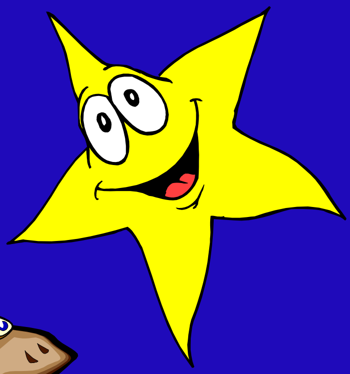
3. Gauge Invariance



- Field Theory
- Classical Electrodynamics
- Quantum Electrodynamics (QED)
- SU(N) Gauge Theory
- Quantum Chromodynamics (QCD)

QUANTUM MECHANICS WAVE EQUATIONS

Non Relativistic:

Schrödinger equation

$$\vec{p} = -i\vec{\nabla} \quad ; \quad E = i\frac{\partial}{\partial t}$$

$$E = \frac{\vec{p}^{\,2}}{2m} \Rightarrow \boxed{i\frac{\partial}{\partial t}\Psi = -\frac{\vec{\nabla}^2}{2m}\Psi}$$

Relativistic:

Klein-Gordon equation

$$p^\mu = i\partial^\mu = i\,g^{\mu\nu}\frac{\partial}{\partial x^\nu}$$

$$E^2 = \vec{p}^{\,2} + m^2 \Rightarrow \boxed{(\Box + m^2)\,\phi = 0}$$

$$\Box \equiv \partial^\mu\partial_\mu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

Spin $\frac{1}{2}$:

$$\boxed{(i\gamma^\mu\partial_\mu - m)\,\psi = 0}$$

Dirac equation

$$-(i\gamma^\nu\partial_\nu + m)\,[(i\gamma^\mu\partial_\mu - m)\,\psi] = 0 \equiv (\Box + m^2)\,\psi \Rightarrow \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

spin matrices $D = (\quad 4 \quad)$

DIRAC ALGEBRA

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}$$

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \quad ; \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; ; \; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij} \quad ; \quad [\sigma^i, \sigma^j] = 2i\varepsilon^{ijk}\sigma^k$$

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

Particle

Antiparticle

} Spinors

Chirality Projectors:

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \quad ; \quad (\gamma_5)^2 = I_4$$

$$P_R \equiv \frac{1+\gamma_5}{2} \quad ; \quad P_L \equiv \frac{1-\gamma_5}{2}$$

$$\mathrm{P}_R^2 = \mathrm{P}_R \quad ; \quad \mathrm{P}_L^2 = \mathrm{P}_L \quad ; \quad \mathrm{P}_R \mathrm{P}_L = \mathrm{P}_L \mathrm{P}_R = 0$$

LAGRANGIAN FORMALISM

$$S = \int d^4x \;\; \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\delta S = 0 \;\; \Rightarrow \;\; \frac{\partial \mathcal{L}}{\partial \phi} - \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \right) = 0$$

Eq. Motion

Klein-Gordon: (spin 0)

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi \;\; \Rightarrow \;\; (\Box + m^2)\,\phi = 0$$

Dirac: (spin $\frac{1}{2}$)

$$\bar{\psi} \equiv \psi^\dagger \gamma^0$$

$$\mathcal{L} = \bar{\psi}\,(i\,\gamma^\mu \partial_\mu - m)\,\psi \;\; \Rightarrow \;\; (i\,\gamma^\mu \partial_\mu - m)\,\psi = 0$$

CLASSICAL ELECTRODYNAMICS

Maxwell´s Equations:

$$\vec{\nabla}\cdot\vec{E} = \rho \quad ; \quad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{J} \quad ; \quad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = \vec{0}$$

Potentials:

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} \quad ; \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

Covariant Notation:

$$A^{\mu} \equiv (V, \vec{A}) \quad ; \quad J^{\mu} \equiv (\rho, \vec{J})$$

$$F^{\mu\nu} \equiv \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \longrightarrow \boxed{\partial_{\mu}F^{\mu\nu} = J^{\nu}}$$

Gauge Invariance:

$$A^{\mu} \longrightarrow A'^{\mu} \equiv A^{\mu} + \partial^{\mu}\Lambda$$

Same Physics described by many different A^{μ}

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\partial_\mu F^{\mu\nu} = \Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = J^\nu$$

CONSERVED ELECTROMAGNETIC CURRENT:

$$\partial_\mu \partial_\nu F^{\mu\nu} = 0 \quad \Rightarrow \quad \partial_\mu J^\mu = 0$$

Lorentz Gauge $(\partial_\mu A^\mu = 0)$ and $J^\mu = 0$ $\Rightarrow$ $\Box A^\mu = 0$

Klein-Gordon equation with $m = 0$ MASSLESS PHOTON

Residual Invariance: $A^\mu \Rightarrow A^\mu + \partial^\mu \Lambda$; $\Box \Lambda = 0$

$\mu = 0,\ 1,\ 2,\ 3,$; 2 arbitrary constraints

$\Rightarrow$ TWO PHOTON POLARIZATIONS

FREE Dirac fermion:

$$\mathcal{L} = \bar{\psi}\,(i\gamma^{\mu}\partial_{\mu} - m)\,\psi$$

Phase Invariance: $\psi \to \psi' = e^{iQ\theta}\,\psi \quad ; \quad \bar{\psi} \to \bar{\psi}' = e^{-iQ\theta}\,\bar{\psi}$

Absolute phases are not observable in Quantum Mechanics

GAUGE PRINCIPLE: $\theta = \theta(x)$

Phase Invariance should hold LOCALLY

BUT $\partial_{\mu}\psi \to e^{iQ\theta}\,(\partial_{\mu} + i\,Q\,\partial_{\mu}\theta)\,\psi$

SOLUTION: Covariant Derivative

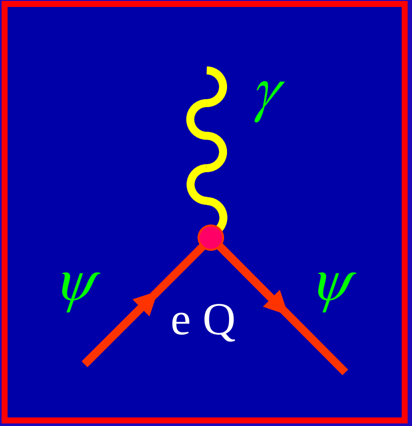
$$D_{\mu}\psi \equiv (\partial_{\mu} - i\,e\,Q\,A_{\mu})\,\psi \to e^{iQ\theta}\,D_{\mu}\psi$$

