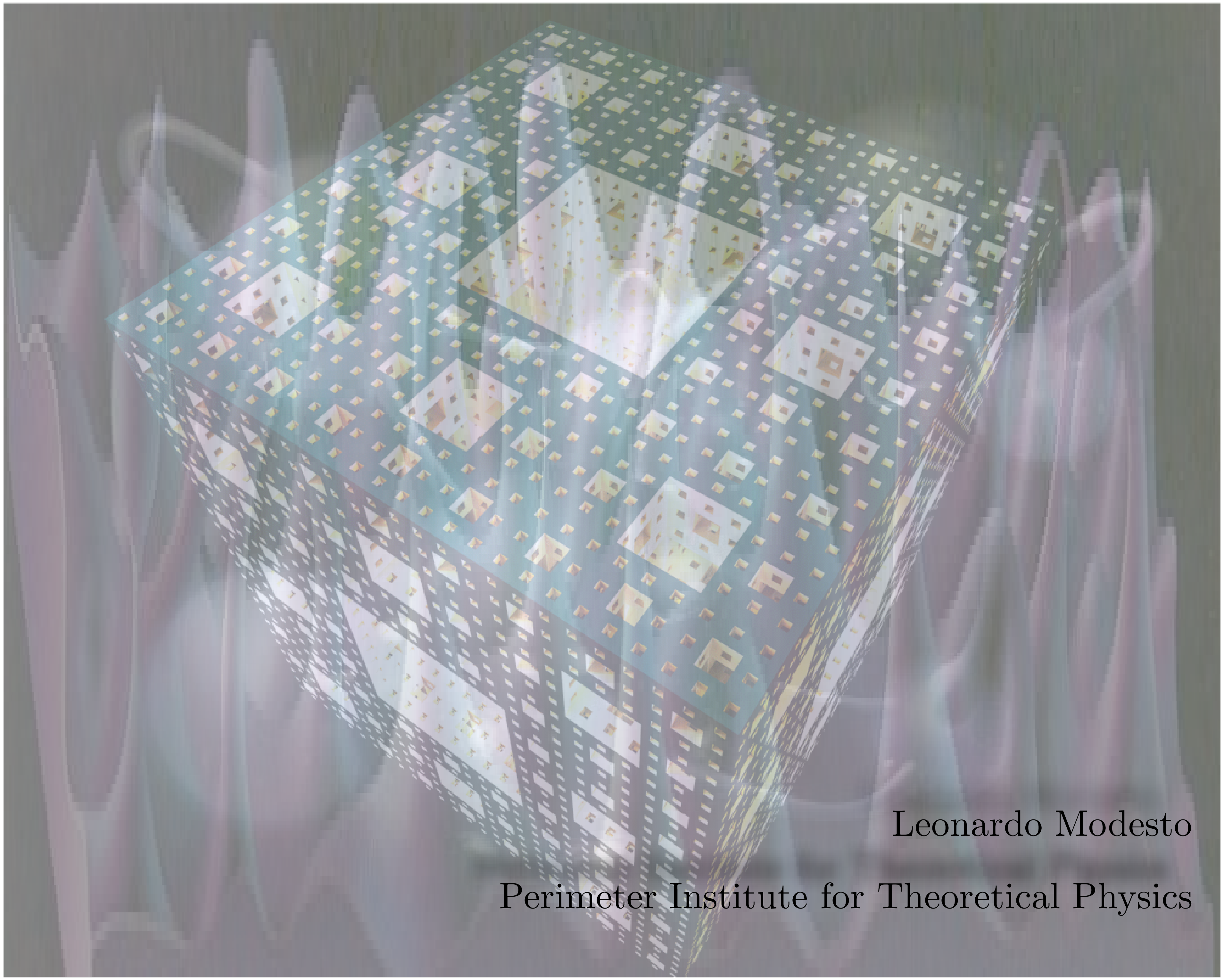
# Super-renormalizable Quantum Gravity



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# Outline

- Einstein Quantum Gravity,
  perturbative theory and non-renormalizability .
- Complete Quantum Gravity,
  perturbative theory and renormalizability,
  unitarity/no poltergeists,
  spectral dimension flow,
  regular multi-horizon black holes solutions.

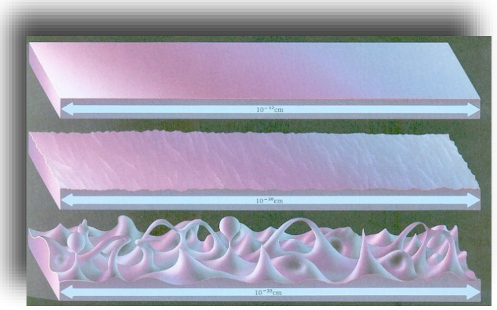
## Einstein Gravity

- Second order differential equations,
- General covariance,
- $\nabla^a T_{ab} = 0$,
- GRAVITY = CURVATURE .

