6th Aegean Summer School

Quantum Gravity and Cosmological Perturbations

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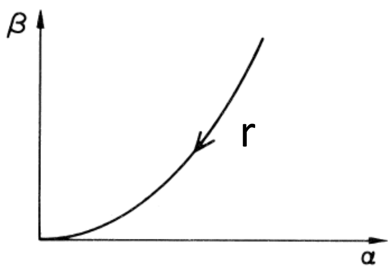
Running Couplings

Renormalization Group Approach in QFT

- $\alpha(q^2)$ QED coupling constant
- $\alpha_s(q^2)$ QCD coupling constant
- $G(q^2)$ QG coupling constant

~ *Vacuum Polarization*

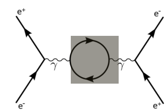
QED



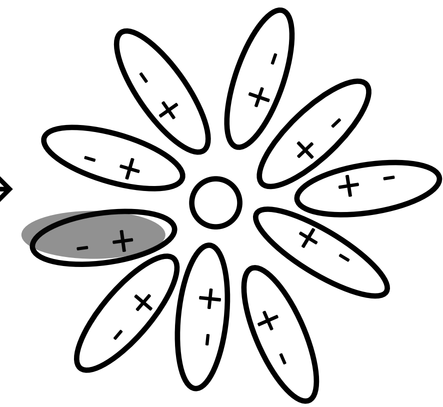
$$\mu\frac{\partial\alpha}{\partial\mu} = \beta(\alpha) = \frac{2\alpha^2}{3\pi} + \ldots > 0$$

screening

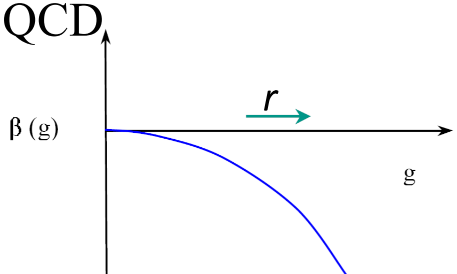
pair creating



←→



QCD



$$\mu\frac{\partial\alpha_s}{\partial\mu} = \beta(\alpha_s) = -\frac{11-\frac{2}{3}n_f}{2\pi}\,\alpha_s^2 + \mathcal{O}(\alpha_s^3) < 0$$

Gross and Wilczek 1973

Asymptotic free theory, antiscreening

for $n_f = 6$ for Standard Model

$\Lambda_{\overline{MS}}$ is generated

QG $\mathcal{L} = -\frac{1}{16\pi G}\sqrt{g}\,R$ $[G] = 2 - d$

How to approach a theory with coupling constant which has 2 – d dim ?

motivation :Non-Linear Sigma Model

E. Brezin J. Zinn-Justin 1975
F. Wegner, 1989
E. Brezin and S. Hikami, 1996

e.g., Ising model (d= 3, N=1)
Heisenberg model (ferromagnetism) (d=3, N=3)

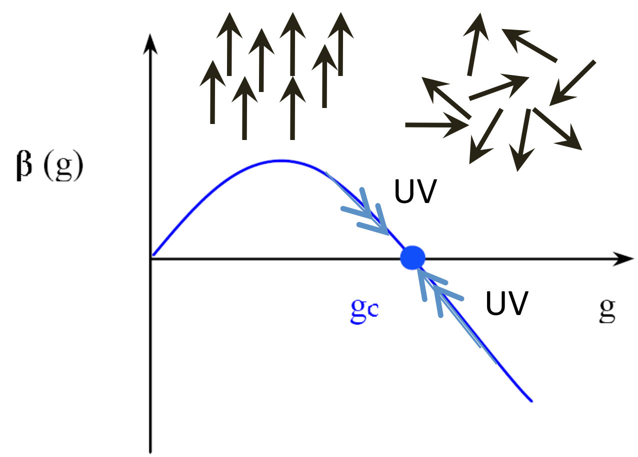
$$Z[J] = \int [d\phi] \prod_x \delta\left[\phi^a(x)\phi^a(x) - 1\right] \exp\left(-\frac{1}{2\,g}\int d^d x \partial_\mu \phi^a(x) \partial^\mu \phi^a(x) + \int d^d x\, J^a(x)\, \phi^a(x)\right)$$

