Towards Canonical Quantum Gravity for Geometries Admitting Maximally Symmetric Two-dimensional Surfaces

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Preliminaries: Kuchař's Quantization of Finite Degrees of Freedom Constraint Systems

- We assume a system described by a Hamiltonian of the form:

$$\begin{aligned} H &\equiv \mu X + \mu^i \chi_i \\ &= \mu \left(\frac{1}{2} G^{AB}(Q^\Gamma) P_A P_B + U^A(Q^\Gamma) P_A + V(Q^\Gamma) \right) \\ &\quad + \mu^i \phi_i^A(Q^\Gamma) P_A \end{aligned}$$

- Primary Constraints: $P_\mu \approx 0$, $P_{\mu^i} \approx 0$
- Secondary Constraints: $X \approx 0$, $\chi_i \approx 0$
- The constraints are assumed to be first class:

$$\{X, X\} = 0, \qquad \{X, \chi_i\} = X C_i + C_i^j \chi_j, \qquad \{\chi_i, \chi_j\} = C_{ij}^k \chi_k$$

- The physical state of the system is unaffected by the "gauge" transformations generated by (X, χ_i), i.e. $\delta_i A = \{A, \chi_i\}$, $\delta_0 A = \{A, X\}$ **but also** under the following three changes:
  (I) Mixing of the super-momenta with a non-singular matrix

$$\bar{\chi}_i = \lambda_i^j(Q^\Gamma)\chi_j$$

  (II) Gauging of the super-Hamiltonian with the super-momenta

$$\bar{X} = X + \kappa^{(Ai}(Q^\Gamma)\phi_i^{B)}(Q^\Gamma)P_A P_B + \sigma^i(Q^\Gamma)\phi_i^A(Q^\Gamma)P_A$$

  (III) Scaling of the super-Hamiltonian

$$\bar{X} = \tau^2(Q^\Gamma)X$$

- The Dirac quantization scheme, $\hat{X}\Psi = 0$, $\hat{\chi}_i = 0$, must be independent of actions (I), (II), (III). This is achieved by:
  (1) Realize the linear operator constraint conditions with the momentum operators to the right

$$\hat{\chi}_i \Psi = 0 \leftrightarrow \phi_i^A(Q^\Gamma) \frac{\partial\, \Psi(Q^\Gamma)}{\partial\, Q^A} = 0,$$

  maintaining the geometrical meaning of the linear constraints and producing the $M - N$ independent solutions $q^\alpha(Q^\Gamma)$, $\alpha = 1, 2, \ldots, M - N$ called physical variables.
  (2) Define the induced structure $g^{\alpha\beta} \equiv G^{AB} \dfrac{\partial\, q^\alpha}{\partial\, Q^A} \dfrac{\partial\, q^\beta}{\partial\, Q^B}$ and realize the quadratic in momenta part of X as the conformal Laplace-Beltrami operator based on $g_{\alpha\beta}$.
- $g^{\alpha\beta}(q^\gamma)$ by virtue of the 1st class algebra.

Possible metrics and Hamiltonian Formulation

The starting point is the line element:

$$ds^2 = \left(-\alpha(t,r)^2 + \frac{\beta(t,r)^2}{\gamma(t,r)^2}\right) dt^2 + 2\,\beta(t,r)\,d\,t\,d\,r + \gamma(t,r)^2\,d\,r^2 + \psi(t,r)^2\,d\,\theta^2 + \psi(t,r)^2\,f(\theta)^2\,d\,\phi^2$$

where:

- $f(\theta) = \sin\theta$: spherical symmetry,
- $f(\theta) = \theta$: plane symmetry,
- $f(\theta) = \sinh\theta$: GBL symmetry.

With the aid of the canonical decomposition, we arrive at the Hamiltonian:

$$H = \int \left(N^o \mathcal{H}_o + N^i \mathcal{H}_i \right) dr,$$

where

$$N^o = \alpha(t,r), \quad N^1 = \frac{\beta(t,r)}{\gamma(t,r)^2}, \quad N^2 = 0, \quad N^3 = 0$$

$$\begin{aligned} \mathcal{H}_o &= \frac{1}{2} G^{\alpha\beta} \pi_\alpha \pi_\beta + V, \\ \mathcal{H}_1 &= -\gamma \pi'_\gamma + \psi' \pi_\psi, \qquad \mathcal{H}_2 = 0, \qquad \mathcal{H}_3 = 0, \end{aligned}$$

and $\{\alpha, \beta\}$ take the values $\{\gamma, \psi\}$ while $' = \frac{d}{dr}$.