## A Unique Gravity Dynamics

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}.$$

# Perturbative Quantum Gravity



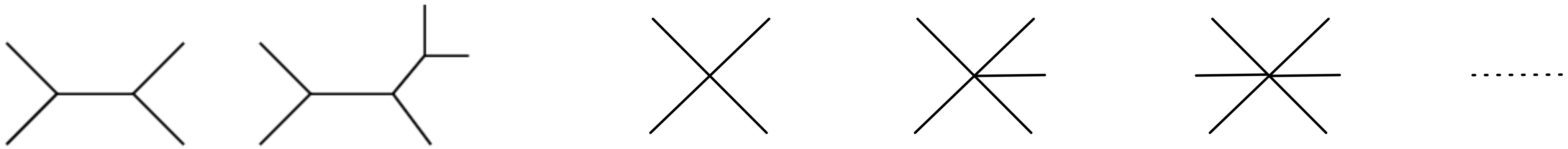
$$L_{EH} = -\frac{1}{16\pi G_N}\sqrt{-g}\,R(g)\,,$$

$$\sqrt{-g}\,g^{\mu\nu} = g^{o\,\mu\nu} + \kappa h^{\mu\nu},$$
$$\kappa^2 = 16\pi G_N.$$

$$L \approx \partial h\,\partial h + \kappa\,h\,\partial h\,\partial h + \kappa^2\,h^2\,\partial h\,\partial h + \kappa^3\,h^3\,\partial h\,\partial h + \cdots + \kappa^n\,h^n\,\partial h\,\partial h\ldots.$$

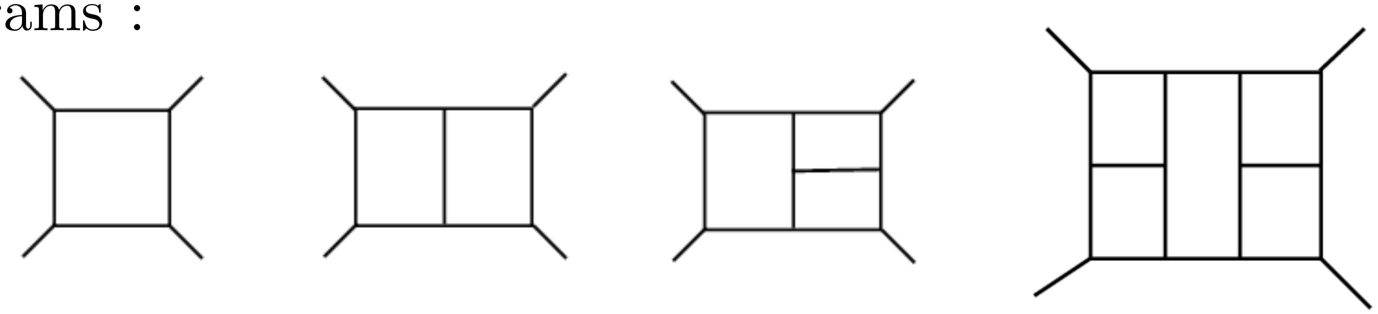
## Quantization of the fluctuation $h^{\mu\nu}$

Tree level amplitudes :



# Perturbative Quantum Gravity

Loops diagrams :



Superficial degree of divergence of a Feynman diagram

$D = L\,d + 2V - 2I,$

$L$ : number of loops,

$V$ : number of vertices,

$I$ : number of internal lines in the graph.

Topological relation between $V$, $I$ and $L$, $L = 1 + I - V$,

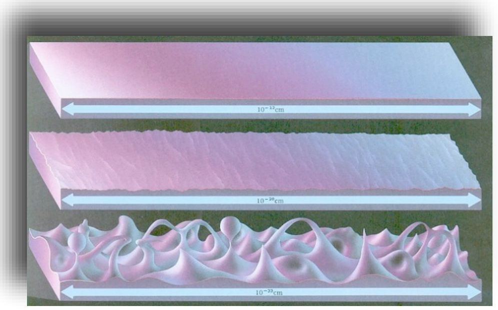
$D = 2 + (d-2)L$. For $d = 4 \;\rightarrow\; D = 2 + 2L$ .

Regularization : $$S = -\int \mathrm{d}^d x \sqrt{g} \left[ \kappa^{-2} R + \frac{\alpha}{\epsilon} \underbrace{\sum_{m,n} \nabla^n R^m}_{n+2m=2+(d-2)L} \right],$$

## Peturbative Quantum Gravity

$$L_G = -\frac{1}{16\pi G_N}\sqrt{-g}\,R(g)$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}\,,$$



$$L_G \approx \partial h \partial h + \sum_n \kappa^n h^n \partial h \partial h$$

**Quantization of $h_{\mu\nu}$**

$$L_{QG} = -\sqrt{-g}\Big[\kappa^{-2}\, R + \alpha_1\, R^2 + \alpha_2\, R_{\mu\nu} R^{\mu\nu} + \alpha_3\, R^3 + \alpha_4\, R_{\mu\nu} R^{\mu\rho} R^{\nu}_{\rho} + \underbrace{\ldots}_{\infty\ \alpha_i}\Big].$$

## Peturbative Quantum Electrodynamics

$$L_{ED} = -\frac{1}{4}F^2(A) + L_m(\psi) + e\, j_\mu(\psi) A^\mu\,,$$

$$A_\mu \to \underbrace{A^{(0)}_\mu}_{=0} + A_\mu\,,$$

**Quantization of $A_\mu$**

$$L_{QED} = -\frac{1}{4}F^2(A') + L_m(\psi') + e'\, j_\mu(\psi') A'^\mu\,.$$

# Complete Quantum Gravity

- Low energy limit $\rightarrow$ Einstein's gravity,
  $M_P \rightarrow +\infty$.
- Regular classical solutions. Ex. Black holes $(r \sim 0)$ :
  $$ds^2 \approx -(1-r^\alpha)dt^2 + \frac{dr^2}{1-r^\alpha} + r^2 d\Omega^{(2)} \ , \ \alpha > 0.$$
- Unitary quantum theory (ghosts free),
  $G(p) \propto \frac{f(p)}{p^2}$ , $f(p)$ does not have poles.
- Renormalizability and/or Super-renormalizability,
  $G(p) \rightarrow \frac{1}{p^n}$ , $n \geqslant 4$ in the UV,
- Decreasing of the "spacetime dimension" at small distances :
  $d_s \leqslant 2$.

