Quantum dynamics of the reduced phase space loop cosmology

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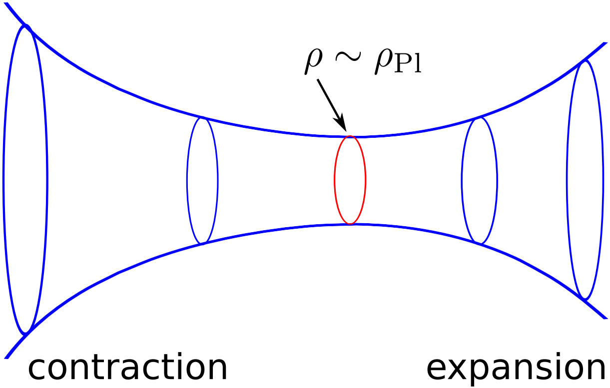
16 September, 2011, Naxos

According to Loop Quantum Cosmolgy (LQC) the Big Bang singularity is replaced by the smooth Big Bounce transition.



The Planck energy density:

$$\rho_{\rm Pl} \equiv \frac{c^5}{\hbar G^2} = 5.16 \cdot 10^{96} \frac{\rm kg}{\rm m^3}.$$

Cosmic amnesia

To get some insight into the nature of the quantum bounce, one studies possible correlation between quantum fluctuations before and after the bounce. In particular, one tries to find the answer to the question: Is the *semiclassicality* of the universe preserved across the bounce? If the answer is 'no', we would not be able to learn what happened before the big bounce. It is called the cosmic forgetfulness or amnesia.

The issue of cosmic amnesia was investigated in the frameworks of effective dynamics[1] and sLQC[2] leading to contradicting predictions. In this talk we address the issue of cosmic amnesia in the framework of reduced phase space LQC[3].

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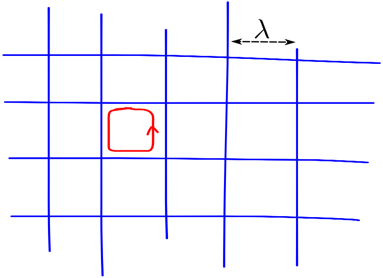
[1] M. Bojowald, "What happened before the Big Bang?", Nature Physics **3** (2007) 523.

[2] A. Corichi and P. Singh, "Quantum bounce and cosmic recall", Phys. Rev. Lett. **100** (2008) 161302.

[3] J. Mielczarek and W. Piechocki, " Evolution in bouncing quantum cosmology", arXiv:1107.4686, J. Mielczarek and W. Piechocki, " Quantum memory of the Universe", arXiv:1108.0005.

Why "loop" ?

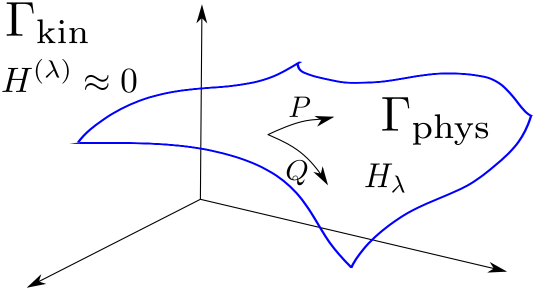
The quantum gravity effects are introduced by holonomies $h_e = \mathcal{P} \exp \int_e A$ of Ashtekar connection, which take value in $\mathfrak{su}(2)$.



The effect of holonomies may be seen as a kind of discretization of continuous space. Here, discretization have a form of the regular cubic lattice with the the elemantary lattice spacing λ. Based on various indications, as renormalization of the perturbative quantum gravity, one may expect that $\lambda \sim l_{\text{Pl}} \equiv \sqrt{\frac{\hbar G}{c^3}} = 1.62 \cdot 10^{-35}$ m. In our considerations, we keep λ as a free parameter.

Idea of the method

By solving the Hamiltonian constraint $H^{(\lambda)} \approx 0$ we determine the physical phase space $\Gamma_{\text{phys}} \subset \Gamma_{\text{kin}}$. The physical phase space Γ_{phys} is parametrized by canonical variables Q and P satisfying the algebra $\{Q, P\} = 1$.



We find the physical Hamiltonian H_λ for the system without constraints. Finally, we quantize the unconstrained system.