$[g] = 2 - d$ → non-renormalizable in dim > 2, *but* renormalizable in 2 – dim.

close to 2 – dim

$$\beta(g) = (d-2)\,g - \frac{N-2}{2\pi} g^2 + \mathcal{O}\left(g^3, (d-2)g^2\right)$$

→ *existence of UV fixed point*

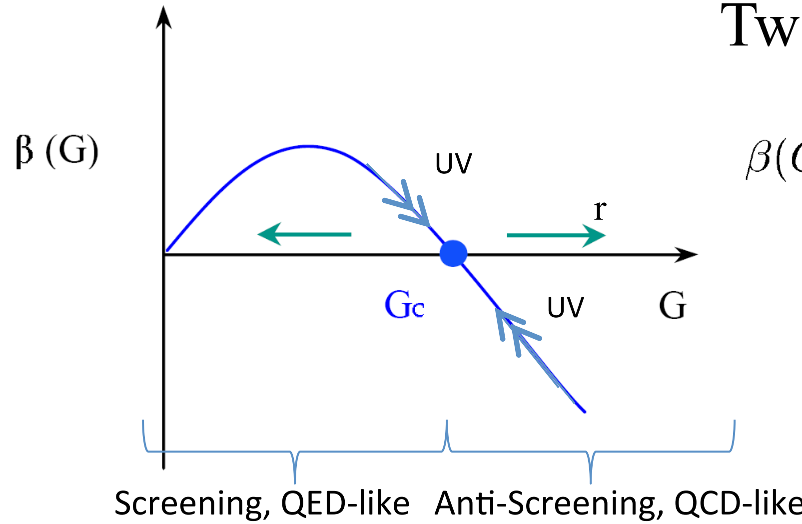


Compare with $2+\varepsilon$ dim , large N expansion, lattice approaches

Non-linear sigma model in $2+\varepsilon$ dim gives the *correct* ***qualitative*** *picture* for the system of 3 dim.

Running G: Quantum Gravity

2 + ε dim



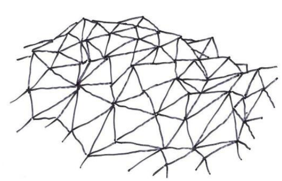
Two phases with non-trivial UV fixed point

$$\beta(G) = \epsilon\, G - \frac{2}{3}\,(25-c)\,G^2 - \frac{20}{3}\,(25-c)^2\,G^3 + \cdots$$

$c = n_s + ½\, n_f$
(the central charge of the Virasoro algebra in *2* dim)
n_s: =scalar fields, n_f :=Majorana fermion fields

Wilson 1973
Weinberg 1977
Kawai, Ninomiya 1995
Kitazawa, Aida 1998

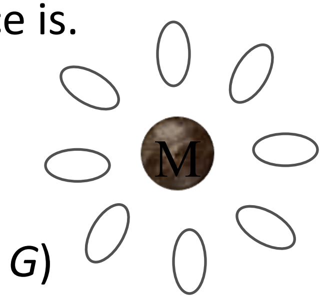
4 dim



Collapses *Smooth (i.e., physical)*

Hamber and Williams 1984

In gravity we do not have opposite charges.
Expect: larger the cloud is,
the stronger the gravitational force is.



→ Antiscreening (β $(g) < 0$)

→ Strong coupling phase ($G_c < G$)

Scale Dependent G

$$G_0 \;\to\; G(q^2) \quad \leftrightarrow \quad G(\Box) \qquad\qquad \Box \;=\; g^{\mu\nu}\nabla_\mu\nabla_\nu$$

Strong coupling phase ($G_c < G$)

In the vicinity of UV fixed point,

$$G(q^2) = G_0 \left(1 + c_0 \left(\frac{m^2}{q^2} \right)^{\frac{1}{2\nu}} + \ldots \right)$$

Weinberg 1979,
Hamber 1984,
Kawai *et al.* 1993,
Reuter 1998

equivalently,

$$G(\Box) = G_0 \left[1 + c_0 \left(\frac{1}{\xi^2 \Box} \right)^{1/(2\nu)} + \cdots \right]$$

HH & R Williams,
Phys Rev D 76 084008 (2007)
Phys Rev D 81 084048 (2010)

$$m^2 = \frac{1}{\xi^2} \sim \lambda \qquad\qquad \nu = \frac{1}{3}$$

(Hamber and Williams, 2004)

$$\lambda \sim \left(10^{-28} cm\right)^2 \sim \left(10^{-30} eV\right)^2$$

- *Cosmological constant λ is scale invariant.* *(similar to $\Lambda_{\overline{MS}}$ in QCD)*

As $q \to 0$, *i.e.*, as larger the distance scale, bigger the effect of the running G.

