Asymptotic Safety,

Fractals,

and Cosmology

Martin Reuter

The 4 Fundamental Interactions: theoretical status

electromagnetic, weak, strong:

- class.: Yang-Mills theories
- quant.: perturbatively renormalizable QFTh.s

gravity:

- class.: General Relativity, with $S[g_{\mu\nu}] \sim \int d^4x \sqrt{-g}\, R$
- quant.: ???

Quantum field theory of spacetime metric $g_{\mu\nu}(x)$ based upon $\int \sqrt{-g}\, R$ is not renormalizable in perturbation theory:

increasing order in pert. th. $\Rightarrow$
" number of counter terms $\Rightarrow$
" " " undetermined parameters

$\rightsquigarrow$ no / very questionable predictivity at high energies ("effective" rather than "fundamental" theory)

The key question:

Is it possible to construct a consistent and predictive quantum field theory of gravity as the (non-perturbative) "continuum" limit of some appropriate system involving an ultraviolet cutoff?

Standard quantization of gravity $\widehat{=}$

degrees of freedom carried by:	$g_{\mu\nu}(x)$
bare action:	$\int d^4x \sqrt{-g}\, R$
calculational method:	$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{8\pi G}\, h_{\mu\nu}$, perturbative quantization, renormalization

What should be given up in order to arrive at a "fundamental" or "microscopic" quantum theory of gravity?

String Theory: d.o.f., action, calc. meth.

Loop Quantum Gravity: d.o.f., calc. meth.

Asymptotic Safety: calc. meth., action

Asymptotic Safety Approach:

- degrees of freedom carried by $g_{\mu\nu}$
- quantization/renormalization is non-perturbative in an essential way
- bare action Γ_* is not an ad hoc assumption, but a <u>prediction:</u>

  $\Gamma_* \sim \int d^4x \sqrt{-g}\, R$ + "more" is a non-Gaussian fixed point of the (∞-dimensional, non-pert.) Wilsonian renormalization group flow
- fixed point "controls" UV divergences

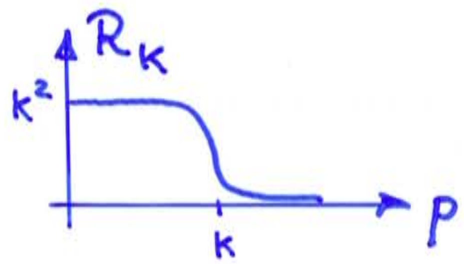
The Effective Average Action $\Gamma_k[g_{\mu\nu}, \cdots]$

- Scale-dependent (coarse grained) effective action functional for the metric

- Defines family of effective field theories:
  $\{\Gamma_k \mid 0 \leq k < \infty\}$

- Built-in IR cutoff: Only metric fluctuations with cov. momentum $p > k$ are integrated out fully.

  Modes with $p < k$ are suppressed by "mass" term added to the bare action:

  $$(\text{mass})^2 = R_k(p^2)$$

- $\Gamma_{k \to \infty} \sim S$ = bare action

  $\Gamma_{k \to 0} = \Gamma$ = standard eff. action

- Γ_k satisfies a FRGE; symbolically:

  $$k \partial_k \Gamma_k = \frac{1}{2} \, \mathrm{STr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1} k \partial_k R_k\right]$$

- Natural (nonperturbative) approximation scheme: project RG flow onto truncated theory space

Construction of Γ_k for Gravity

M.R. 1996

- starting point: $\int \mathcal{D}\gamma_{\mu\nu}\, e^{-S[\gamma_{\mu\nu}]}$

- decompose $\gamma_{\mu\nu} = \underbrace{\bar{g}_{\mu\nu}}_{\substack{\text{arbitrary} \\ \text{backgrd. metric}}} + h_{\mu\nu}$

- add background gauge fixing $S_{gf}[h;\bar{g}]$ + ghost terms

- expand $h_{\mu\nu}$ in $\bar{D}^2$-eigenmodes, and introduce IR cutoff k^2: only modes with generalized momenta ($\bar{D}^2$-eigenvalues) $> k$ are integrated out.