Physical Hamiltonian

The physical Hamiltonian we find is the following

$$H_\lambda := \frac{2}{\lambda\sqrt{G}} P \sin(\lambda Q),$$

The Hamiltonian is positive-definite since $P > 0$ and $Q \in [0, \pi/\lambda]$.
The Hamilton equation takes the following form

$$\frac{df}{dT} = \{f, H_\lambda\},$$

where T is the intrinsic time parameter.
The Q and P variables can be related with the Hubble factor H and volume v as follows

$$H := \frac{1}{3v}\frac{dv}{dt} = \frac{1}{\gamma}\frac{\sin(2\lambda Q)}{2\lambda}, \quad v = 4\pi l_{\mathrm{Pl}}^2 \gamma P,$$

where γ is Barberro-Immirzi parameter.

Quantization

In what follows we use the Hilbert space $\mathcal{H} = L^2([0, \pi/\lambda], dQ)$ so it has the scalar product

$$\langle f|g\rangle := \int_0^{\pi/\lambda} \overline{f} g dQ.$$

The quantum Hamiltonian corresponding to the classical one is defined in a *standard* way to be

$$\hat{H}_\lambda := \frac{m_{\mathrm{Pl}}}{\lambda}\left(\hat{P}\,\widehat{\sin(\lambda Q)} + \widehat{\sin(\lambda Q)}\hat{P}\right).$$

The classical canonical variables Q and P satisfy the algebra $\{Q, P\} = 1$. Choosing the Schrödinger representation for these variables

$$\hat{Q}\phi(Q) := Q\phi(Q), \qquad \hat{P}\phi(Q) := -i\frac{d}{dQ}\phi(Q),$$

where $\phi \in \mathcal{H}$, gives formally $[\hat{Q}, \hat{P}] = i\hat{\mathbb{I}}$.

The eigenvalue problem for $\hat{H}_\lambda$ has the solution

$$\Psi_E(x) = \sqrt{\frac{\lambda\sqrt{G}}{4\pi}\cosh\left(\frac{2}{\sqrt{G}}x\right)}e^{iEx},$$

where $x := \frac{\sqrt{G}}{2}\ln\left|\tan\left(\frac{\lambda Q}{2}\right)\right|$ and $E \in \mathbb{R}$, satisfying $\langle\Psi_E|\Psi_{E'}\rangle = \delta(E' - E)$. The Hamiltonian is essentially self-adjoint on the domain defined to be, $D(\hat{H}_\lambda) := \mathrm{span}(\mathcal{F})$, where $\mathcal{F} := \{\Psi_E | E \in \mathbb{R}\}$. The positive-definite physical Hamiltonian is defined as follows

$$\hat{\mathbb{H}}|\Psi_E\rangle := |E||\Psi_E\rangle,$$

and we have $\langle\Psi|\hat{\mathbb{H}}|\Psi\rangle = \int_{-\infty}^{+\infty} dE|c(E)|^2|E| \geq 0$, where $|\Psi(x)\rangle := \int_{-\infty}^{+\infty} c(E)\Psi_E(x)dE$, and where $c(E)$ is a square integrable function.

Evolution

Making use of the Stone theorem, we define the unitary operator of an *evolution* as follows

$$\hat{U}(s) := e^{-is\hat{\mathbb{H}}},$$

where $s \in \mathbb{R}$ is a 'time' parameter. The state at any moment of time can be found as follows

$$|\Psi(s)\rangle = \hat{U}(s)|\Psi(0)\rangle = e^{-is\hat{\mathbb{H}}}|\Psi(0)\rangle.$$

Let us consider a superposition of the Hamiltonian eigenstates $|\Psi(0)\rangle = \int_{-\infty}^{+\infty} dEc(E)|\Psi_E\rangle$ at $s = 0$. Then evolution of this state is given by

$$|\Psi(s)\rangle = \int_{-\infty}^{+\infty} dEc(E)e^{-is|E|}|\Psi_E\rangle.$$

In what follows we consider the Gaussian packet:

