Canonical simplicial gravity

Philipp Höhn

ITF, Universiteit Utrecht

6th Aegean Summer School
Chora, Naxos

based on B. Dittrich, PH arXiv:1108.1974, 0912.1817 and *wip*

September 12th, 2011

Motivation and Goal

- Thus far mainly covariant formulations for simplicial gravity
- Efforts to construct canonical evolution scheme for Regge Calculus [Piran, Williams '86; Friedman, Jack '86; Gambini, Pullin '03; Bahr, Dittrich '09 etc.]
- Recently first canonical formulation which reproduces exactly covariant theory (for triangulations), however, spatial triangulation preserved [Dittrich, PH '09]

Motivation and Goal

- Thus far mainly covariant formulations for simplicial gravity
- Efforts to construct canonical evolution scheme for Regge Calculus

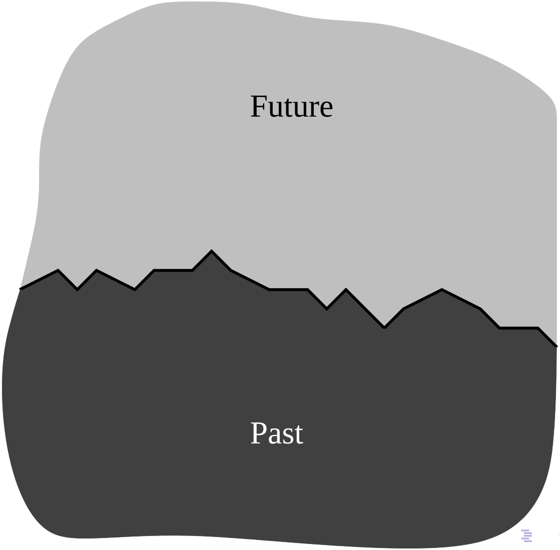
  [Piran, Williams '86; Friedman, Jack '86; Gambini, Pullin '03; Bahr, Dittrich '09 etc.]
- Recently first canonical formulation which reproduces exactly covariant theory (for triangulations), however, spatial triangulation preserved [Dittrich, PH '09]
- How to treat situation where lattice evolves/changes? (as in LQG)
  - Numbers of physical and gauge degrees of freedom may vary
  - Could be interesting numerically, but also for expanding universes etc.
  - may be interesting for quantization

Motivation and Goal

- Thus far mainly covariant formulations for simplicial gravity
- Efforts to construct canonical evolution scheme for Regge Calculus [Piran, Williams '86; Friedman, Jack '86; Gambini, Pullin '03; Bahr, Dittrich '09 etc.]
- Recently first canonical formulation which reproduces exactly covariant theory (for triangulations), however, spatial triangulation preserved [Dittrich, PH '09]
- How to treat situation where lattice evolves/changes? (as in LQG)
  - Numbers of physical and gauge degrees of freedom may vary
  - Could be interesting numerically, but also for expanding universes etc.
  - may be interesting for quantization
- Goal: devise general canonical scheme, reproducing all triangulations
- Requires significant generalization

Evolution in discrete 'multi-fingered' or 'bubble' time

step k



glue (or remove) a single D-simplex, to (or from) a $D-1$-dimensional triangulated hypersurface Σ_k at each elementary step counted by $k \in \mathbb{Z}$

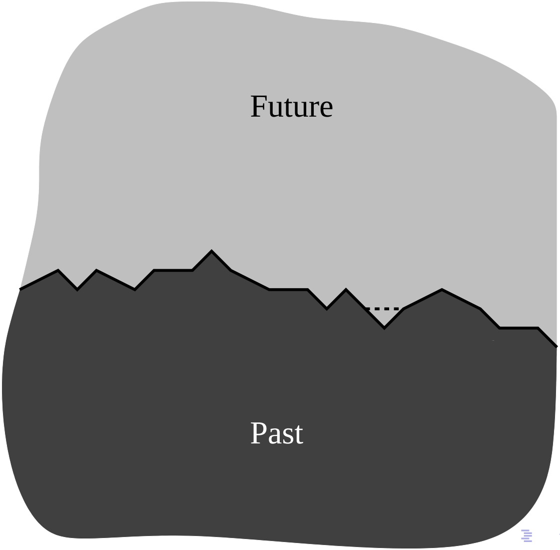
$\Rightarrow$ requires action to be additive

Evolution in discrete 'multi-fingered' or 'bubble' time

step $k+1$

glue (or remove) a single D-simplex, to (or from) a $D-1$-dimensional triangulated hypersurface Σ_k at each elementary step counted by $k \in \mathbb{Z}$

$\Rightarrow$ requires action to be additive

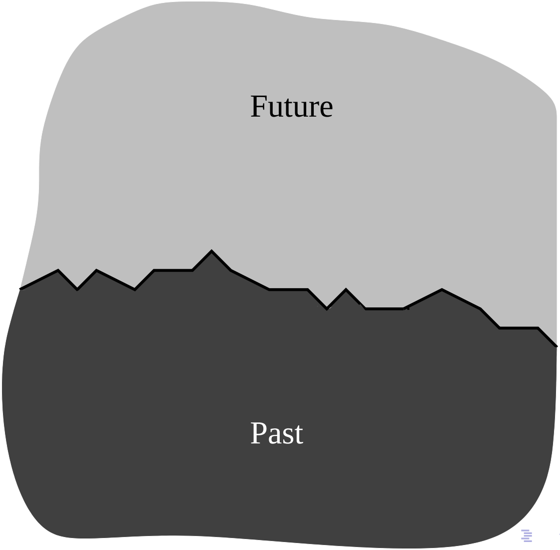


Evolution in discrete 'multi-fingered' or 'bubble' time

step $k+1$

glue (or remove) a single D-simplex, to (or from) a $D-1$-dimensional triangulated hypersurface Σ_k at each elementary step counted by $k \in \mathbb{Z}$