- add sources: generating fctl. $W_k[\text{sources}; \bar{g}]$

$$\Big\downarrow \text{Legendre transf.}$$

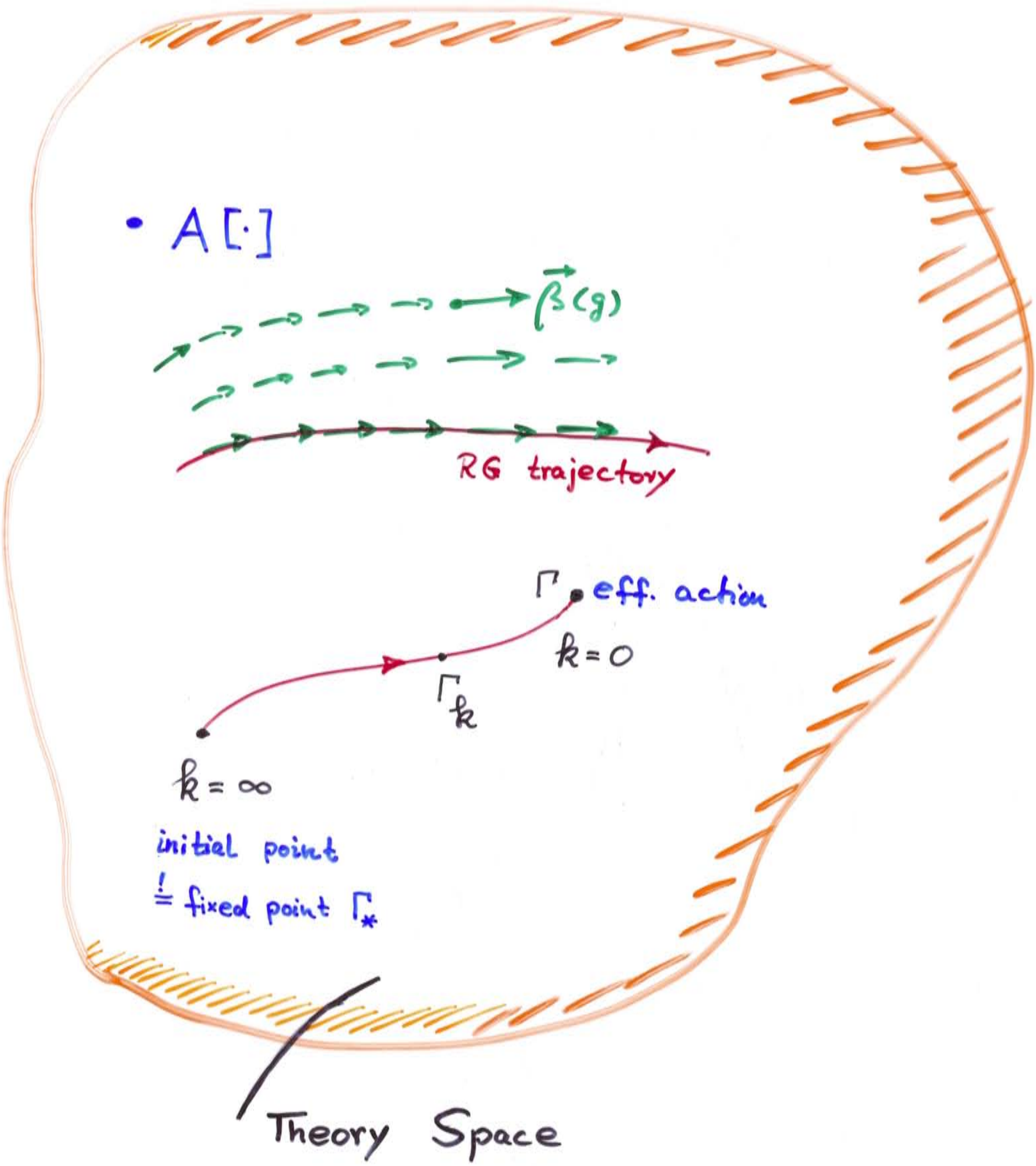
$g_{\mu\nu} \equiv \langle \gamma_{\mu\nu} \rangle$ $\qquad\qquad \Gamma_k[g_{\mu\nu}, \bar{g}_{\mu\nu}, \text{ghosts}]$

- derive exact RG equation from path integral:

$$k \frac{\partial}{\partial k} \Gamma_k[g, \bar{g}, \cdots] = \mathrm{Tr}(\cdots)$$

- "Ordinary" diffeomorphism invariant action:

$$\Gamma_k[g] = \Gamma_k[g, \bar{g} = g, \text{ghosts} = 0]$$



A[·]
$\vec{\beta}(g)$
RG trajectory
Γ eff. action
k = 0
Γ_k
k = ∞
initial point
$\overset{!}{=}$ fixed point Γ_*
Theory Space

Taking the UV-limit in QEG

If there exists a non-Gaussian Fixed Point Γ_*,

$\beta_i(\Gamma_*) = 0$, Quantum Einstein Gravity is

nonperturbatively renormalizable ("asymptotically safe").