$$c(E) := \left(\frac{2\alpha}{\pi}\right)^{1/4} e^{-\alpha\,(E-E_0)^2}.$$

The corresponding wavefunction (assuming that $\sqrt{\alpha}E_0 \gg 1$) is

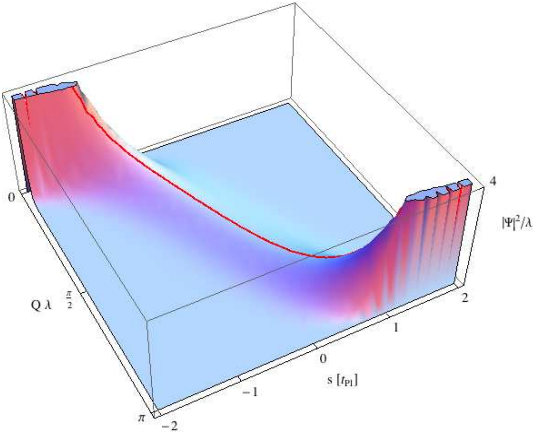
$$\Psi(Q,s) = \langle Q|\Psi(s)\rangle = \sqrt{\frac{\lambda \cosh\left(\frac{2}{\sqrt{G}}x\right)}{\sqrt{8\pi\tilde{\alpha}}}}\, e^{-\frac{(x-s)^2}{4\alpha}}\, e^{iE_0(x-s)},$$

where $\tilde{\alpha} := \alpha/G$. The probability density is easily found to be

$$\frac{|\Psi(Q,s)|^2}{\lambda} = \frac{\exp\left\{-\frac{1}{2\tilde{\alpha}}\left[\frac{1}{2}\ln\left|\tan\left(\frac{\lambda Q}{2}\right)\right| - \frac{s}{t_{\text{Pl}}}\right]^2\right\}}{\sqrt{8\pi\tilde{\alpha}}\sin(\lambda Q)}.$$

For the later purpose we also define $\tilde{E}_0 := E_0\sqrt{G}$.

Probability density function $|\Psi(Q, s)|^2$



Here $\tilde{\alpha} = 0.1$. The red line is the classical trajectory drawn by taking classical intrinsic time T equal to quantum evolution parameter s.

Mean values and dispersions

Based on the state $|\Psi\rangle$ one can now investigate expectation values

$$\langle\hat{\mathcal{O}}\rangle = \langle\Psi|\hat{\mathcal{O}}|\Psi\rangle$$

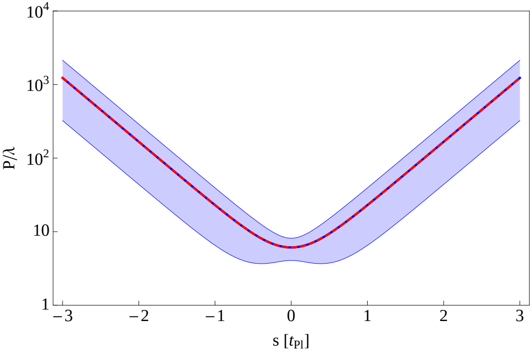
and dispersions

$$\Delta\hat{\mathcal{O}} := \sqrt{\langle\hat{\mathcal{O}}^2\rangle - \langle\hat{\mathcal{O}}\rangle^2}$$

of the observables $\hat{\mathbb{H}}$, $\hat{Q}$ and $\hat{P}$. For $\hat{\mathbb{H}}$:

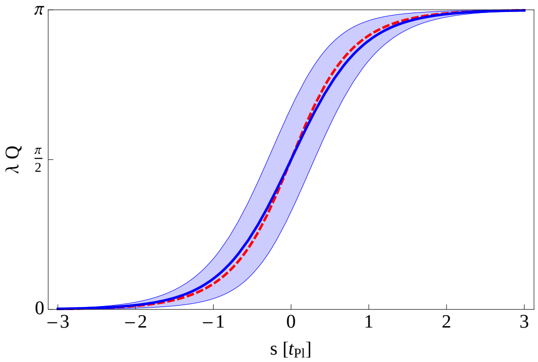
$$\begin{aligned}
\langle\hat{\mathbb{H}}\rangle &= E_0 \,\mathrm{erf}\left(\sqrt{2\alpha}E_0\right) + \frac{e^{-2\alpha E_0^2}}{\sqrt{2\pi\alpha}} \simeq E_0 \quad \text{if} \quad \sqrt{\alpha}E_0 \gg 1, \\
\Delta\hat{\mathbb{H}} &= \left[\frac{1}{4\alpha} + E_0^2\left[1 - \mathrm{erf}^2\left(\sqrt{2\alpha}E_0\right)\right] - \frac{2E_0 e^{-2\alpha E_0^2}}{\sqrt{2\pi\alpha}}\mathrm{erf}\left(\sqrt{2\alpha}E_0\right)\right. \\
&- \left.\frac{e^{-4\alpha E_0^2}}{2\pi\alpha}\right]^{1/2} \simeq \frac{1}{\sqrt{4\alpha}} \quad \text{if} \quad \sqrt{\alpha}E_0 \gg 1.
\end{aligned}$$