- will be looking at *"large scale structure "* in cosmology.

addendum: λ cannot run

Argument 1) **<u>*Compare Equations of Motion*</u>**

Field part

Source

$$R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu}\, R + \lambda\, g_{\mu\nu} \quad = \quad 8\pi G\, T_{\mu\nu}$$

$$\partial^\mu F_{\mu\nu} + \mu^2 A_\nu \quad = \quad 4\pi e\, j_\nu$$

$$\partial^\mu \partial_\mu \phi + m^2 \phi \quad = \quad \frac{g}{3!}\, \phi^3$$

suggests $\lambda \sim m^2 \equiv \xi^{-2}$

Masses (RG invariants) *Running couplings*

Argument 2) **<u>*From Wilson's loop*</u>**

$$W(C) \sim exp\left(-A_c/\xi^2\right)$$

$$W(C) \sim exp\left(-A_c\, R\right) \qquad \textit{for gravity}$$

ξ related to <u>curvature</u>.

($R = 4\,\lambda$ for vacuum)

$$\lambda_{obs} \;\simeq\; +\,\frac{1}{\xi^2}$$

HH & R Williams,
Phys Rev D 76 084008 (2007)
Phys Rev D 81 084048 (2010)

addendum: λ cannot run

Argument 3) ***Without the knowledge of RG***

$$\lambda(\Box) \sim (\xi^2 \, \Box)^{-\sigma}$$

$$\left(\frac{1}{A}\right)^n = \frac{1}{\Gamma(n)} \int_0^\infty dt \; t^{\,n-1} e^{-t\,A}$$

$$A \to -\Box + m^2$$

$$n \to \sigma$$

$$\begin{aligned}
\lambda(\Box)\, g_{\mu\nu} &= \frac{1}{\xi^{2\sigma}} \frac{1}{(-\Box + m^2)^\sigma} g_{\mu\nu} \\
&= \frac{1}{\xi^{2\sigma}} \frac{1}{\Gamma(\sigma)} \int_0^\infty dt \; t^{\,\sigma-1} e^{-t\,(-\Box+m^2)} \; g_{\mu\nu} \\
&= \frac{1}{\xi^{2\sigma}} \left(\frac{1}{m}\right)^{2\sigma} g_{\mu\nu}
\end{aligned}$$

= *constant*

$$\Box = g^{\mu\nu} \, \nabla_\mu \nabla_\nu$$

with $\nabla_\lambda g_{\mu\nu} = 0$

$$\Box g_{\mu\nu} = 0$$

$$\Box^2 g_{\mu\nu} = 0$$

.
.
.

Hamber and Toriumi 2011

Effective Field Equations with $G(\Box)$

$G(\Box)$ ***gives manifestly general covariant non-local equations***

Additional source term due to *vacuum polarization* contribution

Field eqns

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G_0 \left(1 + \frac{\delta G(\Box)}{G_0} + \dots \right) T_{\mu\nu}$$

Energy-momentum conservation

$$\left[\left(1 + \frac{\delta G(\Box)}{G_0} + \dots \right) T \right]^{\mu\nu}{}_{;\nu} = 0$$

G.A. Vilkovisky …
G. Veneziano
HH & R Williams PRD 06,07
Deser & Woodard 2008

Complexities in $\frac{\delta G(\Box)}{G_0} T_{\mu\nu}$.

