Introduction to Dynamical Triangulations

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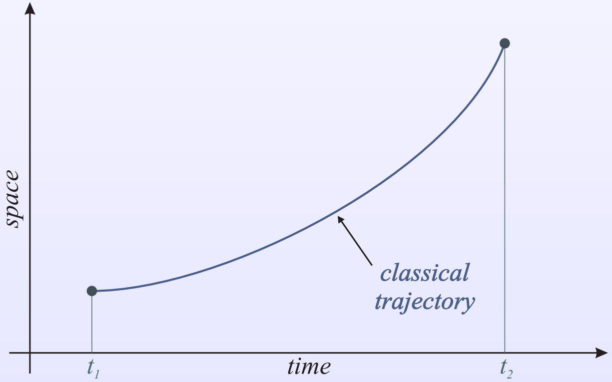
Outline

1. Path integral for quantum gravity
2. Causal Dynamical Triangulations
3. Numerical setup
4. Phase diagram
5. Background geometry
6. Quantum fluctuations

Path integral formulation of quantum mechanics

- A classical particle follows a unique trajectory.
- *Quantum mechanics* can be described by *Path Integrals*: All possible trajectories contribute to the transition amplitude.
- To define the functional integral, we discretize the time coordinate and approximate each path by linear pieces.



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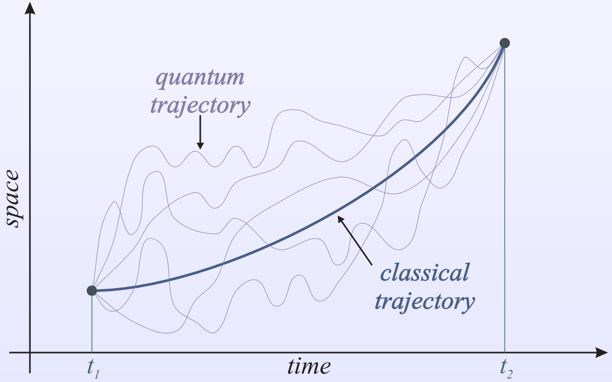

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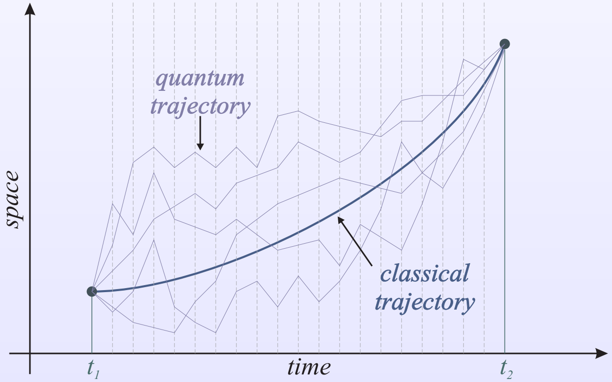
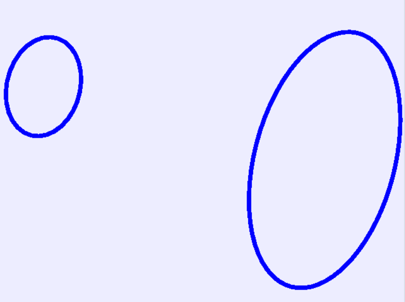


Path integral formulation of quantum gravity

- General Relativity: gravity is encoded in space-time geometry.
- The role of a trajectory plays now the geometry of four-dimensional space-time.
- All space-time histories contribute to the transition amplitude.