# Complete Quantum Gravity

$$S = -\int \mathrm{d}^4x\sqrt{g}\Big[\alpha R_{\mu\nu}R^{\mu\nu} - \beta R^2 + \kappa^{-2}R\Big]$$

$$\Downarrow$$

$$S = -\int d^4x\sqrt{-g}\{R_{\mu\nu}\alpha(\Box_\Lambda)R^{\mu\nu} - R\beta(\Box_\Lambda)R + \kappa^{-2}R\}$$

# Complete Quantum Gravity

A. T. Tomboulis 1997, L. M. 2011.

$$\mathcal{L} = -\sqrt{-g}\left[\frac{\beta}{\kappa^2}R - \beta_2\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) + \beta_0 R^2 \right.$$
$$\left. + \left(R_{\mu\nu}\, h_2\left(-\Box_\Lambda\right)\, R^{\mu\nu} - \frac{1}{3}R\, h_2\left(-\Box_\Lambda\right)\, R\right) - R\, h_0\left(-\Box_\Lambda\right) R\right]$$
$$-\frac{1}{2\xi}F^\mu \omega(-\Box_\Lambda^\eta)F_\mu + \bar{C}^\mu M_{\mu\nu} C^\nu .$$

$$\Box_\Lambda := \frac{\Box}{\Lambda^2} ,$$
$F^\tau = \partial_\mu h^{\mu\tau}$ : gauge fixing function,

$\omega(-\Box_\Lambda)$ : weight functional,

$M^\tau_\alpha = F^\tau_{\mu\nu} D^{\mu\nu}_\alpha$,

$F^\tau_{\mu\nu} = \delta^\tau_\mu \partial_\nu$ , $\delta h^{\mu\nu} = D^{\mu\nu}_\alpha \xi^\alpha = \partial^\mu \xi^\nu + \partial^\nu \xi^\mu - \eta^{\mu\nu}\partial_\alpha \xi^\alpha$.

# Action purely quadratic in the gravitational field $h^{\mu\nu}$

$$S^{(2)} = \int d^4k\, h^{\mu\nu}(-k) K^{\rm kin}_{\mu\nu\rho\sigma}(k)\, h^{\rho\sigma}(k)\,,$$

$$S^{(2)} = \int d^4k\, h^{\mu\nu}(-k)\Big( - [\beta - \beta_2 \kappa^2 k^2 + \kappa^2 k^2 h_2(k^2/\Lambda^2)]\, k^2\, P^{(2)}_{\mu\nu\rho\sigma}(k) + \xi^{-1}\,\omega(k^2/\Lambda^2) P^{(1)}_{\mu\nu\rho\sigma}(k)$$
$$+\{3\,k^2\,[\beta/2 - 3\beta_0\kappa^2k^2 + 3\kappa^2k^2h_0(k^2/\Lambda^2)] + 2\xi^{-1}\omega(k^2/\Lambda^2)\}P^{(0-\omega)}_{\mu\nu\rho\sigma}(k)$$
$$+k^2\,[\beta/2 - 3\beta_0\kappa^2k^2 + 3\kappa^2k^2h_0(k^2/\Lambda^2)]\{P^{(0-s)}_{\mu\nu\rho\sigma}(k) + \sqrt{3}[P^{(0-\omega s)}_{\mu\nu\rho\sigma}(k) + P^{(0-s\omega)}_{\mu\nu\rho\sigma}(k)]\}\Big) h^{\rho\sigma}(k)$$

$$P^{(2)}_{\mu\nu\rho\sigma}(k) = \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma},$$
$$P^{(1)}_{\mu\nu\rho\sigma}(k) = \frac{1}{2}\,(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho})\,,$$
$$P^{(0-s)}_{\mu\nu\rho\sigma}(k) = \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma}\ ,\ P^{(0-\omega)}_{\mu\nu\rho\sigma}(k) = \omega_{\mu\nu}\omega_{\rho\sigma},$$
$$P^{(0-s\omega)}_{\mu\nu\rho\sigma} = \frac{1}{\sqrt{3}}\theta_{\mu\nu}\omega_{\rho\sigma},\ \ P^{(0-\omega s)}_{\mu\nu\rho\sigma} = \frac{1}{\sqrt{3}}\omega_{\mu\nu}\theta_{\rho\sigma},$$
$$\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\ ,\ \omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}.$$

# Graviton Propagator

Gauge : $\xi = 0$ .

$$D_{\mu\nu\rho\sigma}(k) = \frac{-i}{(2\pi)^4}\frac{1}{k^2+i\epsilon}\left(\frac{2P^{(2)}_{\mu\nu\rho\sigma}(k)}{\beta-\beta_2\kappa^2k^2+\kappa^2k^2h_2(k^2/\Lambda^2)} - \frac{4P^{(0-s)}_{\mu\nu\rho\sigma}(k)}{\beta-6\beta_0\kappa^2k^2+\kappa^2k^2h_0(k^2/\Lambda^2)}\right).$$

Assume $h_i(-\Box_\Lambda)$ to be a polynomial

↓

Factorization theorem for polynomial :

$$\frac{1}{k^2(1+p_n(k^2))} = \frac{c_0}{k^2} + \sum_i \frac{c_i}{k^2-M_i^2} \;\rightarrow\; \text{at least one of the } c_i < 0 \;\rightarrow\; \text{GHOST}.$$

↓

$h_i(-\Box_\Lambda)$ CAN NOT be polynomial.

