Quantum geometry in Topological BF model in 3d and 4d

Valentin Bonzom

Perimeter Institute for Theoretical Physics

Topological BF theory [Blau & Thompson, Horowitz '89]

Generalization of Chern-Simons

- Lie group G (internal symmetry)
- gauge field A, curvature strength $F = dA + \frac{1}{2}[A, A]$
- $\mathrm{Ad}(G)$-valued $(d-2)$-form B, Lagrange multiplier for curvature

$$S_{BF}(A, B) = \int \mathrm{tr}\ B \wedge F(A)$$

- Gauge symmetry 1

$$A \mapsto \mathrm{Ad}(g)\, A + g dg^{-1}, \qquad B \mapsto \mathrm{Ad}(g)\, B$$

- Gauge symmetry 2

$$A \mapsto A, \qquad B \mapsto B + d_A\, \eta$$

- Diffeomorphisms

Equations of motion

- Vary B, Lagrange multiplier,

$$F = 0$$

A flat connection. Locally $A = 0$ up to gauge.

Equations of motion

- Vary B, Lagrange multiplier,

$$F = 0$$

  A flat connection. Locally $A = 0$ up to gauge.

- Vary A

$$d_A\ B = 0$$

  B closed, so locally exact: $B = dC$.

- Use gauge symmetry 2, $B' = B - dC = 0$.

Equations of motion

- Vary B, Lagrange multiplier,

$$F = 0$$

A flat connection. Locally $A = 0$ up to gauge.

- Vary A

$$d_A\ B = 0$$

B closed, so locally exact: $B = dC$.

- Use gauge symmetry 2, $B' = B - dC = 0$.

All solutions are gauge equivalent to trivial sol

No local degrees of freedom

Motivations

- This is 3d gravity with degenerate metrics
- Theories written like BF + something
  Yang-mills

$$S_{YM} = S_{BF} + g_{YM}^2 \int \text{tr}\ B \wedge *_\eta B$$

Motivations

- This is 3d gravity with degenerate metrics
- Theories written like BF + something
  Gravity a la Palatini-Cartan

$$S_{PC} = S_{BF} + \int \mathrm{tr}\ \phi B \wedge B$$

Simplicity constraint, $B^{IJ} = \epsilon^{IJ}{}_{KL} e^K \wedge e^L$

Motivations

- This is 3d gravity with degenerate metrics
- Theories written like BF + something
  Gravity a la MacDowell-Mansouri

$$S_{MDM} = S_{BF} + \alpha \int \mathrm{tr}\ \gamma_5 B \wedge *B$$

with $\alpha = G\Lambda$

Motivations

- This is 3d gravity with degenerate metrics
- Theories written like BF + something
  Gravity a la MacDowell-Mansouri

$$S_{MDM} = S_{BF} + \alpha \int \text{tr}\ \gamma_5 B \wedge *B$$

  with $\alpha = G\Lambda$
- Lattice gauge theory
- Topological effects in condensed matter (top insulators)
- Kitaev model and generalizations: topological order, quantum information
- Topological invariants, quantum invariants

Issues

In BF and Gravity, dynamics = gauge symmetries

Vector field ξ, def: $\phi_\xi = i_\xi A$ and $\eta_\xi = i_\xi B$

$$\begin{aligned} \mathcal{L}_\xi A &= d_A \phi_\xi + i_\xi F, \\ \mathcal{L}_\xi B &= d_A \eta_\xi + [\phi_\xi, B] + i_\xi d_A B. \end{aligned}$$

Diffeos = internal gauge + e.o.m. of BF

What we want to do

In Loop quantum gravity context

- Wheeler-DeWitt equation for 3d gravity ??
- Transition amplitudes in a 4d theory ??
- Relations to gauge symmetries in spin foams ??

What we got (I)

Hamiltonian scalar constraint in 3d

- $H|\psi\rangle = 0$ in spin network basis: pentagon identity of group G representation theory
- Enables to solve the theory
- Interesting consequences on splitting diffeo/scalar constraints
- Side-product: Recursion relations on arbitrary Wigner coefficients

Some insights into cosmological case $\Lambda \neq 0$

Relation quantum group/curved spacetime geometry

Collaborators: E.R. Livine, L. Freidel, S. Speziale
On symmetries: Berlin groups B. Dittrich and D. Oriti.
On spin network evaluations: many people in different fields !
R. Littlejohn, S. Garoufalidis, M. Marino, A. Marzuoli

What we got (II)

Loop quantization of 4d BF

- Evaluation of spin networks on flat connections
- Lattice definition of the model
- Gauge symmetry identified (in any dim)
- Non-trivial measure (same as that from path integral)
- Challenge for Ooguri spin foam model

Collaborator: M. Smerlak
Mathematics: C. Frohman, J. Dubois, F. Costantini,...