- Reduced Wheeler-deWitt super-metric $G^{\alpha\beta}$:

$$G^{\alpha\beta} = \begin{pmatrix} \dfrac{\gamma}{4\,\psi^2} & -\dfrac{1}{4\,\psi} \\ -\dfrac{1}{4\,\psi} & 0 \end{pmatrix},$$

while the potential V is

$$V = -2\,\epsilon\,\gamma + 2\Lambda\gamma\,\psi^2 - 2\frac{\psi'^{\,2}}{\gamma} + 4\left(\frac{\psi\,\psi'}{\gamma}\right)'$$

with $\epsilon = \{1, 0, -1\}$ for positive, zero or negative constant curvature, respectively.

- The requirement for preservation, in time, of the primary constraints leads to the secondary constraints:

$$\mathcal{H}_o \approx 0, \qquad \mathcal{H}_1 \approx 0$$

- Open Poisson bracket algebra of these constraints:

$$\begin{aligned} \{\mathcal{H}_o(r), \mathcal{H}_o(\tilde{r})\} &= \left[\frac{1}{\gamma^2(r)}\mathcal{H}_1(r) + \frac{1}{\gamma^2(\tilde{r})}\mathcal{H}_1(\tilde{r})\right]\delta'(r,\tilde{r}), \\ \{\mathcal{H}_1(r), \mathcal{H}_o(\tilde{r})\} &= \mathcal{H}_o(r)\delta'(r,\tilde{r}), \\ \{\mathcal{H}_1(r), \mathcal{H}_1(\tilde{r})\} &= \mathcal{H}_1(r)\delta'(r,\tilde{r}) - \mathcal{H}_1(\tilde{r})\delta(r,\tilde{r})'. \end{aligned}$$

- Under general changes $r \to \tilde{r} = h(r)$, it follows:

$$\tilde{\gamma}(\tilde{r}) = \gamma(r)\,\frac{d\,r}{d\,\tilde{r}}, \qquad \tilde{\psi}(\tilde{r}) = \psi(r), \qquad \frac{d\,\tilde{\psi}(\tilde{r})}{d\,\tilde{r}} = \frac{d\,\psi(r)}{d\,r}\,\frac{d\,r}{d\,\tilde{r}},$$

- With the infinitesimal transformation $r \to \tilde{r} = r - \eta(r)$, the corresponding changes induced on the basic fields are:

$$\delta\,\gamma(r) = (\gamma(r)\,\eta(r))', \qquad \delta\,\psi(r) = \psi'(r)\,\eta(r)$$

(one-dimensional analogue of the appropriate Lie derivatives).

- Action of $\mathcal{H}_1$ on the basic configuration space variables: Generator of spatial diffeomorphisms, i.e.

$$\left\{\gamma(r)\,,\,\int d\tilde{r}\,\eta(\tilde{r})\,\mathcal{H}_1(\tilde{r})\right\} = (\gamma(r)\,\eta(r))',$$
$$\left\{\psi(r)\,,\,\int d\tilde{r}\,\eta(\tilde{r})\,\mathcal{H}_1(\tilde{r})\right\} = \psi'(r)\,\eta(r).$$

Canonical Quantization

- Realize classical momenta as functional derivatives with respect to their corresponding conjugate fields:

$$\hat{\pi}_\gamma(r) = -i\,\frac{\delta}{\delta\,\gamma(r)}, \qquad \hat{\pi}_\psi(r) = -i\,\frac{\delta}{\delta\,\psi(r)}.$$

- Decide on initial state of space vectors:

$$\hat{\pi}_\gamma(r)\gamma(\tilde{r})^2 = -2i\gamma(\tilde{r})\delta(\tilde{r}, r).$$

- Choose as initial collection of states all smooth functionals of the configuration variables $\gamma(r), \psi(r)$ and their derivatives of any order:

$$\hat{\pi}_\gamma(r)\int d\tilde{r}\gamma(\tilde{r})^2 = -2i\int d\tilde{r}\gamma(\tilde{r})\delta(\tilde{r}, r) = -2i\gamma(r)$$

