Gauge Theories, Higgs Mechanism, Standard Model

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- Antonio Pich: Particle Physics: The Standard Model (lectures)
- W. Hollik: Theory of Electroweak Interactions, Corfu Summer Institute, School and Workshop on Standard Model and Beyond, 2013
- Aitchison I J R & Hey A J G: Gauge Theories In Particle Physics Volume 1: From Relativistic Quantum Mechanics To QED
- Francis Halzen-Alan D.Martin:Quarks and Leptons
- Tai-Pei Cheng,Ling-Fong Li: Gauge Theory of Elementary Particle Physics
- Abdelhak Djouadi: The Anatomy of Electro-Weak Symmetry Breaking, Tome I: The Higgs boson in the Standard Model

Principle of Least Action

Classical mechanics of particle:

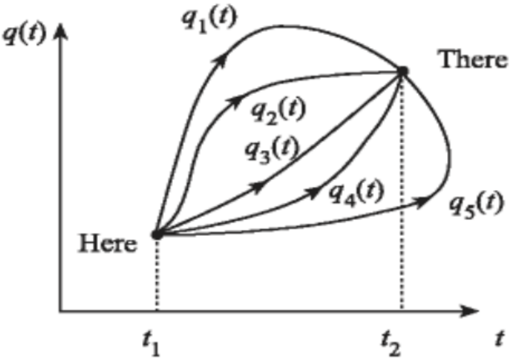
- Newtonian Mechanics
  Equations of motion are inserted axiomatically and involve forces as main ingredients emerging from physics $\rightarrow$ calculation of trajectories.
- Least action perspective
  E.o.M. are not inserted axiomatically, potentials are considered the main ingredients.

Principle of least action:

From all possible trajectories the particle chooses the one that minimizes the action S:

$$S = \int_{t_1}^{t_2} L(q(t), \dot{q}(t))\,\mathrm{d}t,$$

where L is the Lagrangian. For simple cases, it is the difference between the kinetic energy and the potential, $L = T - V$



Euler-Lagrange Equations

Certainly it's not the optimal way to proceed by calculating every possible trajectory and choosing the one with the minimum action . It's too difficult and time consuming. There is another way:

We consider a variation $\delta q(t)$ in the trajectory from $q(t_1)$ to $q(t_2)$.
In the minimum of S, the δS that corresponds to a δq, should vanish ($\delta S = 0$):

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q(t)} \delta q(t) + \frac{\partial L}{\partial \dot{q}(t)} \delta \dot{q}(t) \right) \mathrm{d}t$$

Using:

$$\delta \dot{q}(t) = \frac{d(\delta q(t))}{dt}$$

the above equation gives:

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q(t)} \delta q(t) + \frac{\partial L}{\partial \dot{q}(t)} \frac{d(\delta q(t))}{dt} \right) \mathrm{d}t$$

Then, by integrating the second term by parts, we obtain:

$$\delta S = \int_{t_1}^{t_2} \delta q(t) \left(\frac{\partial L}{\partial q(t)} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}(t)} \right) \mathrm{d}t + \left[\frac{\partial L}{\partial \dot{q}(t)} \delta q(t) \right]_{t_1}^{t_2}$$

However, given that every possible trajectory begins and ends at the moment t_1 and t_2 respectively, we have $\delta q(t_1) = \delta q(t_2) = 0$. So:

$$\delta S = \int_{t_1}^{t_2} \delta q(t) \left(\frac{\partial L}{\partial q(t)} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}(t)} \right) \mathrm{d}t = 0$$

The above equation should be valid for an arbitrary $\delta q(t)$, so:

$$\frac{\partial L}{\partial q(t)} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}(t)} = 0$$

This is the Euler- Lagrange Equation . Its solution is the real trajectory of the particle.

As a simple example we take the Lagrangian $L = T - V = \frac{1}{2} m \dot{x}^2 - V(x)$.
Therefore, the E-L equation becomes:

$$-\frac{\partial V(x)}{\partial x} - \frac{1}{2} m \frac{d}{dt} \frac{\partial \dot{x}^2}{\partial \dot{x}} = 0$$

But

$$-\frac{\partial V(x)}{\partial x} = F \qquad \text{and} \qquad \frac{1}{2} m \frac{d}{dt} \frac{\partial \dot{x}^2}{\partial \dot{x}} = \frac{1}{2} \frac{d}{dt} 2m\dot{x} = m\ddot{x}$$

Finally we obtain:

$$F = m\ddot{x}$$

which is Newton's second law of motion.