$\Rightarrow$ requires action to be additive

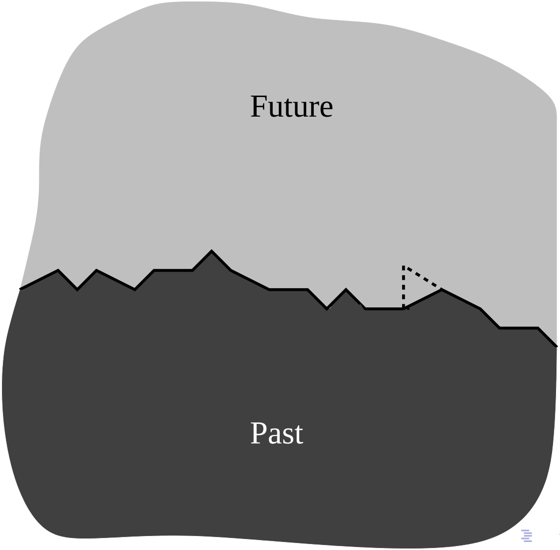


Evolution in discrete 'multi-fingered' or 'bubble' time

step $k+2$

glue (or remove) a single D-simplex, to (or from) a $D-1$-dimensional triangulated hypersurface Σ_k at each elementary step counted by $k \in \mathbb{Z}$

$\Rightarrow$ requires action to be additive

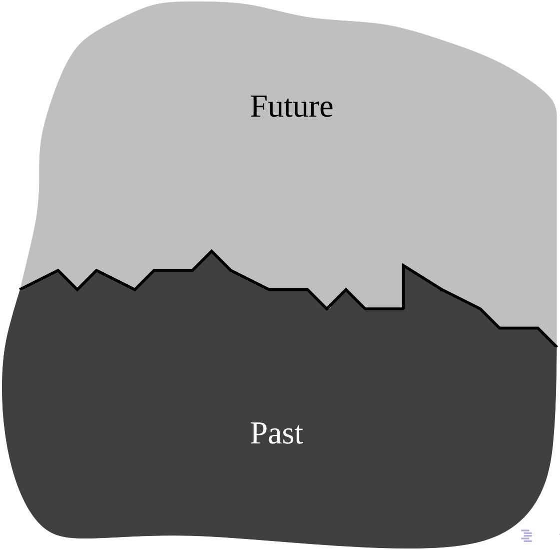


Evolution in discrete ‘multi-fingered’ or ‘bubble’ time

step $k+2$

glue (or remove) a single D-simplex, to (or from) a $D-1$-dimensional triangulated hypersurface Σ_k at each elementary step counted by $k \in \mathbb{Z}$

$\Rightarrow$ requires action to be additive

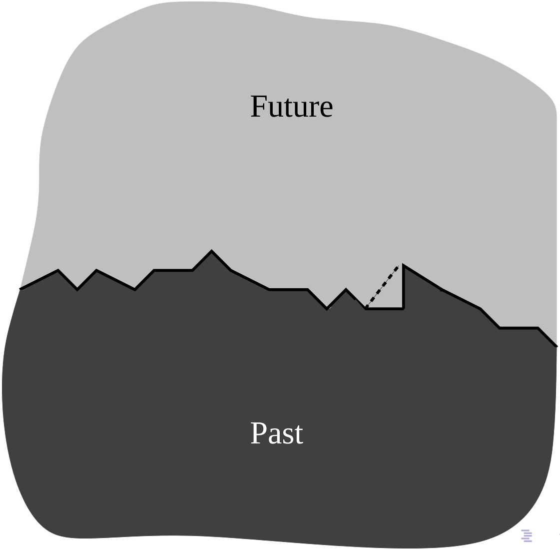


Evolution in discrete ‘multi-fingered’ or ‘bubble’ time

step $k + 3$

glue (or remove) a single D-simplex, to (or from) a $D - 1$-dimensional triangulated hypersurface Σ_k at each elementary step counted by $k \in \mathbb{Z}$

$\Rightarrow$ requires action to be additive

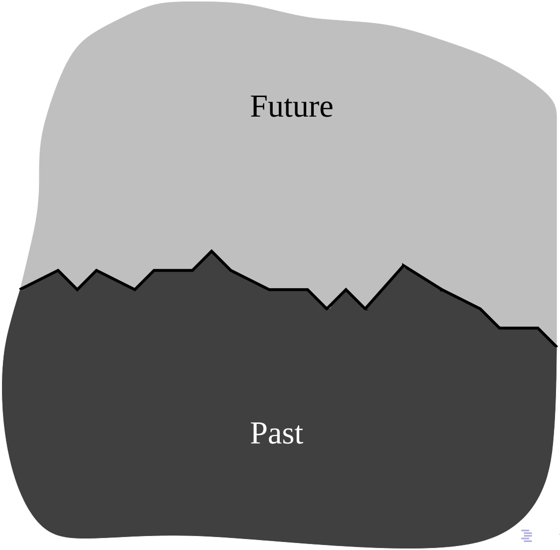


Evolution in discrete 'multi-fingered' or 'bubble' time

step $k+3$

glue (or remove) a single D-simplex, to (or from) a $D-1$-dimensional triangulated hypersurface Σ_k at each elementary step counted by $k \in \mathbb{Z}$