But

$$\hat{\pi}_\gamma(r)\,\hat{\pi}_\gamma(r)\int d\tilde{r}\gamma(\tilde{r})^2 = \hat{\pi}_\gamma(r)(-2i\int d\tilde{r}\gamma(\tilde{r})\delta(\tilde{r},r)) = \hat{\pi}_\gamma(r)(-2i\gamma(r))$$
$$= -2\delta(0)$$

$$\hat{\pi}_\gamma(r)\int d\tilde{r}\gamma''(\tilde{r})^2 = -2i\int d\tilde{r}\gamma''(\tilde{r})\delta''(\tilde{r},r) = -2i\gamma^{(4)}(r)$$

Generalizing the Kuchař-Hajíček quantization yields a quantization scheme of our system which:

(a) avoids the occurrence of $\delta(0)$'s,

(b) reveals the value $n = 1$ as the only *natural* (i.e. without *ad-hoc* cut-offs) possibility to obtain a closed space of state vectors,

(c) extracts a finite-dimensional Wheeler-deWitt equation governing the quantum dynamics.

- **Step 1**: Define the quantum analogue of $\mathcal{H}_1(r) \approx 0$ as:

$$\hat{\mathcal{H}}_1(r)\Phi = 0 \leftrightarrow -\gamma(r)\,(\frac{\delta\,\Phi}{\delta\,\gamma(r)})' + \psi'(r)\,\frac{\delta\,\Phi}{\delta\,\psi(r)} = 0.$$

- General Solutions:

$$\Phi = \int \gamma(\tilde{r})\, L\left(\Psi^{(0)}, \Psi^{(1)}, \ldots, \Psi^{(n)}\right) d\tilde{r}$$

$$\Psi^{(0)} \equiv \psi(\tilde{r}), \quad \Psi^{(1)} \equiv \frac{\psi'(\tilde{r})}{\gamma(\tilde{r})}, \ldots, \Psi^{(n)} \equiv \frac{1}{\gamma(\tilde{r})}\frac{d}{d\tilde{r}}\left(\underbrace{\ldots}_{n-1}\psi(\tilde{r})\right)$$

where L is any function of its arguments.

- **Step 2**: Define the equivalent of Kuchař's induced metric on the so far space of "physical" states Φ: analogues of Kuchař's physical variables q^α.
- Consider **one** initial candidate of the above form.
- Generalize the partial to functional derivatives.
- $\Rightarrow$ Induced metric:

$$g^{\Phi\Phi} = G^{\alpha\beta}\,\frac{\delta\Phi}{\delta\,x^\alpha}\,\frac{\delta\Phi}{\delta\,x^\beta}, \quad \text{where} \quad x^\alpha = \{\gamma, \psi\}$$

- Well-defined metric (it contains only first functional derivatives of the state vectors).

- For the induced metric $g^{\Phi\Phi}$, which is a local function, to be composed out of the "physical" states annihilated by $\hat{\mathcal{H}}_1$, a correspondence between local functions and smooth functionals is necessary.
  Assumption: *We assume that, as part of the renormalization procedure, we are permitted to map local functions to their corresponding smeared expressions e.g.,* $\psi(r) \leftrightarrow \int d\tilde{r}\psi(\tilde{r})$.

- In order to proceed with the generalization of Kuchař's method, we have to demand that:
  Requirement: $L\left(\Psi^{(0)}, \dots, \Psi^{(n)}\right)$ *must be such that* $g^{\Phi\Phi}$ *becomes a general function, say* $F\left(\gamma(r)\, L(\Psi^{(0)}, \dots, \Psi^{(n)})\right)$ *of the integrand of* Φ*, so that it can be considered as a function of this state:*
  $$g^{\Phi\Phi} \overset{Assumption}{\equiv} F\left(\int \gamma(\tilde{r})\, L(\Psi^{(0)}, \dots, \Psi^{(n)}) d\tilde{r}\right) = F(\Phi).$$

Therefore

$$g^{\Phi\Phi} = \ldots - \frac{\gamma}{2\psi}(-1)^{2n-1}\left(\frac{\partial^2 L}{\partial\left(\Psi^{(n)}\right)^2}\right)^2 \Psi^{(1)}\,\Psi^{(2n-1)}\,\Psi^{(2n)},$$

where the ... stand for all other terms, not involving $\Psi^{(2n)}$. According to the aforementioned **Requirement** we need this to be a general function, say $F(\gamma L)$, and for this to happen the coefficient of $\Psi^{(2n)}$ must vanish, i.e.

$$\frac{\partial^2 L}{\partial\left(\Psi^{(n)}\right)^2} = 0$$

$$\Leftrightarrow L = L_1\left(\Psi^{(0)},\ldots,\Psi^{(n-1)}\right)\Psi^{(n)} + L_2\left(\Psi^{(0)},\ldots,\Psi^{(n-1)}\right).$$

