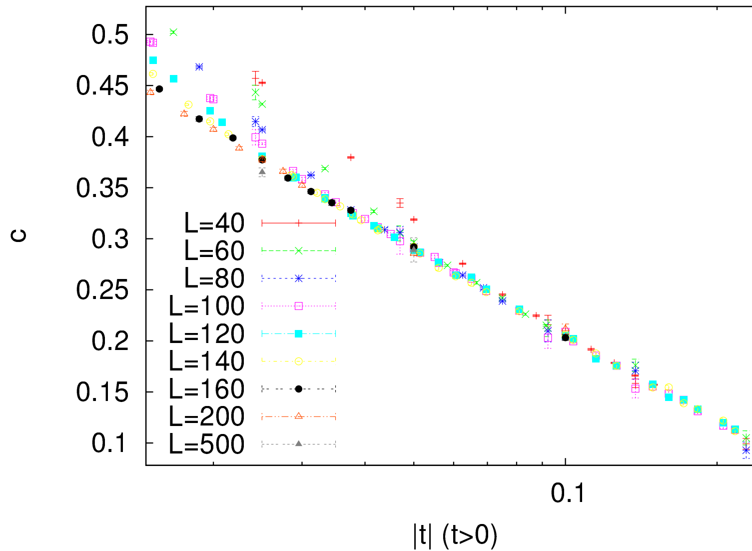


Figure 14.19: The specific heat $c(t, L)$ in the scaling region fitted to equation (14.44). Only the $|t|$ axis is in logarithmic scale. The data is for $\beta < \beta_c(t > 0)$ and the critical point is approached from the disordered phase. The exponent ν is set equal to 1.

14.8 Finite Size Scaling

In this section we will calculate the critical exponents by using relations (14.7)-(14.9), i.e. by using the asymptotic scaling of $\chi(\beta = \beta_c, L)$, $c(\beta = \beta_c, L)$ and $\langle m\rangle(\beta = \beta_c, L)$ with increasing system size L. This is called "finite size scaling".

In order to calculate the exponent γ/ν given by equation (14.7), we calculate the magnetic susceptibility at the known β_c for increasing values of L. We fit the results $\chi(\beta_c, L)$ to the function aL^g and calculate the fitting parameters a and g. Then, we compare the computed value of g to the expected value of $\gamma/\nu = 7/4 = 1.75$. In this procedure we have to decide which values of L should be included in the fits. The most obvious criterion is to obtain reasonable $\chi^2/\text{dof} \lesssim 1$ and that the error in g and a be small. This is not enough: Table 14.4 shows small variations in the obtained values of γ/ν, if we consider different fit ranges. These variations give an estimate of the systematic error which enters in the calculation. Problem 9 is about trying this calculation yourselves. Table 14.4 shows the results, and figure 14.20 shows the corresponding plot.

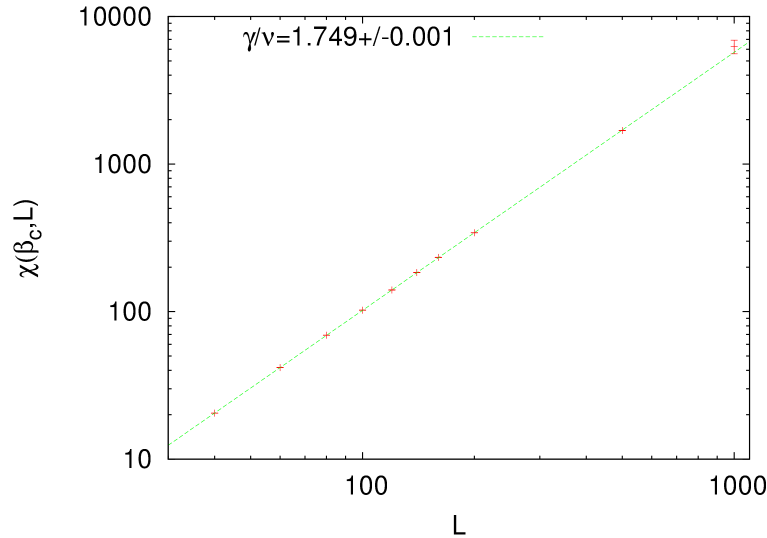


Figure 14.20: The magnetic susceptibility $\chi(\beta_c, L)$ at the critical temperature for different values of L. The axes are in a logarithmic scale and the linear plots are consistent with the power fit $\chi(\beta_c, L) = cL^g$. The value of g computed by the fits is consistent with the critical exponent γ/ν given by equation (14.7).

The final result, which includes also an estimate of the systematic errors, is

$$\frac{\gamma}{\nu} = 1.748 \pm 0.005\,. \tag{14.45}$$

For the calculation of the exponent β/ν given by equation (14.9), we compute the magnetization $\langle m\rangle(\beta_c, L)$ at the critical temperature and repeat the same analysis. The result is

$$\frac{\beta}{\nu} = 0.1245 \pm 0.0006\,. \tag{14.46}$$

Equation (14.9) gives the exponent α/ν. But the expected value $\alpha = 0$ leads, in analogy with equation (14.44), to

$$c(\beta_c, L) \sim \log L\,. \tag{14.47}$$

This relation is shown in figure 14.22. The vertical axis is *not* in a logarithmic scale whereas the horizontal is. The linear plot of the data shows consistency with equation (14.47). Problem 9 asks you to show

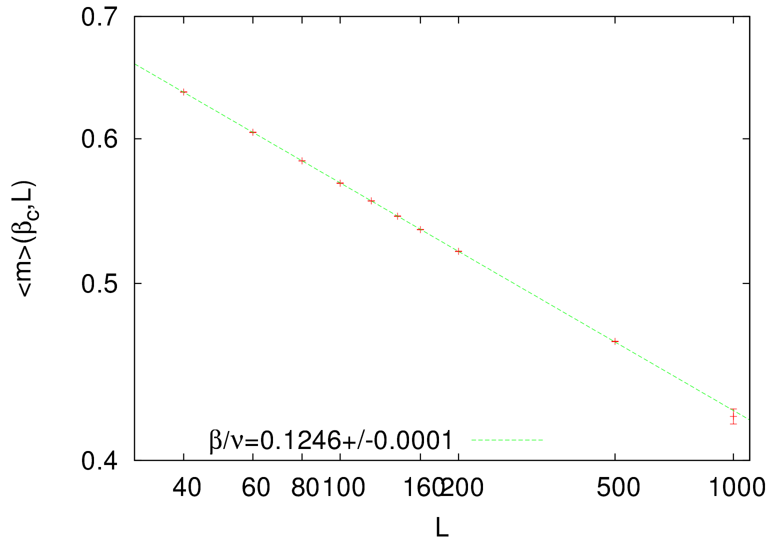


Figure 14.21: The magnetization $\langle m\rangle(\beta_c, L)$ at the critical temperature for different values of L. The axes are in a logarithmic scale and the linear plots are consistent with the power fit $\langle m\rangle(\beta_c, L) = cL^g$. The value of g computed by the fits is consistent with the critical exponent β/ν given by equation (14.9).

whether the logarithmic fit is better than a fit to a function of the form cL^a+b and appreciate the difficulties that arise in this study. By increasing the statistics, and by measuring for larger L, the data in table 14.8 will improve and lead to even clearer conclusions.

We observe that, by using finite size scaling, we can compute the critical exponents more effectively, than by using temperature scaling as in section 14.7. The data follow the scaling relations (14.7)–(14.9) suffering smaller finite size effects[28].

14.9 Calculation of β_c

In the previous sections we discussed scaling in t and L in the critical region. In the calculations we used the exact value of the critical temper-

[28]Remember that the instability of the results with respect to the choice of the fitting range is large for the temperature scaling method. When the exact value of the critical temperature is not known, the superiority of finite size scaling is even higher.

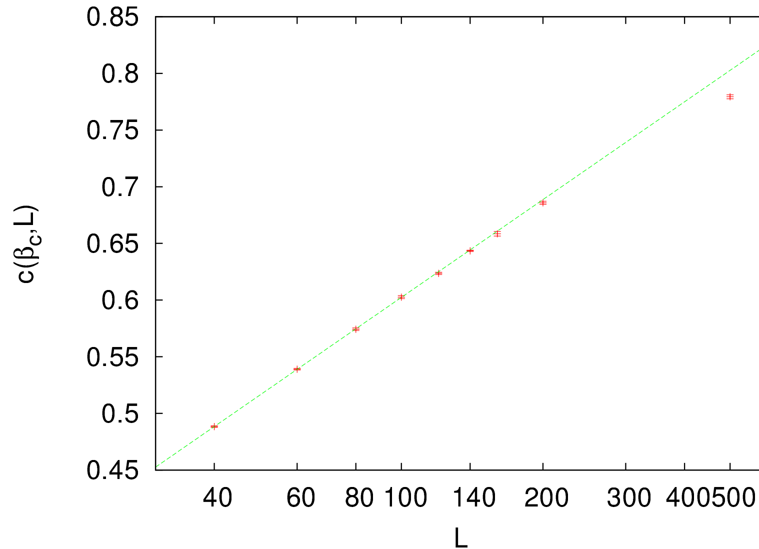


Figure 14.22: The specific heat $c(\beta_c, L)$ at the critical temperature for different values of L. The horizontal axis is in a logarithmic scale and the linear plot is consistent with the scaling relation $c(\beta_c, L) = c \log L$. The result is consistent with the expectation $\alpha = 0$ (see equation (14.8)).

ature $\beta_c = \log(1+\sqrt{2})/2$. When β_c is not known, the analysis becomes harder and its computation contributes to the total error in the value of the critical exponents. When doing finite size scaling using the scaling relations (14.7)–(14.9), one has to choose the values of the temperature at which to calculate the left hand sides. So far, these values were computed *at* β_c. What should we do when β_c is not a priori known? A good choice is to use the pseudocritical temperature $\beta_c(L)$, the temperature where the fluctuations of the order parameter $\chi(\beta)$ are at their maximum. Otherwise, we can compute β_c according to the discussion in this section and use the computed β_c in the finite size scaling analysis.

Both choices yield the same results in the large L limit, even though the finite size effects are different. In fact any value of β in the critical region can be used for this calculation. The reason is that as we approach the critical region for given L, the correlation length becomes $\xi \sim L$ and finite size effects become important. This is the behavior that characterizes the *pseudocritical region* of the finite L system. The pseudocritical region becomes narrower as L becomes larger. Any value of β in this region

L	$\beta_c(L)$	$\chi_{\max}$	$\beta'_c(L)$	$c_{\max}$
40	0.4308(4)	30.68(4)	0.437(1)	0.5000(20)
60	0.4342(2)	62.5(1)	0.4382(7)	0.5515(15)
80	0.4357(2)	103.5(1)	0.4388(5)	0.5865(12)
100	0.4368(1)	153.3(2)	0.4396(2)	0.6154(18)
120	0.4375(1)	210.9(2)	0.4396(4)	0.6373(20)
140	0.43793(13)	276.2(4)	0.4397(5)	0.6554(18)
160	0.4382(1)	349.0(5)	0.4398(4)	0.6718(25)
200	0.43870(7)	516.3(7)	0.4399(2)	0.6974(17)
500	0.43988(4)	2558(5)	0.44038(8)	0.7953(25)
1000	0.44028(4)	8544(10)	0.44054(8)	0.8542(36)

Table 14.5: The pseudocritical temperatures $\beta_c(L)$ and $\beta'_c(L)$ calculated from the maxima of the magnetic susceptibilities $\chi_{\max}$ and the specific heat $c_{\max}$ respectively. The values of the maxima are also shown.

will give us observables that scale at large L, but the best choice is

$$\chi(\beta_c(L), L) \equiv \chi_{\max}(L)\,. \tag{14.48}$$

In this case, the values on the left hand sides of (14.7)–(14.9) should be taken at $\beta = \beta_c(L)$.

The definition of $\beta_c(L)$ in not unique. One could use, for example, the maximum of the specific heat

$$c(\beta'_c(L), L) \equiv c_{\max}(L)\,, \tag{14.49}$$

which defines a different $\beta'_c(L)$. Of course $\lim_{L\to\infty}\beta_c(L) = \lim_{L\to\infty}\beta'_c(L) = \beta_c$ and both choices will yield the same results for large L. The speed of convergence and the errors involved in the calculation of the pseudocritical temperatures are different in each case and there is a preferred choice, which in our case is $\beta_c(L)$.