$\Rightarrow$ requires action to be additive

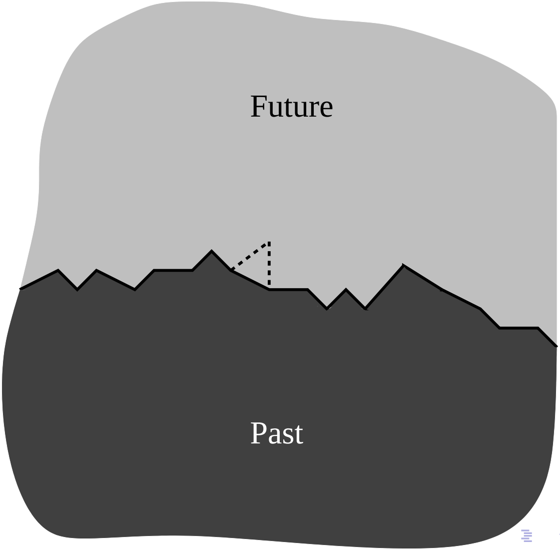


Evolution in discrete ‘multi-fingered’ or ‘bubble’ time

step $k+4$

glue (or remove) a single D-simplex, to (or from) a $D-1$-dimensional triangulated hypersurface Σ_k at each elementary step counted by $k \in \mathbb{Z}$

$\Rightarrow$ requires action to be additive

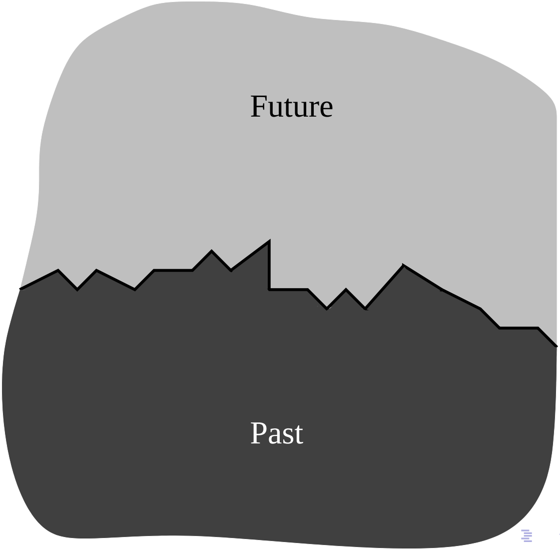


Evolution in discrete 'multi-fingered' or 'bubble' time

step $k+4$

glue (or remove) a single D-simplex, to (or from) a $D-1$-dimensional triangulated hypersurface Σ_k at each elementary step counted by $k \in \mathbb{Z}$

$\Rightarrow$ requires action to be additive

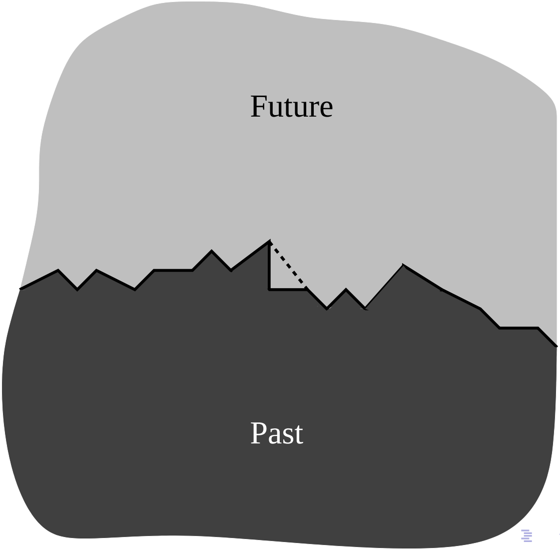


Evolution in discrete ‘multi-fingered’ or ‘bubble’ time

step $k + 5$

glue (or remove) a single D-simplex, to (or from) a $D - 1$-dimensional triangulated hypersurface Σ_k at each elementary step counted by $k \in \mathbb{Z}$

$\Rightarrow$ requires action to be additive

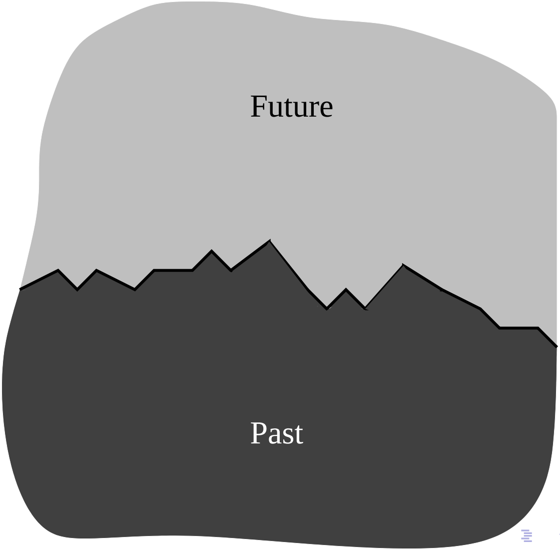


Evolution in discrete ‘multi-fingered’ or ‘bubble’ time

step $k + 5$

glue (or remove) a single D-simplex, to (or from) a $D - 1$-dimensional triangulated hypersurface Σ_k at each elementary step counted by $k \in \mathbb{Z}$