$[\Box T]_{\mu\nu}$ → 1920 terms

$$\nabla_\nu T_{\alpha\beta} = \partial_\nu T_{\alpha\beta} - \Gamma^\lambda_{\alpha\nu} T_{\lambda\beta} - \Gamma^\lambda_{\beta\nu} T_{\alpha\lambda} \equiv I_{\nu\alpha\beta}$$

$$\nabla_\mu (\nabla_\nu T_{\alpha\beta}) = \partial_\mu I_{\nu\alpha\beta} - \Gamma^\lambda_{\nu\mu} I_{\lambda\alpha\beta} - \Gamma^\lambda_{\alpha\mu} I_{\nu\lambda\beta} - \Gamma^\lambda_{\beta\mu} I_{\nu\alpha\lambda}$$

$$\frac{\delta G(\Box)}{G_0} \equiv c_0 \left(\frac{1}{\xi^2 \Box} \right)^{\frac{1}{2\nu}}$$

Need to evaluate to <u>negative fractional power</u>

Cosmological Solutions

Zeroth order in the Fluctuations

Set up {
FRW metric $d\tau^2 = dt^2 - a^2(t)\,\delta_{ij}\,dx^i dx^j$
perfect fluid $T_{\mu\nu} = (p(t)+\rho(t))\,u_\mu\,u_\nu + g_{\mu\nu}\,p(t)$
} *power law for the density* $\rho(t) = \rho_0\,t^\beta$

$$\frac{\delta G(\Box)}{G_0} \equiv c_0 \left(\frac{1}{\xi^2 \Box}\right)^{\frac{1}{2\nu}} \xrightarrow{\text{reduces}} \frac{\delta G(t)}{G_0} = c_t \left(\frac{t}{\xi}\right)^{\frac{1}{\nu}} \qquad c_0 \sim c_t$$

FRW solution acquires *radiation-like* components (Hamber & Williams, 2007)

t-t eq.
$$\frac{\dot a^2(t)}{a^2(t)} = \frac{8\pi G_0}{3}\left[1 + \frac{\delta G(t)}{G_0}\right]\rho(t) + \frac{\lambda}{3}$$

ρ_{vac}

r-r eq.
$$\frac{\dot a^2(t)}{a^2(t)} + 2\frac{\ddot a(t)}{a(t)} = -8\pi G_0\,p(t) - 8\pi G_0\,\frac{1}{3}\frac{\delta G(t)}{G_0}\rho(t) + \lambda$$

p_{vac} : *Induced pressure term even with $p(t)=0$*

$$\rightarrow \quad p_{vac} = \frac{1}{3}\rho_{vac}$$

Similarities to: $p = w\,\rho$ *with* $w_{vac} = \frac{1}{3}$ *(like radiation)*

Effect of $G(\Box)$ is reflected in (ρ, p) as (ρ_{vac}, p_{vac}) in $T^{vac}_{\mu\nu} \equiv \left[\frac{\delta G(\Box)}{G_0}\,T\right]_{\mu\nu}$

Cosmological Solutions

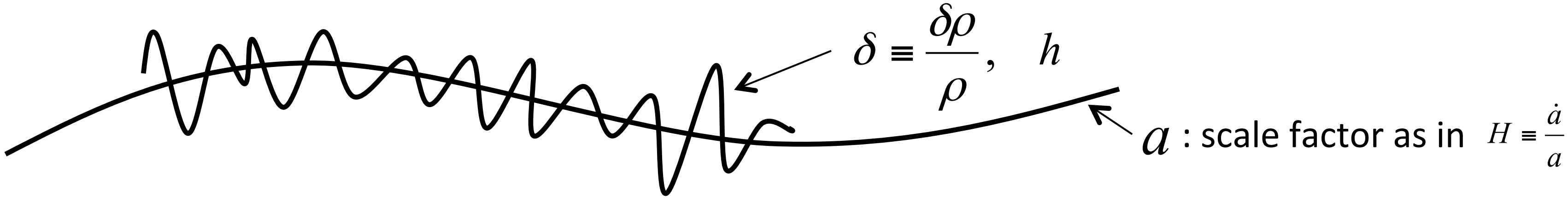
First order in the Fluctuations

$d\tau^2 = dt^2 - a^2 \left(\delta_{ij} + h_{ij}\right) dx^i dx^j$ *fluctuations in metric (gravitational field)*