Evolution of P - the Big Bounce



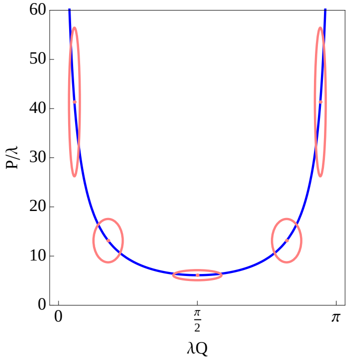
The thick blue line represents mean value $\langle \hat{P} \rangle$ for $\tilde{\alpha} = 0.1$ and $\tilde{E}_0 = 10$. The dashed red line is the classical solution $P = P_0 \cosh(2s/t_{\rm Pl})$. The shadowed region is constrained by $\langle \hat{P} \rangle + \Delta \hat{P}$ from above and by $\langle \hat{P} \rangle - \Delta \hat{P}$ from below.

Evolution of Q



The thick blue line represents mean value $\langle \hat{Q} \rangle$ for $\tilde{\alpha} = 0.1$. The dashed red line is the classical solution $Q = \frac{2}{\lambda} \arctan \exp\left(2s/t_{\text{Pl}}\right)$. The shadowed region is constrained by $\langle \hat{Q} \rangle + \Delta \hat{Q}$ from above and by $\langle \hat{Q} \rangle - \Delta \hat{Q}$ from below.

Phase space evolution



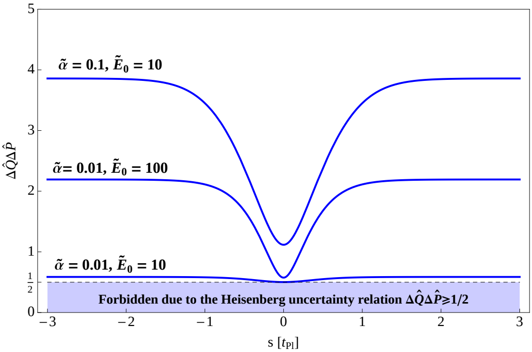
The ellipsoides of uncertainty:

$$\left(\frac{Q-\langle\hat{Q}\rangle}{\Delta\hat{Q}/2}\right)^2+\left(\frac{P-\langle\hat{P}\rangle}{\Delta\hat{P}/2}\right)^2=1.$$

The dispersion of $\hat{Q}$ is largest at the bounce and tends to zero with the increase of $\langle\hat{P}\rangle$. In turn, the dispersion $\Delta\hat{P}$ reaches its minimal value at the bounce and grows with the increase of $\langle\hat{P}\rangle$. It means that quantum uncertainty of the volume (related to $\langle\hat{P}\rangle$) measurement is growing with increase of $\langle\hat{P}\rangle$. The quantum uncertainty of expansion rate (related to $\langle\hat{Q}\rangle$) is decreasing with increase of volume.

Here $\tilde{\alpha}=0.1$ and $\tilde{E}_0=10$.

Heisenberg uncertainty relation



The dispersions satisfy the Heisenberg uncertainty relation $\Delta\hat{Q}\Delta\hat{P} \geq \hbar/2$ at any time. Moreover, the quantity $\Delta\hat{Q}\Delta\hat{P}$ reaches its minimal value at the transition point between the contracting and expanding phases. Therefore, the bounce is the *least quantum* part of the evolution!