Field Mechanics

We shall have in mind the passage from a large number (N) of discrete mass points connected to springs (N degrees of freedom), to the continuous limit of the string (infinite degrees of freedom - field), that is, for $N \to \infty$:

$$\{q_r(t), r = 1, 2, ..., N\} \overset{N\to\infty}{\Longrightarrow} \phi(x, t),$$

where x is now a continuous variable labeling the displacement of the string. At each point x we have an independent degree of freedom $\phi(x, t)$.

In this case, the action will be:

$$S = \int L \,\mathrm{d}t, \quad \text{where} \quad L = \int \mathcal{L} \,\mathrm{d}x,$$

where $\mathcal{L}$ is the Lagrangian density.

We see that $[\phi] = [Length]$, $\quad [\mathcal{L}] = \left[\frac{Energy}{Length}\right]$ $\quad$ or $\quad [\mathcal{L}] = \left[\frac{Energy}{Volume}\right]$

A new feature is that ϕ is continuous, therefore the $\mathcal{L}$ could also depend on $\partial\phi/\partial t$:

$$\mathcal{L} = \mathcal{L}\left(\phi, \dot{\phi}, \frac{\partial\phi}{\partial x}\right)$$

Like before, we demand that $\delta S = 0$:

$$\delta S = \int \mathrm{d}t \int \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\dot{\phi}}\delta\dot{\phi} + \frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial x)}\delta(\partial\phi/\partial x)\right] \mathrm{d}x = 0$$

Working in the same way:

$$\delta S = \int \mathrm{d}t \int \mathrm{d}x \quad \delta\phi \left[\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial x)}\right)\right] = 0,$$

we finally obtain the E-L field equations :

$$\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial x)}\right) = 0 \xrightarrow{3D} \frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) - \vec{\nabla}\left(\frac{\partial\mathcal{L}}{\partial(\vec{\nabla}\phi)}\right) = 0$$

Example: Let's consider the (1-D) Lagrangian density for a string:

$$\mathcal{L}_{string} = \frac{1}{2}\rho\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\rho\upsilon^2\left(\frac{\partial\phi}{\partial x}\right),$$

where ρ is the density and υ the velocity. Applying the E-L equations we obtain the wave equation:

$$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{\upsilon^2}\frac{\partial^2\phi}{\partial t^2} = 0 \quad (\textit{Homework})$$

For relativistic fields, the E-L equations can be written in a relativistically invariant form:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

Also, the action will be written as:

$$S = \int dt \int d^3x \mathcal{L} = \int d^4x \mathcal{L}$$

The action S is Lorentz invariant because d^4x is invariant.

Therefore, in order to construct a relativistic field theory, we have to construct an invariant $\mathcal{L}$ and then make use of the E-L.

Example: The previous Lagrangian density $\mathcal{L}_{string}$ becomes:

$$\mathcal{L}_{string} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi), \quad \rho = \upsilon = 1$$

Using the E-L equations, we obtain the equation of motion (relativistic wave equation):

$$\cancelto{0}{\frac{\partial \mathcal{L}_{string}}{\partial \phi}} - \partial_\mu \left(\frac{\partial \mathcal{L}_{string}}{\partial(\partial_\mu \phi)} \right) = 0 \quad \Rightarrow \quad \partial_\mu \partial^\mu \phi = 0$$

Noether's Theorem

We consider a system that can be described by the Lagrangian:

$$L = \int d^3x \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x))$$

and has E.o.M:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0$$

Every continuous transformation under which the action

$$S = \int L \, \mathrm{d}t$$

is invariant, leads to a conserved current:

$$\partial^\mu J_\mu(x) = 0$$

Proof of Noether's Theorem (for a continuous internal non-abelian symmetry)

Consider that $\mathcal{L}$ is invariant under a symmetry group G, namely under infinitesimal transformations:

$$\phi_i(x) \to \phi_i'(x) = \phi_i(x) + \delta\phi_i(x)\,,$$

where $\delta\phi_i(x) = i\epsilon^a t^a_{ij}\phi_j(x)$, ϵ^a: spacetime-invariant small parameters and t^a a set of matrices that satisfy G's Lie algebra:

$$[t^a, t^b] = iC^{abc}t^c\,,$$

where C^{abc} are the structure constants of G.
Then, the variation of $\mathcal{L}$ becomes:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta(\partial_\mu\phi_i)$$