$\Rightarrow$ requires action to be additive

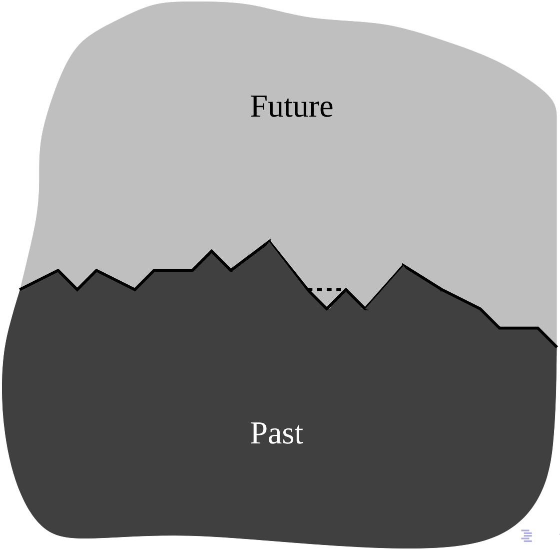


Evolution in discrete 'multi-fingered' or 'bubble' time

step $k + 6$

glue (or remove) a single D-simplex, to (or from) a $D-1$-dimensional triangulated hypersurface Σ_k at each elementary step counted by $k \in \mathbb{Z}$

$\Rightarrow$ requires action to be additive

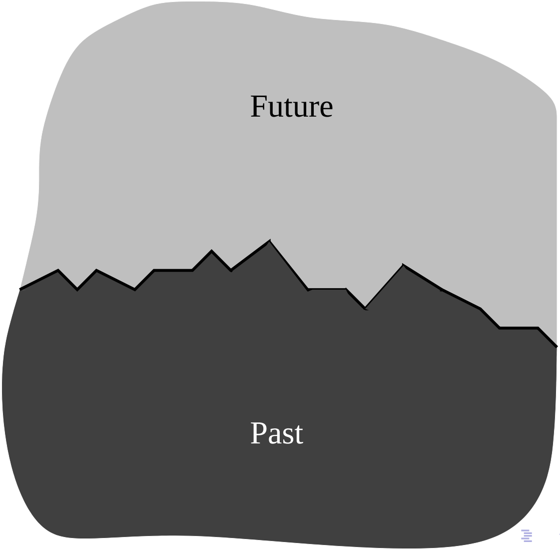


Evolution in discrete 'multi-fingered' or 'bubble' time

step $k+6$

glue (or remove) a single D-simplex, to (or from) a $D-1$-dimensional triangulated hypersurface Σ_k at each elementary step counted by $k \in \mathbb{Z}$

$\Rightarrow$ requires action to be additive

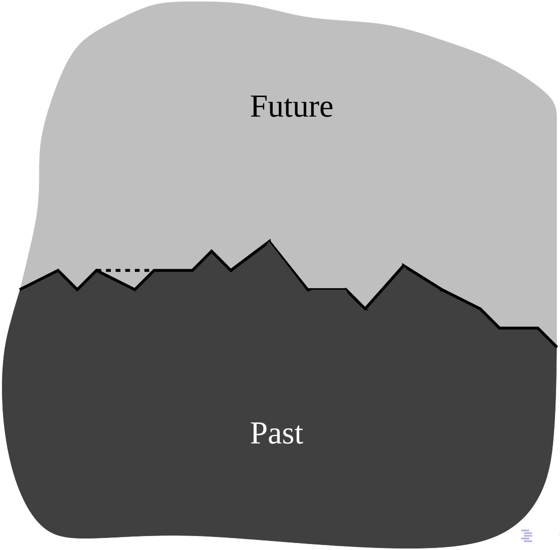


Evolution in discrete ‘multi-fingered’ or ‘bubble’ time

step $k+7$

glue (or remove) a single D-simplex, to (or from) a $D-1$-dimensional triangulated hypersurface Σ_k at each elementary step counted by $k \in \mathbb{Z}$

$\Rightarrow$ requires action to be additive

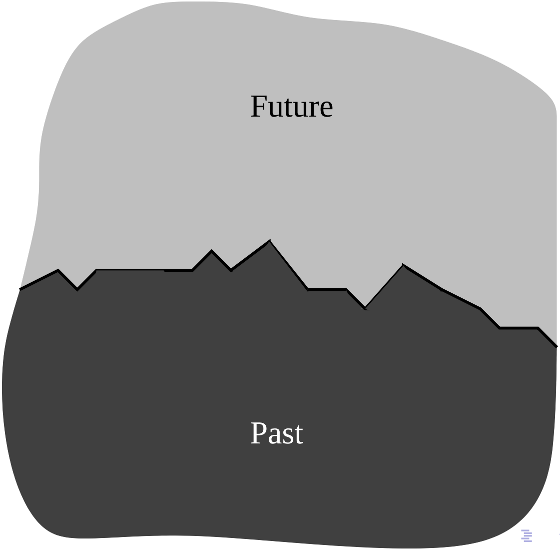


Evolution in discrete 'multi-fingered' or 'bubble' time

step $k + 7$

glue (or remove) a single D-simplex, to (or from) a $D - 1$-dimensional triangulated hypersurface Σ_k at each elementary step counted by $k \in \mathbb{Z}$