One needs a spin-1 field A_{μ} satisfying $A_{\mu} \to A_{\mu} + \frac{1}{e}\partial_{\mu}\theta$

QUANTUM ELECTRODYNAMICS

$$\mathcal{L} = \bar{\psi}\,(i\,\gamma^{\mu} D_{\mu} - m)\,\psi$$

$$= \bar{\psi}\,(i\,\gamma^{\mu}\partial_{\mu} - m)\,\psi \;+\; e\,Q\,A_{\mu}\;(\bar{\psi}\,\gamma^{\mu}\psi)$$



Kinetic term:

$$\mathcal{L}_K = -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \quad \Rightarrow \quad \partial_{\mu}F^{\mu\nu} = -e\,Q\,(\bar{\psi}\,\gamma^{\nu}\,\psi)$$

Maxwell

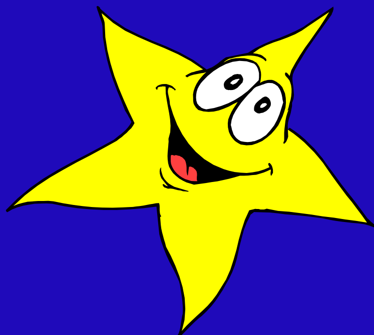
Mass term: [exp: $m_{\gamma} < 2\cdot 10^{-16}$ eV]

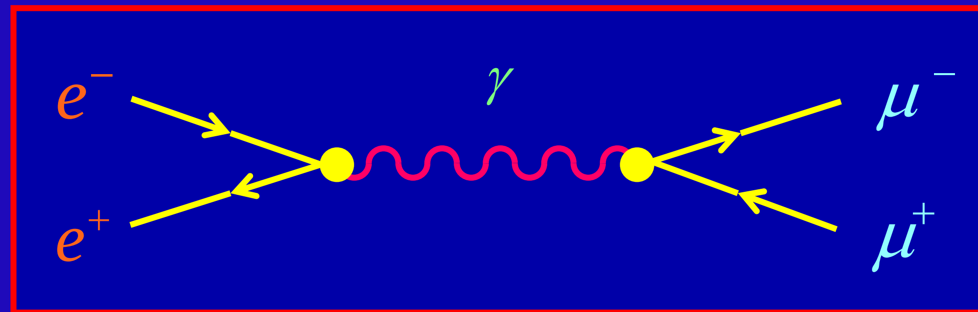
$$\mathcal{L}_M = \frac{1}{2}\,m_{\gamma}^{2}\,A^{\mu}A_{\mu}$$

Not Gauge Invariant $\Rightarrow$ $m_{\gamma} = 0$

Gauge Symmetry $\Rightarrow$ QED Dynamics

Successful Theory

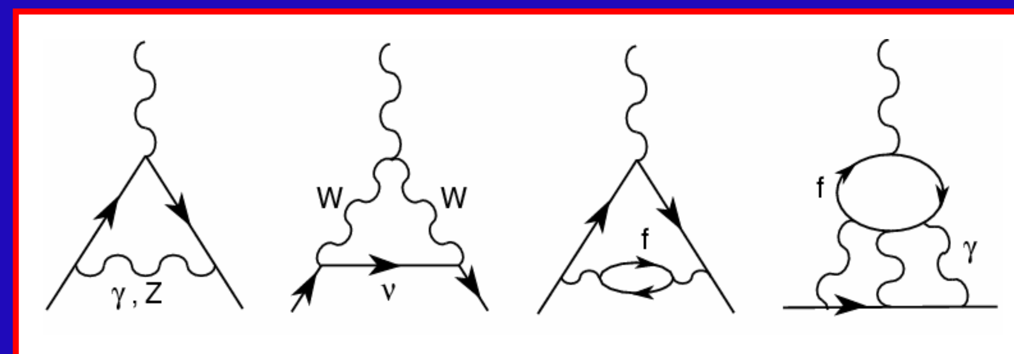




Anomalous Magnetic Moment

$$\mu_l \equiv g_l \frac{e}{2\,m_l}$$

$$a_l \equiv \frac{1}{2}\,(g_l - 2)$$



$$a_e = (115\,965\,218.59 \pm 0.38) \times 10^{-11} \quad \Rightarrow \quad \alpha^{-1} = 137.035\,999\,11 \pm 0.000\,000\,46$$

$$\Rightarrow \quad a_\mu^{\text{th}} = (116\,591\,922 \pm 88) \times 10^{-11} \quad [\,\text{Exp:} \quad (116\,592\,080 \pm 60) \times 10^{-11}\,]$$