First we calculate β_c. When we are in the pseudocritical region we have that $\xi \approx L$, therefore (14.2) gives

$$|t| = \left|\frac{\beta_c - \beta_c(L)}{\beta_c}\right| \sim \xi^{-\frac{1}{\nu}} \sim L^{-\frac{1}{\nu}} \Rightarrow \beta_c(L) = \beta_c - \frac{c}{L^{\frac{1}{\nu}}}\,. \tag{14.50}$$

The calculation is straightforward to do: First we measure the magnetic susceptibility. For each L we determine the pseudocritical region and we

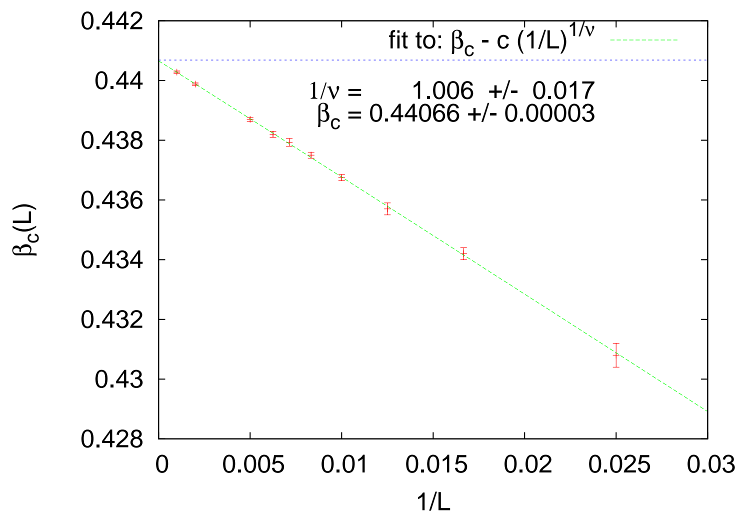


Figure 14.23: Calculation of the critical temperature β_c and the critical exponent ν using relation (14.50). By using the pseudocritical temperatures $\beta_c(L)$ of table 14.5, we fit the data to $a-c(1/L)^b$. From the calculated values of a, b and c we calculate $\beta_c = a$ and $1/\nu = b$. The horizontal line is the exact, known value $\beta_c = \log(1+\sqrt{2})/2 = 0.44069\ldots$.

calculate $\beta_c(L)$ and the corresponding maximum value $\chi_{\rm max}$. In order to do that, we should take many measurements around $\beta_c(L)$. We have to be very careful in determining the autocorrelation time (which increases as $\tau \sim L^z$), so that we can control the number of independent measurements and the thermalization of the system. We use the relation (14.50) and fit the results to $a - c/L^b$, and from the calculated parameters a, b and c we compute $\beta_c = a$, $\nu = 1/b$. In cases where one of the parameters β_c, ν is known independently, then it is kept constant during the fit.

The results are shown in figure 14.23 where we plot the numbers contained in table 14.5. The final result is:

$$\begin{aligned} \beta_c &= 0.44066 \pm 0.00003 \\ \frac{1}{\nu} &= 1.006 \pm 0.017\,. \end{aligned}$$

This can be compared to the known values $\beta_c = \log(1+\sqrt{2})/2 \approx 0.44069$ and $1/\nu = 1$.

This process is repeated for the pseudocritical temperatures $\beta_c'(L)$ and the maximum values of the specific heat $c_{\rm max}$. The results are shown in

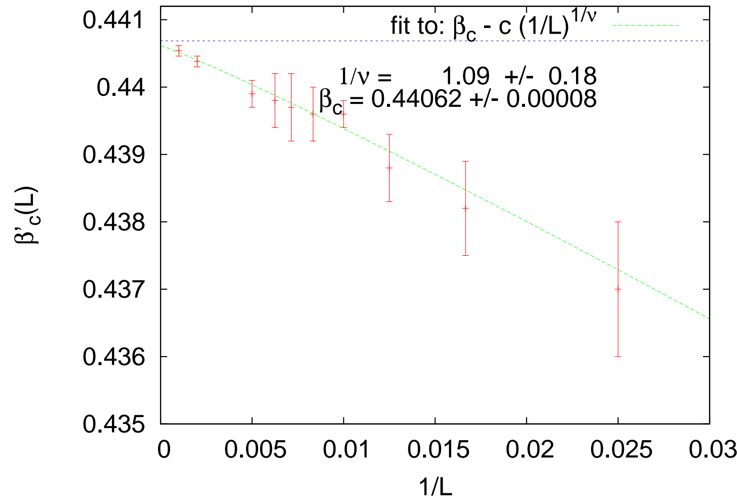


Figure 14.24: Calculation of the critical temperature β_c and the critical exponent ν using relation (14.50). By using the pseudocritical temperatures $\beta'_c(L)$ of table 14.5, we fit the data to $a - c(1/L)^b$. From the calculated values of a, b and c we calculate $\beta_c = a$ and $1/\nu = b$. The horizontal line is the exact, known value $\beta_c = \log(1+\sqrt{2})/2 = 0.44069\ldots$.

figure 14.24. The final result is:

$$\begin{aligned} \beta_c &= 0.44062 \pm 0.00008 \\ \frac{1}{\nu} &= 1.09 \pm 0.18\,. \end{aligned}$$

Figure 14.24 and the results reported above show that the calculation using the specific heat gives results compatible with (14.51), but that they are less accurate. The values of the specific heat around its maximum are more spread and more noisy than the ones of the magnetic susceptibility.

From the maxima of the magnetic susceptibility $\chi_{\max}(L)$ we can calculate the exponent γ/ν. Their values are shown in table 14.5. The data are fitted to aL^b, according to the asymptotic relation (14.9), with a and b being fitting parameters. We find very good scaling, therefore our data are in the asymptotic region. The result is

$$\frac{\gamma}{\nu} = 1.749 \pm 0.001\,, \tag{14.51}$$

which is consistent with the analytically computed value $7/4$.

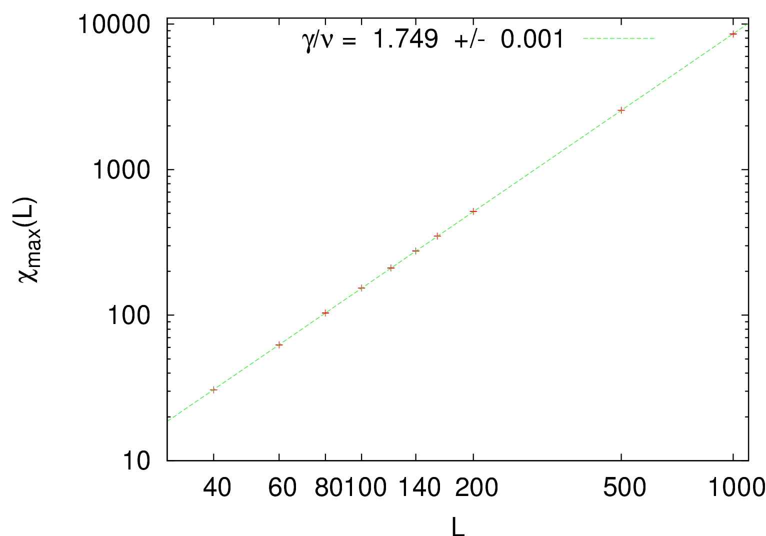


Figure 14.25: Calculation of the critical exponent γ/ν from the maxima of the magnetic susceptibility using the asymptotic scaling (14.9). The values $\chi_{\mathrm{max}}(L)$ are taken from table (14.5) and are fitted to a function of the form aL^b. The result of the fit is $\gamma/\nu = 1.749(1)$.

From the maxima of the specific heat we can calculate the exponent α/ν. Since $\alpha = 0$, the form of the asymptotic behavior is given by (14.47). We find that our results are not very well fitted to the function $a \log L$ and it is possible that the discrepancy is due to finite size effects. We add terms that are subleading in L and find that the fit to the function $a \log L + b - c/L$ is very good[29]. If we attempt to fit the data to the function $aL^d + b - c/L$, the quality of the fit is poor and the result for d is consistent with zero. The results are shown in figure 14.26.

[29]Our ansatz is justified by the analytic calculations in [73], which compute the corrections to the $\log L$ behavior. These corrections are shown to be given by integer powers of $1/L$: $c = a \log L + \sum_{k=0}^{\infty} c_k/L^k$.

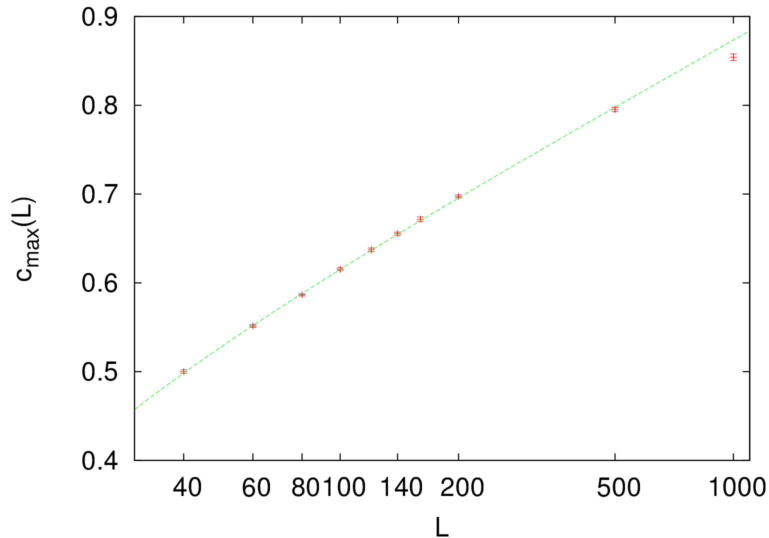


Figure 14.26: Calculation of the critical exponent α/ν from the maxima of the specific heat using the asymptotic scaling (14.8). The values $c_{\max}(L)$ are taken from table (14.5) and are fitted to a function of the form $a \log L + b - c/L$. We obtain $a = 0.107(3)$, $b = 0.13(1)$ and $c = 1.2(3)$ with $\chi^2/\text{dof} = 0.9$ by fitting for $L = 40, \ldots, 500$. The fit to $aL^d + b - c/L$ gives $d = 0.004(97)$, i.e. an exponent consistent with 0 and somehow weird values for the parameters a and b. We conclude that the data is consistent with $\alpha/\nu = 0$.

14.10 Studying Scaling with Collapse

The scaling relations (14.3)–(14.9) are due to the appearance of a unique, dynamical length scale, the correlation length[30] ξ. As we approach the critical point, ξ diverges as $\xi \sim |t|^{-\nu}$, and we obtain universal behavior for all systems in the same universality class. If we consider the magnetic susceptibility $\chi(\beta, L)$, its values depend both on the temperature β, the size of the system L and of course on the details of the system's degrees of freedom and their dynamics. Universality leads to the *assumption* that the magnetic susceptibility of the infinite system in the critical region depends *only* on the correlation length ξ. For the finite system in the pseudocritical region, finite size effects suppress the fluctuations when $\xi \sim L$. The length scales that determine the dominant scaling behavior

[30]Careful: $\xi = \xi(t)$ is the correlation length of the *infinite* system at temperature t and not the correlation length at finite L.

