Spin foam models: Lessons from the canonical theory

Marc Geiller
APC, Université Paris 7

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Introduction

We would like to give a meaning to the formal path integral for gravity

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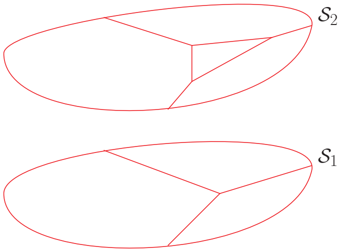

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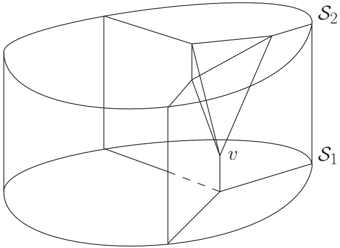

A spin foam model associates an amplitude to a two-complex interpolating between the two boundary kinematical states:

$$W(\Delta) = \int \mathrm{d}\mu_{\{j_f\}} \int \mathrm{d}\mu_{\{i_e\}} \prod_f A_f(j_f) \prod_e A_e(j_{f\supset e}, i_e) \prod_v A_v(j_{f\supset v}, i_{e\supset v}).$$

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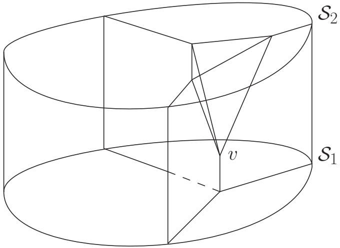

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Is this expression finite? How to define the individual amplitudes? Do we have to sum over different two-complexes?

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Spin network states $\mathcal{S}$ solve the Gauss and diffeomorphism constraints.

To obtain the dynamics, we have to solve the Hamiltonian constraint:

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What is our understanding of the relation between the covariant and canonical quantizations?

3-dimensional quantum gravity ($\Lambda = 0$)

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Covariant quantization

The partition function for the first-order action,

$$\mathcal{Z} = \int \mathrm{d}[e]\mathrm{d}[\omega] \exp\left(i \int_{\mathcal{M}} \mathrm{Tr}(e \wedge F) \right),$$

can be discretized on a two-complex to obtain the Ponzano-Regge spin foam model:

$$\mathcal{Z}(\Delta) = \sum_{\{j\} \to \{f\}} \prod_{f \in \Delta} (2j_f + 1) \prod_{v \in \Delta} \{6j\}_v .$$

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The Hamiltonian theory has two constraints:
Gauss constraint $\rightarrow$ solved by working with spin network states,
Flat curvature constraint $\rightarrow$ imposed by means of a regularized projector.

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Gauss constraint $\rightarrow$ solved by working with spin network states,
Flat curvature constraint $\rightarrow$ imposed by means of a regularized projector.

⋆ The physical inner product is given by the Ponzano-Regge amplitudes.
⋆ There are no (bubble) divergencies.

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To regularize the divergencies of the Ponzano-Regge model, Turaev and Viro suggested to replace $\mathfrak{su}(2)$ with its quantum deformation $\mathrm{U}_q\big(\mathfrak{su}(2)\big)$.

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- ⋆ This introduces a natural regularization when q is a root of unity.
- ⋆ The Turaev-Viro spin foam model is a discretization of

$$S[e,\omega]=\int_{\mathcal{M}}\left(\mathrm{Tr}(e\wedge F)+\frac{\Lambda}{6}\mathrm{Tr}(e\wedge e\wedge e)\right),$$

  as indicated by the results of Witten on SU(2) Chern-Simons theory, and the asymptotic behavior of the q-deformed $6j$ symbol.

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⋆ At the kinematical level, quantization of the non-commutative holonomy $A^{\pm}=A\pm e\sqrt{\Lambda}$ leads to Kauffman's q-deformed binor identity.
⋆ It is necessary to complete the quantization following the $\Lambda=0$ case.

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Covariant quantization

The Plebanski action allows to write gravity as a topological field theory with constraints:

$$S[e, \omega, \Phi] = \int_{\mathcal{M}} \left(B_{IJ} \wedge F^{IJ} + \Phi^{IJKL} B_{IJ} \wedge B_{KL} \right).$$

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Strategy to build 4-dimensional spin foam models:

- ⋆ Discretize the topological theory on a two-complex,
- ⋆ Promote the basic variables to quantum operators,
- ⋆ Impose the (second-class) simplicity constraints on the group theoretical data.

Various ways to impose the constraints lead to different spin foam models.

See Carlo Rovelli's lectures for the EPRL model.

4-dimensional quantum gravity ($\Lambda = 0$)

Canonical quantization

A starting point for the canonical quantization is the Holst-Palatini action

$$S[e,\omega] = \int_{\mathcal{M}} e^I \wedge e^J \wedge \left(\star + \frac{1}{\gamma}\right) F_{IJ}, \qquad \gamma \in \mathbb{R} - \{0\}.$$

With the spacetime connection ω, one can construct three spatial connections:

- ⋆ The commutative $\mathfrak{su}(2)$ Ashtekar-Barbero connection A,
- ⋆ A commutative $\mathfrak{sl}(2,\mathbb{C})$ connection $\mathcal{A}$,
- ⋆ A non-commutative $\mathfrak{sl}(2,\mathbb{C})$ connection $\mathbf{A}$.

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The first two choices are equivalent. In particular, they lead to discrete spectra for the geometric operators, and lead to SU(2) spin network states.

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When working with $\mathcal{A}$, we can solve the second-class simplicity constraints at the classical level (in agreement with the EPRL prescription). Then, the kinematical states agree with the boundary states of the covariant theory, and the spectra of the geometric operators are identical.

Quantizing the theory with $\mathbf{A}$ is still an open problem.

Conclusion and further directions

Several (intermediate) results support the idea of a correspondence between the covariant spin foam approach and canonical loop quantum gravity.

More open questions for the 4-dimensional theory, which can however be approached along other directions:

- ⋆ Spin foam cosmology (see talks by Francesca Vidotto and Mercedes Martín-Benito),
- ⋆ Relation between topological BF theory and LQG (see Valentin Bonzom's talk).

Merci – Ευχαριστώ