QUANTUM CHROMODYNAMICS

FREE QUARKS: $N_C = 3$

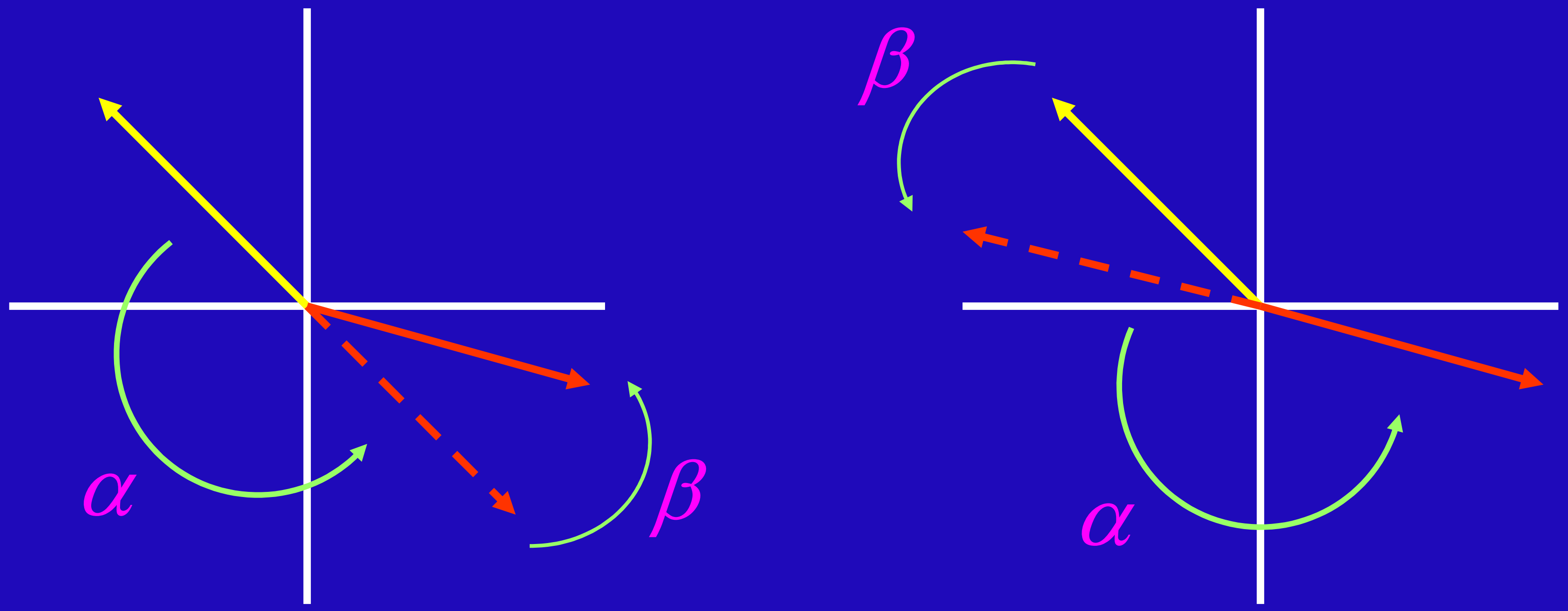
$$\mathcal{L} = \bar{\mathbf{q}}\,[\,i\gamma^\mu\partial_\mu - m\,]\,\mathbf{q} \qquad \mathbf{q} \equiv \begin{pmatrix} q \\ q \\ q \end{pmatrix}$$

SU(3) Colour Symmetry:

$$\mathbf{q} \to \mathbf{U}\,\mathbf{q} \quad ; \quad \bar{\mathbf{q}} \to \bar{\mathbf{q}}\,\mathbf{U}^\dagger$$

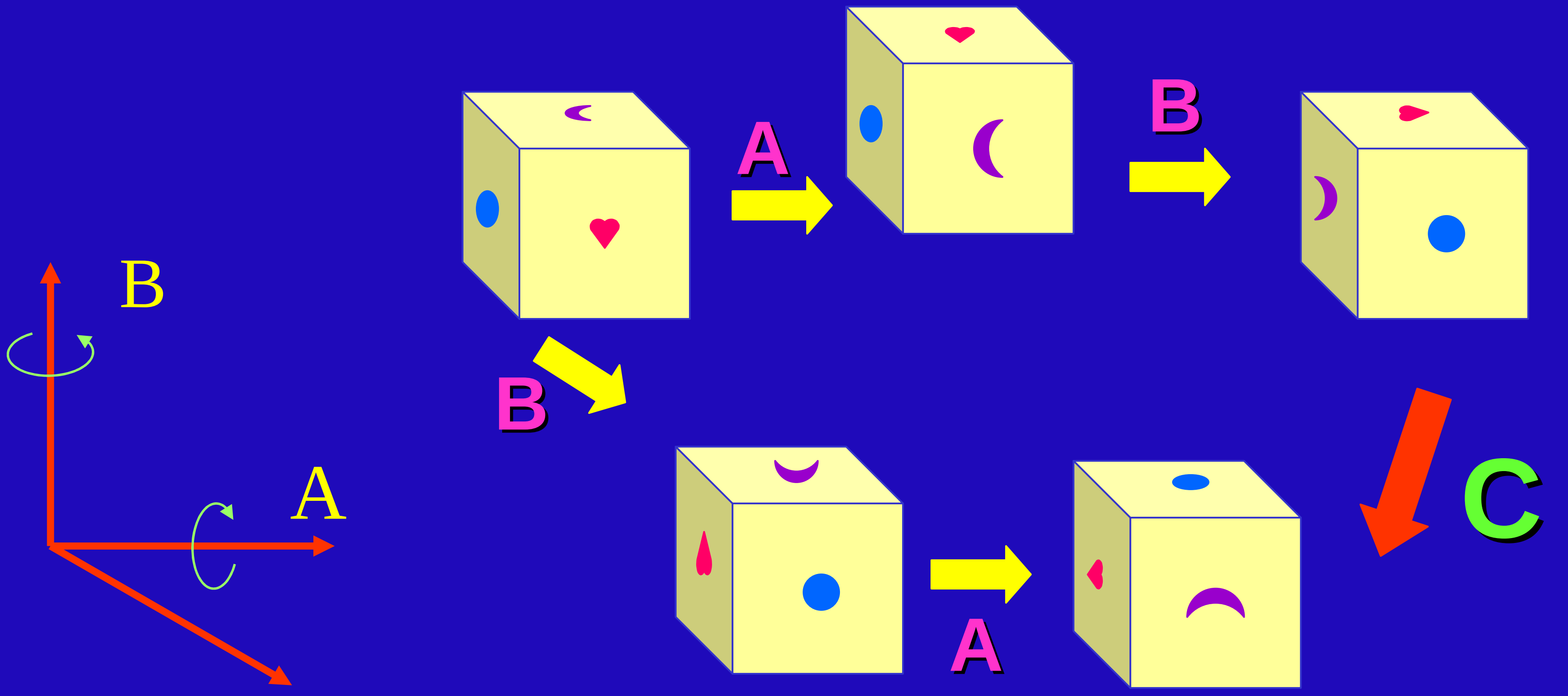
$$\mathbf{U}\,\mathbf{U}^\dagger = \mathbf{U}^\dagger\mathbf{U} = \mathbf{1} \quad ; \quad \det \mathbf{U} = 1$$