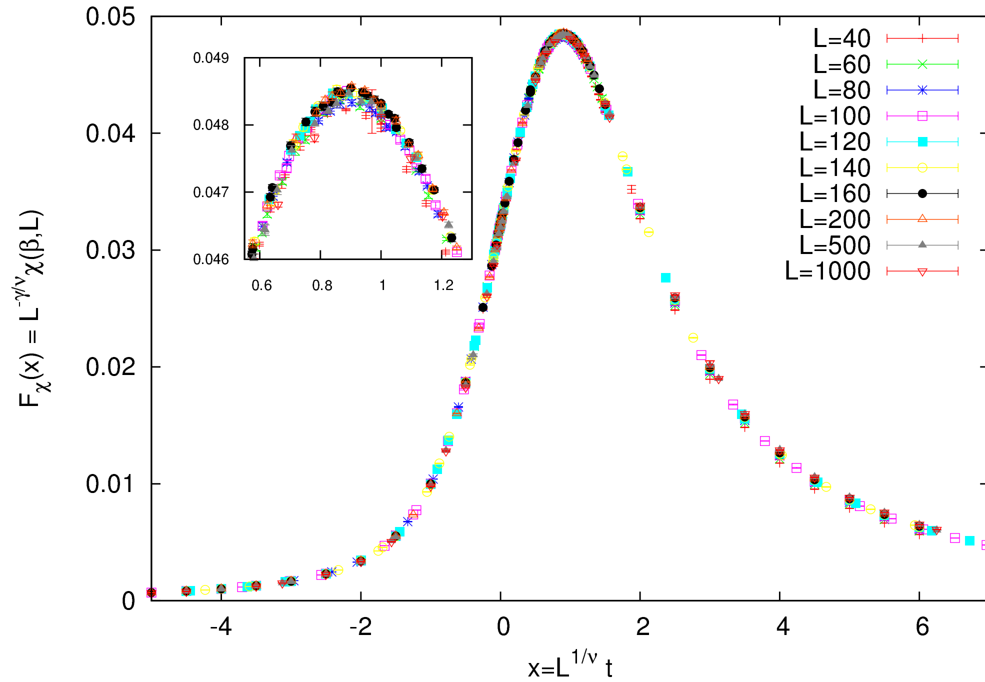


Figure 14.27: Collapse of the plots $\chi(\beta, L)$ for several values of L according to equation (14.59). The known values $\beta_c = \ln(1+\sqrt{2})/2$, $\nu = 1$ and $\gamma/\nu = 7/4$ have been used.

$\chi \sim \xi^{\gamma/\nu}$ are ξ and L, therefore the *dimensionless* variable L/ξ is the only independent variable in the scaling functions. In order to obtain the scaling relation $\chi \sim \xi^{\gamma/\nu}$, valid for the infinite system, we only need to assume that for the finite system[31]

$$\chi = \chi(\beta, L) = \xi^{\gamma/\nu} F_\chi^{(0)}(L/\xi)\,, \tag{14.52}$$

where $F_\chi^{(0)}(z)$ is a function of *one* variable, such that

$$F_\chi^{(0)}(z) = \text{const.} \qquad z \gg 1\,, \tag{14.53}$$

and

$$F_\chi^{(0)}(z) \sim z^{\gamma/\nu} \qquad z \to 0\,. \tag{14.54}$$

[31]For more details see appendix 14.12. The β dependence in $\chi(\beta, L)$ enters through the dependence $\xi(\beta)$ of the correlation length of the infinite system in β.

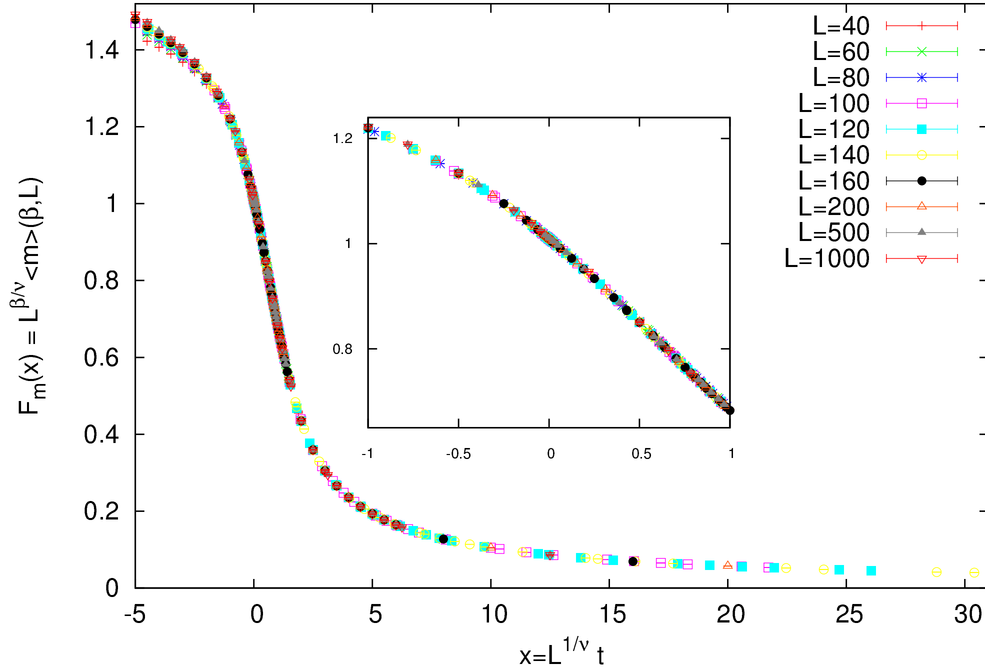


Figure 14.28: Collapse of the plots $\langle m\rangle(\beta, L)$ for several values of L according to equation (14.60). The known values $\beta_c = \ln(1+\sqrt{2})/2$, $\nu = 1$ and $\beta/\nu = 1/8$ have been used.

Indeed, when $1 \ll \xi \ll L$ ($z \gg 1$) the magnetic susceptibility takes values very close to those of the infinite system, and (14.53) gives $\chi \sim \xi^{\gamma/\nu}$. As $\xi \sim L$, finite size effects enter and (14.54) gives $\chi \sim \xi^{\gamma/\nu}(L/\xi)^{\gamma/\nu} = L^{\gamma/\nu}$. The latter is nothing but (14.7) for the maxima of the magnetic susceptibility of the finite system that we studied in figure 14.25. Therefore the function $F_\chi^{(0)}(z)$ describes how the magnetic susceptibility deviates from scaling due to finite size effects.

The function $F_\chi^{(0)}(z)$ can be calculated using the measurements coming from the Monte Carlo simulation. Since the correlation length is not directly calculated, but appears indirectly in the measurements, we define the dimensionless variable

$$x = L^{1/\nu} t \,, \tag{14.55}$$

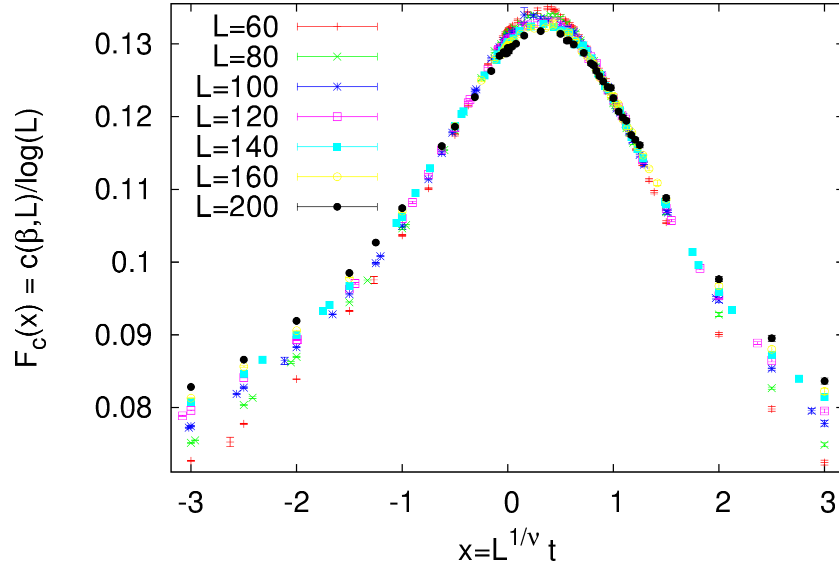


Figure 14.29: Collapse of the plots $c(\beta, L)$ for several values of L according to equation (14.61). The known values $\beta_c = \ln(1+\sqrt{2})/2$ and $\nu = 1$ have been used.

where $|x| \sim (L/\xi)^{1/\nu}$ since[32] $\xi \sim |t|^{-\nu}$. We define $F_\chi(x) \propto x^{-\gamma} F_\chi^{(0)}(x^\nu)$ so that (14.52) becomes

$$\chi = L^{\gamma/\nu} F_\chi(x) = L^{\gamma/\nu} F_\chi(L^{1/\nu} t)\,. \tag{14.56}$$

The asymptotic properties of the scaling function $F_\chi(x)$ are determined by the relations (14.53) and (14.54). When $x = L^{1/\nu} t \gg 1$, equation (14.53) is valid for $F_\chi^{(0)}(x^\nu)$ and we obtain $F_\chi^{(0)}(x^\nu) = \text{const}$. From the definition $F_\chi(x) = x^{-\gamma} F_\chi^{(0)}(x^\nu)$ we obtain $F_\chi(x) \sim x^{-\gamma} = (L/\xi)^{-\gamma/\nu}$ and we confirm the scaling property of the magnetic susceptibility in the thermodynamic limit $\chi \sim L^{\gamma/\nu} F_\chi(x) \sim L^{\gamma/\nu}(L/\xi)^{-\gamma/\nu} = \xi^{\gamma/\nu}$. Therefore

$$F_\chi(x) \sim x^{-\gamma} \qquad x \gg 1\,. \tag{14.57}$$

When $x \to 0$, (14.54) is valid and we have that $F_\chi^{(0)}(x^\nu) \sim (x^\nu)^{\gamma/\nu} = x^\gamma$. Then we obtain $F_\chi(x) \propto x^{-\gamma} F_\chi^{(0)}(x^\nu) \sim x^{-\gamma} x^\gamma = \text{const}$. Therefore, we

[32] The absolute value is dropped in the definition of x so that we have a convenient notation for temperatures above and below the critical temperature.

confirm that, when finite size effects are dominant ($x \to 0$), we have that $\chi = L^{\gamma/\nu} F_\chi(x) \sim L^{\gamma/\nu}$. Therefore

$$F_\chi(x) \sim \text{const.} \qquad |x| \ll 1 \,. \tag{14.58}$$

By inverting equation (14.56), we can calculate the scaling function from the measurements of the magnetic susceptibility

$$F_\chi(L^{1/\nu} t) = L^{-\gamma/\nu} \chi(\beta, L) \,, \tag{14.59}$$

where $\chi(\beta, L)$ are measurements for temperatures in the pseudocritical region for several values of L. When equation (14.59) is valid, then all the measurements fall onto the same curve $F_\chi(x)$ independently of the size L! Of course deviations due to finite size effects are expected, especially when L is small. But, as we will see, convergence is quite fast.

Using the above procedure, we can determine the critical temperature β_c, the exponent ν and the ratio γ/ν simultaneously! In order to check (14.59), we have to compute the variable $x = L^{1/\nu} t$, for which it is necessary to know β_c ($t = (\beta_c - \beta)/\beta_c$) and the exponent ν. For the calculation of F_χ it is necessary to know γ/ν that appears on the right hand side of (14.59). Relation (14.59) depends quite sensitively on the parameters β_c, ν and γ/ν and this way we obtain an accurate method for their calculation.

In order to do the calculation, we set initial values for the parameters $(\beta_c, \nu, \gamma/\nu)$. Using L, β, β_c and ν, we calculate the scaling variable $x = L^{1/\nu} t = L^{1/\nu}(\beta_c - \beta)/\beta_c$. Using $\chi(\beta, L)$ and γ/ν, we calculate $F_\chi = \chi(\beta, L)/L^{\gamma/\nu}$ and plot the points $(x_i, F_\chi(x_i))$ near the critical region $t \approx 0$. Then we vary $(\beta_c, \nu, \gamma/\nu)$ until the curves for different L collapse onto each other. The optimal collapse determines $(\beta_c, \nu, \gamma/\nu)$.

The collapse of the curves that are constructed from the points $(L_i^{1/\nu}(\beta_c - \beta_i)/\beta_c, L_i^{-\gamma/\nu} \chi(\beta_i, L_i))$ for different L is the most efficient method for studying scaling in the critical region. Figure 14.27 shows the function $F_\chi(x)$ for the known values of the parameters $(\beta_c, \nu, \gamma/\nu) = (\ln(1 + \sqrt{2})/2, 1, 7/4)$. Small variations of the parameters lead to a sharp change of the quality of the collapse. We can make a quick and dirty estimate of the accuracy of the method by varying *one* of the parameters, and look for a visible deviation from all data collapsing onto a single curve. The

result is

$$\begin{aligned}\beta_c &= 0.44069 \pm 0.00001 \\ \nu &= 1.00 \pm 0.01 \\ \frac{\gamma}{\nu} &= 1.750 \pm 0.002\,,\end{aligned}$$

Notice that, this crude estimate yields results whose accuracy is comparable to the previously calculated ones!