- Repeated imposition of the **Requirement** leads to:

$$\Phi \equiv \int \gamma(\tilde{r}) L\left(\psi, \Psi^{(1)}\right) d\tilde{r}, \quad \text{and}$$

- we end up with the three scalar functionals:

$$y^1 = \int \gamma(\tilde{r}) d\tilde{r}, \quad y^2 = \int \gamma(\tilde{r})\psi(\tilde{r})^2 d\tilde{r}, \quad y^3 = \int d\tilde{r}\, \gamma(\tilde{r})\, L(\Psi^{(1)})$$

- Any other functional, say

$$y^4 = \int d\tilde{r}\, \gamma(\tilde{r})\, K\left[\psi(\tilde{r}), \Psi^{(1)}(\tilde{r})\right] \equiv y^1\, K\left[\sqrt{\frac{y^2}{y^1}}, L^{-1}\left(\frac{y^3}{y^1}\right)\right].$$

- It is interesting that the induced metric is

$$g^{AB}_{ren}(y^1, y^2, y^3) = -\frac{1}{4}\begin{pmatrix} -\frac{(y^1)^2}{y^2} & y^1 & -\frac{y^1 y^3}{y^2} \\ y^1 & 3\,y^2 & y^3 \\ -\frac{y^1 y^3}{y^2} & y^3 & -\frac{(y^3)^2}{y^2} + \frac{4\,(y^1)^2 F(\frac{y^3}{y^1})^2}{3\,y^2 F'\left[F(\frac{y^3}{y^1})\right]^2} \end{pmatrix}$$

where

$$g^{33}_{ren} = \int \frac{\gamma}{4\psi^2} W(L(\Psi^{(1)}))dr$$

or, equivalently, via the **Assumption**

$$g^{33}_{ren} = \frac{(y^1)^2}{4y^2} W(\frac{y^3}{y^1})$$

- The quantity $W(L(\Psi^{(1)}))$ is parameterized as

$$L\left(\Psi^{(1)}\right)^2 - \frac{4\,F[L\left(\Psi^{(1)}\right)]^2}{3\,F'[F[L\left(\Psi^{(1)}\right)]]^2}\,.$$

  and g_{ren}^{AB} satisfies the **Requirement** for any choice of $L(\Psi^{(1)})$.
- More interestingly and unexpectedly, the freedom in the choice of L translates into a pure g.c.t. freedom of the above renormalized metric.

The quantum analogue of the quadratic constraint is taken as:

$$\hat{\mathcal{H}}_o \Psi \equiv [-\frac{1}{2}\,\Box_c + V_{ren}]\,\Psi(Y^1, Y^2, Y^3) = 0$$

with

$$\Box_c = \Box + \frac{d-2}{4\,(d-1)}\,R\,,$$

where (Y^1, Y^2, Y^3) are given by the transformation

$$(y^1, y^2, y^3) = \left(e^{-\frac{1}{8}(5\,Y^1+3\,Y^3)}, e^{Y^1+Y^2+Y^3}, e^{-\frac{1}{8}(5\,Y^1+3\,Y^3)} F^{-1}(e^{\frac{1}{24}(-9\,Y^1+8\,Y^2-15\,Y^3)}\right.$$

and the renormalized metric is now given by:

$$g_{AB\,ren}(Y^1, Y^2, Y^3) = \begin{pmatrix} e^{Y^1+Y^2+Y^3} & 0 & 0 \\ 0 & -\frac{4}{3}\,e^{Y^1+Y^2+Y^3} & 0 \\ 0 & 0 & -e^{Y^1+Y^2+Y^3} \end{pmatrix}$$

The renormalized potential is

$$V_{ren} = -2\,\epsilon\, e^{-\frac{1}{8}(5\,Y^1+3\,Y^3)} - \\ -2\,e^{-\frac{1}{8}(5\,Y^1+3\,Y^3)}\left[L^{-1}\left(F^{-1}(e^{\frac{1}{24}(-9\,Y^1+8\,Y^2-15\,Y^3)})\right)\right]^2 + \\ +2\,\Lambda\, e^{Y^1+Y^2+Y^3}.$$

Exploiting the previously mentioned freedom in the choice of L, we can simplify the potential by the choice

$$L(\Psi^{(1)}) = m + \int \frac{(\Psi^{(1)})^{3/2}}{((\Psi^{(1)})^2-\epsilon)^{13/16}}\, e^{k-\frac{3\,\epsilon}{16((\Psi^{(1)})^2-\epsilon)}}\, d\Psi^{(1)}$$