1+1D Example: State of system: one-dimensional spatial geometry



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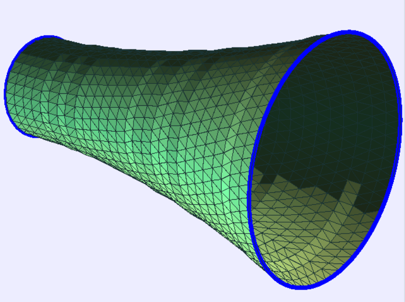
1+1D Example: Evolution of one-dimensional closed universe



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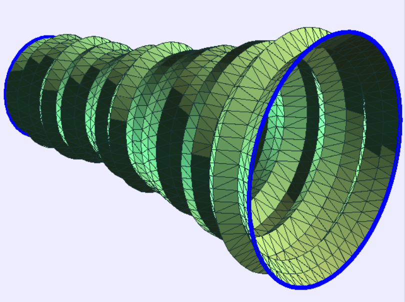
Sum over all two-dimensional surfaces joining the in- and out-state



Transition amplitude

Our aim is to calculate the amplitude of a transition between two geometric states:

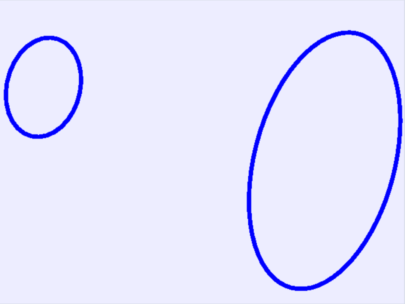
$$G(\mathbf{g}_i, \mathbf{g}_f, t) \equiv \int\limits_{\mathbf{g}_i \to \mathbf{g}_f} \mathrm{D}[g] \mathrm{e}^{iS^{EH}[g]}$$

To define this path integral we have to specify the *measure* $\mathrm{D}[g]$ and the *domain of integration* - **a class of admissible space-time geometries** joining the in- and out- geometries.

Regularization by triangulation. Example in 2D

Dynamical **T**riangulations uses one of the standard regularizations in QFT: **discretization**.

1. One-dimensional **state** with a topology S^1 is built from *links* with length a.
2. 2D **space-time** surface is built from equilateral triangles.
3. **Curvature** (angle deficit) is localized at vertices.



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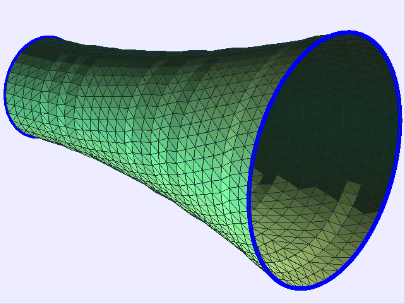

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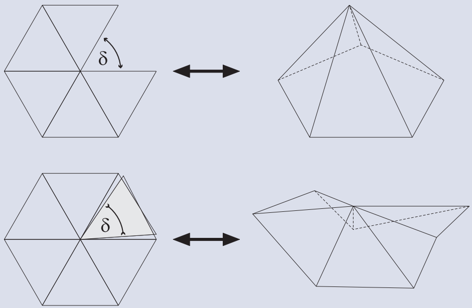
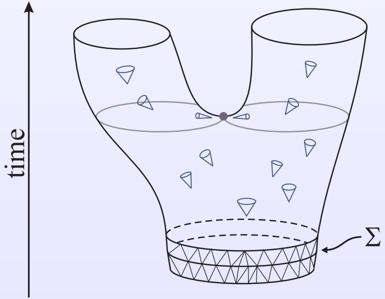


- **Causal D**ynamical **T**riangulations assume global proper-time foliation. Spatial slices (leaves) have fixed topology and are not allowed to split in time.
- Foliation distinguishes between time-like and spatial-like links.
- In **Euclidean DT** one cannot avoid introducing causal singularities, which lead to creation of baby universes.
- **EDT** and **CDT** differ in a class of admissible space-time geometries.