$\Rightarrow$ requires action to be additive

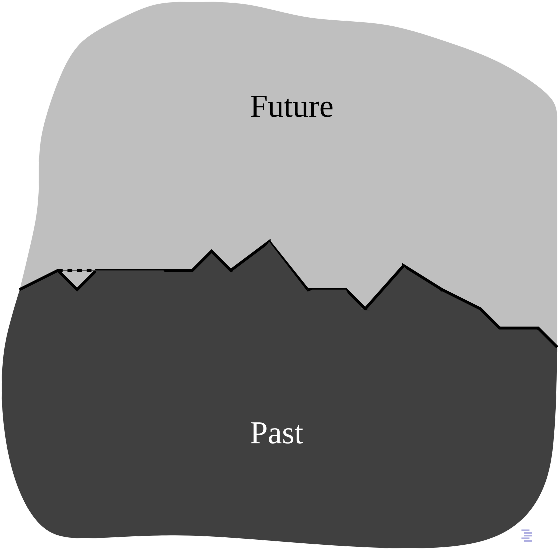


Evolution in discrete 'multi-fingered' or 'bubble' time

step $k+8$

glue (or remove) a single D-simplex, to (or from) a $D-1$-dimensional triangulated hypersurface Σ_k at each elementary step counted by $k \in \mathbb{Z}$

$\Rightarrow$ requires action to be additive

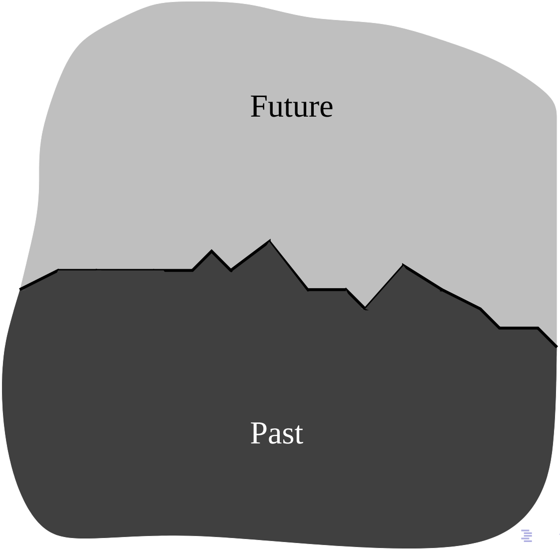


Evolution in discrete ‘multi-fingered’ or ‘bubble’ time

step $k + 8$

glue (or remove) a single D-simplex, to (or from) a $D - 1$-dimensional triangulated hypersurface Σ_k at each elementary step counted by $k \in \mathbb{Z}$

$\Rightarrow$ requires action to be additive

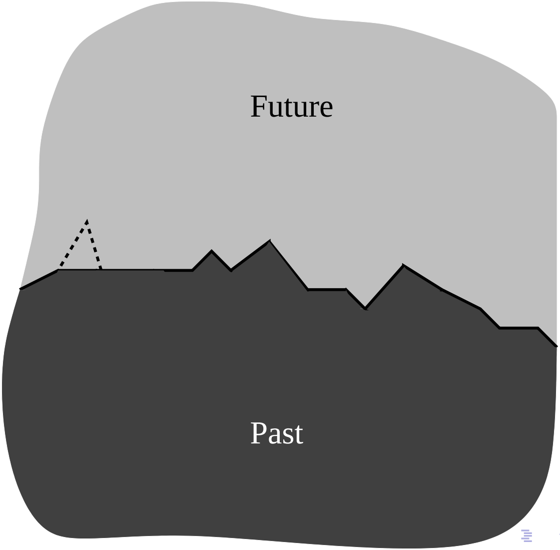


Evolution in discrete 'multi-fingered' or 'bubble' time

step $k + 9$

glue (or remove) a single D-simplex, to (or from) a $D - 1$-dimensional triangulated hypersurface Σ_k at each elementary step counted by $k \in \mathbb{Z}$

$\Rightarrow$ requires action to be additive

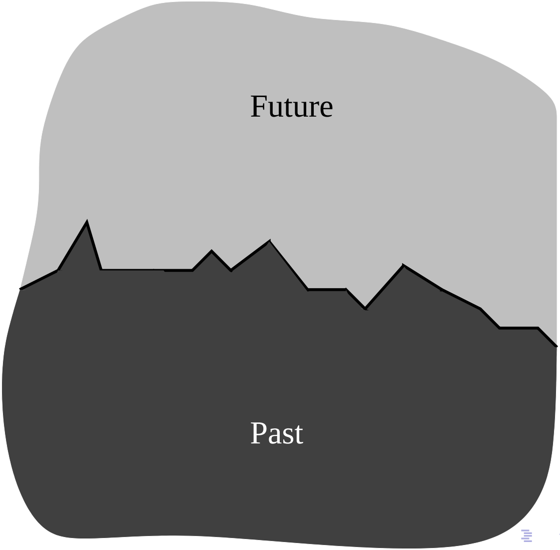


Evolution in discrete 'multi-fingered' or 'bubble' time

step $k + 9$

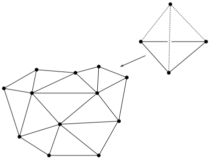
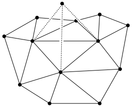
glue (or remove) a single D-simplex, to (or from) a $D-1$-dimensional triangulated hypersurface Σ_k at each elementary step counted by $k \in \mathbb{Z}$

$\Rightarrow$ requires action to be additive



Interpretation within $D-1$ hypersurface: $D-1$ Pachner moves

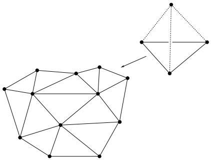
3D Example: gluing of tetrahedron onto single triangle