From $\delta(\partial_\mu\phi_i) \equiv \partial_\mu\phi_i' - \partial_\mu\phi_i = \partial_\mu(\delta\phi_i)$ and the E.o.M.

$$\delta\mathcal{L} = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\partial_\mu(\delta\phi_i) = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i\right] = \epsilon^a\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}it^a_{ij}\phi_j\right]$$

Therefore, if $\delta\mathcal{L} = 0$:

$$\partial^\mu J^a_\mu = 0\,, \quad \text{where} \quad J^a_\mu = -i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}t^a_{ij}\phi_j$$

If we define as charge:

$$Q(t) \equiv \int d^3x J_0(x)$$

It follows that:

$$\frac{dQ}{dt} = 0 \qquad \rightarrow \text{it's a constant of motion}$$

(Taking under consideration that a "surface" term is zero in the infinity)

In the case of invariance under continuous internal symmetry Lie group, the conserved charges:

$$Q^\alpha = \int d^3x J_0^\alpha(x)$$

are the generators of the symmetry group.

Homework: $\mathcal{L}$ invariance under transformations $x^\mu \rightarrow x^{\mu'} = x^\mu + \alpha^\mu \implies$ energy and momentum conservation

Lagrangian density for two real fields

The Lagrangian density for two free, real fields ϕ_1, ϕ_2 of same mass

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m^2\phi_2^2$$

is invariant under the transformation:

$$\begin{aligned} \phi_1' &= \cos\alpha\phi_1 - \sin\alpha\phi_2 \\ \phi_2' &= \sin\alpha\phi_1 + \cos\alpha\phi_2 \end{aligned}$$

By simple substitution of ϕ_1', ϕ_2', we obtain:

$$\mathcal{L}(\phi_1', \phi_2') = \mathcal{L}(\phi_1, \phi_2) \implies SO(2) \text{ symmetry}$$

We would like to find which is the conserved current in this case. It is simple (and sufficient) to consider infinitesimal rotation ϵ, so the above field transformation becomes:

$$\begin{aligned} \phi_1' &= \phi_1 - \epsilon\phi_2 \\ \phi_2' &= \epsilon\phi_1 + \phi_2 \end{aligned}$$

Since $\delta\mathcal{L} = 0$ under these transformations and $\mathcal{L} = \mathcal{L}(\phi_1, \phi_2, \partial_\mu\phi_1, \partial_\mu\phi_2)$, we have:

$$0 = \delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_1}\delta\phi_1 + \frac{\partial\mathcal{L}}{\partial\phi_2}\delta\phi_2 + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\delta(\partial_\mu\phi_1) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)}\delta(\partial_\mu\phi_2)$$

We can use the E.o.M to show that:

$$\begin{aligned} 0 &= \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)}\delta\phi_1 + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)}\partial_\mu(\delta\phi_1) + \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)}\delta\phi_2 + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)}\partial_\mu(\delta\phi_2) \\ &= \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)}\delta\phi_1 + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)}\delta\phi_2\right] \end{aligned}$$

In this case, we have:

$$\begin{aligned} \delta\phi_1 &\equiv \phi_1' - \phi_1 = -\epsilon\phi_2 \\ \delta\phi_2 &\equiv \phi_2' - \phi_2 = \epsilon\phi_1 \end{aligned}$$

so:

$$\epsilon\partial_\mu[(\partial^\mu\phi_2)\phi_1 - (\partial^\mu\phi_1)\phi_2] = 0$$

and since ϵ is arbitrary:

$$\partial_\mu \underbrace{[(\partial^\mu\phi_2)\phi_1 - (\partial^\mu\phi_1)\phi_2]}_{\equiv j^\mu} = 0 \implies \partial_\mu j^\mu = 0$$

Note: Introducing a complex field $\phi = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$, $\phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, the previous Lagrangian density becomes: $\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi$

Dirac Lagrangian density

The Lagrangian density for a free Dirac particle of mass m

$$\mathcal{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi$$

is invariant under the transformation $\psi(x) \to e^{i\alpha}\psi(x)$, where α is a real constant. The invariance derives easily, observing that:

$$\begin{array}{rcl} \partial_\mu \psi & \to & e^{i\alpha}\partial_\mu \psi \\ \bar{\psi} & \to & e^{-i\alpha}\bar{\psi} \end{array}$$

The transformations $U(\alpha) \equiv e^{i\alpha}$ form the unitary abelian group $U(1)$.