A similar procedure can be followed for other scaling observables, like the specific heat and the magnetization. Equations (14.8) and (14.9) generalize to[33]

$$\langle m\rangle(\beta, L) = L^{-\beta/\nu} F_m(L^{1/\nu} t)\,, \qquad (14.60)$$

and

$$c(\beta, L) = L^{\alpha/\nu} F_c(L^{1/\nu} t) = \log(L) F_c(L^{1/\nu} t)\,, \qquad (14.61)$$

since $\alpha = 0$. The results are shown in figures 14.28 and 14.29 respectively.

Below, we list a `gnuplot` program in order to construct plots like the ones shown in figures 14.27–14.29. If we assume that the data are in a file named `all` in the following format[34]:

```
# ##########################################################
# e L beta <e>   +/- err c   +/- err
# m L beta <m>   +/- err chi +/- err
# n L beta <n>/N +/- err
# ----------------------------------------------------------
....
e 1000 0.462721 -0.79839031 7.506e-07 0.290266 0.00027
m 1000 0.462721  0.82648701 1.384e-06 2.137    0.00179
....
```

where the lines starting with the character `m` contain $(L, \beta, \langle m\rangle, \delta\langle m\rangle, \chi, \delta\chi)$ whereas the ones starting with `e` contain $(L, \beta, \langle e\rangle, \delta\langle e\rangle, c, \delta c)$. The program can be found in the file `scale_gamma.gpl`:

```
# Usage:
# Ls = "40 60 80 100 120 140 160 200 500 1000"
```

[33] In this relation β on the left hand side is the temperature, whereas on the right hand side the critical exponent in (14.5).

[34] This file can be found in the accompanying software, also named `all`.

```
# bc = bcc; nu = 1 ; gnu = 1.75; load "scale_gamma.gpl";
# Ls:  the values of L used in the collapse
# bc:  the critical temperature used in the calculation of
#      t=(beta_c-beta)/beta_c
# nu:  the exponent used in the calculation of x=L^{1/nu} t
# gnu: the exponent used in the calculation of
#      F_chi = L^{-gnu} chi(beta,L)

#the exact critical temperature (use bc=bcc is you wish):
bcc   = 0.5*log(1.0+sqrt(2.0));
NLs   = words(Ls);  # The number of lattice sizes
LL(i) = word (Ls,i);# Returns the i_th lattice size
cplot(i) = sprintf("\
   <grep 'm %s ' all|\
   sort -k 3,3g|\
   awk -v L=%s -v bc=%f -v nu=%f -v gnu=%f \
   '{print L^(1.0/nu)*(bc-$3)/bc,L^(-gnu)*$6,L^(-gnu)*$7}'\
                ",LL(i),LL(i),bc,nu,gnu);

set macros
set term qt enhanced

set title sprintf("b_c= %f nu= %f g/n= %f",bc,nu,gnu)
set xlabel "x=L^{1/nu} t"
set ylabel "F(x) = L^{-g/n} chi({/Symbol b},L)"

plot for[i=1:NLs] cplot(i) u 1:2:3 w e t sprintf("L=%s",LL(i))
```

In order to use the above program, we give the `gnuplot` commands

```
gnuplot> Ls = "40 60 80 100 120 140 160 200 500 1000"
gnuplot> bc = 0.4406868; nu = 1 ; gnu = 1.75;
gnuplot> load "scale_gamma.gpl"
```

The first two lines define the parameters of the plot. The variable `Ls` contains all the lattice sizes that we want to study, each value of L separated from another by one or more spaces. The variables `bc`, `nu`, `gnu` are the parameters β_c, ν and γ/ν that will be used in the scaling relation (14.59). The third command calls the program that makes the plot. If we need to vary the parameters, then we redefine the corresponding variables and ... repeat.

In order to dissect the above program, look at the online help manual

of `gnuplot`[35]. We will concentrate on the construction of the `awk` filter that computes the points in the plot properly normalized. The value of the function `cplot(i)` is a *string* of characters which varies with L (the index `i` corresponds to the `i`-th word of the variable `Ls`). For each `i`, this string is substituted in the `plot` command, and it is of the form `"< grep ... L**(-gnu)*$7}'"`. The values of the parameters are passed using the function `sprintf()`, which is called each time with a different value of `i`. The dirty work is done by `awk`, which calculates each point (columns 6 and 7: `$6, $7` are χ and its error $\delta\chi$). For *a given value* of L, the `grep` component of the filter prints the lines of the file `all` which contain the magnetization. The `sort` component sorts data in the order of increasing temperature (column 3: `-k3,3g`)

Can we make the above study more systematic and apply quantitative criteria for the quality of the collapse, which will help us estimate the error in the results? One crude way to estimate errors is to split the data in n_b bins and work independently on each set of data. Each bin will give an optimal set of parameters $(\beta_c, \nu, \gamma/\nu)$ which will be assumed to be an independent measurement. The errors can be calculated using equation (13.39) (for $n = n_b$).

In order to provide a quantitative measure of the quality of the collapse, we define a χ^2/dof similar to the one used in data fitting, as discussed in appendix 13.7. When the distance between the collapsing curves increases, this χ^2/dof should increase. Assume that our measurements consist of N_L sets of measurements for $L = L_1, L_2, \ldots, L_{N_L}$. After setting the parameters $\mathbf{p} \equiv (\beta_c, \nu, \gamma/\nu)$ and an interval $\Delta x \equiv [x_{\text{min}}, x_{\text{max}}]$, we calculate the data sets $\{(x_{i,k}, F_\chi(x_{i,k}; \mathbf{p}, L_i))\}_{k=1,\ldots,n_i}$, for all $x_{i,k} \in \Delta X$, using our measurements. The data sets consist of n_i points of the data for $L = L_i$, for which the x_k are in the interval Δx. For each point x_k, we calculate the scaling function $F_\chi(x_{i,k}; \mathbf{p}, L_i) = L_i^{-\gamma/\nu}\chi(\beta_{i,k}, L_i)$, which depends on the chosen parameters $\mathbf{p}$ and the lattice size L_i. Then, we have to choose an interpolation method, which will define the interpolation *functions* $F_\chi(x; \mathbf{p}, L_i)$[36], so that we obtain a *good* estimate of the scaling function between two data points. Then, each point $\{(x_{i,k}, F_\chi(x_{i,k}; \mathbf{p}, L_i))\}_i$

[35] In order to look for help from `gnuplot`'s online help system, use the commands `help word, help words, help macros, help sprintf, help plot iteration`

[36] This can be a polynomial interpolation, a cspline interpolation or one of its generalizations or multihistogramming. The last one is slightly more involved but carries smaller systematic errors.

in a data set has a well defined distance from every other data set j ($j = 1, \ldots, N_L$ and $j \neq i$), which is defined by the distance from the interpolating function of the other sets. This is equal to[37] $|F_\chi(x_{i,k}; \mathbf{p}, L_i) - F_\chi(x_{i,k}; \mathbf{p}, L_j)|$. We define the quantity

$$\chi^2(\mathbf{p}; \Delta x) = \frac{1}{N_L(N_L-1)n_{\text{points}}} \times \sum_{i=1}^{N_L} \sum_{\{j=1, j\neq i\}}^{N_L} \sum_{k=1}^{n_i} \frac{(F_\chi(x_{i,k}; \mathbf{p}, L_i) - F_\chi(x_{i,k}; \mathbf{p}, L_j))^2}{(\delta F_\chi(x_{i,k}; \mathbf{p}, L_i))^2}, \tag{14.62}$$

where $n_{\text{points}} = \sum_{i=1}^{N_L} n_i$ is the number of terms in the sum. The normalization constant $N_L(N_L-1)$ is used, because this is the number of pairs of curves in the sum. Each term is weighted by its error $\delta F_\chi(x_{i,k}; \mathbf{p}, L_i)$, so that points with small error have a larger contribution than points with large error. This is the definition used in [74], but you can see other approaches in [4], [72].

The $\chi^2(\mathbf{p}; \Delta x)$ depends on the parameters $\mathbf{p}$ and the interval Δx. Initially, we keep Δx fixed and perform a minimization with respect to the parameters $\mathbf{p}$. The minimum is given by the values $\mathbf{p}_{\text{min}}$ and these values are the estimators that we are looking for. In order to calculate the errors $\delta\mathbf{p}$ we can bin our data according to the discussion on page 682. Alternatively, we may assume a χ^2 distribution of the measurements, and if the minimum $\chi^2 \lesssim 1$, then the intervals of the parameter values that keep $\chi^2 \lesssim 2$ give an estimate of the errors in the parameters.

The results depend on the chosen interval Δx. Usually [72], this is chosen so that its center is at the maximum of $F_\chi(x)$ so that $\Delta x = [x_{\text{max}} - \delta x, x_{\text{max}} + \delta x]$. If δx is larger than it should, then $\chi^2(\mathbf{p}_{\text{min}}; \Delta x)$ is large and we don't have good scaling. If it is too small, then the errors $\delta\mathbf{p}$ will be large. By taking the limit $\delta x \to 0$, we calculate $\mathbf{p}$ by studying the convergence of $\mathbf{p}_{\text{min}}$ to stable, optimal with respect to error, values (see figure 8.7, page 238 in [4] as well as [72]).

[37] You can see the necessity of the interpolation, since the value $x_{i,k}$ most likely doesn't exist in the data set j.

14.11 Binder Cumulant

Up to now, we have studied fluctuations of observables by computing second order cumulants[38]. The calculation of the critical temperature, the order of the phase transition and the critical exponents can also be done by computing higher order cumulants and sometimes this calculation gives more accurate and clear results.The most famous one is the Binder cumulant which is a fourth order cumulant[39], and its name derives from Kurt Binder who studied it first [75,76],

$$U = 1 - \frac{\langle m^4 \rangle}{3\langle m^2 \rangle}\,. \tag{14.63}$$

Appendix 14.12 discusses its properties in detail. For a continuous phase transition

$$U = \begin{cases} 0 & \beta \ll \beta_c \\ U^* & \beta = \beta_c \\ \frac{2}{3} & \beta \gg \beta_c \end{cases}\,, \tag{14.64}$$

where for the Ising model on a square lattice $U^* = 0.610690(1)$ [76]. The value $U = 0$ corresponds to the Gaussian distribution, whereas the value $U = 2/3$ corresponds to two Gaussian distributions of small width around two symmetric values $\pm\langle m \rangle$ (see problems 14 and 15).

It practice, it is found that finite size corrections to U^* are small, therefore the calculation of $U(\beta, L)$ gives an accurate measurement of the critical temperature β_c. The curves $U(\beta, L)$ intersect at the point (β_c, U^*) for different L and this point gives a very good estimate of β_c.

Figure 14.30 shows our measurements for $U(\beta, L)$. The intersection of the curves in the figure at a single point (β_c, U^*) is impressively accurate. Table 14.6 shows an attempt to calculate β_c systematically by computing the critical temperature from the intersection of the curves $U(\beta, L)$ for three values of L. By taking into account all the measurements for $L = 100 - 1000$ the computed result is

$$\beta_c = 0.440678(9) \qquad U^* = 0.6107(4)\,, \tag{14.65}$$

[38] http://en.wikipedia.org/wiki/Cumulant,
http://mathworld.wolfram.com/Cumulant-GeneratingFunction.html

[39] In statistics, the 4th order cumulant of a random variable x is equal to $\kappa_4 = \langle (x - \langle x \rangle)^4 \rangle - 3\langle (x - \langle x \rangle)^2 \rangle$ and $\kappa_2 = \langle (x - \langle x \rangle)^2 \rangle$, so that $U = -\kappa_4/3\kappa_2$ for $x = m$ and $\langle m \rangle = 0$.