Relative fluctuations

The relative fluctuations

$$\delta(\mathcal{O}) := \frac{\Delta\hat{\mathcal{O}}}{\langle\hat{\mathcal{O}}\rangle}$$

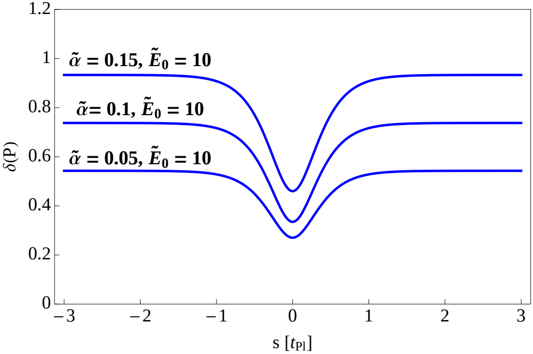
are measure of the semi-classicality of a quantum state. We say that $|\Psi\rangle$ is *semiclassical* if $\delta(\mathcal{O}) \ll 1$, and *quantum* if $\delta(\mathcal{O}) \sim 1$. It is clear that the semiclassicality notion is not at all defined uniquely.

For $\hat{\mathbb{H}}$, the relative dispersion

$$\delta(\mathbb{H}) \simeq \frac{1}{2\sqrt{\alpha}E_0} \ll 1.$$

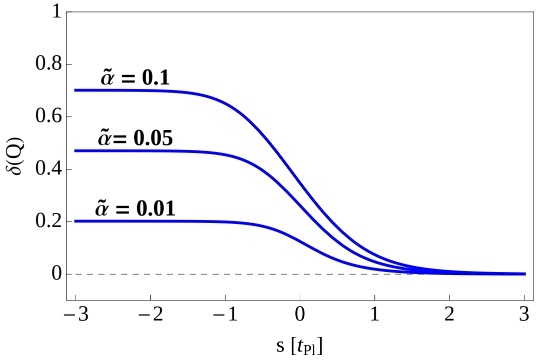
Therefore, the earlier requirement $\sqrt{\alpha}E_0 \gg 1$ is equivalent with the semiclassicalit condition $\delta(\mathbb{H}) \ll 1$.

The relative fluctuations $\delta(P)$ are shown below.



They saturate on $\delta(P)|_{\max}$ while $s \to \pm\infty$ and reach the minimum at the bounce ($s = 0$). Therefore, if the semiclassicality condition $\delta(P) \ll 1$ is imposed in the expanding phase, it constraints the rest of the evolution.

The relative fluctuations $\delta(Q)$ are shown below.



They are *asymmetric* with respect to the bounce and only become symmetric when $\tilde{\alpha} \to 0$. For $s \to -\infty$, the $\delta(Q)$ saturates at $\delta(Q)|_{\text{max}} = \sqrt{e^{4\tilde{\alpha}} - 1}$, while tends to zero for $s \to +\infty$. Since the directions of evolution parameter s and cosmological time t are opposite, the relative fluctuations $\delta(Q)$ monotonically *grow* in the cosmological time.

From the point of view of the possible *observability* (detection) of the amnesia, the observable $\hat{Q}$ is favorite because in the classical limit: $Q = \gamma H$, where H is the Hubble parameter and γ is Barbero-Immirzi parameter. Therefore, relative fluctuations of $\hat{Q}$ *can be* constrained observationally! In contrast, it is hard to put any constraint on $\delta(P)$, because P is linked to the physical volume of space $v = 4\pi G\gamma P$, which is not measurable.

However, the difficulties related with the lack of possibility of constraining $\delta(P)$ can be partially bypassed. Namely, one can prove that, if the condition $\delta(\mathbb{H}) \ll 1$ is fulfilled, we have the following implication:

$$\left(\delta(\hat{Q})\Big|_{\text{max}} < 1 \right) \implies \left(\delta(\hat{P})\Big|_{\text{max}} < 1 \right)$$

Therefore, the observational constraint $\delta(\hat{Q})\Big|_{\text{max}} \ll 1$ ensures that also $\delta(\hat{P})\Big|_{\text{max}} \ll 1$.

Observational constraints

The relative fluctuations $\delta(Q)$ at the given time cannot be greater than the relative uncertainty of measurement.

In particular, from the present value of the Hubble factor $H_0 = 70.2 \pm 1.4$ km s^{-1} Mpc^{-1} we have the constraint $\delta(Q) < \frac{\sigma(H_0)}{H_0} \approx 0.02$. Another constraint can be derived for the phase of cosmic inflation: $\delta(Q) < \frac{\sigma(H_*)}{H_*} \approx 0.19$.