$\Box(g) \; = \; \Box^{(0)} + \Box^{(1)}(h) + O(h^2)$: Now $\Box$ contributes to the fluctuations

$$\rightarrow \quad G(\Box) = G_0 \left[1 + \frac{c_0}{\xi^{1/\nu}} \left(\left(\frac{1}{\Box^{(0)}} \right)^{1/2\nu} - \frac{1}{2\,\nu} \frac{1}{\Box^{(0)}} \cdot \Box^{(1)}(h) \cdot \left(\frac{1}{\Box^{(0)}} \right)^{1/2\nu} + \ldots \right) \right]$$

$$\rightarrow \quad \delta\rho_{vac}(t) = \frac{\delta G(t)}{G_0} \delta\rho(t) + \frac{1}{2\,\nu} c_h \frac{\delta G(t)}{G_0} h(t) \bar{\rho}(t) \qquad c_h \simeq +7.927 \qquad \frac{\delta G(t)}{G_0} = c_t \left(\frac{t}{\xi} \right)^{\frac{1}{\nu}}$$



(Need to assume background is slowly varying : $\dot{h}/h \gg \dot{a}/a$)

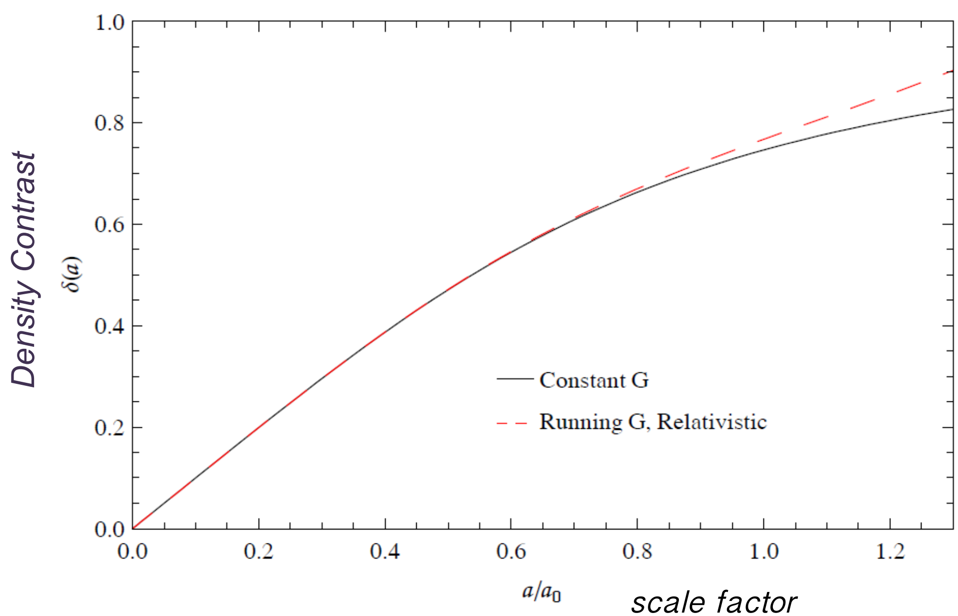
Cosmological Solutions

Density Contrast, $\delta \equiv \frac{\delta\rho}{\bar{\rho}}$

Single ODE for density perturbation, from covariant field equations with running $G(\Box)$:

$$\ddot{\delta}(t) + \left[\left(2\,\frac{\dot{a}(t)}{a(t)} - \frac{1}{3}\,\frac{\delta\dot{G}(t)}{G_0} \right) - \frac{1}{2\nu}\cdot 2\,c_h \cdot \left(\frac{\dot{a}(t)}{a(t)}\,\frac{\delta G(t)}{G_0} + 2\,\frac{\delta\dot{G}(t)}{G_0} \right) \right] \dot{\delta}(t)$$
$$+ \left[-\,4\pi\,G_0 \left(1 + \frac{7}{3}\,\frac{\delta G(t)}{G_0} - \frac{1}{2\nu}\cdot 2\,c_h \cdot \frac{\delta G(t)}{G_0} \right) \bar{\rho}(t) \right.$$
$$\left. -\frac{1}{2\nu}\cdot 2\,c_h \cdot \left(\frac{\dot{a}^2(t)}{a^2(t)}\,\frac{\delta G(t)}{G_0} + 3\,\frac{\dot{a}(t)}{a(t)}\,\frac{\delta\dot{G}(t)}{G_0} + \frac{\ddot{a}(t)}{a(t)}\,\frac{\delta G(t)}{G_0} + \frac{\delta\ddot{G}(t)}{G_0} \right) \right] \delta(t) = 0\ .$$