3D perspective:  $\longrightarrow$ 

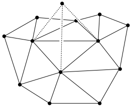
Interpretation within $D-1$ hypersurface: $D-1$ Pachner moves

3D Example: gluing of tetrahedron onto single triangle

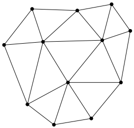
3D perspective:



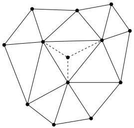
$\longrightarrow$



2D perspective:

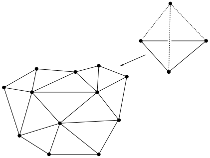
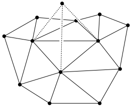


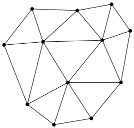
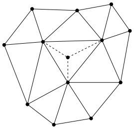
$\longrightarrow$



Interpretation within $D-1$ hypersurface: $D-1$ Pachner moves

3D Example: gluing of tetrahedron onto single triangle

3D perspective:  $\longrightarrow$ 

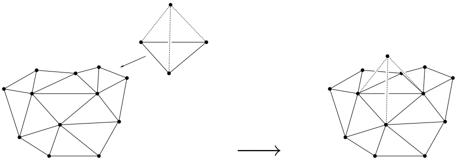
2D perspective:  $\longrightarrow$ 

$\Rightarrow$ 1–3 Pachner move (other moves in 3D and 4D similarly) $\Rightarrow$ Pachner moves are ergodic

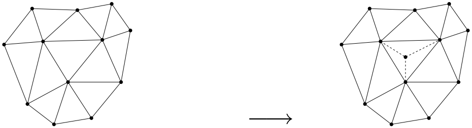
Interpretation within $D-1$ hypersurface: $D-1$ Pachner moves

3D Example: gluing of tetrahedron onto single triangle

3D perspective:



2D perspective:

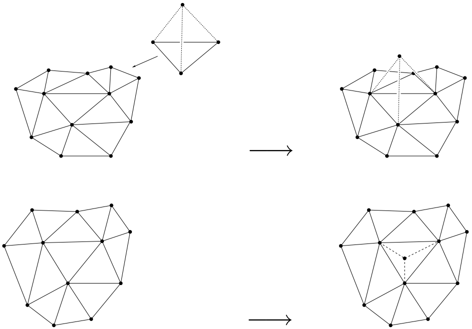


In general, face 'problems':

Interpretation within $D-1$ hypersurface: $D-1$ Pachner moves

3D Example: gluing of tetrahedron onto single triangle

3D perspective:



2D perspective:

In general, face 'problems':

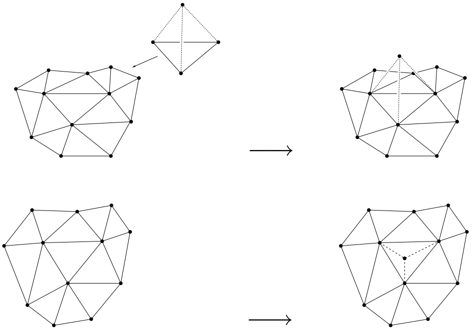
(a) subsets of variables coincide at different steps

Interpretation within $D-1$ hypersurface: $D-1$ Pachner moves

3D Example: gluing of tetrahedron onto single triangle

3D perspective:

2D perspective:

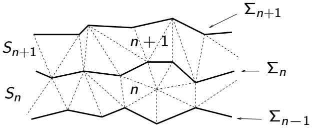


In general, face 'problems':

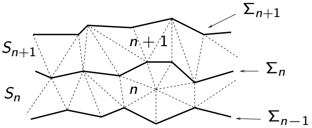
(a) subsets of variables coincide at different steps

(b) numbers of variables differ (phase space dim. varies) from step to step

Discrete Legendre transformation and canonical momenta



choose fat slicing, count fat slices by n, elementary moves by k



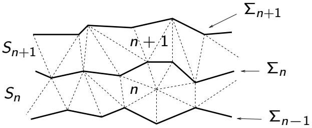
choose fat slicing, count fat slices by n, elementary moves by k

- $S_n(l^e_n, l^i_n, l^{e'}_{n-1})$ as 'generating function' $\Rightarrow$ defines conjugate momenta

[Marsden, West '01; Gambini, Pullin '03; Dittrich, PH '09, '11; etc....]

$$^+p^n_e := \frac{\partial S_n}{\partial l^e_n} \qquad {}^-p^{n-1}_e := -\frac{\partial S_n}{\partial l^e_{n-1}}$$

$$^+p^n_i := \frac{\partial S_n}{\partial l^i_n} \qquad {}^-p^{n-1}_i := -\frac{\partial S_n}{\partial l^i_{n-1}} = 0.$$



choose fat slicing, count fat slices by n, elementary moves by k

- $S_n(l_n^e, l_n^i, l_{n-1}^{e'})$ as 'generating function' $\Rightarrow$ defines conjugate momenta

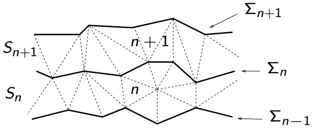
[Marsden, West '01; Gambini, Pullin '03; Dittrich, PH '09, '11; etc....]

$$ {}^{+}p_e^n := \frac{\partial S_n}{\partial l_n^e} \qquad {}^{-}p_e^{n-1} := -\frac{\partial S_n}{\partial l_{n-1}^e} $$

$$ {}^{+}p_i^n := \frac{\partial S_n}{\partial l_n^i} \qquad {}^{-}p_i^{n-1} := -\frac{\partial S_n}{\partial l_{n-1}^i} = 0. $$