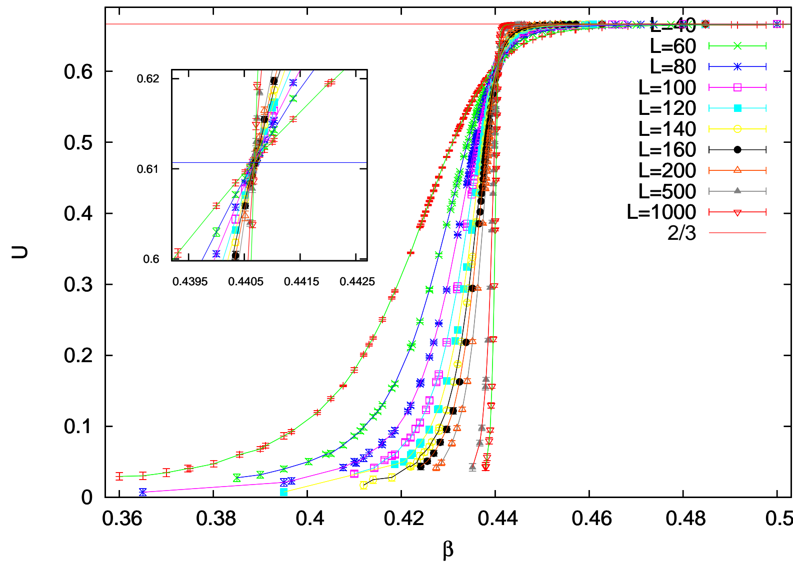


Figure 14.30: Binder cumulant for the Ising model on the square lattice for different temperatures and lattices sizes. The horizontal line is the expected value $U^* = 0.610690(1)$ [76].

which is in a very good agreement with the expected values $\beta_c = 0.44068679\ldots$, $U^* = 0.610690(1)$. Notice that, in the calculation of U^* the systematic error due to finite size effects decreases with increasing L, whereas the statistical error increases due to the increase of the slope of the curves $U(\beta, L)$ near the point $\beta = \beta_c$. But the accuracy of the calculation of β_c turns out to be better with increasing L.

Finite size scaling can also be applied to the Binder cumulant in order to calculate β_c and $1/\nu$. From equation (14.90) (14.119) of appendix 14.12, we expect that U scales as

$$U = F_U(x) = F_U(L^{1/\nu}t)\,. \tag{14.66}$$

This is confirmed in figure 14.31. From the value $F_U(x = 0)$, we obtain $U^* = 0.6107(4)$, which is consistent with the result (14.65).

The numerical calculation of critical exponents, and especially $1/\nu$, can be hard in the general case. Therefore it is desirable to cross check results using several observables which have known scaling behavior.

L			β_c	U^*
40	60	80	0.44069(4)	0.6109(5)
60	80	100	0.44069(4)	0.6108(7)
80	100	120	0.44068(4)	0.6108(7)
100	120	140	0.44069(4)	0.6108(11)
120	140	160	0.44069(4)	0.6109(20)
140	160	200	0.44067(3)	0.6102(12)
160	200	500	0.44067(2)	0.6105(10)
200	500	1000	0.44068(1)	0.6106(9)

Table 14.6: The calculation of β_c and U^* from the intersection of the curves $U(\beta, L)$ for fixed L shown in figure 14.30. Each calculation uses three values of L. The expected values from the theory and the bibliography [76] are $\beta_c = 0.44068679\ldots$ and $U^* = 0.610690(1)$ respectively.

We discuss some of them below[40]. They involve the correlations of the magnetization with the energy.

The derivative of the Binder cumulant is

$$D_U = \frac{\partial U}{\partial \beta} = \frac{\langle m^4 E\rangle\langle m^2\rangle + \langle m^4\rangle\left(\langle m^2\rangle\langle E\rangle - 2\langle m^2 E\rangle\right)}{3\langle m^2\rangle^3}\,. \tag{14.67}$$

Its scaling is given by equation (14.120)

$$D_U = L^{1/\nu} F_{D_U}(x) = L^{1/\nu} F_{D_U}(L^{1/\nu} t)\,, \tag{14.68}$$

which us plotted in figure 14.32. Notice that D_U defines a pseudocritical region around its maximum. The scaling of the maximum as well as the scaling of its position can be used in order to compute $1/\nu$, as we did in figures 14.23 and 14.25 for the magnetic susceptibility.

It could also turn out to be useful to study correlation functions of the form

$$D_{\ln m^n} = \frac{\partial \ln\langle m^n\rangle}{\partial \beta} = \langle E\rangle - \frac{\langle E m^n\rangle}{\langle m^n\rangle}\,, \tag{14.69}$$

whose scaling properties are given by equation (14.126) of appendix 14.12,

$$D_{\ln m^n} = L^{1/\nu} F_{D_{\ln m^n}}(x) = L^{1/\nu} F_{D_{\ln m^n}}(L^{1/\nu} t)\,. \tag{14.70}$$

[40]These have been particularly successful in the study of the 3d Ising model [78].

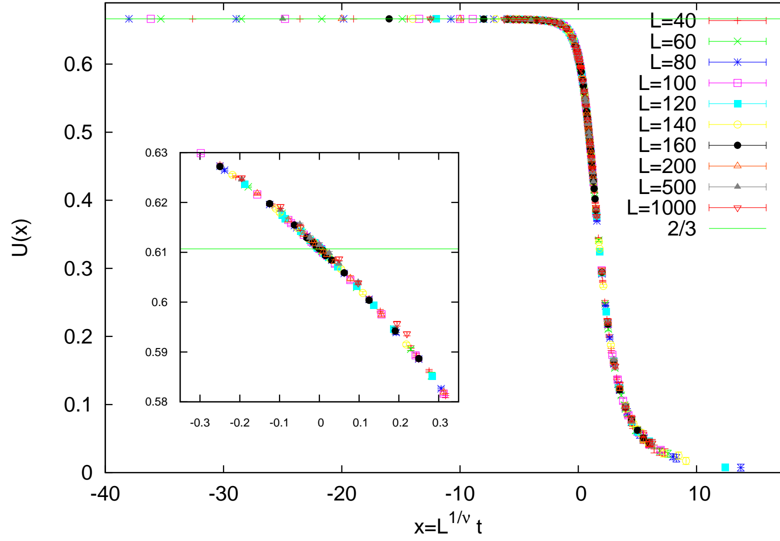


Figure 14.31: Scaling of the Binder cumulant for $1/\nu = 1$ and by using the exactly known critical temperature β_c in $t = (\beta_c - \beta)/\beta_c$. The inset zooms in the critical region. The horizontal line is the expected result $U^* = 0.610690(1)$ [76].

In particular we are interested in the case $n = 1$

$$D_{\ln|m|} = \frac{\partial \ln\langle|m|\rangle}{\partial\beta} = \langle E\rangle - \frac{\langle E|m|\rangle}{\langle|m|\rangle}\,, \tag{14.71}$$

and $n = 2$

$$D_{\ln m^2} = \frac{\partial \ln\langle m^2\rangle}{\partial\beta} = \langle E\rangle - \frac{\langle Em^2\rangle}{\langle m^2\rangle}\,. \tag{14.72}$$

We also mention the energy cumulant V

$$V = 1 - \frac{\langle e^4\rangle}{3\langle e^2\rangle^2}\,. \tag{14.73}$$

In [79], it is shown that for a second order phase transition $V^* = 2/3$, whereas for a first order phase transition, we obtain a non trivial value. Therefore, this parameter can be used in order to determine whereas a system undergoes a first order phase transition. This is confirmed in

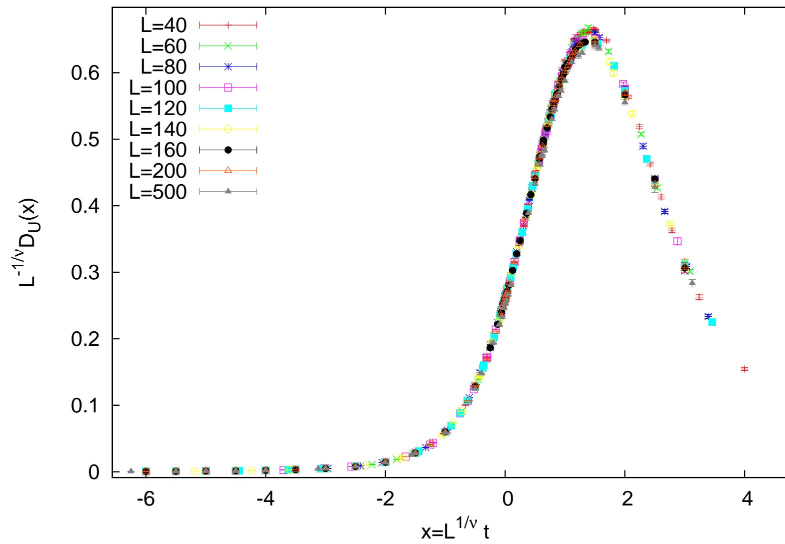


Figure 14.32: Scaling of the derivative of the Binder cumulant D_U (see equation (14.67)) for $1/\nu = 1$ and β_c equal to its known value in $t = (\beta_c - \beta)/\beta_c$.

figure 14.35. The minima of the curves $V(\beta, L)$ converge to the critical temperature according to (14.50).

14.12 Appendix: Scaling

14.12.1 Binder Cumulant

In section 14.11, we studied the scaling properties of the Binder cumulant

$$U = 1 - \frac{\langle m^4 \rangle}{3\langle m^2 \rangle^2} \tag{14.74}$$

numerically. In this appendix, we will use the general scaling properties of a system that undergoes a continuum phase transition near its critical temperature, in order to derive the scaling properties of U and its derivatives. For more details, the reader is referred to [76], [6].

The values of U are trivial in two cases: When the magnetization follows a Gaussian distribution, which is true in the high temperature,

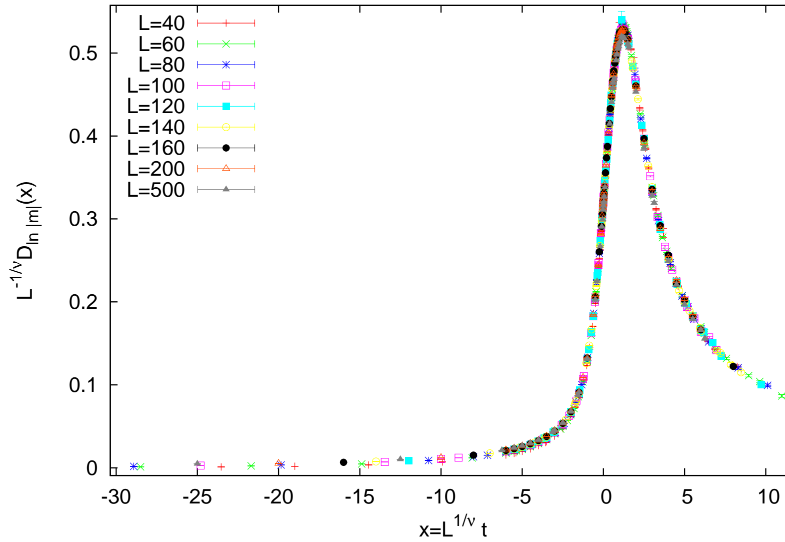


Figure 14.33: Scaling of $D_{\ln|m|}$ (see equation (14.71)) for $1/\nu = 1$ using the exact value of β_c in $t = (\beta_c - \beta)/\beta_c$.

disordered phase, we have that $U = 0$. When we are in the low temperature, ordered phase, we have that $U = 2/3$. The proof is easy and it is left as an exercise (see problems 14 and 15).

According to the discussion in chapter 14, when the critical temperature β_c of a continuum phase transition is approached, the system exhibits scaling properties due to the diverging correlation length ξ. If we approach β_c from the high temperature phase, then we expect that the distribution function of the magnetization per site s (not its absolute value) is approximately of the form

$$P(L,s) = \frac{1}{\sqrt{2\pi\langle s^2\rangle}} \mathrm{e}^{-\frac{s^2}{2\langle s^2\rangle}} = \left(\frac{\beta L^d}{2\pi\chi}\right)^{\frac{1}{2}} \mathrm{e}^{-s^2\frac{L^d\beta}{2\chi}}\,, \tag{14.75}$$

which is a Gaussian with standard deviation $\sigma^2 = \langle s^2\rangle = \chi/(\beta L^d)$. We have temporarily assumed that the system is defined on a d–dimensional hypercubic lattice of edge L.