ABELIAN ROTATIONS



NON ABELIAN ROTATIONS

AB - BA = C



SU(N) ALGEBRA

$N \times N$ matrices: $\mathbf{U}\,\mathbf{U}^\dagger = \mathbf{U}^\dagger\,\mathbf{U} = \mathbf{1}$; $\det \mathbf{U} = 1$

$\mathbf{U} = \exp\left\{ i\,\mathbf{T}^a \theta_a \right\}$; $\mathbf{T}^a = \mathbf{T}^{a\dagger}$; $\mathrm{Tr}\,(\mathbf{T}^a) = 0$; $a = 1, \cdots, N^2 - 1$

Commutation Relation: $\left[\mathbf{T}^a , \mathbf{T}^b \right] = i\, f^{abc}\, \mathbf{T}^c$

Structure Constants f^{abc} real , antisymmetric

Fundamental Representation: $\mathbf{T}^a_F = \frac{1}{2} \lambda^a$ $N \times N$

Adjoint Representation: $(\mathbf{T}^a_A)_{bc} = -i\, f^{abc}$ $(N^2-1) \times (N^2-1)$

$\mathrm{Tr}\,(\mathbf{T}^a_F\, \mathbf{T}^b_F) = T_F\, \delta_{ab}$; $\mathrm{Tr}\left(\mathbf{T}^a_A\, \mathbf{T}^b_A \right) = C_A\, \delta_{ab}$; $\left(\mathbf{T}^a_F\, \mathbf{T}^a_F \right)_{\alpha\beta} = C_F\, \delta_{\alpha\beta}$

$T_F = \frac{1}{2}$; $C_A = N$; $C_F = \frac{N^2-1}{2N}$

SU(2)

2×2 matrices: $\quad \mathbf{U}\,\mathbf{U}^{\dagger} = \mathbf{U}^{\dagger}\,\mathbf{U} = \mathbf{1} \quad ; \quad \det \mathbf{U} = 1$

$$\mathbf{U} = \exp\left\{ i\, \mathbf{T}^{a}\, \theta_{a} \right\} \quad ; \quad \mathbf{T}^{a} = \mathbf{T}^{a\dagger} \quad ; \quad \mathrm{Tr}\,(\mathbf{T}^{a}) = 0 \quad ; \quad \mathrm{a} = 1, \cdots, 3$$

Commutation Relation: $\left[\mathbf{T}^{a}, \mathbf{T}^{b}\right] = i\,\varepsilon^{abc}\,\mathbf{T}^{c}$

Fundamental Representation: $\mathbf{T}_{F}^{a} = \frac{1}{2}\sigma^{a}$ **Pauli**

$$\sigma^{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \sigma^{2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \sigma^{3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

SU(3)

$$[\mathbf{T}^a, \mathbf{T}^b] = i\, f^{abc}\, \mathbf{T}^c$$

Fundamental Representation:

$$\mathbf{T}_F^a = \frac{1}{2}\lambda^a$$

Gell-Mann

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;;\; \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;;\; \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;;\; \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \;;\; \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \;;\; \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \;;\; \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\frac{1}{2} f^{123} = f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{\sqrt{3}} f^{458} = \frac{1}{\sqrt{3}} f^{678} = \frac{1}{2}$$

QUANTUM CHROMODYNAMICS

FREE QUARKS: $N_C = 3$

$$\mathbf{q} \equiv \begin{pmatrix} q \\ q \\ q \end{pmatrix}$$

$$\mathcal{L} = \bar{\mathbf{q}}\,[\,i\gamma^\mu \partial_\mu - m\,]\,\mathbf{q}$$

SU(3) Colour Symmetry:

$$\mathbf{q} \to \mathbf{U}\,\mathbf{q} = \exp\left\{ i\,\frac{\lambda^a}{2}\,\theta_a \right\}\,\mathbf{q}$$

Gauge Principle: Local Symmetry $\theta_a = \theta_a(x)$

$$\mathbf{D}^\mu \mathbf{q} \equiv (\mathbf{I}_3\,\partial^\mu - i\,g_s\,\mathbf{G}^\mu)\,\mathbf{q} \;\to\; \mathbf{U}\;\mathbf{D}^\mu \mathbf{q}$$

$$\mathbf{D}^\mu \to \mathbf{U}\;\mathbf{D}^\mu\;\mathbf{U}^\dagger \quad ; \quad \mathbf{G}^\mu \to \mathbf{U}\,\mathbf{G}^\mu\,\mathbf{U}^\dagger - \frac{i}{g_s}\,(\partial^\mu \mathbf{U})\;\mathbf{U}^\dagger$$

$$[\mathbf{G}^\mu]_{\alpha\beta} \equiv \frac{1}{2}\,(\lambda^a)_{\alpha\beta}\;G^\mu_a(x)$$

8 Gluon Fields

Infinitesimal SU(3) Transformation:

$$\delta \mathrm{q}^{\alpha} = i\ \delta\theta_a \left(\frac{\lambda^a}{2}\right)_{\alpha\beta} \mathrm{q}^{\beta} \qquad ; \qquad \delta G^{\mu}_a = \frac{1}{g_s}\, \partial^{\mu}(\delta\theta_a) - f^{abc}\, \delta\theta_b\, G^{\mu}_c$$

Non Abelian Group:

$$f^{abc} \neq 0$$

- δG^{μ}_a depends on G^{μ}_a
- Universal g_s
- No Colour Charges

Kinetic Term:

$$\mathbf{G}^{\mu\nu} \equiv \frac{i}{g_s}[\mathbf{D}^{\mu}, \mathbf{D}^{\nu}] = \partial^{\mu}\mathbf{G}^{\nu} - \partial^{\nu}\mathbf{G}^{\mu} - i\, g_s [\mathbf{G}^{\mu}, \mathbf{G}^{\nu}] \quad \rightarrow \quad \mathbf{U}\; \mathbf{G}^{\mu\nu}\; \mathbf{U}^{\dagger}$$

$$\mathbf{G}^{\mu\nu} \equiv \frac{\lambda^a}{2} G_a^{\mu\nu} \quad ; \quad G_a^{\mu\nu} = \partial^{\mu} G_a^{\nu} - \partial^{\nu} G_a^{\mu} + g_s\, f^{abc}\, G_b^{\mu}\, G_c^{\nu}$$

$$\mathcal{L}_K = -\frac{1}{2}\,\mathrm{Tr}\,(\mathbf{G}^{\mu\nu}\mathbf{G}_{\mu\nu}) = -\frac{1}{4}\, G_a^{\mu\nu}\, G^a_{\mu\nu}$$