- similarly, ${}^{-}p_e^n = -\frac{\partial S_{n+1}}{\partial l_n^e}$

Discrete Legendre transformation and canonical momenta



choose fat slicing, count fat slices by n, elementary moves by k

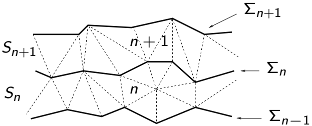
- $S_n(l^e_n, l^i_n, l^{e'}_{n-1})$ as 'generating function' $\Rightarrow$ defines conjugate momenta

  [Marsden, West '01; Gambini, Pullin '03; Dittrich, PH '09, '11; etc....]

$$
\begin{aligned}
{}^+p^n_e &:= \frac{\partial S_n}{\partial l^e_n} & \qquad {}^-p^{n-1}_e &:= -\frac{\partial S_n}{\partial l^e_{n-1}} \\
{}^+p^n_i &:= \frac{\partial S_n}{\partial l^i_n} & \qquad {}^-p^{n-1}_i &:= -\frac{\partial S_n}{\partial l^i_{n-1}} = 0.
\end{aligned}
$$

- similarly, ${}^-p^n_e = -\frac{\partial S_{n+1}}{\partial l^e_n}$
- eom $\frac{\partial S_n}{\partial l^e_n} + \frac{\partial S_{n+1}}{\partial l^e_n} = 0 \Leftrightarrow$ *momentum matching* ${}^+p^n_e = {}^-p^n_e$

Discrete Legendre transformation and canonical momenta



choose fat slicing, count fat slices by n, elementary moves by k

- $S_n(l^e_n, l^i_n, l^{e'}_{n-1})$ as 'generating function' $\Rightarrow$ defines conjugate momenta

[Marsden, West '01; Gambini, Pullin '03; Dittrich, PH '09, '11; etc....]

$$ {}^+p^n_e := \frac{\partial S_n}{\partial l^e_n} \qquad {}^-p^{n-1}_e := -\frac{\partial S_n}{\partial l^e_{n-1}} $$

$$ {}^+p^n_i := \frac{\partial S_n}{\partial l^i_n} \qquad {}^-p^{n-1}_i := -\frac{\partial S_n}{\partial l^i_{n-1}} = 0. $$

- similarly, ${}^-p^n_e = -\frac{\partial S_{n+1}}{\partial l^e_n}$
- eom $\frac{\partial S_n}{\partial l^e_n} + \frac{\partial S_{n+1}}{\partial l^e_n} = 0 \Leftrightarrow$ *momentum matching* ${}^+p^n_e = {}^-p^n_e$
- likewise, for internal variables l^i, eom $\frac{\partial S}{\partial l^i} = 0 \Leftrightarrow p_i = 0$
  constraints as equations of motion

Pachner moves as canonical transformations

Solve 'problems'

- solve 'problem' (a) by *momentum updating*: for all edges occurring in both Σ_k and Σ_{k+1}

$$l^e_{k+1} = l^e_k \qquad p_e^{k+1} = p_e^k + \frac{\partial S_\sigma}{\partial l^e_k}$$

Pachner moves as canonical transformations

Solve 'problems'

- solve 'problem' (a) by *momentum updating*: for all edges occurring in both Σ_k and Σ_{k+1}

$$l^e_{k+1} = l^e_k \qquad p^{k+1}_e = p^k_e + \frac{\partial S_\sigma}{\partial l^e_k}$$

- solve 'problem' (b) by phase space extension: 'add' variables l^n_k, l^o_k of edges occurring only 'to the future' or only 'to the past' of hypersurface Σ_k $\Rightarrow$ eoms require constraints $p^k_n = 0 = p^k_o$

Pachner moves as canonical transformations

Solve 'problems'

- solve 'problem' (a) by *momentum updating*: for all edges occurring in both Σ_k and Σ_{k+1}

$$l^e_{k+1} = l^e_k \qquad p^{k+1}_e = p^k_e + \frac{\partial S_\sigma}{\partial l^e_k}$$

- solve 'problem' (b) by phase space extension: 'add' variables l^n_k, l^o_k of edges occurring only 'to the future' or only 'to the past' of hypersurface $\Sigma_k \Rightarrow$ eoms require constraints $p^k_n = 0 = p^k_o$
  e.g. 1–3 Pachner move, use $S_\tau(l^n_{k+1}, ...)$ as type 1 generating function (trivial dependence on l^n_k)

$$p^k_n = 0 \qquad p^{k+1}_n = \frac{\partial S_\tau}{\partial l^n_{k+1}}$$

Pachner moves as canonical transformations

Solve 'problems'

- solve 'problem' (a) by *momentum updating*: for all edges occurring in both Σ_k and Σ_{k+1}

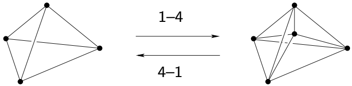
$$l^e_{k+1} = l^e_k \qquad p^{k+1}_e = p^k_e + \frac{\partial S_\sigma}{\partial l^e_k}$$