From $1+1$ to $3+1$ dimensions

- 2D triangles are replaced by higher-dimensional simplices.
- Spatial states are 3D geometries with a topology S^3. Discretized states are build from equilateral **tetrahedra**.
- 4D simplicial manifold can be obtained by gluing pairs of 4-simplices along their 3-faces.
- The metric is **flat** inside each 4-simplex.
- Length of time links a_t and space links a_s is constant.
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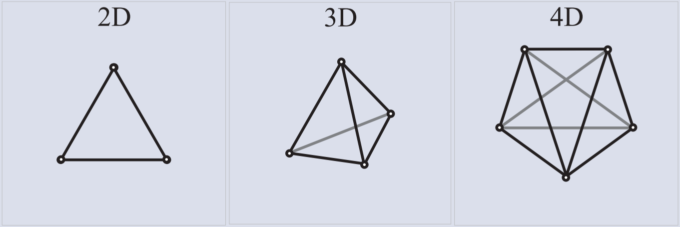
Fundamental building blocks of *Euclidean DT*



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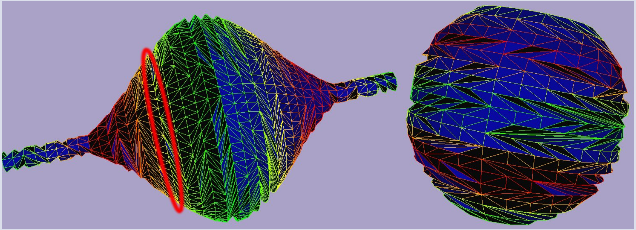
3D spatial slices with topology S^3



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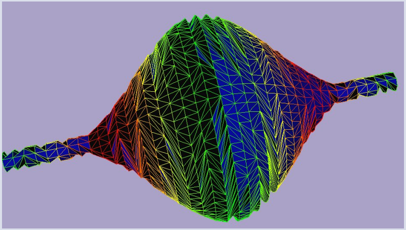
4D space-time with topology $S^3 \times S^1$



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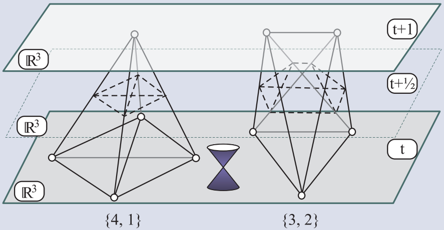
Fundamental building blocks of $4\mathrm{D}$ CDT

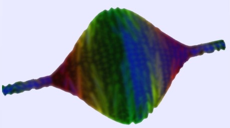


Regge action

The Einstein-Hilbert action has a natural realization on piecewise linear geometries called Regge action

$$S^E[g] = -\frac{1}{G}\int \mathrm{d}t \int \mathrm{d}^D x \sqrt{g}(R - 2\Lambda)$$

N_0	number of vertices
N_4	number of simplices
N_{14}	number of simplices of type $\{1,4\}$
K_0 K_4 Δ	bare coupling constants $(G, \Lambda, a_t/a_s)$



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$$S^R[\mathcal{T}] = -K_0 N_0 + K_4 N_4 + \Delta(N_{14} - 6N_0)$$

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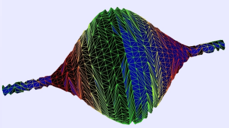

Causal Dynamical Triangulations (CDT) is a background independent approach to quantum gravity.

- The partition function of quantum gravity is defined as a formal integral over all geometries weighted by the Einstein-Hilbert action.

$$Z = \int \mathrm{D}[g] e^{iS^{EH}[g]}$$

- To make sense of the gravitational path integral one uses the standard method of regularization - discretization.
- The path integral is written as a nonperturbative sum over all causal triangulations $\mathcal{T}$.
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Numerical setup

- To calculate the **expectation value of an observable**, we approximate the path integral by a sum over a finite set of Monte Carlo configurations