Hamber and Toriumi
Phys. Rev. D 82, 043518 (2010)



$$\frac{\delta G(a)}{G_0} = c_a \left(\frac{a}{a_\xi} \right)^{\zeta} \qquad \zeta = \frac{3}{2\,\nu}$$

$$\delta_0(a) = a \cdot {}_2F_1 \left(\frac{1}{3}, 1; \frac{11}{6}; -a^3\,\theta \right)$$

Standard GR
(*e.g.* Peebles 1993)

$$\theta = 8\pi G \rho_0 / \lambda$$

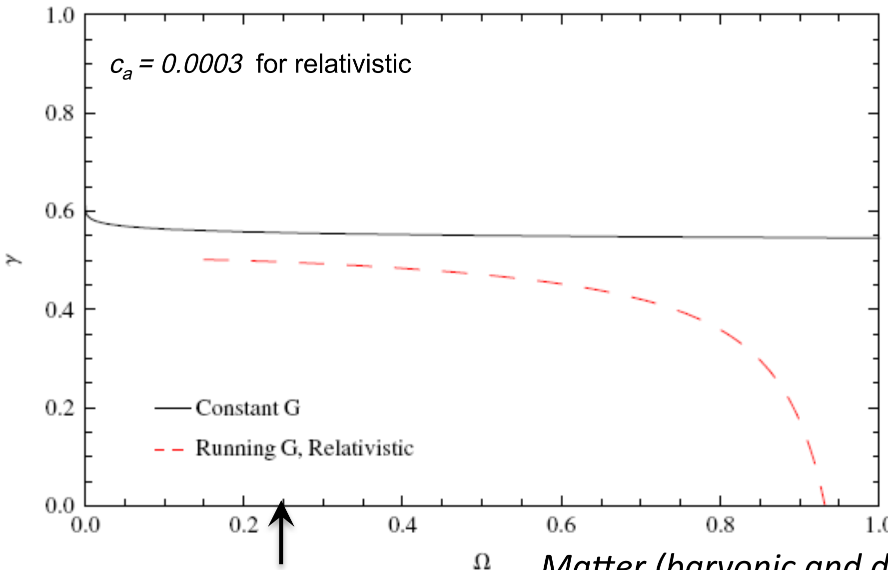
Cosmological Solutions

Growth Indices

$$f\,(a = a_0) \equiv \left.\frac{\partial \ln \delta(a)}{\partial \ln a}\right|_{a=a_0} \equiv \Omega^{\gamma}$$

$$\Omega + \Omega_\lambda = 1$$

Growth Index



today

Matter (baryonic and dark) density

$$\frac{\delta G(\Box)}{G_0} \equiv c_0 \left(\frac{1}{\xi^2 \Box}\right)^{\frac{1}{2\nu}}$$

$$\frac{\delta G(a)}{G_0} = c_a \left(\frac{a}{a_\xi}\right)^{\zeta}$$

$$\frac{\delta G(t)}{G_0} = c_t \left(\frac{t}{\xi}\right)^{\frac{1}{\nu}}$$

At today ($\Omega \approx 0.25$)

$$\gamma = 0.5562 - \underbrace{(1.60 + 7.20\, c_h)}_{\sim 59}\; c_t + \mathcal{O}\left(c_t^2\right)$$

$$c_h = 7.927$$

↑ classical GR

Correction is negative; significant uncertainty in magnitude of c_t

$\gamma = 0.44 \pm 0.16$ for clusters of galaxies, Vikhlinin, Jan 2010

Cosmological Solutions in cN gauge

conformal Newtonian gauge (ψ, ϕ) $ds^2 = a^2\{-(1+2\psi)\,dt^2 + (1-2\phi)\,dx^i\,dx_i\}$

ij Field eqn in cN gauge with G_0 but with general en. mom. tensor

Ma & Bertschinger 1995

$$k^2(\phi - \psi) = 12\pi G a^2(\bar{\rho} + \bar{P})\sigma$$

← Anisotropic stress

↑ Measure of deviation from traditional GR with perfect fluid + fluctuations $(\psi = \varphi \because \sigma = 0)$