(where $c_1\,m + c_2 + c_3\,e^k = 0$)
which satisfies the relation:

$$F^{-1}\left(e^{\frac{1}{24}(-9\,Y^1+8\,Y^2-15\,Y^3)}\right) = L\left(\sqrt{e^{\frac{1}{24}(-9\,Y^1+8\,Y^2-15\,Y^3)}-\epsilon}\right)$$

Gathering all the pieces together, we end up with the following Wheeler-deWitt equation:

$$2\,\Lambda\, e^{2(Y^1+Y^2+Y^3)}\Psi(Y^1,Y^2,Y^3) -$$
$$2\, e^{\frac{4}{3}\,Y^2}\Psi(Y^1,Y^2,Y^3) - \frac{3}{128}\Psi(Y^1,Y^2,Y^3) -$$
$$\frac{1}{4}\frac{\partial\Psi(Y^1,Y^2,Y^3)}{\partial Y^1} + \frac{3}{16}\frac{\partial\Psi(Y^1,Y^2,Y^3)}{\partial Y^2} +$$
$$\frac{1}{4}\frac{\partial\Psi(Y^1,Y^2,Y^3)}{\partial Y^3} - \frac{1}{2}\frac{\partial^2\Psi(Y^1,Y^2,Y^3)}{\partial\left(Y^1\right)^2} +$$
$$\frac{3}{8}\frac{\partial^2\Psi(Y^1,Y^2,Y^3)}{\partial\left(Y^2\right)^2} + \frac{1}{2}\frac{\partial^2\Psi(Y^1,Y^2,Y^3)}{\partial\left(Y^3\right)^2} = 0.$$

which is readily solved via separation of variables

$$\Psi(Y^1, Y^2, Y^3) = \Psi^1(Y^1)\,\Psi^2(Y^2)\,\Psi^3(Y^3)$$

leading to the solutions:

$$\begin{aligned}
\Psi^1(Y^1) &= c_1\, e^{\frac{1}{4}\left(-1-\sqrt{1+32\,m}\right)Y^1} + c_2\, e^{\frac{1}{4}\left(-1+\sqrt{1+32\,m}\right)Y^1} \\
\Psi^2(Y^2) &= c_3\, e^{-Y^2/4}\, I_{-\frac{\sqrt{3}}{8}\sqrt{3+128\,n}}\left(2\sqrt{3}\, e^{2\,Y^2/3}\right) \\
&\quad + c_4\, e^{-Y^2/4}\, I_{\frac{\sqrt{3}}{8}\sqrt{3+128\,n}}\left(2\sqrt{3}\, e^{2\,Y^2/3}\right) \\
\Psi^3(Y^3) &= c_5\, e^{\frac{1}{8}\left(-2-\sqrt{7+128\,m-128\,n}\right)Y^3} + c_6\, e^{\frac{1}{8}\left(-2+\sqrt{7+128\,m-128\,n}\right)Y^3}
\end{aligned}$$

Summary

- The proper imposition of quantum analogues of the linear (momentum) constraint reduces an initial collection of state vectors to all scalar smooth functionals.
- The demand that the midi-superspace metric must define on the space of these states an induced metric whose components are given in terms of the same states, which is made possible through an appropriate renormalization **Assumption** and **Requirement**, severely reduces possible state vectors to three unique smooth scalar functionals.
- The quantum analogue of the Hamiltonian constraint produces a Wheeler-deWitt equation based on the reduced manifold of states, which is completely integrated.

Open Questions

- Before a probability can be assigned to each of these geometries, two problems remain to be solved:
  (1) Render finite the three smooth functionals y^1, y^2, y^3.
  (2) Select an appropriate inner product.
- The first will need a final regularization of y^1, y^2, y^3, but the detailed way to do this will depend upon the particular geometry under consideration.
- For the second, a natural choice could be the determinant of the induced renormalized metric, although the problem with the positive definiteness may dictate another choice.

Conceptual Query

- If the above quantization scheme trully achieves general coordinate invariant characterization of the wave function, then there must be a way to classify classical geometries using only **first derivatives** of the metrics.
- The current state of knowledge for the subject is the Cartan-Karlhede equivalence classification scheme, which requires up to seven derivatives of the Riemann tensor.
- BUT... we are working on it!!!!!!!!!!!!