SIXTH AEGEAN SUMMER SCHOOL NAXOS
QUANTUM GRAVITY AND QUANTUM COSMOLOGY

Spherically Symmetric Solutions in Covariant Horava-Lifshitz Gravity

J. Alexandre and P. Pasipoularides, Spherically symmetric solutions in Covariant Horava-Lifshitz Gravity, Phys. Rev. D 83 (2011) 084030 [arXiv:1010.3634 [hep-th]].

Horava-Lifshitz Gravity

i. It is a **power counting renormalizable**, higher order gravity model.

ii. In order to achieve perturbative renormalizability (and keep time derivatives up to second in order to achieve unitarity of the model) we have **to sacrifice the standard 4D diffeoromorphism** of General Relativity.

iii. The UV behavior of the model is governed by a **Lifshitz fixed point**, which is characterized by an **anisotropy between Space and time** coordinates.

iv. In the **IR limit** General Relativity should be recovered.

Anisotropic Scaling

$$x \to bx,\ \ t \to b^{z} t$$

Z=dynamical critical exponent

$$[x] = -1, \quad [t] = -z$$

Anisotropic Scaling

- $z=1$ corresponds to Gaussian fixed point (General Relativity).
- $z\neq 1$ corresponds to a Lifshitz fixed point.

1. 3+1 HL Gravity → $z=3$ the model is renormalizable.
2. 3+1 HL Gravity, $z<3$ nonrenormalizable.
3. 3+1 HL Gravity, $z>3$ superenormalizable

ADM decomposition of the metric

$$ds^2 = -N^2 c^2 dt^2 + g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right)$$

$$\mathrm{g_{ij}}\left(\mathrm{t}, \mathrm{x}^{\mathrm{k}}\right) = \text{3d metric}$$

$$\mathrm{N}\left(\mathrm{t}, \mathrm{x}^{\mathrm{k}}\right) = \text{lapse function}$$

$$\mathrm{N_i}\left(\mathrm{t}, \mathrm{x}^{\mathrm{k}}\right) = \text{Shift function}$$

$$[\mathrm{N}] = 0, \qquad [\mathrm{N}_i] = \mathrm{z} - 1, \quad [\mathrm{c}] = \left[\frac{\mathrm{dx_i}}{\mathrm{dt}}\right] = \mathrm{z} - 1$$

Foliation-preserving diffeomorphism → Diff(M,F)

$$\delta t = f(t), \qquad \delta x^i = \zeta^i(t, x^k)$$

$$\delta g_{ij} = \partial_i \xi^k g_{ik} + \partial_j \xi^k g_{ik} + \xi^k \partial_k g_{ij} + f \dot{g}_{ij}$$

$$\delta N_i = \partial_i \xi^j N_j + \xi^j \partial_j N_i + \xi^j g_{ij} + \dot{f} N_i + f \dot{N}_i$$

$$\delta N = \partial_j \xi^j N + \dot{f} N + f \dot{N}$$

The action

$$S = \frac{1}{16\pi G c}\int dt\,dx^d \sqrt{g}\,N\left(T - V\left(g_{ij}\right)\right)$$

Kinetic term

Potential term

$$K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right), \quad i,j = 1,2,3$$

Extrinsic curvature

$$T = K_{ij}K^{ij} - \lambda\left(g_{ij}K^{ij}\right)^2$$

λ is a new running coupling

In GR **λ=1** due to 4D diffeoromorphism

The potential term

$$V = V_{z=0} + V_{z=1} + V_{z=2} + V_{z=3}$$

Relevant deformations

UV Lifshitz fixed point

$$\mathrm{V}_{z=0} + \mathrm{V}_{z=1} = \mathrm{V}_{\mathrm{IR}} = -\mathrm{c}^2\left(\mathrm{R} - 2\Lambda\right)$$

$$\mathrm{V}_{z=2} = -\alpha_1 \mathrm{R}^2 - \alpha_2 \mathrm{R}_{\mathrm{ij}} \mathrm{R}^{\mathrm{ij}}, \quad \left[a_i\right] = 2, \ \mathrm{i} = 1,2$$