# Properties of the transcendental entire functions $h_i$

Deff. :

$$\bar{h}_2(z) := \beta - \beta_2\kappa^2\Lambda^2 z + \kappa^2\Lambda^2 z\, h_2(z),$$
$$\bar{h}_0(z) := \beta - 6\beta_0\kappa^2\Lambda^2 z + 6\kappa^2\Lambda^2 z\, h_0(z),$$
$$z := -\Box_\Lambda.$$

1. $\bar{h}_i(z)$ is real and positive on the real axis, it has no zeroes on the whole complex plane $|z| < +\infty$. This requirement implies that there are no gauge-invariant poles other than the transverse massless physical graviton pole.

2. $|h_i(z)|$ has the same asymptotic behavior along the real axis at $\pm\infty$.

3.There exists $\Theta > 0$ such that

$$\lim_{|z|\to+\infty} |h_i(z)| \to |z|^\gamma \,, \quad \gamma \geqslant 2$$

for the argument of $z$ in the cones :

$C = \{z \mid -\Theta < \arg z < +\Theta\,,\ \pi - \Theta < \arg z < \pi + \Theta\}$ for $0 < \Theta < \pi/2$.

# Renormalizability

Propagator and Interaction in the UltraViolet :

$$D(k) \sim \frac{1}{k^{2\gamma+4}}\,,$$

$$L^{(n)} \ \sim \ h^n \,\Box_\eta h \ h_i(-\Box_\Lambda)\,\Box_\eta h \ \ \rightarrow \ \ h^n\,\Box_\eta h \underbrace{\left(\frac{\Box_\eta + h^m\,\partial h \partial}{\Lambda^2}\right)}_{k^{2\gamma+4}}{}^{\gamma} \,\Box_\eta h\ .$$

Superficial degree of divergence in 4d :

$$D = 4L - (2\gamma+4)I + (2\gamma+4)V \ = \ 4 - 2\gamma(L-1)\,.$$

$$I = V + L - 1$$

if $\gamma \geqslant 3$ only 1-loop divergences exist and the theory is super-renormalizable.

# Renormalized Lagrangian

$$\mathcal{L}_{\mathrm{Ren}} = -\sqrt{-g}\Big\{\frac{\beta\, Z}{\kappa^2}R + \lambda Z_\lambda - \beta_2 Z_2 (R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2) + \beta_0 Z_0 R^2$$
$$+ \left(R_{\mu\nu}\, h_2\left(-\Box_\Lambda\right)\, R^{\mu\nu} - \frac{1}{3}R\, h_2\left(-\Box_\Lambda\right)\, R\right) - R\, h_0\left(-\Box_\Lambda\right) R\Big\}\,.$$

- All the coupling must be understood as renormalized at an energy scale $\mu_0$ ,
- $h_i(z) = \sum_{r=0}^{+\infty} a_r z^r$ and the coefficients $a_r$ are not renormalized.

## $h_2(-\Box_\Lambda)$ & $h_0(-\Box_\Lambda)$ functions

Imposing the conditions (1)-(3) we have :

$$h_2(z) = \frac{\bar{h}_2(z) - \alpha + \alpha_2 z}{\kappa^2 \Lambda^2 z},$$
$$h_0(z) = \frac{\bar{h}_0(z) - \alpha + \alpha_0 z}{6\kappa^2 \Lambda^2 z},$$

$$\bar{h}_i(z) = \alpha\, e^{H(z)},$$

$H(z)$ entire function that exhibits logarithmic asymptotic behavior in the conical region C.

The theory is renormalized at some scale $\mu_0$ :

$$\alpha = \beta(\mu_0),$$
$$\frac{\alpha_2}{\kappa^2\Lambda^2} = \beta_2(\mu_0),$$
$$\frac{\alpha_0}{6\kappa^2\Lambda^2} = \beta_0(\mu_0).$$

Propagator :

$$D_{\mu\nu\rho\sigma}(k) = \frac{-i}{(2\pi)^4} \frac{e^{-H(k^2/\Lambda^2)}}{\alpha\,(k^2 + i\epsilon)} \Big(2P^{(2)}_{\mu\nu\rho\sigma}(k) - 4P^{(0-s)}_{\mu\nu\rho\sigma}(k)\Big).$$

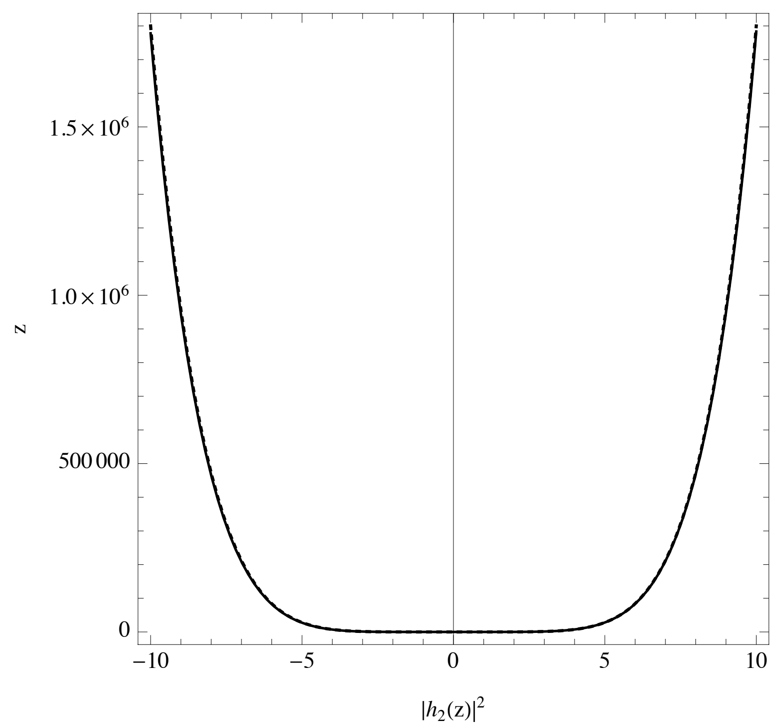
# An explicit realization

$$H(z) = \frac{1}{2}\left[\gamma_E + \Gamma\left(0, z^{2\gamma+2}\right)\right] + \log(z^{\gamma+1})\,,$$

$$\mathrm{Re}(z^{2\gamma+2}) > 0\,,$$

or

$$H(z) = \sum_{n=1}^{+\infty} (-1)^{n-1}\,\frac{p_{\gamma+1}(z)^{2n}}{2n\,n!}\,.$$