Now with $G(\Box)$, $\sigma \neq 0$

Hamber & Toriumi 2011

"Slip function"

$$\frac{\psi - \phi}{\phi} = \frac{16}{3\nu}\frac{\delta G}{G_0}\log\left[\frac{a}{a_\xi}\right]$$

$a_\xi \sim 1.15 > a_0 = 1$ $\frac{\delta G}{G_0} = c_t\left(\frac{t}{\xi}\right)^{\frac{1}{\nu}}$

$$= -1.49\,c_t - 6.42\,c_t\,z + 30.07\,c_t\,z^2 + \cdots$$

negative and scale dependent

$$= \varpi_0(1+z)^{-3} \quad \begin{cases} \varpi_0 = 1.7^{+4.0}_{-2.0} \\ \varpi_0 = 0.09^{+0.74}_{-0.59} \\ \varpi_0 = -0.07^{+0.13}_{-0.16} \end{cases}$$

(WMAP)

(supernovae, weak lensing, CMB) Daniel, S. *et al*, 2009

(future, mock data (CMB, weak lensing))

Conclusions

QFT motivated generally covariant scale dependent G

$$\frac{\delta G(\Box)}{G_0} \equiv c_0 \left(\frac{1}{\xi^2 \Box} \right)^{\frac{1}{2\nu}} \quad \rightarrow \quad \frac{\delta G}{G_0} = c_t \left(\frac{t}{\xi} \right)^{\frac{1}{\nu}}$$

comoving frame

cosmological parameters that measure deviations from classical GR:

$$\gamma = 0.5562 - \underbrace{(1.60 + 7.20\, c_h)}_{\sim 59} c_t + \mathcal{O}\left(c_t^2\right)$$

$$\frac{\psi - \phi}{\phi} = 0 - 1.49\, c_t - 6.42\, c_t\, z + 30.07\, c_t\, z^2 + \cdots$$

classical GR Corrections are negative and scale dependent

$$ratio: \frac{\Delta \gamma}{\Delta \frac{\psi - \phi}{\phi}} \approx +40$$

Hamber and Toriumi 2011

Infrared Regulate $G(\Box) = G_0\left[1 + c_0\left(\frac{1}{\xi^2\Box}\right)^{1/2\nu} + \ldots\right]$

$$
\left\{
\begin{aligned}
\frac{\delta G(k^2)}{G_0} &= c_0 \left(\frac{m^2}{k^2 + m^2}\right)^{\frac{1}{2\nu}} \\
\frac{\delta G(\Box)}{G_0} &= c_0 \left(\frac{1}{-\xi^2\Box + 1}\right)^{\frac{1}{2\nu}} \\
\frac{\delta G(t)}{G_0} &= c_0 \left(\frac{1}{\left(\frac{c_0}{c_t}\right)^{2\nu} \left(\frac{\xi}{t}\right)^2 + 1}\right)^{\frac{1}{2\nu}} \\
\frac{\delta G(a)}{G_0} &= c_0 \left(\frac{1}{\left(\frac{c_0}{c_a}\right)^{2\nu} \left(\frac{a_\xi}{a}\right)^3 + 1}\right)^{\frac{1}{2\nu}}
\end{aligned}
\right.
$$

$$c_\xi = \frac{c_t}{c_0} \sim 0.618$$

$$c_a \sim c_t$$

$$c_0 \sim 33$$

(with big uncertainty, but expect ~ O (1))

$$\frac{\psi - \phi}{\phi} = -0.77\, c_t - 4.11\, c_t\, z + 12.19\, c_t\, z + \cdots$$