Mass Term:

$$\mathcal{L}_M = \frac{1}{2}\, m_G^2\; G_a^{\mu}\, G^a_{\mu}$$

Not Gauge Invariant ➡ $m_G = 0$

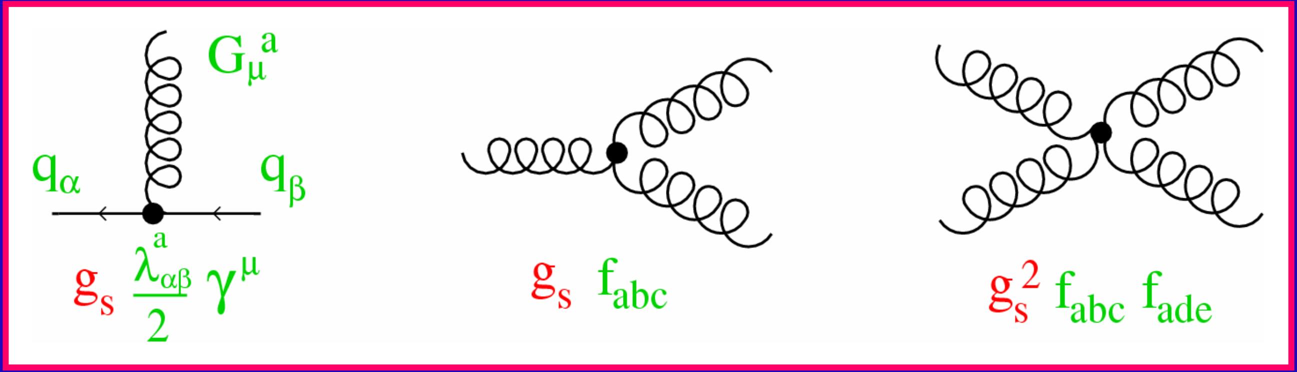
Massless Gluons

$$\mathcal{L}_{\rm QCD} = -\frac{1}{2}\,\mathrm{Tr}\,(\mathbf{G}^{\mu\nu}\,\mathbf{G}_{\mu\nu}) + \bar{\mathbf{q}}\,[\,i\,\gamma^{\mu}\,\mathbf{D}_{\mu} - m_q\,]\,\mathbf{q}$$

$$= -\frac{1}{4}\left(\partial^{\mu}G_a^{\nu} - \partial^{\nu}G_a^{\mu}\right)\left(\partial_{\mu}G_{\nu}^{a} - \partial_{\nu}G_{\mu}^{a}\right) + \sum_q \bar{q}_{\alpha}\,[\,i\,\gamma^{\mu}\partial_{\mu} - m_q\,]\,q_{\alpha}$$

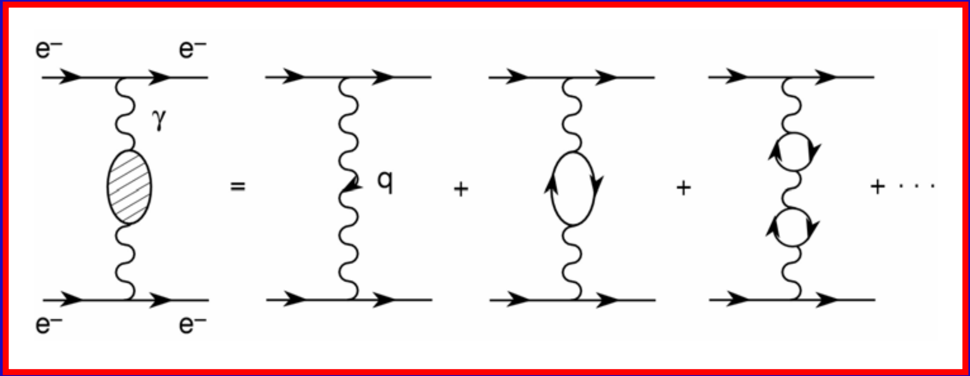
$$+ \frac{1}{2}\sum_q g_s\,[\bar{q}_{\alpha}\,(\lambda^{a})_{\alpha\beta}\,\gamma^{\mu}q_{\beta}\,]\,G_{\mu}^{a}$$

$$- \frac{1}{2}\,g_s\,f_{abc}\,(\partial_{\mu}G_{\nu}^{a} - \partial_{\nu}G_{\mu}^{a})\,G_b^{\mu}\,G_c^{\nu} - \frac{1}{4}\,g_s^2\,f_{abc}\,f_{ade}\,G_b^{\mu}\,G_c^{\nu}\,G_{\mu}^{d}\,G_{\nu}^{e}$$



- **Gluon Self – interactions** G^3 , G^4
- **Universal Coupling** g_s **(No Colour Charges)**

QUANTUM CORRECTIONS



$$\mathbf{T}(Q^2) \sim \frac{\alpha}{Q^2}\left\{1 + \Pi(Q^2) + \Pi(Q^2)^2 + \cdots\right\} \sim \frac{\alpha(Q^2)}{Q^2}$$

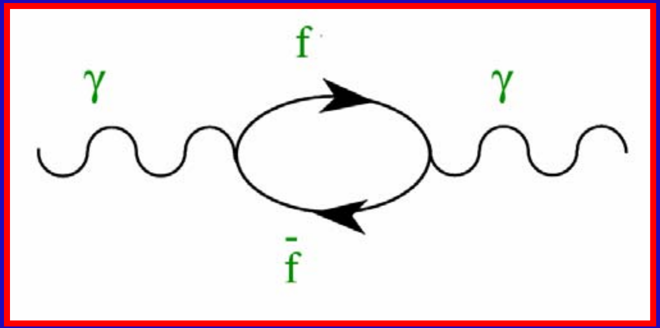
Effective *(Running)* Coupling:

$$\alpha(Q^2) = \frac{\alpha}{1-\Pi(Q^2)} \approx \frac{\alpha}{1-\frac{\alpha}{3\pi}\log\left(\frac{Q^2}{m^2}\right)}$$