The potential term

$$V_{z=3} = -\beta_1 R^3 - \beta_2 R R_{ij} R^{ij} - \beta_3 R_i^{\ j} R_j^{\ k} R_k^{\ i} - \beta_4 R \nabla^2 R - \beta_5 \nabla_i R_{jk} \nabla^i R^{jk}$$

$$[\beta_i] = 0, \ \ i = 1,2...,5$$

Detailed balance action

$$V = E^{ij} G_{ijkl} E^{kl}, \qquad E^{ij} = \frac{1}{\sqrt{|g|}} \frac{\delta W}{\delta g_{ij}}$$

Super potential

De Witt metric

$$G_{ijkl} = \frac{1}{2}\left(g^{ik} g^{jl} + g^{il} g^{jk}\right) - \lambda g^{ij} g^{kl}$$

$$E^{ij} = -\frac{1}{w^2} C_{ij} - \frac{\mu}{2}\left(R^{ij} - \frac{1}{2} R g^{ij} + \Lambda_w g^{ij} \right)$$

Cotton tensor

IR LIMIT

$$\lambda \to 1, \quad \alpha_i \to 0, \quad \beta_i \to 0, \quad \Lambda = 0$$

$$x^0 = ct, \quad [x^0] = -1 \quad R^{(4)} = K_{ij}K^{ij} - K^2 + R^{(3)}$$

$$S = \frac{1}{16\pi Gc}\int dt d^3x \sqrt{g^{(3)}} N\left(K_{ij}K^{ij} - \lambda K^2 + c^2\left(R^{(3)} - 2\Lambda\right)\right)$$

$$= \frac{1}{16\pi G}\int dx^0 d^3x \sqrt{g^{(4)}} R^{(4)}$$ General Relativity

Projectable and Non-Projectable HL Gravity

- In <u>Non-Projectable</u> version the Lapse function is allowed to be a function of space and time coordinates both.

- In <u>Projectable</u> version the Lapse function is only a function of the time coordinate.

Possible problems in HL Gravity

- **The RG behavior in the IR has not been studied.**
- **Problems in the canonical structure** (only for Non-projectable version)
- **Ghosts** (there are no ghosts if $\lambda \geq 1$)
- **Classical Instabilities** (they set constraints to RG flow)
- **Strong coupling problem**

Strong Coupling Problem

$$N = 1, \quad N_i = \partial_i B + n_i, \quad g_{ij} = (1 + 2\varphi)\delta_{ij} + h_{ij}$$

Scalar Graviton

Tensor Graviton

The couplings between the Scalar Graviton φ and the Tensor Graviton $\mathbf{h_{ij}}$ blows up in the IR limit λ->1.

This prevent us from recovering general relativity in the IR

Covariant HL Gravity by Horava and Melby-Tomson

$$S=\frac{1}{16\pi Gc}\int \mathrm{dt}\,\mathrm{dx}^{\mathrm{d}}\sqrt{\mathrm{g}}\left\{N\left(\mathrm{t}\right)\left[\mathrm{T}-\mathrm{V}+\nu\Theta_{\mathrm{ij}}\left(2\mathrm{K}_{\mathrm{ij}}+\nabla_{\mathrm{i}}\nabla_{\mathrm{j}}\nu\right)\right]-\mathrm{A}\left(\mathrm{R}-2\Omega\right)\right\}$$

$$\mathrm{T}=K_{ij}K^{ij}-\lambda\mathrm{K}^{2}$$

Projectability Condition

$$\Theta^{\mathrm{ij}}=R^{ij}-\frac{1}{2}Rg^{ij}+\Omega g^{ij}$$

The above action contains two additional auxiliary non-dynamical space-time fields:

1) the **potential A(x,t)** , and
2) the **prepotential ν(x,t)**

Extended Gauge Symmetry

$$\text{diff}(M,F)\times U(1)$$

$$\delta_\alpha N(t)=0,\quad \delta_\alpha g_{ij}=0$$ New Gauge Symmetry

$$\delta_\alpha N_i(x,t)=N\nabla_i\alpha$$

$$\delta_\alpha A=\dot{\alpha}-N^i\nabla_i\alpha,\qquad \delta_\alpha \nu=\alpha$$

Indeed the new Gauge symmetry eliminates the scalar graviton, hence Covariant HL Gravity avoids strong coupling problems. However, this new symmetry can not forced λ to be equal to one (as it shown by DaSilva), so λ remains a running coupling constant.