When the critical temperature is approached, the distribution function

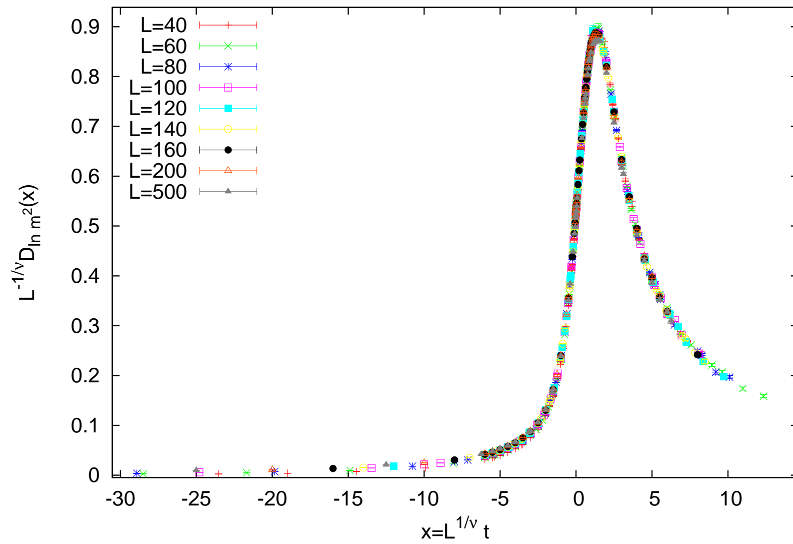


Figure 14.34: Scaling of $D_{\ln m^2}$ (see equation (14.72)) for $1/\nu = 1$ using the exact value of β_c in $t = (\beta_c - \beta)/\beta_c$.

$P(L, s)$ scales according to the relation [76]

$$P(L, s) = L^x p_0 \tilde{P}(aL^y s, \frac{L}{\xi}) \,, \tag{14.76}$$

where $\xi = \xi(t) = \lim_{L\to\infty} \xi(\beta, L)$, $t = (\beta_c - \beta)/\beta_c$, is the correlation length *in the thermodynamic limit*. As we approach the critical point, $\lim_{t\to 0} \xi(t) = +\infty$, in such a way that $\xi \sim |t|^{-\nu}$. Equation (14.76) is a *scaling hypothesis* which plays a fundamental role in the study of critical phenomena.

In order to calculate the exponents in equation (14.76), we apply the

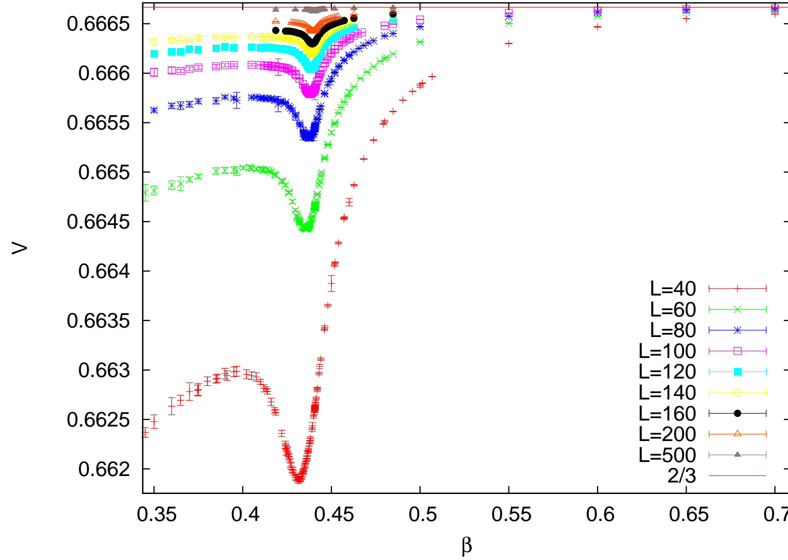


Figure 14.35: The energy cumulant defined by equation (14.73). As L is increased, its value converges to $2/3$, as expected for a second order phase transition. The position of the minima converge to the critical temperature as $L^{-1/\nu}$

normalization condition of a probability distribution function

$$\begin{aligned} 1 &= \int_{\infty}^{+\infty} ds\, P(L,s) \\ &= L^x p_0 \int_{\infty}^{+\infty} ds\, \tilde{P}(aL^y s, \frac{L}{\xi}) \\ &= L^x p_0 \frac{1}{aL^y} \int_{\infty}^{+\infty} dz\, \tilde{P}(z, \frac{L}{\xi}) \\ &= L^{x-y} p_0 \frac{1}{a} \int_{\infty}^{+\infty} dz\, \tilde{P}(z, \frac{L}{\xi})\,, \end{aligned} \tag{14.77}$$

where we set $z = aL^y s$. For the left hand side to be equal to one, we must have that $x = y$,

$$C_0 \equiv \int_{\infty}^{+\infty} dz\, \tilde{P}(z, \frac{L}{\xi}) < \infty\,, \tag{14.78}$$

and $p_0 = a/C_0$. $C_0 = C_0(L/\xi)$, $p_0 = p_0(L/\xi)$ and $p_0 C_0 = a$ is a constant

independent of L and ξ. Finally, we obtain

$$P(L,s) = \frac{a}{C_0} L^y \tilde{P}(aL^y s, \frac{L}{\xi})\,. \tag{14.79}$$

The moments of the distribution of the spins $\langle s^k\rangle$ are

$$\begin{aligned}
\langle s^k\rangle &= \int_{\infty}^{+\infty} ds\, s^k\, P(L,s) \\
&= \frac{a}{C_0} L^y \int_{\infty}^{+\infty} ds\, s^k\, \tilde{P}(aL^y s, \frac{L}{\xi}) \\
&= \frac{a}{C_0} L^y \frac{1}{a^{k+1} L^{(k+1)y}} \int_{\infty}^{+\infty} dz\, z^k\, \tilde{P}(z, \frac{L}{\xi}) \\
&= L^{-ky} F_k\left(\frac{L}{\xi}\right)\,,
\end{aligned} \tag{14.80}$$

where the last line is the definition of the function $F_k(x)$. When we *first* take the thermodynamic limit $L\to\infty$ and *then* approach the critical temperature $t\to 0$, the correlation length $\xi\to\infty$ diverges in such a way that $L/\xi\to\infty$. In the region $\beta<\beta_c$ ($\langle m\rangle=0$) we have that $\chi=\beta L^d\langle s^2\rangle$, and by using the relations

$$\left.\begin{array}{rl} \chi = & \chi_+ t^{-\gamma} \\ \xi = & \xi_+ t^{-\nu} \end{array}\right\} \Rightarrow \chi = \frac{\chi_+}{\xi_+^{\gamma/\nu}}\,\xi^{\gamma/\nu} \sim \xi^{\gamma/\nu}\,, \tag{14.81}$$

we obtain

$$\langle s^2\rangle = \beta^{-1} L^{-d}\chi = \frac{\chi^+}{\beta L^d \xi_+^{\gamma/\nu}}\,\xi^{\gamma/\nu} \sim \xi^{\gamma/\nu}\,. \tag{14.82}$$

In the above equations, we introduced the universal amplitudes χ_+ and ξ_+, which are universal constants (i.e. they are the same within a universality class) and they are defined from equation (14.81). In this limit, in order for (14.80) to have consistent scaling for $k=2$ on the right and left hand sides[41], we obtain (compare with equation (14.57))

$$F_2\left(\frac{L}{\xi}\right) \sim \left(\frac{L}{\xi}\right)^{-\gamma/\nu} \qquad \text{for} \qquad \frac{L}{\xi} \gg 1\,. \tag{14.83}$$

[41]i.e. both sides should scale w.r.t to the correlation length as $\sim\xi^{\gamma/\nu}$.

In order to compute the L-scaling, we substitute the above equations to (14.80) for $k = 2$, and we obtain

$$\frac{\chi^+}{\beta L^d \xi_+^{\gamma/\nu}} \xi^{\gamma/\nu} \sim L^{-2y} \left(\frac{L}{\xi}\right)^{-\gamma/\nu} . \tag{14.84}$$

Then, we obtain

$$L^{-d} \sim L^{-2y} L^{-\gamma/\nu} \Rightarrow d = 2y + \frac{\gamma}{\nu} \Rightarrow y = \frac{d\nu - \gamma}{2\nu} = \frac{\beta}{\nu} , \tag{14.85}$$

where we used the known[42] hyperscaling relation

$$d\,\nu = \gamma + 2\beta . \tag{14.86}$$

Finally, we obtain the equations $\beta < \beta_c$,

$$P(L, s) = \frac{a}{C_0} L^{\beta/\nu} \tilde{P}(a L^{\beta/\nu} s, \frac{L}{\xi}) , \tag{14.87}$$

$$\langle s^2 \rangle = L^{-2\beta/\nu} F_2 \left(\frac{L}{\xi}\right) , \tag{14.88}$$

$$\langle s^4 \rangle = L^{-4\beta/\nu} F_4 \left(\frac{L}{\xi}\right) , \tag{14.89}$$

which are valid in the disordered phase $\beta < \beta_c$. From equation (14.74), we find that the critical behavior of the Binder cumulant is

$$U \sim 1 - \frac{L^{-4\beta/\nu} F_4\left(\frac{L}{\xi}\right)}{3L^{-4\beta/\nu} F_2\left(\frac{L}{\xi}\right)^2} = 1 - \frac{1}{3} \frac{F_4\left(\frac{L}{\xi}\right)}{F_2\left(\frac{L}{\xi}\right)^2} . \tag{14.90}$$

Finite size effects dominate in the pseudocritical region, in which case we take the thermodynamic limit $L \to \infty$ keeping L/ξ finite, and the fluctuations get suppressed, rendering the functions $F_k(x)$ finite. Therefore, we obtain[43]

$$\lim_{L\to\infty} U(t = 0, L) \equiv U^* = 1 - \frac{1}{3} \frac{F_4(0)}{F_2(0)^2} = \text{const.} , \tag{14.91}$$

[42] See e.g. [77], equations 3.35, 3.36, 3.53.

[43] At $t = 0$ we have $\xi(0) = +\infty$, therefore for finite L we have that $L/\xi = 0$.

which shows why the value of U at the critical temperature turned out to be almost independent of the system size L. U^* is found to depend on the boundary conditions and on the anisotropy of the interaction. For the Ising model on the square lattice we have that [76] (Kamieniarz+Blöte)

$$U^* = 0.610690(1) \tag{14.92}$$

14.12.2 Scaling

Consider a change of length scale on a lattice so that

$$\xi \to \frac{\xi}{b}\,, \tag{14.93}$$

where ξ is the dimensionless correlation length in the thermodynamic limit and b is the scaling factor. Then, the basic assumption for the scaling of thermodynamics quantities in the region of a continuous phase transition is that the free energy is changed according to [44]

$$f(t,h) = b^{-d} f(tb^{y_t}, hb^{y_h})\,, \tag{14.94}$$

where t is the reduced temperature and h is the external magnetic field[45]. The above relation summarizes the *scaling hypothesis*, and it is a relation similar to (14.76). This relation can be understood through the renormalization group approach, and the fundamental assumption is the appearance of a unique dynamical length scale that diverges as we approach the critical point. The arguments tb^{y_t} and hb^{y_h} give the change in the coupling constants t and h under the change in length scale in order that the equation remains valid.