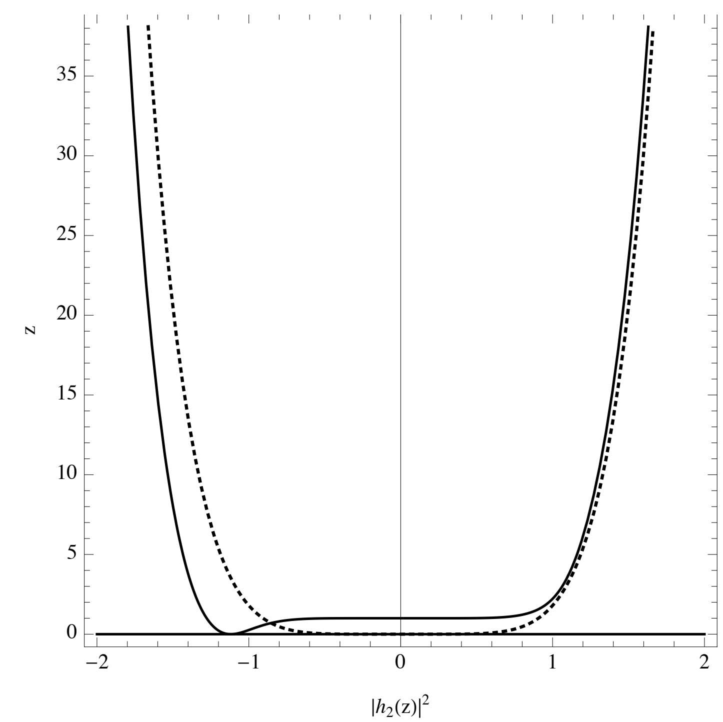
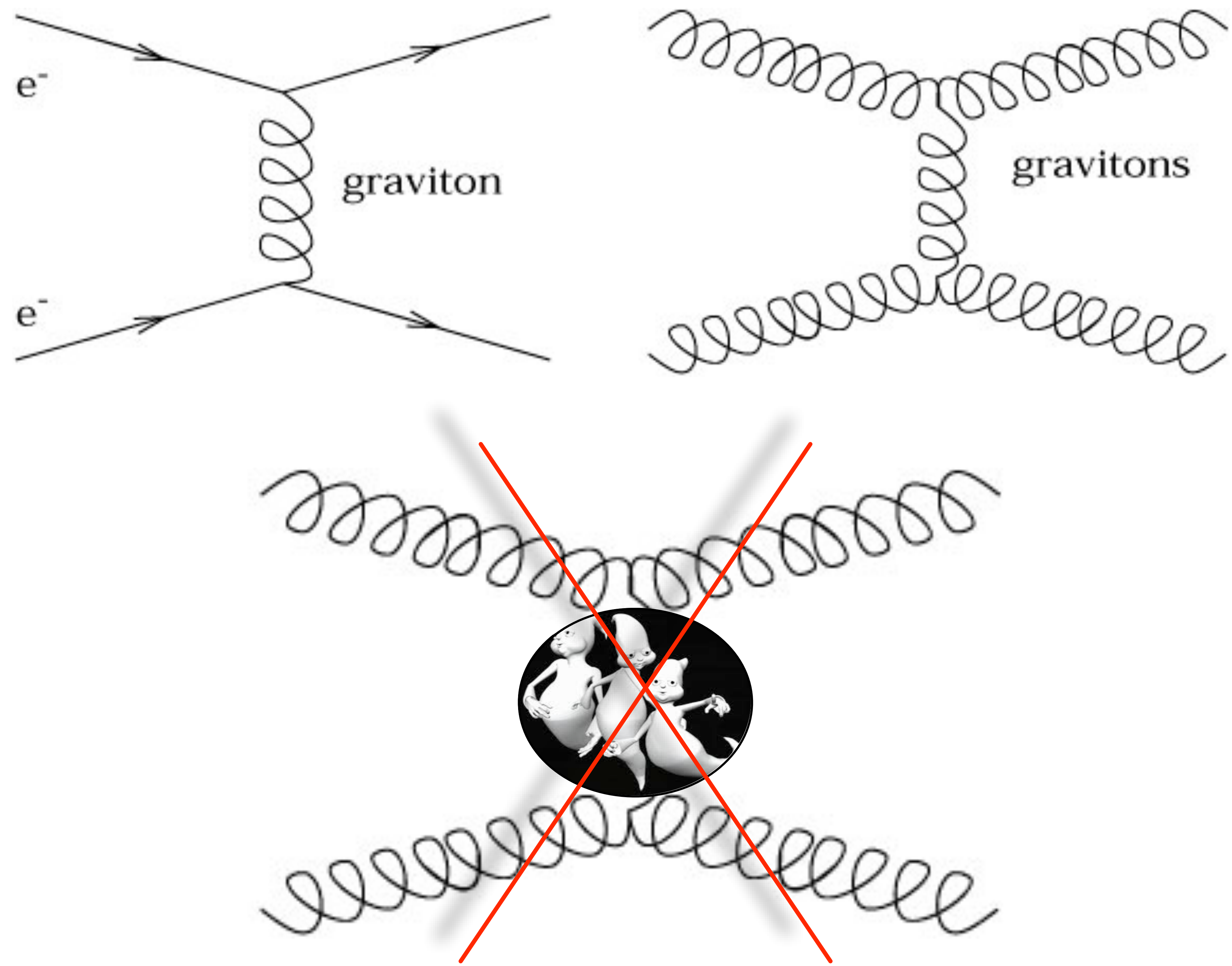