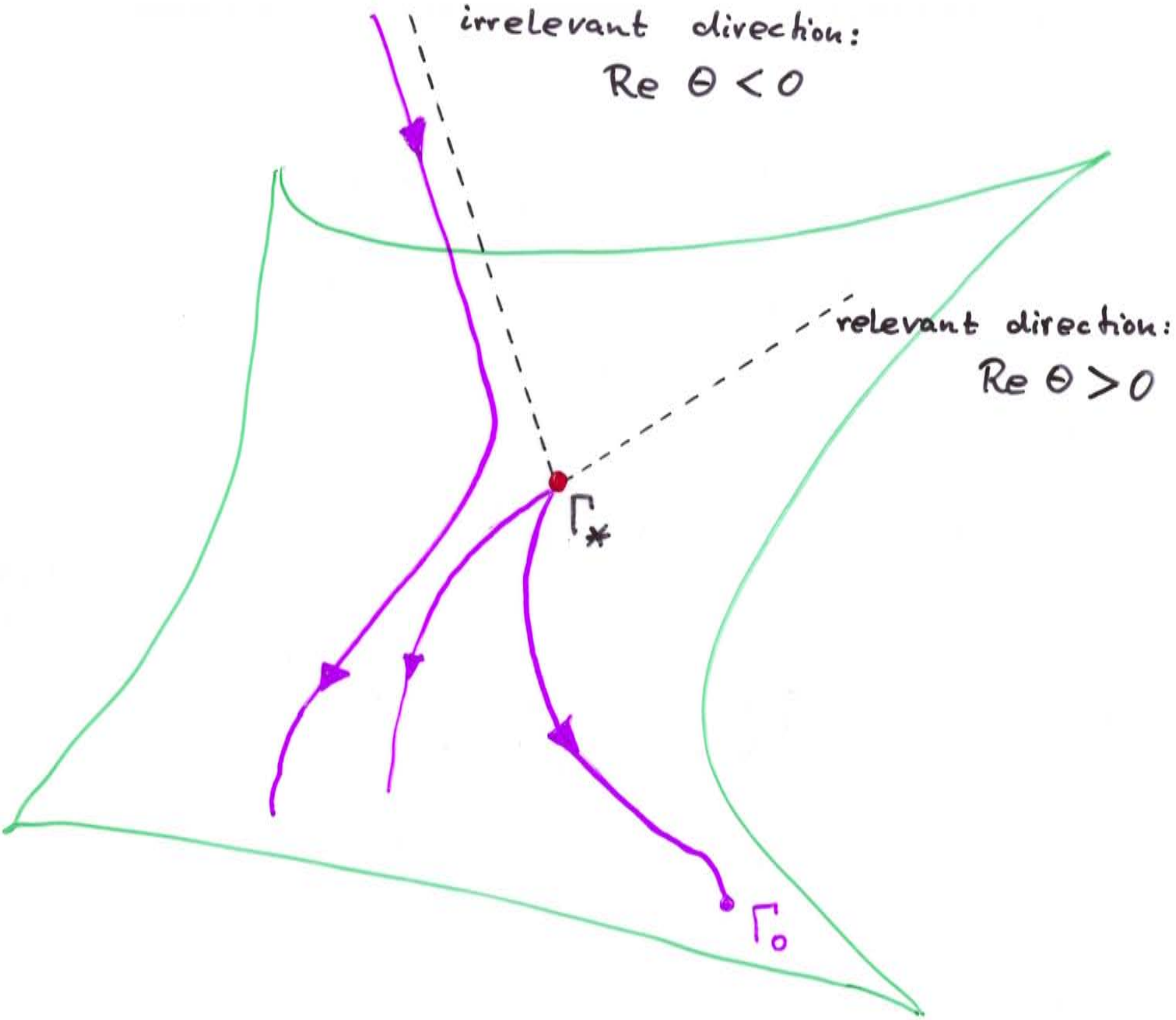
Weinberg 1979

Quantum theory is defined by a RG trajectory running inside the UV-critical hypersurface of the FP, with

initial point $= \Gamma_{k\to\infty} \stackrel{!}{=}$ action infinitesimally close to Γ_*

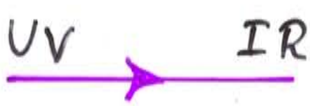
end point $= \Gamma_0 \equiv \Gamma$

The UV-critical hypersurface $\mathcal{S}_{UV}$:



$\Delta_{UV} \equiv \dim \mathcal{S}_{UV}$ = # relevant directions

= # free parameters in the a.s. quantum field theory



Θ : critical exponent (neg. eigenvalue of lin. flow)

The Einstein-Hilbert Truncation

(M.R., 1996)

ansatz:

$$\Gamma_k = -\frac{1}{16\pi G_k} \int d^dx \sqrt{g} \left\{ R - 2\Lambda_k \right\}$$

+ classical gauge fixing and ghost terms

two running parameters:

Newton constant G_k, dimensionless: $g(k) = k^{d-2} G_k$

cosmological constant Λ_k, dimensionless: $\lambda(k) = \Lambda_k / k^2$

insert ansatz into flow equation, expand

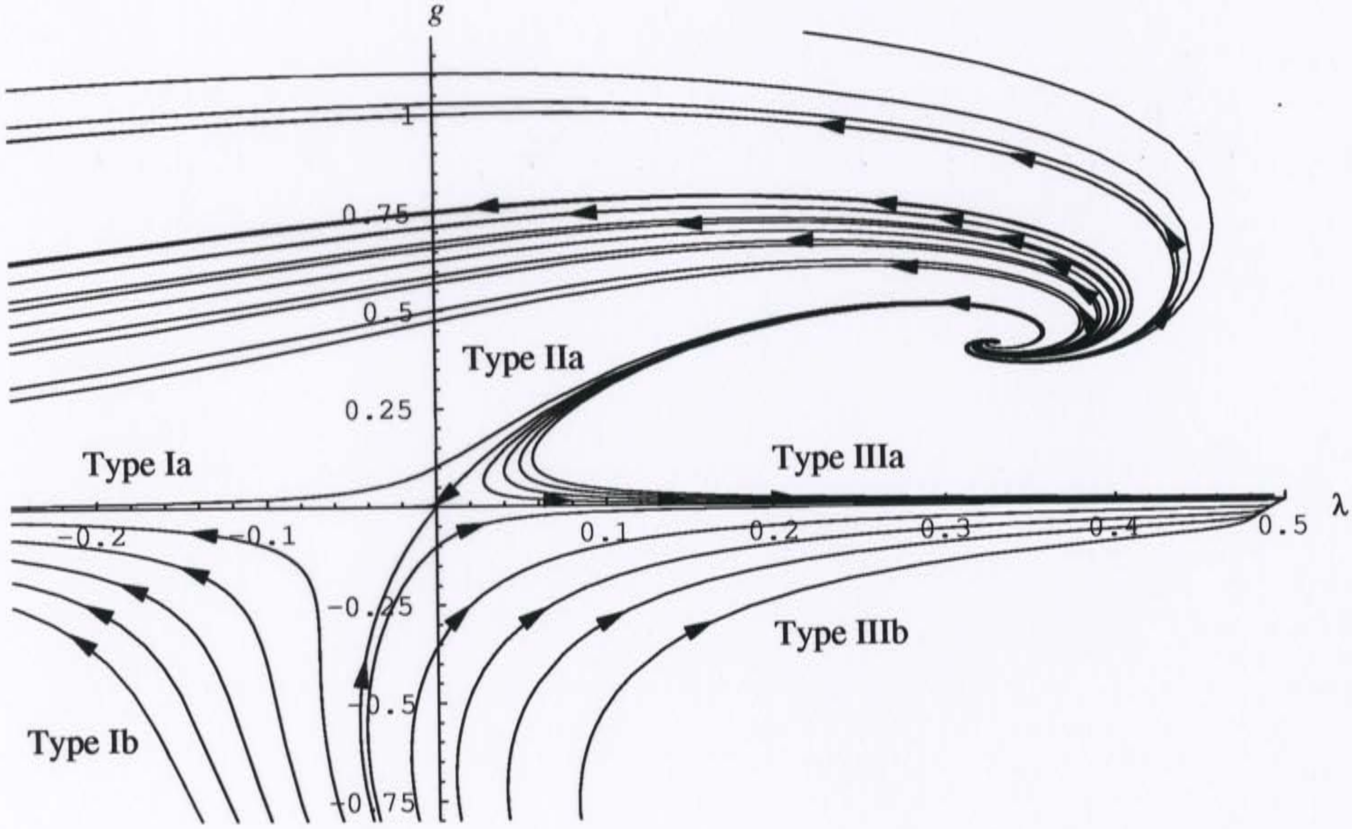
$$\mathrm{Tr}[\cdots] = (\cdots) \int \sqrt{g} + (\cdots) \int \sqrt{g} R + \cdots$$

$\rightsquigarrow$

$$\boxed{\begin{aligned} k\,\partial_k\, g(k) &= \beta_g(g, \lambda) \\ k\,\partial_k\, \lambda(k) &= \beta_\lambda(g, \lambda) \end{aligned}}$$

Einstein - Hilbert Truncation :

RG Flow on the $g-\lambda$ plane

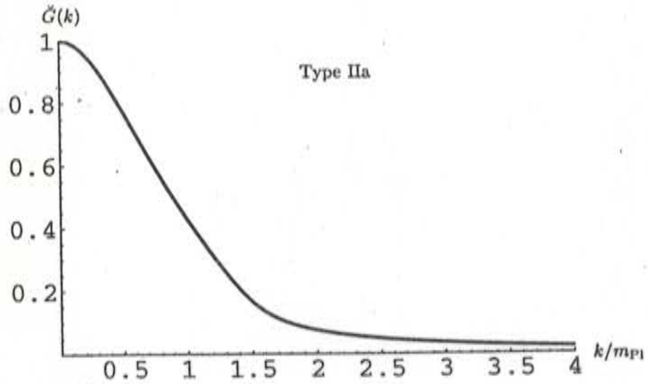


M.R., F. Saueressig, hep-th/0110054

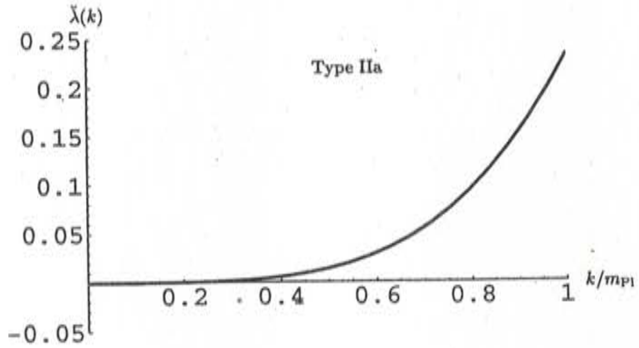
The Separatrix:

cross-over from the non-Gaussian to the Gaussian fixed point

G



Λ



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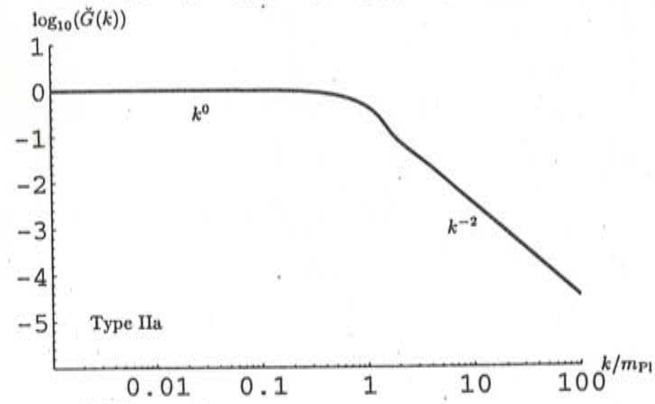