An abelian group has the property:

$$U(\alpha_1)U(\alpha_2) = U(\alpha_2)U(\alpha_1)$$

We would like to find which conservation law is related to this symmetry. Again, it's sufficient to study the invariance of $\mathcal{L}$ under an infinitesimal $U(1)$ transformation:

$$\psi \to (1 + i\alpha)\psi$$

Following the same steps as before, we get the conserved current:

$$j^\mu = -e\bar{\psi}\gamma^\mu\psi \quad (Homework)$$

Global Invariance in Non-Abelian Symmetries

The generalization to the non-abelian case occurs if we consider together more than one wavefunctions or states.
Quantum mechanics: When we have degenerate states in energy (or mass), there is no unique way to determine them. Every linear combination of an initially selected set of degenerate states will be as good as any if the normalization conditions are satisfied.

Example: The neutron and proton masses are almost equal (939.553 MeV - 938.259 MeV), so we could consider (Heisenberg 1932) the two states as degenerate. Thus, linear combinations of the proton and neutron eigenstates are completely equivalent i.e. redefinitions of the proton and neutron eigenfunctions, such as:

$$\begin{aligned}\psi_p &\rightarrow \psi'_p = \alpha\psi_p + \beta\psi_n \\ \psi_n &\rightarrow \psi'_n = \gamma\psi_p + \delta\psi_n \ ,\end{aligned}$$

for complex coefficients $\alpha, \beta, \gamma, \delta$, are permitted.
Since ψ_p, ψ_n are degenerate states i.e. $H\psi_p = E\psi_p$ and $H\psi_n = E\psi_n$, we obtain:

$$\begin{aligned}H\psi'_p &= H(\alpha\psi_p + \beta\psi_n) = E(\alpha\psi_p + \beta\psi_n) = E\psi'_p \\ H\psi'_n &= \ldots = E\psi'_n\end{aligned}$$

The redefined states describe 2 states with the same energy degeneracy.

The double degeneracy suggests a resemblance to spin 1/2 - systems in absence of magnetic field ($s_z = \pm 1/2$ are both degenerate).

This analogy suggests that we can introduce the 2-component nucleon isospinor:

$$\psi^{\left(\frac{1}{2}\right)} = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} = \psi_p \chi_p^{\left(\frac{1}{2}\right)} + \psi_n \chi_n^{\left(\frac{1}{2}\right)}\ , \quad \text{where} \quad \underbrace{\chi_p^{\left(\frac{1}{2}\right)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\text{isospin up}}, \quad \underbrace{\chi_n^{\left(\frac{1}{2}\right)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{isospin down}}$$

The ψ_p, ψ_n are coefficients from which the $|\psi_p|^2, |\psi_n|^2$ give the probability of the nucleon to have isospin $\chi_p^{\left(\frac{1}{2}\right)}$ or $\chi_n^{\left(\frac{1}{2}\right)}$.

Let's write a redefinition of the isospinor as:

$$\psi^{\left(\frac{1}{2}\right)} \to \psi^{\left(\frac{1}{2}\right)'} = U \psi^{\left(\frac{1}{2}\right)},$$

where U is a complex 2×2 matrix.

Heisenberg postulated that the strong interaction physics is invariant under these transformations.

Rephrasing: There is a symmetry (in absence of E/M interactions).

Let us examine the restrictions imposed on U

$$\begin{pmatrix} \psi'_p \\ \psi'_n \end{pmatrix} = U \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}$$

In order to maintain the normalization (probability), we demand that U is unitary:

$$UU^\dagger = U^\dagger U = \mathbb{1}$$

Obviously, this property is not limited to the 2-states case.

Clearly, the coefficients of the transformation of n degenerate states will form the elements of a $n \times n$ matrix.

Trivial case: $n = 1 \to U$ becomes a phase factor indicating that the previous examples will be included as a special case of these more general transformations.