FIG. 1: Plot of $|h(z)|^2$ for $z$ real and $\alpha = \alpha_2 = 1$ (solid line). In the first plot $z \in [-10, 10]$ and in the second one $z \in [-2, 2]$. The dashed line represents the asymptotic limit for large real positive and negative values of $z$. The asymptotic behavior is $|h(z)|^2 \approx 1.8\, z^6$.


e⁻
e⁻
graviton
gravitons

# Spectral Dimension

Diffusion of a probe particle on a d-dimensional manifold :

$$K_g(x, x'; T) = \langle x' | \exp(T\, \Delta_g^{\mathrm{eff}}) | x \rangle.$$

(probability to diffuse from x$'$ to x in a time $T$),

$$\partial_T K_g(x, x'; T) = \Delta_g^{\mathrm{eff}} K_g(x, x'; T).$$

Average return probability :

$$P_g(T) \equiv V^{-1} \int \mathrm{d}^d x \sqrt{g(x)}\, K_g(x, x; T) \equiv V^{-1}\, \mathrm{Tr}\, \exp(T\, \Delta_g^{\mathrm{eff}}) \quad \rightarrow \quad d_s \equiv -2 \frac{\mathrm{d} \ln P_g(T)}{\mathrm{d} \ln T}\,.$$

Propagator ⟺ Heat-kernel

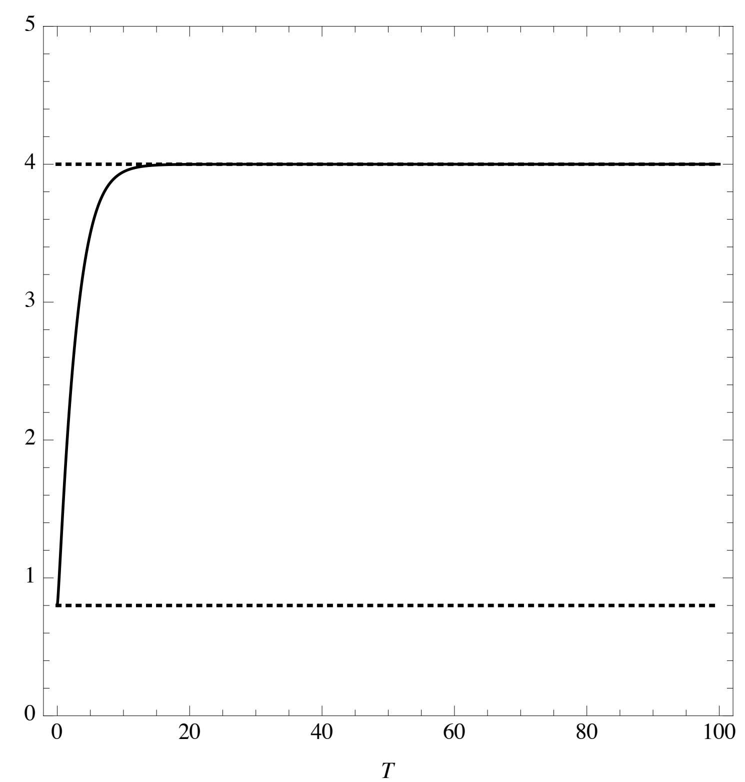
$$G(x, x') = \int_0^{+\infty} \mathrm{d}T\, K_g(x, x'; T) \propto \int \mathrm{d}^4 k\, e^{ik(x-x')} \int_0^{+\infty} \mathrm{d}T\, K_g(k; T)\,.$$

## Spectral Dimension Flux

$$D(k) \propto \frac{1}{k^2\,\bar{h}(k^2/\Lambda^2)} \underset{\text{UV}}{\rightarrow} \frac{1}{k^{2\gamma+4}} \quad \Rightarrow \quad K_g(k;T) \propto e^{-k^2\,\bar{h}(k^2/\Lambda^2)\,T}$$

$$P_g(T) \propto \int \mathrm{d}^4k\, e^{-k^2\,\bar{h}(k^2/\Lambda^2)\,T} \quad \Rightarrow \quad \text{In the UV} \quad d_s = \frac{4}{\gamma+2}\,.$$

$d_s(T)$
for $\gamma = 3$.