Spherically Symmetric Solutions for λ=1

$$ds^2 = -N^2(t)\ c^2 dt^2 + \frac{1}{f(r)}\left(dr + n(r)dt\right)^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

Nonzero radial shift function

$$A = A(r), \quad \nu(r) = 0$$

U(1) Gauge Fixing

Equations of motion

Constant of integration

$$\frac{\delta S}{\delta A}=0\Rightarrow R^{(3)}=0\Rightarrow f(r)=1-\frac{2B}{r}$$

Momentum constraint

$$\frac{\delta S}{\delta n}=0\Rightarrow f'(r)n(r)=0\Rightarrow f(r)=1 \text{ or } n(r)=0$$

$$\frac{\delta S}{\delta f}=0\Rightarrow A'+\frac{A}{2r}\left(1-\frac{1}{f}\right)+4\frac{fn\left(\sqrt{r}n\right)'}{N\sqrt{r}}=OV$$

$$\frac{\delta S}{\delta N}=0\Rightarrow\int_0^{+\infty}\frac{r^2}{\sqrt{f(r)}}(T+V)dr=0$$

Hamiltonian Constraint

Two Classes of Solutions

1. Solutions with nonzero radial shift function n(r) and f(r)=1, B=0.

2. Solutions with zero radial shift function n(r)=0 and f(r)=1-2B/r, B different from zero.

We will examine these cases separately

First class of solutions: Non zero radial shift function n(r) (and **f(r)=1)**

$$n^2(r) = \frac{\tilde{C}_M}{r} - \frac{1}{2}A(r) - \frac{1}{2r}\int_0^{+\infty} A(\rho)\,d\rho$$

The choice of A(r) is arbitrary, but it should satisfy the Hamiltonian Constraint

$$\int_0^{+\infty} A(\rho)\,d\rho = 0$$

The minimal choice A(r)=0

$$n(r) = \pm\sqrt{\frac{\tilde{C}_M}{r}},\ \tilde{C}_M = 2GMc^2$$

$$ds^2 = -c^2 dt_{PG}^2 + \left(dr \pm \sqrt{\frac{2GMc^2}{r}} dt_{PG}\right)^2 + r^2\left(d\theta^2 + \sin^2(\theta) d\varphi^2\right)$$

Schwarzschild metric expressed in Painleve-Gullstrant coordinates

The Newtonian potential is recovered by the Nonzero radial shift function

$$\varphi_{NP}(r) = -\frac{n(r)^2}{2c^2} = -\frac{GM}{r}$$

Solutions with nonzero A(r): two cases

I. A(r) determines the sub-leading behavior of the radial shift function n(r).

II. A(r) determines the leading behavior of the radial shift function n(r).

- Solar system tests require:

$$A(r) \approx \frac{C_A}{r^b}, \quad r \to \infty, \ b \geq 3\,(\text{Case I}), \ \ b \approx 1\,(\text{Case II})$$

J. Greenwald, V. H. Satheeshkumar and A. Wang, `Black holes, compact objects and solar system tests in non-relativistic general covariant theory of gravity,'' arXiv:1010.3794 [hep-th].

Case I: Solutions with nonzero A and $\tilde{C}_M \neq 0$

$$n^2(r) = \frac{2GMc^2}{r} - \frac{1}{2}A(r) - \frac{1}{2r}\int_0^{+\infty} A(\rho)d\rho$$

Sub-leading asymptotic behavior

$$A(r) = \frac{C_A}{1+r^{b_1}}\left(1 - \gamma_3 r^{b_2}\right), \quad b_1 \geq 3, \quad b_1 - b_2 \geq 3, \quad b_2 > -1$$

$$\gamma_3 = \frac{\sin\left(\frac{\pi}{b_1} + \frac{\pi b_2}{b_1}\right)}{\sin\left(\frac{\pi}{b_1}\right)}$$

$$\int_0^{+\infty} A(\rho)d\rho = 0$$

Hamiltonian constraint

Case II: Solutions with nonzero A and $\tilde{C}_M = 0$

$$n^2(r) = -\frac{1}{2}A(r) - \frac{1}{2r}\int_0^{+\infty} A(\rho)d\rho \qquad A(r) \approx \frac{C_A}{r^b}, \quad r \to \infty, \quad b \approx 1$$