SCREENING

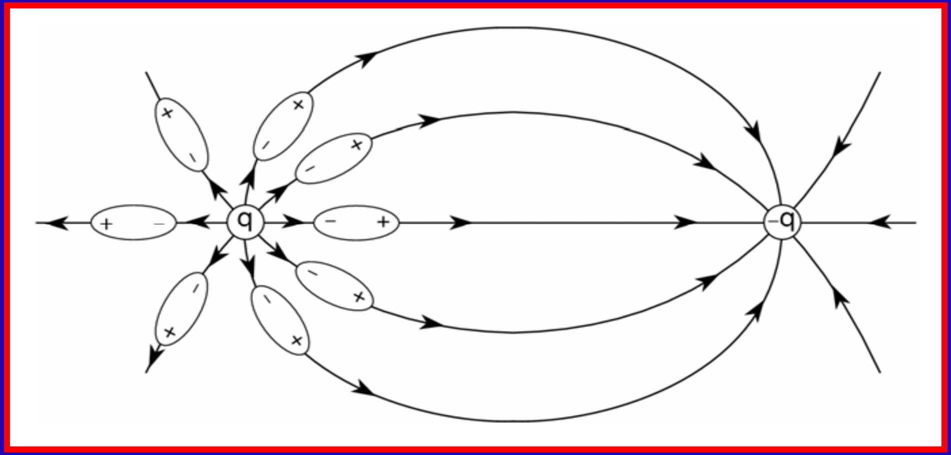
$\alpha(Q^2)$ Increases with $Q^2 \equiv -q^2$ → **Decreases at Large Distances**

VACUUM POLARIZATION



The Photon Couples to *Virtual* $f\,\bar{f}$ *Pairs*

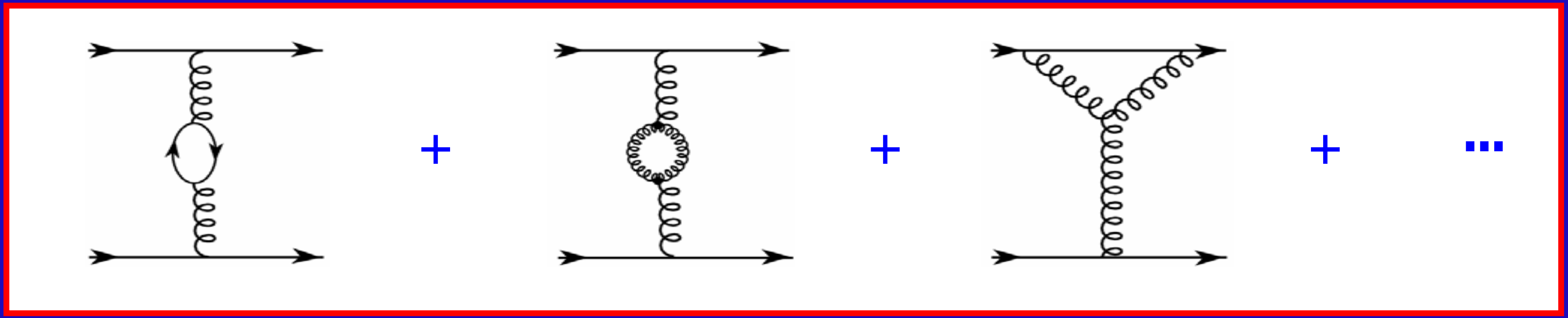
Vacuum ⟷ Polarized Dielectric Medium



$$\alpha^{-1} = \alpha(m_e^2)^{-1} = 137.03599911\ (46) \quad ; \quad \alpha\,(M_Z^2)^{-1} = 128.95 \pm 0.05$$

($l^- \, l^+$ and $q\,\bar{q}$ contributions included)

QCD RUNNING COUPLING



$$\alpha_s(Q^2) = \frac{\alpha_s(Q_0^2)}{1 - \beta_1 \frac{\alpha_s(Q_0^2)}{2\pi} \log\left(\frac{Q^2}{Q_0^2}\right)}$$

$$\beta_1 = \frac{1}{3} N_F - \frac{11}{6} N_C$$

quarks gluons

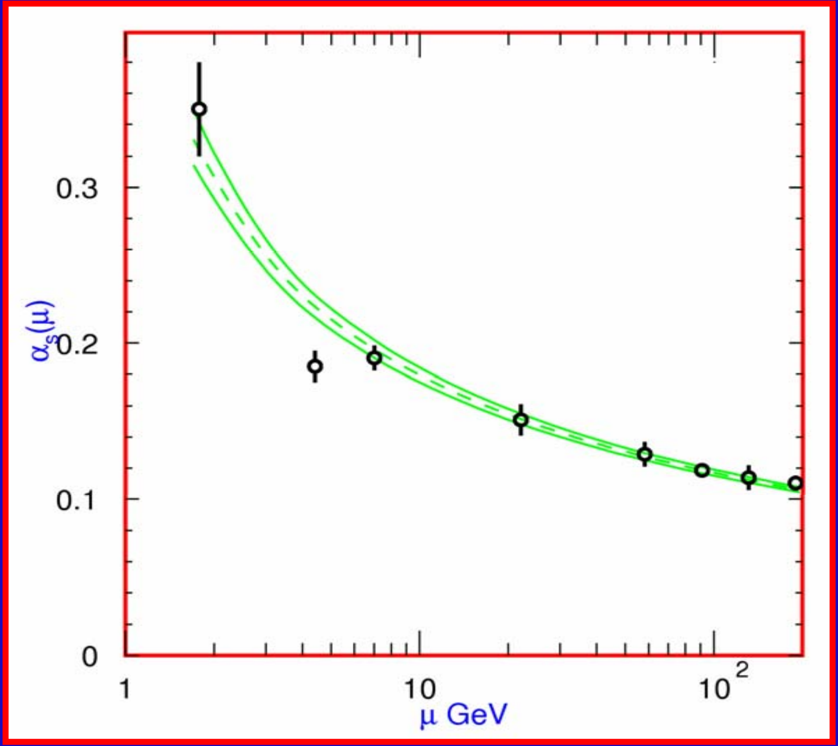
$N_C = 3\ ,\ N_F = 6$ ➡ $\beta_1 < 0$; $Q^2 > Q_0^2$ ➡ $\alpha_S(Q^2) < \alpha_S(Q_0^2)$