By applying the above relation n times we obtain

$$f(t,h) = b^{-nd} f(tb^{ny_t}, hb^{ny_h})\,. \tag{14.95}$$

If we take $n \to \infty$, $t \to 0$, keeping the product $tb^{ny_t} = t_0 = \mathcal{O}(1)$ fixed, we obtain

$$\begin{aligned} f(t,h) &= t^{d/y_t} f(t_0, ht^{-y_h/y_t}) \\ &\equiv t^{d/y_t}\Psi(ht^{-y_h/y_t}) \\ &= t^{2-\alpha}\Psi(ht^{-y_h/y_t})\,, \end{aligned} \tag{14.96}$$

[44] More precisely the *singular part* of the free energy.
[45] See e.g. chapter 3 in [77].

where we substituted $b^n \sim t^{-1/y_t}$ and *defined* the scaling function $\Psi(z)$ and the critical exponent

$$\alpha = 2 - \frac{d}{y_t}\,. \tag{14.97}$$

By applying the same reasoning to (14.93) for the correlation length we obtains

$$\xi(t,h) = b^{-1}\xi(tb^{y_t}, hb^{y_h}) = \ldots = b^{-n}\xi(tb^{ny_t}, hb^{ny_h})\,. \tag{14.98}$$

By taking the limit $n \to \infty$, $t \to 0$, keeping the product $tb^{ny_t} \sim \mathcal{O}(1)$, the left hand side will give a finite value, e.g. $\xi_0 < \infty$ whereas the right hand side will give

$$\xi_0 = t^{1/y_t}\xi(t_0, ht^{-y_h/y_t})\,. \tag{14.99}$$

By considering the case $h = 0$, and by comparing to the known relation (14.4) $\xi \sim t^{-\nu}$, we obtain

$$\xi = \xi_0 t^{-1/y_t} \Rightarrow \nu = \frac{1}{y_t}\,. \tag{14.100}$$

By taking the derivative of (14.96) with respect to the temperature, we obtain

$$\frac{\partial f}{\partial t} \sim t^{1-\alpha}\Psi(ht^{-y_h/y_t}) + t^{2-\alpha}ht^{-y_h/y_t-1}\Psi'(ht^{-y_h/y_t})\,. \tag{14.101}$$

We use the notation $\sim$ whenever we neglect terms that are not related to the scaling properties of a function.

By taking the derivative once more, and by setting $h = 0$, we obtain the specific heat

$$c \sim \frac{\partial^2 f}{\partial t^2} \sim t^{-\alpha}\Psi(0)\,. \tag{14.102}$$

Therefore, the critical exponent α is nothing but the critical exponent of the specific heat defined in equation (14.4).

The magnetic susceptibility can be obtained in a similar way by taking the derivative of (14.96) with respect to h

$$\frac{\partial f}{\partial h} \sim t^{d/y_t}t^{-y_h/y_t}\Psi'(ht^{-y_h/y_t}) \sim t^{\nu d-\nu y_h}\Psi'(ht^{-\nu y_h})\,. \tag{14.103}$$

By taking the derivative once more time and by setting $h = 0$ we obtain the magnetic susceptibility

$$\chi \sim \frac{\partial^2 f}{\partial h^2} \sim t^{\nu d - 2\nu y_h} \Psi'(0)\,, \tag{14.104}$$

and, by comparing to (14.3) $\chi \sim t^{-\gamma}$, we obtain

$$\gamma = 2\nu y_h - \nu d \Leftrightarrow y_h = \frac{1}{2}(d + \frac{\gamma}{\nu}) = d - \frac{\beta}{\nu} = \frac{\beta + \gamma}{\nu} \tag{14.105}$$

In the last two equations we used the hyperscaling relations

$$\nu d = \gamma + 2\beta\,. \tag{14.106}$$

14.12.3 Finite Size Scaling

We will now extend the analysis of the previous section to the case of a system of finite size. We will assume that the system's degrees of freedom are located on a lattice whose linear size is $l = La$ (the volume is $V = l^d$, d is the number of dimensions), where L is the (dimensionless) number of lattice sites and a is the lattice constant. We consider the limit $L \to \infty$ and $a \to 0$, so that l *remains constant*. By changing the L-scale

$$L \to \frac{L}{b} \Leftrightarrow L^{-1} \to bL^{-1}\,, \tag{14.107}$$

and

$$a \to ba\,, \tag{14.108}$$

equation (14.94) generalizes to

$$f(t, h, L^{-1}) = b^{-nd} f(tb^{ny_t}, hL^{ny_h}, b^n L^{-1})\,. \tag{14.109}$$

By taking the limit $t \to 0$, $n \to \infty$ and $tb^{ny_t} = t_0 < \infty \Rightarrow b^n \sim t^{-1/y_t}$ (approach of the critical point), the above relation becomes

$$f(t, h, L^{-1}) = t^{d/y_t} f(t_0, ht^{-y_h/y_t}, t^{-1/y_t} L^{-1}) = t^{d/y_t} \Psi(ht^{-y_h/y_t}, t^{-1/y_t} L^{-1})\,. \tag{14.110}$$

By differentiating and setting $h = 0$ as in the previous section we obtain[46]

$$\chi(t, L^{-1}) = \left.\frac{\partial^2 f}{\partial h^2}\right|_{h=0} = t^{-\gamma}\phi_2(L^{-1}t^{-\nu}) = t^{-\gamma}\phi_2(\frac{\xi}{L}) \,, \tag{14.111}$$

where we set $y_t = 1/\nu$, $\phi_2(x) = \Psi^{(2,0)}(0,x) = \partial^2\Psi(z,x)/\partial z^2|_{z=0}$.

The thermodynamic limit is obtained for $L \gg \xi$ where $\phi_2(\frac{\xi}{L}) \to \phi_2(0) < \infty$, which yields the known relation $\chi \sim t^{-\gamma}$.

When L is comparable to ξ, finite size effects dominate. The large fluctuations are suppressed and the magnetic susceptibility has a maximum at a crossover (pseudocritical) temperature $t_X \equiv (\beta_c - \beta_c(L))/\beta_c$, where $t_X \sim L^{-1/\nu}$. The last relation holds because $L \sim \xi \sim t^{-\nu}$ by assumption. We obtain

$$\chi_{\max} \sim t_X^{-\gamma}\phi_2(L^{-1}t_X^{-\nu}) \sim L^{\gamma/\nu}\phi_2(L^{-1}L) \sim L^{\gamma/\nu}\phi_2(1) \sim L^{\gamma/\nu} \,. \tag{14.112}$$

In the region of the maximum, we obtain the functional form

$$\chi(t, L^{-1}) = L^{\gamma/\nu}F_\chi(L^{1/\nu}t) \,, \tag{14.113}$$

which is nothing but equation (14.59). The function $F_\chi(x)$ is analytic in its argument $x = L^{1/\nu}t$, since for a finite system $\chi(t, L^{-1})$ is an analytic function of the temperature[47]. In the thermodynamic limit ($L \to \infty$ and $|t| > 0$, therefore $x \to \infty$)

$$F_\chi(x) \sim x^{-\gamma} \qquad x \gg 1 \,, \tag{14.114}$$

so that $\chi(t, L^{-1}) = L^{\gamma/\nu}F_\chi(L^{1/\nu}t) \sim L^{\gamma/\nu}\,(L^{1/\nu}t)^{\gamma} \sim t^{-\gamma}$. Near the pseudocritical point

$$F_\chi(x) = F_{\chi,0} + F_{\chi,1}x + F_{\chi,2}x^2 + \ldots \qquad x \ll 1 \,, \tag{14.115}$$

and we expect that for $L^{1/\nu}t \ll 1$ we have that

$$\chi(t, L^{-1}) = L^{\gamma/\nu}\left(1 + \chi_1 L^{1/\nu}t + \chi_2 L^{2/\nu}t^2 + \ldots\right) \,. \tag{14.116}$$

The above relations lead to the following conclusions:

[46] We stress again that $\xi \sim t^{-\nu}$ in (14.111) is the correlation length in the thermodynamic limit and not at finite L.

[47] This is because the partition function is an analytic function of the temperature. Therefore it is $x = L^{1/\nu}t$ which is the scaling variable and not a power of it, such as $\tilde{x}$ used in (14.53) and (14.54).

- The pseudocritical point shifts as $\sim L^{-1/\nu}$ (equation (14.50))
- The peak of the magnetic susceptibility increases as $\chi_{\max} \sim L^{\gamma/\nu}$
- The direction of the shifting of the maximum of the magnetic susceptibility depends on the boundary conditions:
  - Periodic boundary conditions suppress the effects of the fluctuations, since the wave vectors are limited by $\frac{2\pi}{L}n$. This increases the pseudocritical temperature $T_c(L)$ ($\beta_c(L) < \beta_c \Rightarrow c > 0$ in (14.50)).
  - Free boundary conditions lead to free fluctuations on the boundary, which decrease the pseudocritical temperature $T_c(L)$ ($\beta_c(L) > \beta_c \Rightarrow c < 0$ in (14.50))
  - Frozen (fixed) spins on the boundary lead to increased order in the system. This increases the pseudocritical temperature $T_c(L)$ ($\beta_c(L) < \beta_c \Rightarrow c > 0$ in (14.50)).

We conclude that $F_\chi(L^{1/\nu}t)$ depends on the boundary conditions and the geometry of the lattice.

Similarly, we obtain

$$\langle m^k \rangle \sim L^{-d}\frac{\partial^k f}{\partial h^k} \sim L^{-d}t^{d/y_t - k y_h/y_t}\phi_k(L^{-1}t^{-\nu}) \sim L^{-d}t^{\nu d - k\nu y_h}\phi_k\left(\frac{\xi}{L}\right)\,, \tag{14.117}$$

and by following similar arguments leading to (14.113), we obtain

$$\langle m^k \rangle \sim L^{-d}L^{-d+ky_h}F_k(L^{1/\nu}t) \sim L^{k\frac{\beta+\gamma}{\nu}}F_k(L^{1/\nu}t)\,. \tag{14.118}$$

For the Binder cumulant we obtain

$$U = 1 - \frac{\langle m^4 \rangle}{3\langle m^2 \rangle^2} \sim 1 - \frac{L^{4y_h}F_4(L^{1/\nu}t)}{3(L^{2y_h}F_2(L^{1/\nu}t))^2} \sim U_* + U_1\cdot(L^{1/\nu}t) + U_2\cdot(L^{1/\nu}t)^2 + \ldots\,, \tag{14.119}$$

where in the last equality we expanded the analytic functions $F_{2,4}(L^{1/\nu}t)$ for small $L^{1/\nu}t$. Then, we see that

$$\frac{\partial U}{\partial \beta} \sim \partial_t U \sim L^{1/\nu}\,. \tag{14.120}$$

By differentiating (14.110) with respect to the temperature we obtain

$$\begin{aligned}
\frac{\partial f}{\partial t} &\sim t^{d/y_t-1}\Psi(ht^{y_h/y_t}, L^{-1}t^{-1/y_t}) \\
&\quad + t^{d/y_t}(ht^{y_h/y_t-1})\Psi^{(1,0)}(ht^{y_h/y_t}, L^{-1}t^{-1/y_t}) \\
&\quad + t^{d/y_t}L^{-1}t^{-1/y_t-1}\Psi^{(0,1)}(ht^{y_h/y_t}, L^{-1}t^{-1/y_t}) \\
&\sim t^{\nu d-1}\Psi(ht^{\nu y_h}, L^{-1}t^{-\nu}) \\
&\quad + ht^{\nu d+\nu y_h-1}\Psi^{(1,0)}(ht^{\nu y_h}, L^{-1}t^{-\nu}) \\
&\quad + L^{-1}t^{\nu d-1-\nu}\Psi^{(0,1)}(ht^{\nu y_h}, L^{-1}t^{-\nu}), \qquad (14.121)
\end{aligned}$$

where we used the notation $\Psi^{(n,m)}(x,z) = \partial^{n+m}\Psi(x,z)/\partial x^n \partial z^m$. The term proportional to h vanishes when we set $h = 0$. In the pseudocritical region, where $t_X \sim L^{-1/\nu}$, the first and third term are of the same order in L and we obtain

$$\left.\frac{\partial f}{\partial t}\right|_{h=0} = L^{-d+\frac{1}{\nu}} F^1(L^{1/\nu}t), \qquad (14.122)$$

and by successive differentiation

$$\left.\frac{\partial^k f}{\partial t^k}\right|_{h=0} = L^{-d+\frac{k}{\nu}} F^k(L^{1/\nu}t). \qquad (14.123)$$

The derivatives

$$\left.\frac{\partial^2 f}{\partial t \partial h}\right|_{h=0} = L^{-d+y_h+\frac{1}{\nu}} F^1_1(L^{1/\nu}t) = L^{\frac{1-\beta}{\nu}} F^1_1(L^{1/\nu}t), \qquad (14.124)$$

$$\left.\frac{\partial^{1+k} f}{\partial t \partial h^k}\right|_{h=0} = L^{-d+ky_h+\frac{1}{\nu}} F^1_k(L^{1/\nu}t). \qquad (14.125)$$