From the properties of determinants we obtain:

$$\det UU^\dagger = \det U \det U^\dagger = |\det U|^2 = 1 \implies \det U = \exp(i\theta), \quad \theta - \text{real}$$

We can distinguish such total phase factor from the transformations of $p - n$ mixing because it corresponds to a phase rotation of both p and n by the same quantity.

The invariance of the theory under such transformations corresponds to the conservation law of the total number of p and n.

The new physics will be found in the rest of the transformations that satisfy:

$$\det U = +1$$

Such a matrix is called special unitary. The set of all matrices U forms the Lie group $SU(2)$.

An important property of Lie groups is that their physical consequences can be found considering *infinitesimal transformations*, i.e. U matrices that differ infinitesimally form the "no-change" state that corresponds to $U = \mathbb{1}$.

$$U = \mathbb{1} + i\xi\,,$$

where ξ is a 2×2 matrix whose elements are all small quantities of first order. Omitting the second order terms, from the condition $UU^\dagger = \mathbb{1}$, we obtain :

$$(\mathbb{1} + i\xi)(\mathbb{1} - i\xi) = \mathbb{1} \implies \xi = \xi^\dagger \rightarrow \quad \xi \ \ is \ \ a \ \ 2 \times 2 \ \ Hermitian \ \ matrix,.$$

and from the condition $\det U = +1$:

$$\mathrm{Tr}\xi = 0 \qquad (\text{since} \quad u = e^{i\xi} \rightarrow \det u = e^{\mathrm{Tr}\xi},\ \xi \ \text{Hermitian})$$

Counting the restrictions imposed by the conditions $\xi = \xi^\dagger$, $\mathrm{Tr}\xi = 0$, we see that ξ has 3 independent parameters. Therefore, it can be expressed in a general way as:

$$\xi = \frac{\vec{\epsilon} \cdot \vec{\tau}}{2}$$

as the angular momentum operators of spin 1/2, while the $\vec{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3)$ are 3 small quantities of first order and $\vec{\tau}$ the Pauli matrices:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which satisfy:

$$\left[\frac{\tau_i}{2}, \frac{\tau_j}{2}\right] = i\epsilon_{ijk}\frac{\tau_k}{2}$$

while the basic wavefunctions $\chi_p^{(\frac{1}{2})}, \chi_n^{(\frac{1}{2})}$ are eigenfunctions of the total "spin" and its third component:

$$\begin{aligned} \frac{1}{4}\vec{\tau}^2\chi_p^{(\frac{1}{2})} &= \frac{3}{4}\chi_p^{(\frac{1}{2})} \quad , \quad \frac{1}{2}\tau_3\chi_p^{(\frac{1}{2})} = \frac{1}{2}\chi_p^{(\frac{1}{2})} \\ \frac{1}{4}\vec{\tau}^2\chi_n^{(\frac{1}{2})} &= \frac{3}{4}\chi_n^{(\frac{1}{2})} \quad , \quad \frac{1}{2}\tau_3\chi_n^{(\frac{1}{2})} = -\frac{1}{2}\chi_n^{(\frac{1}{2})} \end{aligned}$$

This double degree of freedom in the "charge space" of $p - n$ is called isospin.

An infinitesimal $SU(2)$ transformation of the doublet $p - n$ is:

$$\begin{pmatrix} \psi'_p \\ \psi'_n \end{pmatrix} = \left(\mathbb{1} + i\frac{\vec{\epsilon} \cdot \vec{\tau}}{2} \right) \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}$$

As an example, let's consider the Lagrangian that describes 2 free fermions u and d that have the same mass m:

$$\begin{aligned} \mathcal{L} &= \bar{\psi}_u (i \not\partial - m) \psi_u + \bar{\psi}_d (i \not\partial - m) \psi_d \\ \not\partial &\equiv \gamma^\mu \partial_\mu \end{aligned}$$

$\mathcal{L}$ is invariant under the unitary $SU(2)$ transformations

$$\begin{pmatrix} \psi'_u \\ \psi'_d \end{pmatrix} = \exp\left(-i\frac{\vec{\alpha} \cdot \tau}{2} \right) \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}$$

Homework: Which is the conserved Noether current? Which are the conserved charges? Show that the conserved charges are indeed the generators of $SU(2)$.