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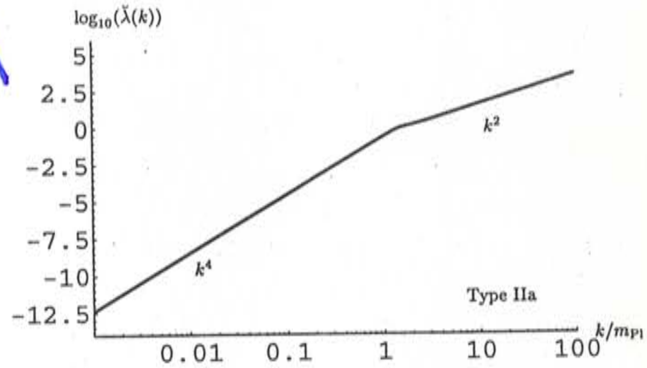

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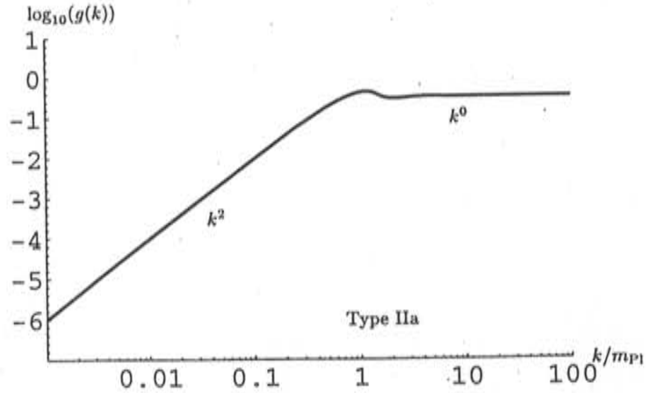
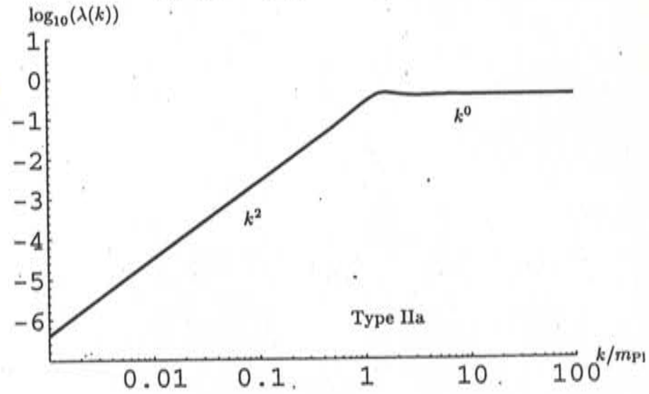


λ



Properties of QEG

- Background-independent quantization scheme:

  No special <u>metric</u> plays any distinguished role!

  <u>The background field method</u>:

  a) Fix arbitrary $\bar{g}_{\mu\nu}$

  b) Quantize (nonlinear) fluctuations $h_{\mu\nu} \equiv \gamma_{\mu\nu} - \bar{g}_{\mu\nu}$ in the backgrd. of $\bar{g}_{\mu\nu}$

  c) Adjust $\bar{g}_{\mu\nu}$ such that $\langle h_{\mu\nu} \rangle = 0$

  $$\rightsquigarrow \quad g_{\mu\nu} \equiv \langle \gamma_{\mu\nu} \rangle = \bar{g}_{\mu\nu}$$

- Fundamental action $S \approx \Gamma_*$ is a prediction:

  No special <u>action</u> plays any distinguished role!

  <u>The only input</u>: field contents + symmetries
  $\widehat{=}$ theory space

  <u>The output</u>: $\Gamma_* = S_{\text{Einstein-Hilbert}}$ + "more"

Einstein-Hilbert action is often a reliable approximation, but not distinguished conceptually.

Fractal properties of

QEG spacetimes

M.R., O. Lauscher

Γ_k as an Effective Field Theory

→ Non-gauge theories in flat space:

- $\Gamma_k[\phi]$ generates correlation fcts. of fields averaged over spacetime volume of radius $\ell \approx k^{-1}$
- In particular: $\frac{\delta \Gamma_k}{\delta \phi(x)}[\langle\phi\rangle_k] = 0$

  yields mean field $\langle\phi\rangle_k \equiv$ field "seen" by microscope with resolving power ℓ

- For observable $\mathcal{O}(\phi)$ involving only momenta $\approx k$:

$$\langle \mathcal{O}(\hat{\phi}) \rangle \approx \mathcal{O}(\langle\phi\rangle_k)$$

→ In gravity:

- averaging replaced by cutting-off of $\bar{D}^2$ - eigenvalues
- relationship $\ell = \ell(k)$ more complicated in general

Metrics on QEG Spacetimes

- Fix a quantum theory by picking a specific RG trajectory $k \mapsto \Gamma_k[\cdot]$
- Solve eff. field eq. at any k:

$$\frac{\delta \Gamma_k}{\delta g_{\mu\nu}}\left[\langle g \rangle_k\right] = 0$$

scale dependent mean field

- A single trajectory gives rise to infinitely many "on-shell metrics":

$$\left\{ \langle g_{\mu\nu}(x) \rangle_k \;\middle|\; k = 0, \cdots, \infty \right\}$$

- Metric structure of "QEG spacetime" is described by <u>infinitely many</u> classical Riemannian metrics.