The model we consider applies to the Planck's epoch, however if assuming that the quantum fluctuations are not decreasing thereafter, the derived observational bound can be used to constraint $\delta(Q)|_{\max}$. From the first constraint $\delta(Q)|_{\max} < 0.02$ which translates into $\tilde{\alpha} < 10^{-4}$ and from the second one $\delta(Q)|_{\max} < 0.19$, so $\tilde{\alpha} < 9 \cdot 10^{-3}$. Both constraints suggest that the semiclassicality condition was indeed fulfilled. Therefore, because constraint $\delta(Q)|_{\max} < 1$ implies $\delta(P)|_{\max} < 1$, one can conclude that there is no cosmic amnesia.

Conclusions and outlook

- We have performed RPS quantization of the FRW loop cosmology with a free scalar field.
- Our method may be generalized to sophisticated cosmological models including the Bianchi type universes.
- We have shown that the bounce transition is the least quantum part of the evolution.
- We have shown that the issue of cosmic amnesia can be observationally probed.
- The preliminary estimations based on astronomical data indicate the semiclassicality is preserved across the bounce - there is no cosmic amnesia.
- The Universe remembers its quantumness across the bounce.
- Further analysis can be done by calculating an evolution of $\delta(Q)$ for the FRW model with a scalar field potential. This would enable obtaining more accurate constraints from the Cosmic Microwave Background observations.

Our lack of an information about a system may be measured in terms of entropy. The increase of the entropy means the increase of our uncertainty. Let us apply the notion of the entropy to our quantum system.

The minimal volume of phase space which can be occupied be the quantum system is obtained by saturating Heisenberg relation so it is equal to $\Gamma_0 = \hbar/2$. Therefore

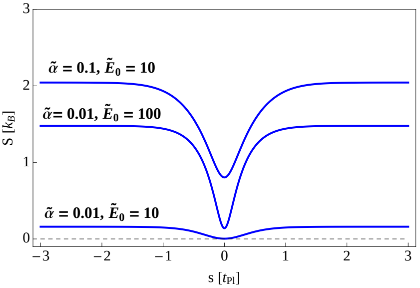
$$\Omega = \Gamma/\Gamma_0,$$

where $\Gamma = \int\int dQdP \approx \Delta\hat{Q}\Delta\hat{P}$, is the number of the elementary cells Γ_0 occupied by the system. We propose to interpret this number as a number of *microstates* in analogy with the microcanonical ensemble. For such a system we propose to apply the Boltzmann definition of entropy

$$S = k_B \ln \Omega,$$

where k_B is the Boltzmann constant.

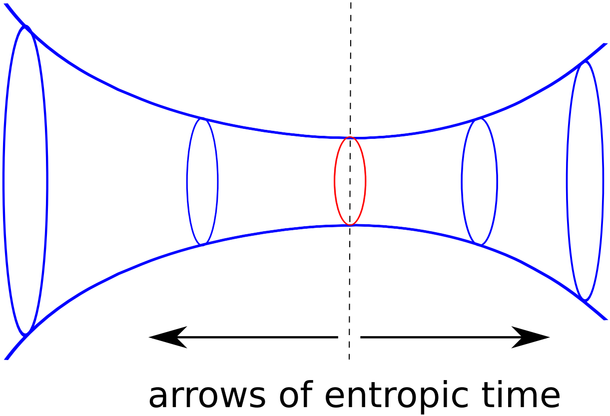
The entropy reaches the minimal value at the bounce ($s = 0$) while saturate for $s \to \pm\infty$. This result is in the agreement with the earlier qualitative predictions[4].



Surprisingly, the entropy is decreasing in the contracting phase, that can be interpreted as the violation of the second law of thermodynamics. This peculiar behavior may be linked to the arrow of time problem in cosmology.

[4] E.g. C. Kiefer, "Quantum Cosmology and the Arrow of Time", Braz. J. Phys. **35** (2005) 296.

The arrows of entropic time (given by the gradient of entropy) are directed outward the bounce. At the transition point (bounce) the arrow of entropic time is undefined.



In the entropic time the Universe evolves only from the high to the low energy densities.