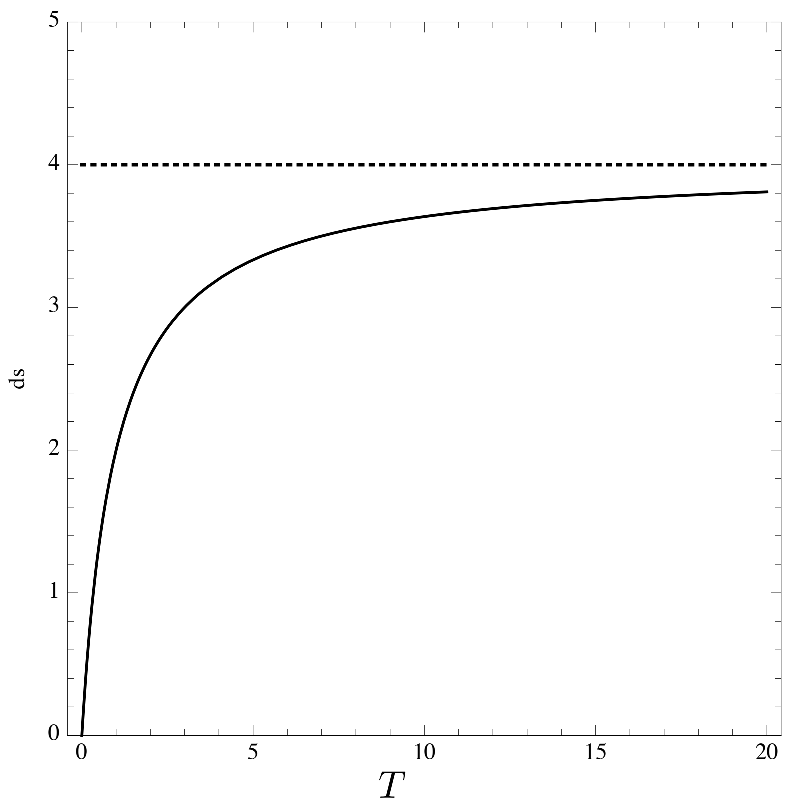


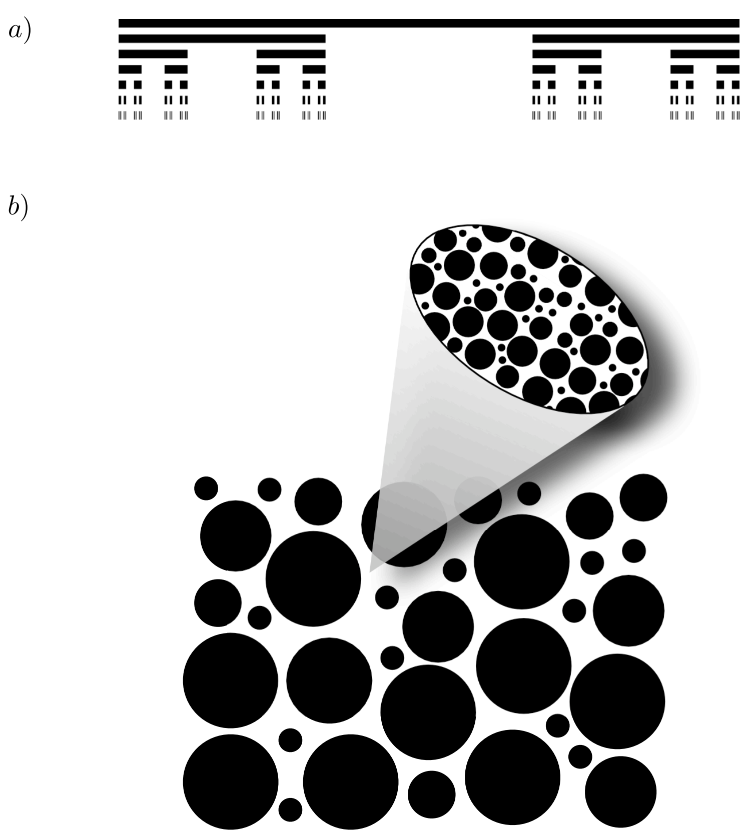
# An Analytically Solvable Example

P. Nicolini, L.M.

$$\bar{h}_2(z) = \bar{h}_0(z) \propto e^z \quad \Rightarrow \quad D(k) \propto \frac{e^{-k^2/\Lambda^2}}{k^2} \quad \Rightarrow \quad K(x, x'; T) = \frac{e^{-\frac{(x-x')^2}{4(T+1/\Lambda^2)}}}{\left[4\pi\left(T + 1/\Lambda^2\right)\right]^2}\,.$$

$$d_s(T) = \frac{4\,T}{T + 1/\Lambda^2}\,.$$



# Multi-horizons Black Holes

$$G_{\mu\nu} = 8\pi G_N \, e^{-H(-\Box_\Lambda)} \, T_{\mu\nu},$$
$$\nabla^\mu (e^{-H} T_{\mu\nu}) = 0\,.$$

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2\Omega^2,$$
$$F(r) = 1 - \frac{2m(r)}{r}.$$