<u>Interpretation</u>: Observing spacetime under a microscope of resolving power ℓ_1 one sees a classical manifold with metric

$$\langle g_{\mu\nu} \rangle_{k_1} \quad \text{where} \quad \ell_1 = \ell(k_1)$$

- Metric is scale dependent!

<u>Analogy</u>: The length of the coast line of England depends on the size of the yardstick used to measure it.

B. Mandelbrot

QEG Spacetimes are Fractals

O. Lauscher, M.R., 2002

... on scales at which the RG trajectory is near the FP 

$$\Lambda(k) = \lambda_* k^2, \quad G(k) = g_* k^{2-d}, \ldots$$

Effective field eqs. in the Einstein-Hilbert approx.:

$$\boxed{R_{\mu\nu}(\langle g \rangle_k) = \Lambda(k) \langle g_{\mu\nu} \rangle_k}$$

- no dim.ful constants of integration

  $\rightsquigarrow$ solution has radius of curvature

  $$r_c(k) \propto \Lambda(k)^{-\frac{1}{2}} \propto k^{-1}$$

- $k \propto 1/\ell$ in the FP regime

$\Rightarrow$ Radius of curvature detected by "microscope of resolution ℓ":

$$\boxed{r_c(\ell) \propto \ell}$$

"Zooming" deeper into the spacetime structure (lowering ℓ) does not change the image seen by the microscope.

Dynamical Dimensional Reduction

O. Lauscher,
M.R., 2002

$$\Gamma_k \supset \frac{1}{G(k)} \int d^dx \sqrt{g}\, R \qquad \rightsquigarrow$$

graviton propagator in d-dim. flat space $\propto \dfrac{G(k)}{p^2}$

Physical cutoff scale is $k = \sqrt{p^2}$ if $p^2 \in$ FP-regime.

$\Longrightarrow$ dressed propagator (in $\Gamma \equiv \Gamma_0$):

$$\boxed{\mathcal{G}(p) \propto \frac{G(\sqrt{p^2})}{p^2}}$$

Anomalous dimension: $\quad \eta \equiv k \dfrac{d}{dk} \ln G(k)$

Example: FP regime

$$G(k) = g_* k^{2-d}$$

$$\Longrightarrow \quad \boxed{\eta = \eta_* \equiv 2-d} = \text{const}$$

$$d=4: \quad \eta_* = -2$$

If η approximately constant

$$\Rightarrow \quad G(k) \propto k^{\eta}$$

$$\Rightarrow \quad \mathcal{G}(p) \propto \left(\frac{1}{p^2}\right)^{1-\eta/2}$$

$\eta \neq \eta_*$:

$$\mathcal{G}(x,y) \propto \frac{1}{|x-y|^{d+\eta-2}}$$

$\eta = \eta_*$:

$$\mathcal{G}(x,y) \propto \ln|x-y|$$

$\Rightarrow$ η has standard interpretation ($\to$ crit. phenomena)

$\Rightarrow$ effective dimensionality $= d + \eta(k)$

- At the FP: 2D-like logarithmic propagator

$$\boxed{d + \eta_* = d + (2-d) = 2 \quad \forall d}$$

- In the classical regime:

$$\boxed{d + \eta \approx d}$$

$\Rightarrow$ **Dynamical change of dimension $2 \to d$ during the RG evolution from the fixed point to the classical regime.**

- Holds true also in the exact theory!

The Spectral Dimension

Random walk (diffusion) of scalar test particle on classical, d-dimensional Riemannian manifold:

$$\partial_T K_g(x,x';T) = \Delta_g K_g(x,x';T)$$

$$\Delta_g \phi \equiv g^{-\frac{1}{2}} \partial_\mu \left(g^{1/2} g^{\mu\nu} \partial_\nu \phi \right)$$

Average return probability:

$$P_g(T) = \frac{1}{V} \int d^dx \sqrt{g(x)}\, K_g(x,x;T) = \frac{1}{V} \operatorname{Tr} e^{T\Delta_g}$$

$$\underset{\text{as. exp.}}{=} \left(\frac{1}{4\pi T} \right)^{d/2} \sum_{n=0}^{\infty} A_n T^n$$