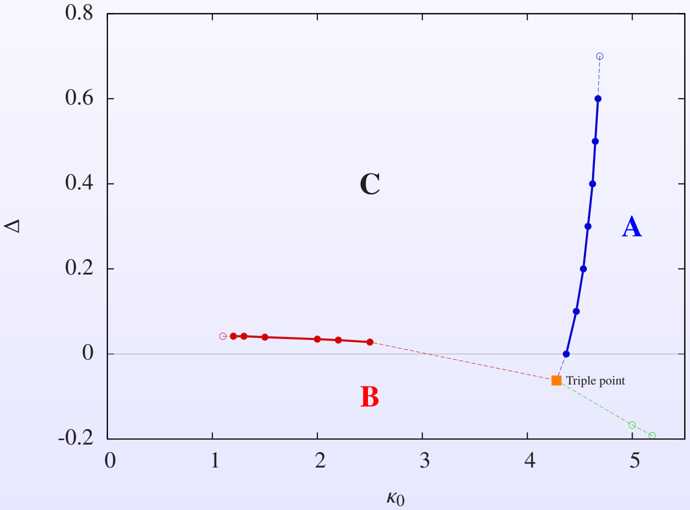
$$\langle \mathcal{O}[g] \rangle = \frac{1}{Z} \int \mathcal{D}[g] \mathcal{O}[g] e^{-S[g]} \quad \rightarrow \quad \langle \mathcal{O}[\mathcal{T}] \rangle = \frac{1}{Z} \sum_{\mathcal{T}} \mathcal{O}[\mathcal{T}] e^{-S[\mathcal{T}]}$$

$$\swarrow$$

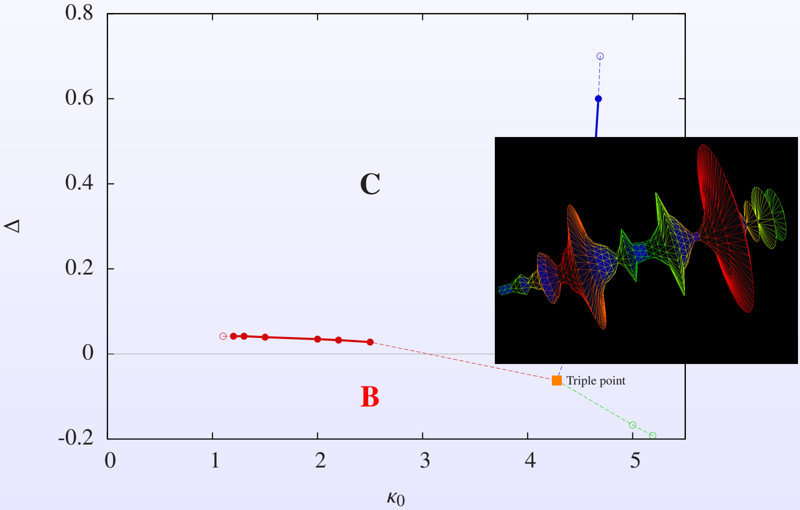
$$\langle \mathcal{O}[\mathcal{T}] \rangle \quad \approx \quad \frac{1}{K} \sum_{i=1}^{K} \mathcal{O}[\mathcal{T}^{(i)}]$$

- **Monte Carlo** algorithm probes the space of configurations with the probability $P[\mathcal{T}] = \frac{1}{Z} e^{-S[\mathcal{T}]}$.
- The simplest observable, giving any information about the geometry, is the **spatial volume** $N(i)$ defined as a number of tetrahedra building a three-dimensional slice $i = 1 \ldots T$.
- Restricting our considerations to the spatial volume $N(i)$ we reduce the problem to one-dimensional quantum mechanics.

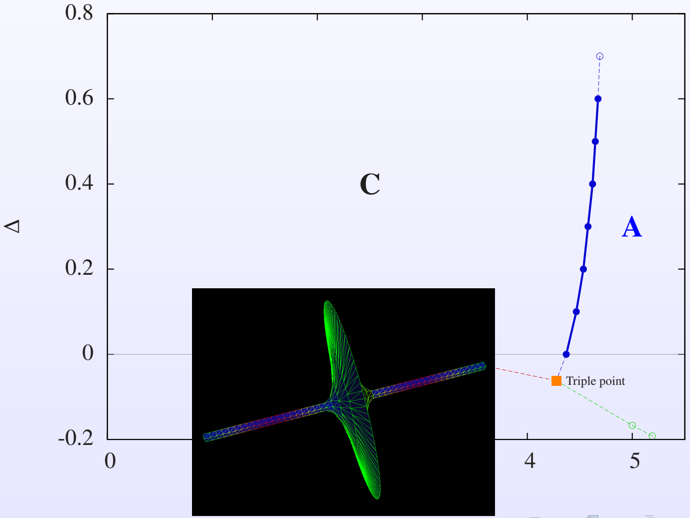
CDT phase diagram:



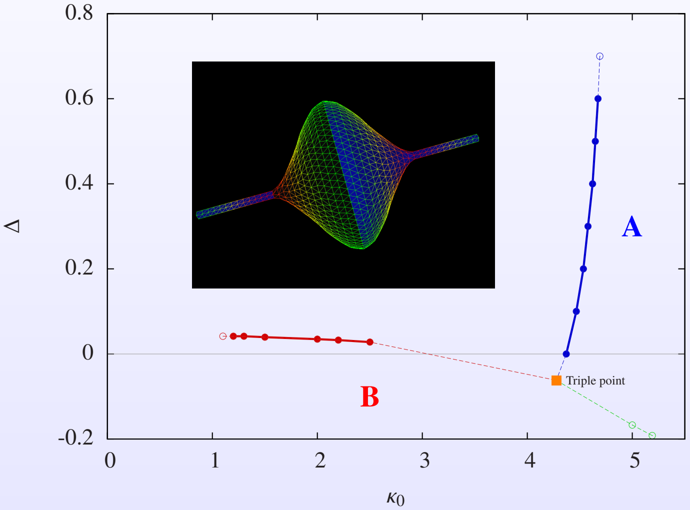
CDT phase diagram: Phase A



CDT phase diagram: Phase B

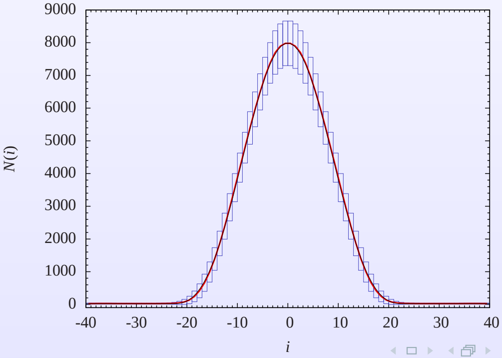


CDT phase diagram: Phase C



- Among the three phases, the de Sitter phase (**C**) is physically most interesting. The distribution $N(i)$ is bell-shaped.
- The average volume $\langle N(i) \rangle$ describes Euclidean **de Sitter** space (S^4), a classical **vacuum solution** ($\Lambda > 0$, $\mathrm{d}s^2 = \mathrm{d}\tau^2 + a^2(\tau)\, \mathrm{d}\Omega_3^2$):

$$\langle N(i) \rangle \propto \cos^3 (i/W)$$



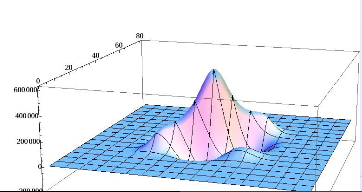
Quantum fluctuations

- We can measure correlations of spatial volume fluctuations around the classical solution:

$$\mathbf{C}_{ij} \equiv \langle (N(i) - \langle N(i) \rangle)(N(j) - \langle N(j) \rangle) \rangle$$

- The propagator **C** appears in the **semiclassical** expansion of the **effective action** describing quantum fluctuations of spatial volume.
- The recovered effective action **agrees** with the **minisuperspace** action:

$$S[N = \bar{N} + \eta] = S[\bar{N}] + \frac{1}{2} \sum_{i,j} \eta_i \, \mathbf{C}^{-1}_{ij} \, \eta_j + O(\eta^3),$$



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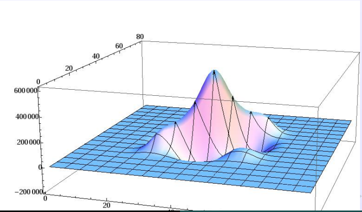
$$S = \Gamma \int \mathrm{d}\tau \left(-6a\dot{a}^2 - 6a + 2\Lambda a^3 \right), \quad N(i) \propto a^3(\tau)$$



Summary

Causal **D**ynamical **T**riangulations is a **background independent** approach to quantum gravity.