$\alpha_S(Q^2)$ Decreases at Short Distances **ANTI-SCREENING**

ASYMPTOTIC FREEDOM

$\alpha_s(M_Z^2) = 0.119 \pm 0.002$

$$\alpha_s(Q^2) = \frac{\alpha_s(Q_0^2)}{1 - \beta_1 \frac{\alpha_s(Q_0^2)}{2\pi} \log\left(\frac{Q^2}{Q_0^2}\right)}$$



$\beta_1 < 0 \Rightarrow \lim_{Q^2 \to \infty} \alpha_s(Q^2) = 0$

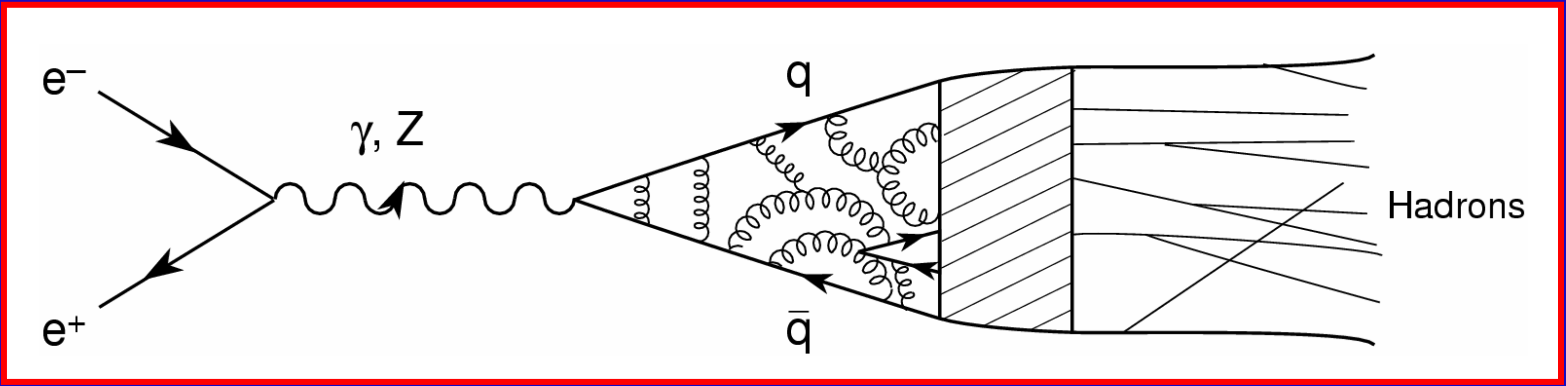
$\alpha_s(Q^2)$ increases at low energies

$\alpha_s \simeq O(1)$ at 1 GeV

Non–Perturbative Region

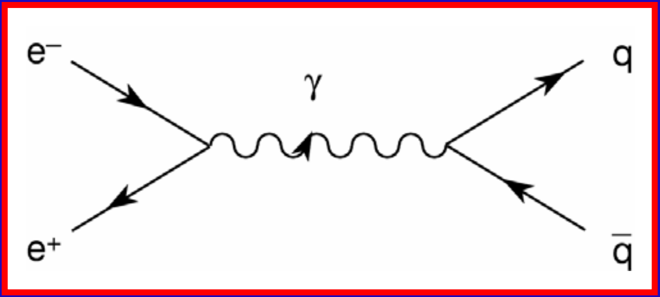
$\Rightarrow$ CONFINEMENT ?

Confinement ⟷ Probability Hadronization = 1



$$\sigma(e^+e^- \to \text{hadrons}) = \sigma(e^+e^- \to q\bar{q} + q\bar{q}G + q\bar{q}GG + q\bar{q}q\bar{q} + \cdots)$$