Recover d from P_g: $\quad d = -2 \dfrac{d \ln P_g(T)}{d \ln T}$

Motivates the following definition for the spectral dimension of a QEG spacetime:

$$P(T) \equiv \langle P_\gamma(T) \rangle$$

$$\equiv \int \mathcal{D}\gamma_{\mu\nu}\, \mathcal{D}C\, \mathcal{D}\bar{C}\; P_\gamma(T)\; e^{-S[\gamma, C, \bar{C}]}$$

$$\boxed{\mathcal{D}_s \equiv -2 \frac{d \ln P(T)}{d \ln T}}$$

Evaluation of $\mathcal{D}_S$ in the RG framework

O. Lauscher
M.R. 2005

hep-th/0508202

(1) Solution of $R_{\mu\nu}(\langle g\rangle_k) = \Lambda(k)\langle g_{\mu\nu}\rangle_k$ satisfying

$$\langle g_{\mu\nu}\rangle_k = \frac{\Lambda(k_0)}{\Lambda(k)}\langle g_{\mu\nu}\rangle_{k_0}$$

leads to "running Laplacian" $\Delta(k) \equiv D^2(\langle g\rangle_k)$ satisfying

$$\boxed{\Delta(k) = \frac{\Lambda(k)}{\Lambda(k_0)}\,\Delta(k_0)}$$

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(2) • Γ_k = eff. field theory valid at k $\rightsquigarrow$

$$\langle \mathcal{O}(\gamma_{\mu\nu})\rangle \approx \mathcal{O}(\langle g_{\mu\nu}\rangle_k)$$

if $\mathcal{O}$ involves only momenta $\approx k$.

• Apply to $\mathcal{O} = \Delta(k)\,K(\cdots)$;

How to choose k ?

(3) If the diffusion process involves only momenta near a single, fixed k:

$$\partial_T K(x,x';T) = \Delta(k)\, K(x,x';T)$$

solved by

$$K(x,x';T) = \sum_n \phi_n(x)\, \phi_n(x')\, e^{-F(k^2)\mathcal{E}_n T}$$

where
$$-\Delta(k_0)\phi_n = \mathcal{E}_n \phi_n$$
$$F(k^2) \equiv \Lambda(k)/\Lambda(k_0)$$

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(4) Time evolution of initial probability distribution $p(x;0)$:

$$p(x;T) = \int d^dx' \sqrt{g_0(x')}\, K(x,x';T)\; p(x';0)$$

If $p(x;0) = \sum_n C_n \phi_n(x)$ then

$$p(x,T) = \sum_n C_n\, \phi_n(x)\, e^{-F(k^2)\mathcal{E}_n T}$$

If $C_n \neq 0$ only for a single $\mathcal{E}_N$:

single-scale problem, obvious choice is $k = \sqrt{\mathcal{E}_N}$

(5) If $C_n \neq 0$ for many different $\mathcal{E}_n$'s :

multi-scale problem,
choose mode-depent scale $\boxed{k = \sqrt{\mathcal{E}_n}}$ $\Rightarrow$

$$p(x,T) = \sum_n C_n \phi_n(x) \, e^{-F(\mathcal{E}_n)\mathcal{E}_n T}$$

- <u>Example flat space</u>:

$$\phi_n \equiv \phi_p \sim e^{i p_\mu x^\mu} \, , \quad \mathcal{E}_n \equiv \mathcal{E}_p = p_\mu^2$$

Mode ϕ_p has "resolving power" $\ell \approx \frac{1}{|p|} = \frac{1}{\sqrt{\mathcal{E}_p}} \overset{!}{=} \frac{1}{k}$

and therefore "sees" metric $\langle g_{\mu\nu} \rangle_k$ with $k = \sqrt{\mathcal{E}_p}$.

- $k = \sqrt{\mathcal{E}_n}$ correct also in curved space since, by construction, k^2 cuts off the spectrum of D^2.

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(6) Traced propagation Kernel (av. return probability):

$$P(T) = \frac{1}{V} \sum_n e^{-F(\mathcal{E}_n)\,\mathcal{E}_n T}$$

$$= \frac{1}{V} \operatorname{Tr} e^{F(-\Delta(k_0))\,\Delta(k_0)\,T}$$

Let k_0 = macroscopic scale, with $\Lambda(k_0) \ll m_{Pl}^2$,
and $\langle g_{\mu\nu} \rangle_{k_0} \approx$ flat metric :

$$\boxed{P(T) = \int \frac{d^d p}{(2\pi)^d}\, e^{-p^2 F(p^2)\,T}}$$

$$F(p^2) \equiv \Lambda\left(k = \sqrt{p^2}\right) / \Lambda(k_0)$$

$T \to \infty$: Long random walks, probe spacetime structure at large distances; governed by $F(p^2)$ with $p^2 \to 0$: <u>classical regime</u>

$T \to 0$: short random walks, probe spacetime structure at small distances; governed by $F(p^2)$ with $p^2 \to \infty$: <u>fixed point regime</u>

Spectral Dimension $\mathcal{D}_s = -2 \frac{d \ln P(T)}{d \ln T}$

O. Lauscher, M.R., 2005

• <u>classical regime</u>

no running, $\Lambda(k) \approx \Lambda(k_0)$, $F \approx 1$

$\Rightarrow \quad P(T) \propto T^{-d/2}$

$\Rightarrow \quad \boxed{\mathcal{D}_s = d}$

• <u>Fixed Point regime</u>

$\Lambda(k) \propto k^2$, $F(p^2) \propto p^2$

$\Rightarrow \quad P(T) \propto T^{-d/4}$

$\Rightarrow \quad \boxed{\mathcal{D}_s = \frac{1}{2} d}$

$d = 4$:

QEG, based upon <u>any</u> RG trajectory, predicts a continuous change of the fractal dimensionality of spacetime from $\mathcal{D}_s = 4$ at macroscopic to $\mathcal{D}_s = 2$ at microscopic distances.