In particular

$$\frac{\langle E\, m^k\rangle}{\langle m^k\rangle} = \frac{\left.\frac{\partial^{1+k} f}{\partial t \partial h^k}\right|_{h=0}}{\left.\frac{\partial^k f}{\partial h^k}\right|_{h=0}} \sim \frac{L^{-d+ky_h+\frac{1}{\nu}}}{L^{-d+ky_h}} \sim L^{1/\nu} \qquad (14.126)$$

$$\frac{\langle e^4\rangle}{\langle e^2\rangle^2} = \frac{L^{-d} \left.\frac{\partial^4 f}{\partial t^4}\right|_{h=0}}{\left(L^{-d} \left.\frac{\partial^2 f}{\partial t^2}\right|_{h=0}\right)^2} \sim \frac{L^{-\frac{4}{\nu}}}{(L^{-\frac{2}{\nu}})^2} \sim \text{const.} \qquad (14.127)$$

14.13 Appendix: Critical Exponents

14.13.1 Definitions

$$\begin{array}{lllll} \alpha: & c \sim t^{-\alpha}, & c_{\text{max}} \sim L^{\alpha/\nu}, & c(t,L) = L^{\alpha/\nu} F^2(L^{1/\nu} t) \\ \beta: & m \sim t^{\beta}, & m \sim L^{-\beta/\nu}, & m(t,L) = L^{-\beta/\nu} F_1(L^{1/\nu} t) \\ \gamma: & \chi \sim t^{\gamma}, & \chi_{\text{max}} \sim L^{\gamma/\nu}, & \chi(t,L) = L^{\gamma/\nu} F_2(L^{1/\nu} t) \\ \nu: & \xi \sim t^{-\nu}, & \xi \sim L, & \\ \delta: & M \sim h^{1/\delta} & & \\ z: & \tau \sim \xi^z & & \end{array} \tag{14.128}$$

The scaling relation

$$f(t,h) = t^{d/y_t} \Psi(h t^{y_h/y_t}) \,, \tag{14.129}$$

defines the exponents y_t, y_h. The relation

$$G(\mathbf{r}, t=0) \sim \frac{1}{r^{d-2+\eta}} \,, \tag{14.130}$$

defines the exponent η coming from the two point correlation function $G(\mathbf{r}, t) = \langle \mathbf{s}(\mathbf{r}) \cdot \mathbf{s}(\mathbf{0}) \rangle$.

14.13.2 Hyperscaling Relations

From the definitions and the hyperscaling relations we have that

$$\begin{aligned} \alpha + 2\beta + \gamma &= 2 \\ \gamma + 2\beta &= \nu d \\ 2 - \nu d &= \alpha \\ \alpha + \beta(1+\delta) &= 2 \\ \nu(2-\eta) &= \gamma \end{aligned} \tag{14.131}$$

$$y_t = \frac{1}{\nu} = \frac{d}{2-\alpha} \qquad y_h = \frac{\beta+\gamma}{\nu} = \frac{1}{2}\left(d + \frac{\gamma}{\nu}\right) = d - \frac{\beta}{\nu} \tag{14.132}$$

$$\alpha = 2 - \frac{d}{y_t} \qquad \beta = \frac{d - y_h}{y_t} \qquad \gamma = \frac{2y_h - d}{y_t} \qquad \delta = \frac{y_h}{d - y_h} \tag{14.133}$$

$$\eta = d + 2 - 2y_h \Leftrightarrow d - 2 + \eta = 2(d - y_h) \tag{14.134}$$

Model	ν	α	β	γ	δ	η	y_t	y_h
q=0 Potts (2d) [69]	∞	$-\infty$	$\frac{1}{6}$	∞	∞	0	0	2
q=1 Potts (2d) [69]	$\frac{4}{3}$	$-\frac{2}{3}$	$\frac{5}{36}$	$2\frac{7}{18}$	$18\frac{1}{5}$	$\frac{5}{24}$	$\frac{3}{4}$	$\frac{91}{48}$
Ising (2d) [69]	1	0	$\frac{1}{8}$	$\frac{7}{4}$	15	$\frac{1}{4}$	1	$\frac{15}{8}$
q=3 Potts (2d) [69]	$\frac{5}{6}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{13}{9}$	14	$\frac{4}{15}$	$\frac{6}{5}$	$\frac{28}{15}$
q=4 Potts (2d) [69, 81]	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{12}$	$\frac{7}{6}$	15	$\frac{1}{4}$	$\frac{3}{2}$	$\frac{15}{8}$
classical (4d) [80]	$\frac{1}{2}$	0	$\frac{1}{2}$	1	3	0	2	3
Spherical (3d) [80]	1	-1	$\frac{1}{2}$	2	5	0	1	$\frac{5}{2}$
Ising (3d) [80]	–	$\frac{1}{8}$	$\frac{5}{16}$	$\frac{5}{4}$	5	–	–	–
Ising (3d) [84]	0.631	$0.108(5)$	$0.327(4)$	$1.237(4)$	$4.77(5)$	0.039	–	–
Heisenberg (3d) [82]	0.70	-0.1	0.36	1.4	5	0.03	–	–
XY (3d) [83]	0.663	–	–	$1.327(8)$	–	–	–	–
AF q=3 Potts (3d) [85]	0.66	-0.011	0.351	1.309	4.73	–	–	–

Table 14.7: Critical exponents of the models referred to in the first column. Whenever the value is shown as a floating point number, the exponents are approximate. For the approximate values we don't apply the hyperscaling relations, but we simply mention the values reported in the bibliography. The values for the 3d Ising model in [80] are a conjecture. For the 3d Ising see also [45] p. 244. 3d XY and 3d AF q=3 Potts are conjectured to belong to the same universality class.

14.14 Problems

The files `all` and `allem` in the accompanying software contain measurements that you can use for your data analysis or compare them with your own measurements.

1. Compute the average acceptance ratio $\bar{A}$ for the Metropolis algorithm as a function of the temperature for $L = 10, 40, 100$. Compute the average size $\langle n \rangle$ of the Wolff clusters at the same values of the temperatures. Then calculate the number of Wolff clusters that are equivalent to a Metropolis sweep. Make the plots of all of your results and connect the points corresponding to the same L.

2. Make the plots in figures 14.6–14.10 and add data for $L = 50$, 120, 140, 160, 180, 200.

3. Make the plots in figures 14.11–14.12 and add data for $L = 50$, 90, 130, 150, 190, 250. Recalculate the dynamic exponent z using your data.

4. Make the plot in figure 14.13 and add data for $L = 30$, 50, 70, 90. Recalculate the dynamic exponent z using your data.

5. Reproduce the results shown in table 14.1. Add a 6th column computing $\tau^A_{m,\mathrm{Metropolis}} = \tau_{m,\mathrm{Metropolis}}\bar{A}$, where $\bar{A}$ is the average acceptance ratio of the Metropolis algorithm. This changes the unit of time to N *accepted* spin flips. These are the numbers that are directly comparable with τ_m.

6. Simulate the 2d Ising model on the square lattice for $L = 10$, 20, 40, 80, 100. Choose appropriate values of β, so that you will be able to determine the magnetic susceptibility and the specific heat with an accuracy comparable to the one shown in table 14.5. In each case, check for the thermalization of the system and calculate the errors.

7. Make the fits that lead to the results (14.38), (14.39), (14.41) and (14.43)

8. Study the scaling of the specific heat as a function of the temperature. Compare the quality of the fits to the functions $a \log|t|$ and

L	$\chi(\beta_c, L)$		$\langle m\rangle(\beta_c, L)$		$c(\beta_c, L)$	
40	20.50	0.02	0.6364	0.0001	0.4883	0.0007
60	41.78	0.08	0.6049	0.0002	0.5390	0.0008
80	69.15	0.09	0.5835	0.0001	0.5743	0.0012
100	102.21	0.25	0.5673	0.0002	0.6026	0.0014
120	140.18	0.11	0.5548	0.0001	0.6235	0.0010
140	183.95	0.33	0.5442	0.0002	0.6434	0.0006
160	232.93	0.55	0.5351	0.0001	0.6584	0.0020
200	342.13	0.72	0.5206	0.0001	0.6858	0.0014
500	1687.2	4.4	0.4647	0.0002	0.7794	0.0018
1000	6245	664	0.4228	0.0040	–	–

Table 14.8: $\chi(\beta_c, L)$, $\langle m\rangle(\beta_c, L)$ and $c(\beta_c, L)$ at the critical temperature for different L used in problem 9.

$a\,|t|^{\alpha}$ by computing the χ^2/dof according to the discussion in appendix 13.7 after page 600.

9. Consider the table 14.8 showing the measurements of $\chi(\beta_c, L)$, $\langle m\rangle(\beta_c, L)$ and $c(\beta_c, L)$. Use the values in this table in order to make the fits which give the exponents γ/ν, β/ν and α as described in the text. For the exponent α, try fitting to a power and a logarithm and compare the results according to the discussion in the text.

10. Consider the table 14.5 which gives the results of the measurements of L, $\beta_c(L)$, χ_{max}, $\beta_c'(L)$ and c_{max}. Make the appropriate fits in order to calculate the exponents $1/\nu$, γ/ν, α/ν and the critical temperature β_c as described in the text. For the exponent α, try fitting to a power and a logarithm and compare the results according to the discussion in the text.

11. Reproduce the collapse of the curves shown in figures 14.27-14.29. Use the data in the file `all` from the accompanying software. Set the appropriate values to the parameters and calculate the scaling functions $F_{\chi,m,c}$. Vary each parameter separately, so that the collapse becomes not satisfactory and use its variation as an estimate of its error. Determine the range in $x = L^{1/\nu}t$ that gives satisfactory collapse of the curves. Repeat your calculation by performing measurements

for $L = 10, 20$, and using the data for $L = 10, 20, 40, 80, 120$. Compare the new results with the previous ones and comment on the finite size effects.

12. Prove that for every observable $\mathcal{O}$ we have that $\partial\langle\mathcal{O}\rangle/\partial\beta = -\langle E\mathcal{O}\rangle + \langle\mathcal{O}\rangle\langle E\rangle = -\langle(E - \langle E\rangle)(\mathcal{O} - \langle\mathcal{O}\rangle)\rangle$. Using this relation calculate the derivative of the Binder cumulant D_U and prove equation (14.67).

13. Use the maximum of the derivative of the Binder cumulant D_U in order to calculate the critical exponent $1/\nu$ according to the analysis shown in figures 14.23 and 14.25 for the magnetic susceptibility.

14. Show that for a Gaussian distribution $f(x) = a\mathrm{e}^{-x^2/2\sigma^2}$ we have that $\langle x^2\rangle = \sigma^2$ and $\langle x^4\rangle = 3\sigma^4$. Conclude that $1 - \langle x^2\rangle/(3\langle x^4\rangle) = 0$.

15. Consider the distribution given by the probability density distribution
$$f(x) = a\left(\mathrm{e}^{-\frac{(x-m)^2}{2\sigma^2}} + \mathrm{e}^{-\frac{(x+m)^2}{2\sigma^2}}\right) .$$
Plot this function and comment on the fact that it looks, qualitatively, like the distribution of the magnetization in the low temperature phase $\beta \gg \beta_c$. Show that $\langle x^4\rangle = m^4 + 6m^2\sigma^2 + 3\sigma^2$ and $\langle x^2\rangle = m^2 + \sigma^2$. Interpret your results, i.e. the meaning of each expectation value. Show that for $\sigma \ll m$ we obtain $U \approx 2/3$. Convince yourself that the approximation used concerns the system in the low temperature phase.

16. Calculate the derivative $\partial U/\partial\beta$ as a function of $\langle em^4\rangle$, $\langle em^2\rangle$, $\langle m^4\rangle$ and $\langle m^2\rangle$. Apply finite size scaling arguments and prove equation (14.120).

17. Use equations (14.131) and $y_t = 1/\nu$, $\gamma = (2y_h - d)/y_t$ in order to prove the other relations in (14.132) and (14.133).

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