For b suitably closely to unity (b~1) this class of solutions may pass solar system tests

$$\boxed{A(r) = -\frac{C_A}{1+r^b}(r-\gamma_2),} \quad b > 1 \qquad \gamma_2 = \frac{\sin\left(\frac{\pi}{b+1}\right)}{\sin\left(\frac{2\pi}{b+1}\right)}$$

$$\int_0^{+\infty} A(\rho)d\rho = 0$$

Hamiltonian constraint

Second class of solutions: Zero radial shift function

$$N=1,\quad n(r)=0,\quad f(r)=1-\frac{B}{r}\quad A(r)=A_{IR}(r)+A_{UV}(r)$$

$$A_{IR}(r)=c^2\left(1-\sqrt{1-2x}\right),\quad x=\frac{B}{r},\ B=GM$$

$$A_{UV}(r)=-\left(\frac{\alpha_2}{5M^2}-\frac{4\beta_3}{77M^4}+\frac{12\beta_5}{77M^4}\right)\sqrt{1-2x}$$

$$-\frac{\alpha_2}{5M^2}\left(-2+2x+x^2+x^3\right)+\frac{2\beta_2}{M^4}x^6$$

$$+\frac{\beta_3}{11M^4}\left(-\frac{7}{4}+\frac{7}{4}x+\frac{2}{7}x^2+\frac{2}{7}x^3+\frac{5}{14}x^4+\frac{x^5}{2}+75x^6\right)$$

$$-\frac{3\beta_5}{11M^4}\left(-\frac{7}{4}+\frac{7}{4}x+\frac{2}{7}x^2+\frac{2}{7}x^3+\frac{5}{14}x^4+\frac{x^5}{2}+20x^6\right)$$

The potential interpretation of A

The U(1) symmetry is promoted to a spacetime symmetry in the IR

$$t' = t + \frac{\varepsilon(x,t)}{c^2}, \quad x' = x, \qquad \varepsilon(x,t) = \frac{\alpha(x,t)}{N}$$

$$N_i' = N_i + N^2 \nabla_i \varepsilon, \;\; A' = A + \dot{\varepsilon} N + \varepsilon \dot{N} - N N_i \nabla_i \varepsilon$$

$$ds_{eff}^2 = -c^2 \left(N^2 - \frac{N_i N^i - 2 A_{IR} N}{c^2} \right) dt^2 + 2 N_i dx^i dt + g_{ij} dx^i dx^j$$

Newtonian potential

$$\varphi_N = \frac{N_i N^i - 2 N A_{IR}}{2c^2} = -\frac{A_{IR}}{2c^2} = -\frac{GM}{r} + O\left(\frac{GM}{r}\right)^2$$

Hamiltonian Constraint

$$\int_L^{+\infty} dr \frac{r^2}{\sqrt{1-\frac{2M}{r}}} V(r) = 0, \qquad L \geq 2M$$

Blows up for r=0

$$V(r) = \frac{6\alpha_2 M^2}{r^6} - \frac{6\beta_3 M^3}{r^9} - \frac{90\beta_5 M^2}{r^9}(r-2M)$$

In order to satisfy the Hamiltonian constraint in the second Class of solutions we have to introduce lower limit L for the radial coordinate r . However, the physical meaning of L is not clear.

Conclusions

- We have examined the most general case of spherically symmetric vacuum solutions in the framework of covariant HL gravity for $\lambda=1$.
- Solutions can be separated to two classes i) solutions with nonzero radial shift and ii) solutions with zero radial shift function .
- In the case i) Schwarzschild geometry is recovered in the IR but there is an arbitrariness in the choice of the non dynamical field A, which should satisfy the Hamiltonian constraint.
- In the case ii) we need the additional assumption of Horava and Melby-Tomson for the U(1) as a space-time symmetry, in order to recover Schwarzschild geometry. Also there are serious problems when we try to satisfy the Hamiltonian constraint.

Topics for feature investigation

- **Spherically symmetric solutions, Newton's Law and IR limit λ-> 1, in Covariant Horava Lifshitz Gravity.** Jean Alexandre, Pavlos Pasipoularides, . Aug 2011. e-Print: **arXiv:1108.1348** [hep-th].
- Spherically symmetric solutions for non-projectable covariant HL gravity.
- **U(1) symmetry and elimination of spin-0 gravitons in Horava-Lifshitz gravity without the projectability condition.** Tao Zhu, Qiang Wu, Anzhong Wang, Fuwen Shu, . Aug 2011. Temporary entry. e-Print: **arXiv:1108.1237** [hep-th]