Spin networks

Let G be a compact Lie group.
A spin network is a decorated graph:

- a closed oriented graph Γ,
- an irreducible representation ρ_l of G attached to each link l,
- an intertwiner attached to each node: invariant vector in $\bigotimes_{l \text{ meeting at } v} \rho_l$
- $G = \mathrm{SO}(3)$, 3-valent node, intertwiner = Clebsch-Gordan coeff

Spin networks II

- They span the Hilbert space:

$$\mathcal{H}_\Gamma = L^2\left(G^E/G^V\right)$$

  Functions of group elements (g_e) on links (Wilson lines), invariant under translation by G at each node.

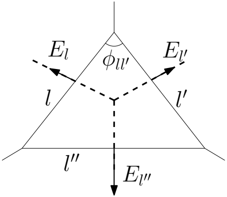
- Quantization on a classical phase space
  Wilson lines of gauge field A
  Flux E^i_l of some non-Abelian electric field.

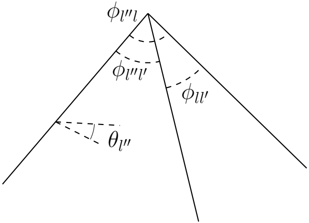
Lattice Yang-Mills phase space !

$$\{E^i_l, g_l\} = g_l \tau^i, \qquad \{E^i_l, E^j_l\} = \epsilon^{ij}{}_k\, E^k_l$$

  Fluxes act as left-invariant derivatives.

Geometry on the dual triangulation, $d = 3$, $G = \mathrm{SO}(3)$





- Triangles embedded in flat 3-space
- Fluxes $E_l \in \mathbb{R}^3$: normals to edges
  $E_l^2 = \ell_l^2, \quad E_l \cdot E_{l'} = \ell_l \ell_{l'} \ \cos\phi_{ll'}$
- Dihedral angles $\theta_l(g_l)$
- Hamiltonian relates extrinsic to intrinsic geometry

Hamiltonian scalar constraint

- In general relativity (Ashtekar-Barbero variables)

$$H = \epsilon^{ij}{}_{k} \; E_i^a \, E_j^b \, F(A)^k_{ab} + \text{others}$$

(at the quantum level: Thiemann's proposal, recently improved by Alesci-Rovelli.)

Hamiltonian scalar constraint

- In general relativity (Ashtekar-Barbero variables)

$$H = \epsilon^{ij}{}_{k}\ E^a_i\, E^b_j\, F(A)^k_{ab} + \text{others}$$

(at the quantum level: Thiemann's proposal, recently improved by Alesci-Rovelli.)

- 3d gravity: flat gauge field, $F(A)^k_{ab} = 0$.
Quantization well-known (Witten, Noui-Perez).

Hamiltonian scalar constraint

- In general relativity (Ashtekar-Barbero variables)

$$H = \epsilon^{ij}{}_k \; E_i^a \, E_j^b \, F(A)_{ab}^k + \text{others}$$

(at the quantum level: Thiemann's proposal, recently improved by Alesci-Rovelli.)

- 3d gravity: flat gauge field, $F(A)_{ab}^k = 0$.
  Quantization well-known (Witten, Noui-Perez).
- F^k has only one space component, F_{12}^k.
  Project it on the normal $\vec{n} = \vec{e}_1 \times \vec{e}_2$,

$$H = \epsilon^{ij}{}_k \; E_i^a \, E_j^b \, F(A)_{ab}^k = \left(\vec{n} \cdot \vec{F}\right) / |\vec{n}|$$