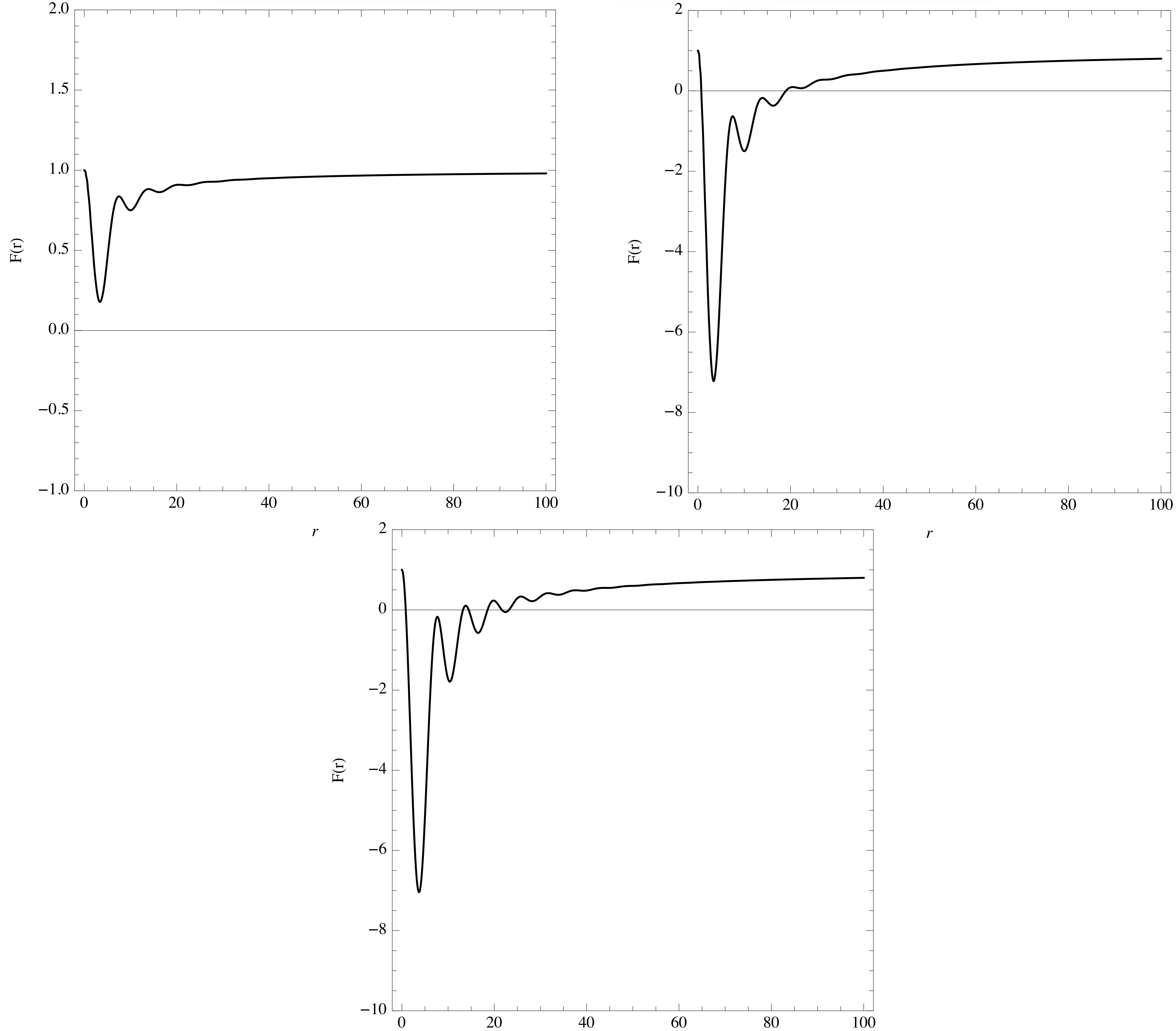
$$\rho^{\rm e}(\vec{x}) := -\tilde{h}^{-1}(-\Box(x)_\Lambda) T^0{}_0 = m\,\tilde{h}^{-1}(-\Box(x)_\Lambda)\,\delta(\vec{x})$$
$$= m \int \frac{d^3k}{(2\pi)^3}\, e^{-H(k^2/\Lambda^2)} e^{i\vec{k}\cdot\vec{x}} = \frac{2m}{(2\pi)^2\, r^3} \int_0^{+\infty} e^{-H(p^2/r^2\Lambda^2)} p \,\sin(p)\, dp\,,$$
$$m(r) = 4\pi \int_0^r dr' r'^2 \; \rho^{\rm e}(r').$$

For $r \approx 0$, $F(r) \approx 1 - c\, m\, \Lambda^{2\gamma+2}\, r^{2\gamma+1}$.

$$R = c\, m\, \Lambda^{2\gamma+2}\,(2\gamma+2)(2\gamma+3)\, r^{2\gamma-1},$$
$$R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} =$$
$$= 4\, c^2\, m^2\, \Lambda^{4\gamma+4}\, \left(4\gamma^4 + 4\gamma^3 + 5\gamma^2 + 4\gamma + 2\right) r^{4\gamma}\,.$$

$\gamma \geqslant 3 \;\rightarrow\;$ No Singularity.

# Exact Solution



# An Analytical Solution

$$G_{\mu\nu} + O(R^2_{\mu\nu}) + O(\nabla_\mu \nabla_\nu R_{\rho\sigma}) = 8\pi G_N \tilde{h}^{-1}(-\Box/\Lambda^2)\, T_{\mu\nu}\,.$$

$$\tilde{h}(-\Box/\Lambda^2) := e^{H(-\Box/\Lambda^2)}.$$

$$G_{\mu\nu} = 8\pi G_N\, \tilde{h}^{-1}(-\Box/\Lambda^2)\, T_{\mu\nu}\,,$$

$$\nabla^\mu \left( \tilde{h}^{-1}(-\Box/\Lambda^2)\, T_{\mu\nu} \right) = 0\,.$$

$$\tilde{h}^{-1}(-\Box/\Lambda^2) = e^{\Box/\Lambda^2}\,,$$

$$ds^2 = -\left(1 - \frac{2m(r)}{r}\right) + \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2 d\Omega^2\,, \qquad m(r) = M \frac{\gamma(3/2; \Lambda^2 r^2/4)}{\Gamma(3/2)}.$$

Non-commutative black hole, P. Nicolini and E. Spallucci.

# Conclusions

- A perturbative super-renormalizable quantum gravity theory,
- no poltergeists,
- spectral dimension flow from $d_s < 1$ in UV to $d_s = 4$ in the IR,
- quasi-polinomial non-locality (string theory, ADS/CFT, LQG),
- regular multi-horizon black holes solutions.