1. Only geometric invariants like length and angles are involved. While no coordinates are introduced, the model is manifestly **diffeomorphism-invariant**.
2. Phase diagram consists of three phases. In phase **C** emerges a **four-dimensional** universe with well defined time and space extent.
3. The **background** geometry corresponds to the Euclidean **de Sitter** space, i.e. **classical solution** of the *minisuperspace* model.
4. **Quantum fluctuations** of the spatial volume are also properly described by this simple model.

Thank You!

Monte Carlo simulations - Alexander moves

- We construct a starting space-time manifold with given topology ($S^3 \times S^1$) and perform a random walk over configuration space.

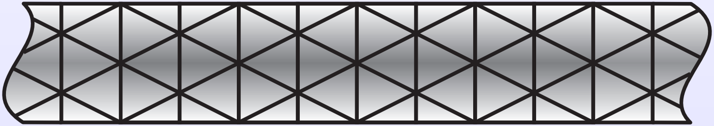
Ergodicity In the dynamical triangulation approach all possible configurations are generated by the set of Alexander moves.

Fixed topology The moves don't change the topology.

Causality Only moves that preserve the foliation are allowed.

4D CDT We have 4 types of moves.

Minimal configuration



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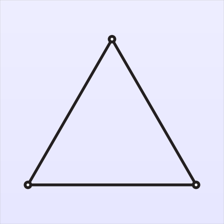

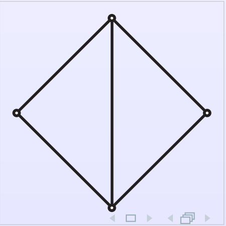

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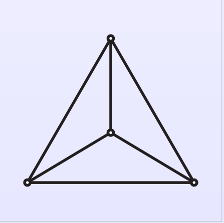

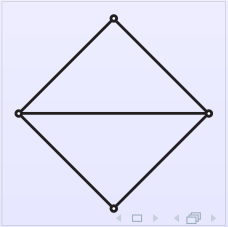

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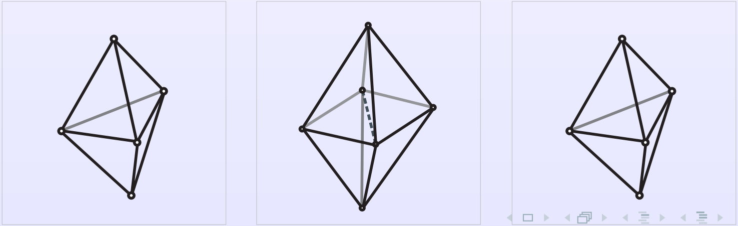

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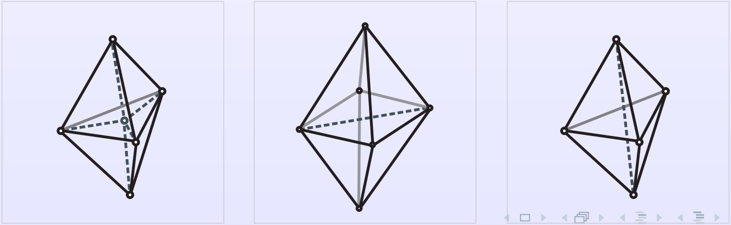

Monte Carlo Markov Chain

- We perform a random walk in the phase-space of configurations (space of piecewise linear geometries).
- Each step is one of the 4D CDT moves.
- The weight (acceptance probability) $W(\mathcal{A} \to \mathcal{B})$ of a move from configuration $\mathcal{A}$ to $\mathcal{B}$ is determined (not uniquely) by the detailed balance condition:

$$P(\mathcal{A})W(\mathcal{A} \to \mathcal{B}) = P(\mathcal{B})W(\mathcal{B} \to \mathcal{A})$$

- The Monte Carlo algorithm ensures that we probe the configurations with the probability $P(\mathcal{A})$.
- After sufficiently long time, the configurations are independent.
- All we need, is the probability functional for configurations $P(\mathcal{A})$.