- solve 'problem' (b) by phase space extension: 'add' variables l^n_k, l^o_k of edges occurring only 'to the future' or only 'to the past' of hypersurface Σ_k $\Rightarrow$ eoms require constraints $p^k_n = 0 = p^k_o$
  e.g. 1–3 Pachner move, use $S_\tau(l^n_{k+1}, \ldots)$ as type 1 generating function (trivial dependence on l^n_k)

$$p^k_n = 0 \qquad p^{k+1}_n = \frac{\partial S_\tau}{\partial l^n_{k+1}}$$

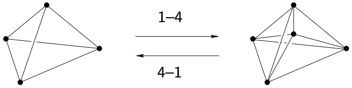
then Pachner moves implemented as canonical transformation

Pachner moves for 4D simplicial gravity

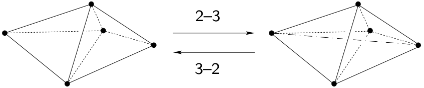


- 1–4 move: introduces 4 new edges, momenta satisfy $C_n^{k+1} = p_n^{k+1} - \frac{\partial S_\sigma}{\partial l_{k+1}^n} = 0$ (post-constraints)

Pachner moves for 4D simplicial gravity

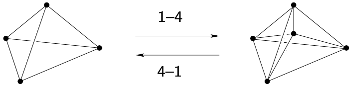


- 1–4 move: introduces 4 new edges, momenta satisfy $C_n^{k+1} = p_n^{k+1} - \frac{\partial S_\sigma}{\partial l_{k+1}^n} = 0$ (post-constraints)

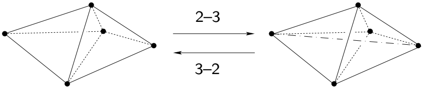


- 2–3 move: introduces 1 edge, renders 1 triangle internal, no new internal edge ⇒ freely choosable curvature generated, new momentum $C_n^{k+1} = p_n^{k+1} - \frac{\partial S_\sigma}{\partial l_{k+1}^n} = 0$ (post-constraint)
  - all new edges can be *a priori* freely chosen, but conjugate momenta constrained by post-constraints

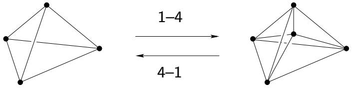
Pachner moves for 4D simplicial gravity



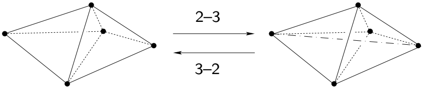
- 1–4 move: introduces 4 new edges, momenta satisfy $C_n^{k+1} = p_n^{k+1} - \frac{\partial S_\sigma}{\partial l_{k+1}^n} = 0$ (post-constraints)
- 4–1 move: removes 4 old edges, momenta satisfy $C_o^k = p_o^k + \frac{\partial S_\sigma}{\partial l_k^o} = 0$ (pre-constraints)



- 2–3 move: introduces 1 edge, renders 1 triangle internal, no new internal edge $\Rightarrow$ freely choosable curvature generated, new momentum $C_n^{k+1} = p_n^{k+1} - \frac{\partial S_\sigma}{\partial l_{k+1}^n} = 0$ (post-constraint)



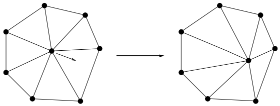
- 1–4 move: introduces 4 new edges, momenta satisfy $C_n^{k+1} = p_n^{k+1} - \frac{\partial S_\sigma}{\partial l_{k+1}^n} = 0$ (post-constraints)
- 4–1 move: removes 4 old edges, momenta satisfy $C_o^k = p_o^k + \frac{\partial S_\sigma}{\partial l_k^o} = 0$ (pre-constraints)



- 2–3 move: introduces 1 edge, renders 1 triangle internal, no new internal edge $\Rightarrow$ freely choosable curvature generated, new momentum $C_n^{k+1} = p_n^{k+1} - \frac{\partial S_\sigma}{\partial l_{k+1}^n} = 0$ (post-constraint)
- 3–2 move: removes 1 old edge, momentum satisfies $C_o^k = p_o^k + \frac{\partial S_\sigma}{\partial l_k^o} = 0$ (pre-constraint)

Constraint business

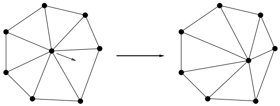
- post-constraints *a priori* do *not* generate gauge transformations of the action (vertex displacement), despite *a priori* forming an abelian Poisson-algebra



- reflect lack of information in hypersurface about full 4D-Regge triangulation $\Rightarrow$ non-uniqueness of solutions given initial data

Constraint business

- post-constraints *a priori* do *not* generate gauge transformations of the action (vertex displacement), despite *a priori* forming an abelian Poisson-algebra



- reflect lack of information in hypersurface about full 4D-Regge triangulation ⇒ non-uniqueness of solutions given initial data
- *a posteriori* 1st or 2nd class nature depends on pre-constraints of 3–2 and 4–1 moves:
  if no complete *constraint matching*, pre-constraints may *a posteriori* fix free lengths of 1–4 and 3–2 moves [Dittrich, PH '11 and to appear]
- if some lengths remain free, obtain proper gauge transformations (in general, gauge symmetry broken in presence of curvature [Rocek, Williams '84; Bahr, Dittrich '09; Dittrich, PH '09; etc.])

Conclusions and Outlook

- devised general canonical framework for simplicial gravity via gluings/removals of single simplices
- interpretation as Pachner moves in hypersurfaces (fixed spatial topology)
- implemented Pachner moves as canonical transformations via phase space extension
- PS-extension controlled by constraints which are equations of motion
- in linearized 4D Regge gravity can count and describe gauge and graviton modes generated/evolved/annihilated by individual Pachner moves [to appear]
- quantization: action as generating function $\Rightarrow$ direct connection between canonical framework and path integral