$$\mathbf{T}\left(e^+e^- \to q\,\bar{q}\right) \quad =$$



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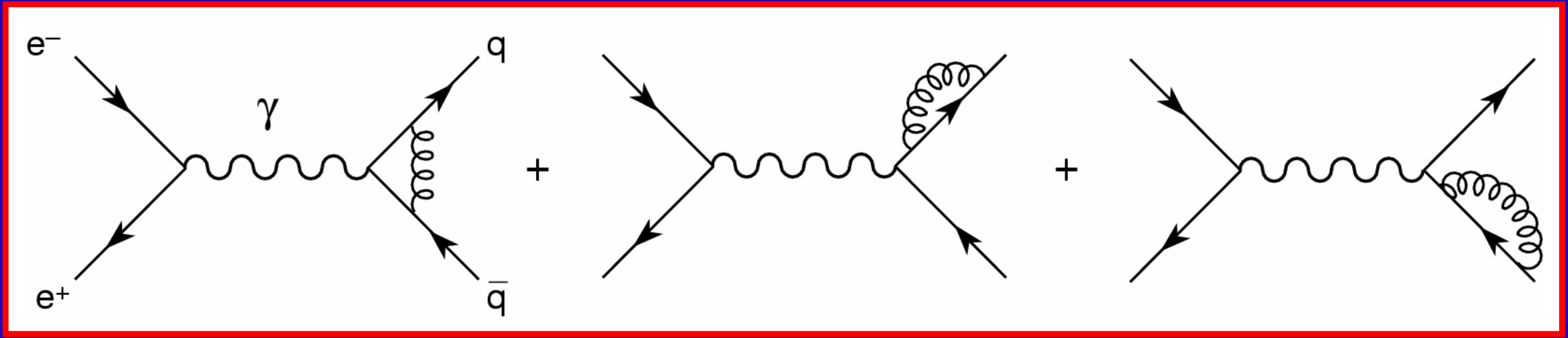
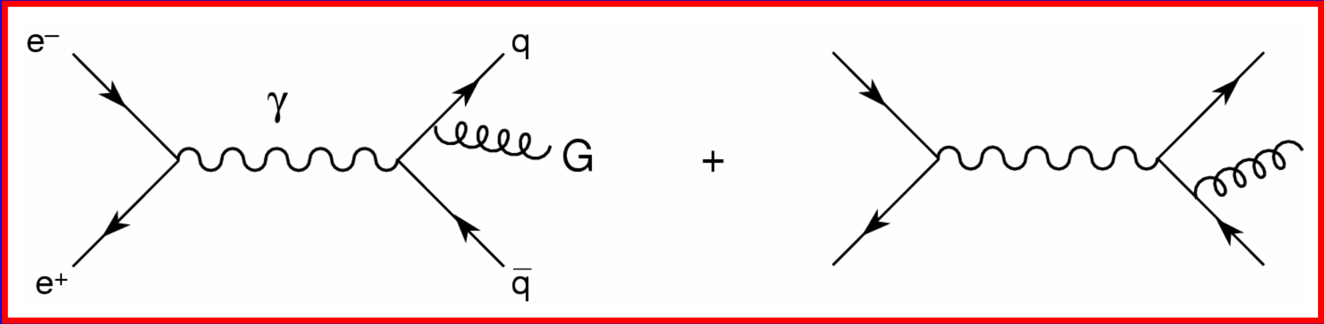
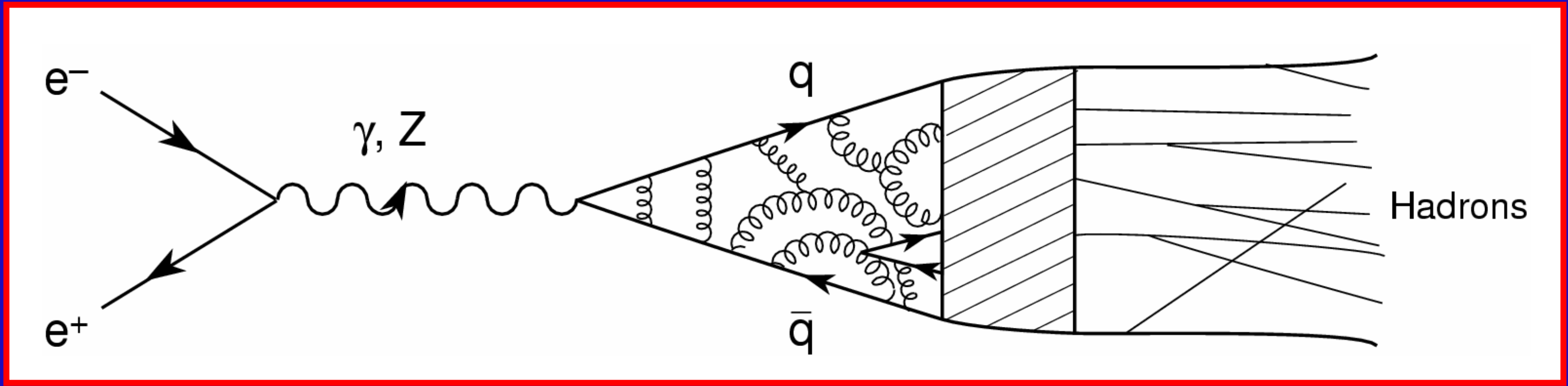


$$\mathbf{T}\left(e^+e^- \to q\,\bar{q}\,G\right) \quad =$$



Confinement ⟷ Probability Hadronization = 1

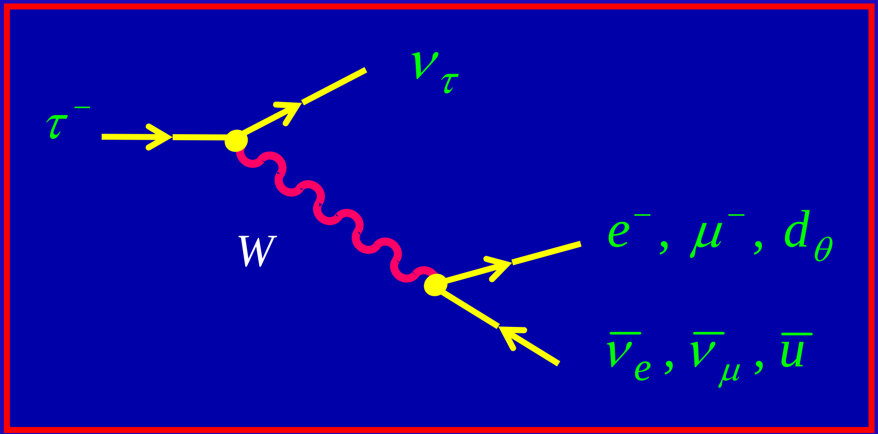


$$\sigma(e^+e^- \to \text{hadrons}) = \sigma(e^+e^- \to q\bar{q} + q\bar{q}G + q\bar{q}GG + q\bar{q}q\bar{q} + \cdots)$$

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_q Q_q^2 \; N_C \; \left\{ 1 + \frac{\alpha_s(s)}{\pi} + \cdots \right\}$$

$$R_Z \equiv \frac{\Gamma(Z \to \text{hadrons})}{\Gamma(Z \to e^+e^-)} = R_Z^{EW} \; N_C \; \left\{ 1 + \frac{\alpha_s(M_Z^2)}{\pi} + \cdots \right\}$$

$\tau^- \to \nu_\tau + \text{Hadrons}$



$$d_\theta = \cos\theta_C\, d + \sin\theta_C\, s$$

$$B_l \equiv \mathrm{Br}(\tau^- \to \nu_\tau\, l^- \bar{\nu}_l) \approx \frac{1}{2+N_C} = \frac{1}{5} = 20\%$$

$$B_e = (17.84 \pm 0.06)\,\%$$

$$B_\mu = (17.36 \pm 0.06)\,\%$$

QCD:

$$R_\tau \equiv \frac{\Gamma\left(\tau^- \to \nu_\tau + \text{Hadrons}\right)}{\Gamma\left(\tau^- \to \nu_\tau\, e^- \bar{\nu}_e\right)} = N_C \left\{1 + \frac{\alpha_s(m_\tau^2)}{\pi} + \cdots\right\}$$

$$R_\tau = 3.484 \pm 0.024 \Rightarrow \alpha_s(m_\tau^2) = 0.345 \pm 0.020 > \alpha_s(M_Z^2) = 0.119 \pm 0.002$$

S. Bethke

MEASUREMENTS OF α_s

$$\alpha_s(M_Z^2) = 0.1182 \pm 0.0027$$