Tentative

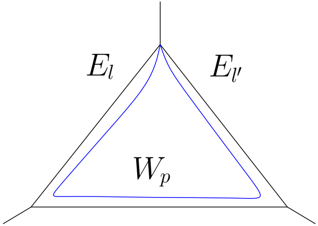
- Know we can restrict to a single cell decomposition
- H not graph changing
- Expect H to shift spins of spin networks.
- Regularize curvature. $W_p \in \mathrm{SO}(3)$ Wilson loop around a face,

$$\epsilon^{ij}{}_{k} F^{k}_{ab} \quad \longrightarrow \quad \delta^{ij} - \left(W_p\right)^{ij}$$

  Flatness: $W_p = \mathbb{1}$
- Usual spin ambiguity: here spin 1 natural.
- Proposal along cycle c, node n where 2 and 6 meet

$$H_{n,c} = \vec{E}_l \cdot \vec{E}_{l'} - \vec{E}_l \cdot W_p \vec{E}_{l'}$$

Proposal to mimic the scalar constraint



$$H_{p,n} = E_l^i \left(\mathbb{1}_{ij} - (W_p)_{ij}\right) E_{l'}^j$$

Some properties, classically

- At least three constraints per face, 3 independent
- Looking at matrix W_p in suitable basis: spanned by fluxes
- Algebra closes: generate gauge symmetry
- No need for splitting vector/scalar constraints !
- Tension

  Curvature regularized around vertices vs. $H = \vec{n} \cdot \vec{F}$

  What is the normal to a vertex in 3-space ?

Some properties, classically

- At least three constraints per face, 3 independent
- Looking at matrix W_p in suitable basis: spanned by fluxes
- Algebra closes: generate gauge symmetry
- No need for splitting vector/scalar constraints !
- Tension
  Curvature regularized around vertices vs. $H = \vec{n} \cdot \vec{F}$

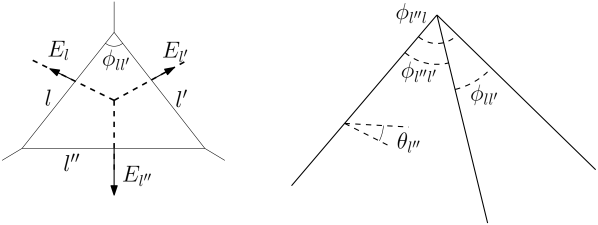
  What is the normal to a vertex in 3-space ?

Some properties, classically

- At least three constraints per face, 3 independent
- Looking at matrix W_p in suitable basis: spanned by fluxes
- Algebra closes: generate gauge symmetry
- No need for splitting vector/scalar constraints !
- Tension
  Curvature regularized around vertices vs. $H = \vec{n} \cdot \vec{F}$

  What is the normal to a vertex in 3-space ?
  One normal per adjacent face (at least 3)

Flat geometry



$$H_{p,n} = 0 \qquad \sim \qquad \cos\theta_{l''} = \frac{\cos\phi_{ll'} - \cos\phi_{l''l}\,\cos\phi_{l''l'}}{\sin\phi_{l''l}\,\sin\phi_{l''l'}}$$

Quantization of flat geometry !

Questions

What is the Wheeler-DeWitt equation ?

$$\widehat{H}_{n,p}\psi = 0,$$

in the spin network basis.

Can we solve it and reproduce known results in 3d ?

Questions

What is the Wheeler-DeWitt equation ?

$$\widehat{H}_{n,p}\psi = 0,$$

in the spin network basis.

Can we solve it and reproduce known results in 3d ?

Yes

The Biedenharn-Elliott identity

Triangulation of S^2 by the boundary of a tetrahedron.
A single physical state satisfying flatness: $\prod_{\mathrm{plaquettes}} \delta(W_p)$, or:

$$|\psi\rangle = \sum_{\{j_i\}} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} |j_i\rangle$$

The Biedenharn-Elliott identity

Triangulation of S^2 by the boundary of a tetrahedron.
A single physical state satisfying flatness: $\prod_{\text{plaquettes}} \delta(W_p)$, or:

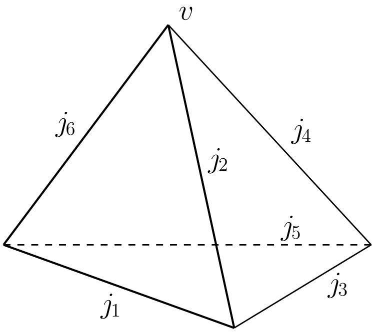
$$|\psi\rangle = \sum_{\{j_i\}} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} |j_i\rangle$$

How can we characterize Wigner 6j-symbol ?