Remarks:

- Result for $\mathcal{D}_s$ is actually an **exact** consequence of asymptotic safety and does not rely on any truncation.

- $d = 4$ is special:

	$d+\eta$	$\mathcal{D}_s$
macroscopic	d	d
microscopic	2	$\frac{d}{2}$

agrees iff $d=4$!

- RG prediction of 2 dimensional fractal $(d=4)$ supported by Monte Carlo simulations (**CDT**):

$$\mathcal{D}_s(T \to \infty) = 4.02 \pm 0.1$$

$$\mathcal{D}_s(T \to 0) = 1.80 \pm 0.25$$

Ambjørn, Jurkiewicz, Loll, 2005

- **EDT** Simulations:

$$\mathcal{D}_s(T \to \infty) = 4.04 \pm 0.26$$

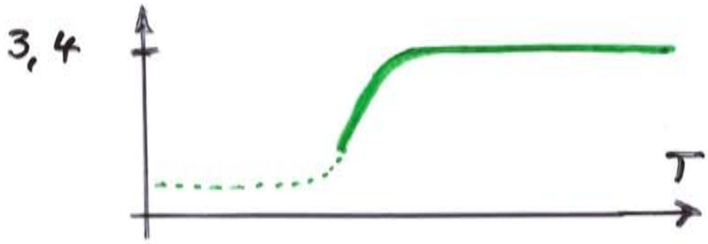
$$\mathcal{D}_s(T \to 0) = 1.457 \pm 0.064$$

Laiho, Coumbe, 2011

CDT simulations:

$\mathcal{D}_s = (4.02 \pm .01) \longrightarrow (1.80 \pm 0.25)$ — Ambjørn, Jurkiewicz, Loll; 2005

$\mathcal{D}_s = (3.05 \pm .04) \longrightarrow (2.04 \pm 0.10)$ — Benedetti, Henson; 2009



no NGFP plateau yet, requires $a \ll \ell_{Pl}$!

LQG (spatial section) and Spin-Foams:

$\mathcal{D}_s = 3 \longrightarrow 1.5 \longrightarrow (1.5 \text{ or } 2)$

$\mathcal{D}_s = 4 \longrightarrow 2$

Modesto, 2008

3D Spin-Foams (P.R. / T.V.):

$\mathcal{D}_s = 3 \longrightarrow 1.5 \longrightarrow 2$

Modesto, Caravelli, 2009

Strong Coupling limit of W-dW. eq.: $4 \rightarrow 2$ — Carlip, 2009

Models: Quantum sphere, κ-Minkowski:

$\mathcal{D}_s = 4 \longrightarrow (< 4)$

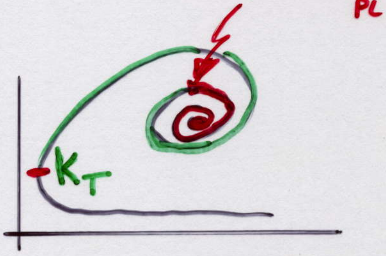
Benedetti, 2008

QFT on given fractal — Calcagni, 2009

A possible explanation for the observed dimensional reduction in Monte Carlo simulations with

$$a \not\lesssim \ell_{Pl} \quad :$$

- $\mathcal{D}_s < d$ requires a significant scale dependence of $\langle g_{\mu\nu} \rangle_k$.



- In the NGFP regime :

$$\langle g_{\mu\nu} \rangle_k \sim \frac{1}{k^2}$$

Follows <u>exactly</u> from the existence of a UV-FP.

- If (i) Einstein-Hilbert truncation is a reliable approximation

  (ii) No matter is present

then $\frac{\delta \Gamma_k}{\delta g_{\mu\nu}} = 0 \rightsquigarrow R_{\mu\nu}(\langle g_{..} \rangle_k) = \Lambda(k) \langle g_{\mu\nu} \rangle_k$

$$\rightsquigarrow \langle g_{\mu\nu} \rangle_k \sim \frac{1}{\Lambda(k)}$$

- In the "k^4-regime" : $\Lambda(k) \approx c_4 \, k^4$

$$\boxed{\text{Perhaps } \mathcal{D}_s < d \text{ already near the turning point } k_T \ll m_{Pl} \text{ ?}}$$

$$\rightsquigarrow \langle g_{\mu\nu} \rangle \sim \frac{1}{k^4}$$

$$\Rightarrow \mathcal{D}_s = \frac{4}{3} \;!$$

Summary

A QEG spacetime "manifold" carries infinitely many metrics $\langle g_{\mu\nu} \rangle_k$ which describe its metric structure on different coarse graining scales.

Spacetime has fractal properties in all regimes with nontrivial RG running of the gravitational couplings.

As a direct and exact consequence of asymptotic safety (existence of a FP) one finds a continuous change of its fractal dimension, from 4 at macroscopic distances to 2 on microscopic scales.