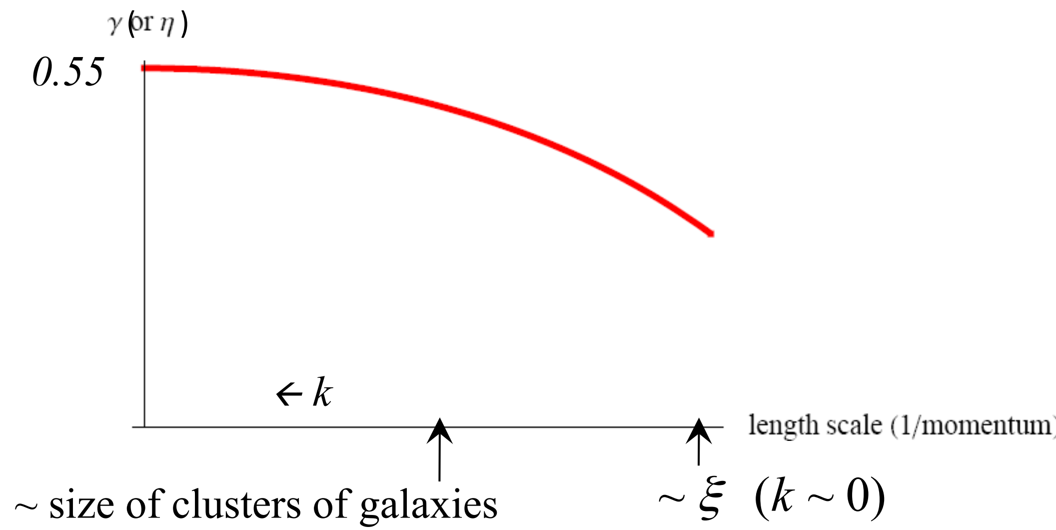
negative and scale dependent

(with infrared regulated $\frac{\delta G(t)}{G_0} = c_t / c_\xi \left(\frac{1}{\left(\frac{1}{c_\xi}\right)^{2\nu} \left(\frac{\xi}{t}\right)^2 + 1}\right)^{\frac{1}{2\nu}}$)

From lattice calculation, but with a significant uncertainty…

$$c_0 \sim 33 \leftrightarrow \; c_t \sim 20$$

$$c_t \simeq 0.618\, c_0$$



addendum: λ cannot run

Argument 3) ***Redefinition of the field and physical coupling G***

QG in 2 + ε Dimensions

Kawai, Ninomiya 1990

$$\mathcal{L} = -\frac{1}{16\pi G_0}\sqrt{g}R + \lambda_0\sqrt{g}$$

Fixing the gauge: $\mathcal{L}_{gf} = \frac{1}{2}\alpha\sqrt{g}\,g_{\nu\rho}\left(\nabla_\mu h^{\mu\nu} - \frac{1}{2}\beta g^{\mu\nu}\nabla_\mu h\right)\left(\nabla_\lambda h^{\lambda\rho} - \frac{1}{2}\beta g^{\lambda\rho}\nabla_\lambda h\right)$

$$\mathcal{L} \longrightarrow \mathcal{L} + \mathcal{L}_{gf} + \mathcal{L}_{ghost}$$

Radiative corrections

- Tadpole type: $\lambda_0 \to \lambda_0\left[1 - \left(\frac{a_1}{\epsilon} + \frac{a_2}{\epsilon^2}\right)G\right]$
- Charge renormalization: $\frac{\mu^\epsilon}{16\pi G} \to \frac{\mu^\epsilon}{16\pi G}\left(1 - \frac{b}{\epsilon}G\right)$

Gauge dependent parameters

- $a_1(\alpha, \beta)$
- $a_2(\beta)$
- $b(\beta)$

continued

addendum: λ cannot run

Argument 3) ***Redefinition of the field and physical coupling G***

QG in 2 + ε Dimensions

Field redefinition: $g_{\mu\nu} = \left[1 - \left(\frac{a_1}{\epsilon} + \frac{a_2}{\epsilon^2}\right) G\right]^{-2/d} g'_{\mu\nu}$

$$\mathcal{L} = -\frac{1}{16\pi G_0}\sqrt{g}\,R + \lambda_0\sqrt{g} \quad \rightarrow \quad -\frac{\mu^\epsilon}{16\pi G}\left[1 - \frac{1}{\epsilon}\left(b - \tfrac{1}{2}a_2\right)G\right]\sqrt{g'}\,R' + \lambda_0\sqrt{g'}$$

$$\left(b - \tfrac{1}{2}a_2\right) = \tfrac{2}{3}\cdot 19$$

Gauge independent!

i.e., $$\frac{1}{G} \rightarrow \frac{1}{G}\left[1 - \frac{1}{\epsilon}\left(b - \tfrac{1}{2}a_2\right)G\right]$$

Therefore G is the physical coupling
and
The running of λ can be absorbed into the redefinition of the field.