Well-known 2nd order recursion relation:

$$A_+(j_1) \begin{Bmatrix} j_1+1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} + A_0(j_1) \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} + A_-(j_1) \begin{Bmatrix} j_1-1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = 0.$$

Triangulation of S^2 by the boundary of a tetrahedron



The recurrence relation

$A_+(j) = j_1 E(j_1+1)$ and $A_-(j) = (j_1-1)E(j_1)$, with:

$$E(j_1) = \Big[\big((j_2+j_3+1)^2 - j_1^2\big)\big(j_1^2 - (j_2-j_3)^2\big) \times \big((j_5+j_6+1)^2 - j_1^2\big)\big(j_1^2 - (j_5-j_6)^2\big)\Big]^{\frac{1}{2}}$$

A_0 is:

$$A_0(j) = (2j_1+1)\Big\{2\big[j_2(j_2+1)j_5(j_5+1) + j_6(j_6+1)j_3(j_3+1) - j_1(j_1+1)j_4(j_4+1)\big] - \big[j_2(j_2+1)+j_3(j_3+1)-j_1(j_1+1)\big]\big[j_5(j_5+1)+j_6(j_6+1)-j_1(j_1+1)\big]\Big\}$$

The Wheeler-DeWitt equation

$H_{n,c} = \vec{E}_l \cdot \vec{E}_{l'} - \vec{E}_l \cdot W_p \vec{E}_{l'}$ becomes an operator on the boundary Hilbert space to the tetrahedron.

- $E_l \cdot E_{l'}$ is diagonal on spin network functions
- The matrix W_p produces shifts.

The Wheeler-DeWitt equation

$H_{n,c} = \vec{E}_l \cdot \vec{E}_{l'} - \vec{E}_l \cdot W_p \vec{E}_{l'}$ becomes an operator on the boundary Hilbert space to the tetrahedron.

- $E_l \cdot E_{l'}$ is diagonal on spin network functions
- The matrix W_p produces shifts.
- $\hat{H} \ \sum_{\{j_l\}} \psi(j_l)|j_l\rangle = 0$ gives the equation:

$$A_+(j_1)\psi(j_1+1, j_{l'}) + A_0(j_1)\psi(j_1, j_{l'}) + A_-(j_1)\psi(j_1-1, j_{l'}) = 0.$$

- solution: $\left\{ \begin{smallmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{smallmatrix} \right\}$.

4d lift

- Lift the Hamiltonian to 4d, get difference equations solved by 15j-symbols of the Ooguri model !

Asymptotics

- In 3d, LQG geometries are Regge, and usual equation:

$$[\Delta_{j_l} + 2\,(1 - \cos\theta_l(j))]\,\psi = 0$$

$\psi \sim \cos(S_{\mathrm{Regge}} + \frac{\pi}{4})/\sqrt{12\pi V}$ [Schulten & Gordon '75]

4d lift

- Lift the Hamiltonian to 4d, get difference equations solved by 15j-symbols of the Ooguri model !

Asymptotics

- In 3d, LQG geometries are Regge, and usual equation:

$$[\Delta_{j_l} + 2\,(1 - \cos\theta_l(j))]\,\psi = 0$$

$\psi \sim \cos(S_{\mathrm{Regge}} + \frac{\pi}{4})/\sqrt{12\pi V}$ [Schulten & Gordon '75]

- In 4d, LQG describes twisted geometries. Different cases: geometric (Regge) sector, and others.
  Geometric sector: the same equation.
  But the Hamiltonian is defined on the whole phase space, and the Wheeler-DeWitt equation makes sense everywhere on it !

Remarks

- Reproduce the expected result, without divergencies even for spherical topology !
- Define H with W_p in spin 2 produces shifts $\pm 2, \pm 1$
  More initial conditions to specify !
- What is the quantum algebra of constraints ???
- More recursions from group representation have been derived, like closure of simplex

Path integral

- Naively, B Lagrange multiplier,

$$Z = \int DA\, \delta(F(A)) = \sum_{\substack{\phi,\ \text{flat connections} \\ \text{(up to gauge)}}} \frac{1}{\left|\frac{\delta F}{\delta A}[\phi]\right|}$$

  Integral peaked on flat connections

- After gauge-fixing

$$Z = \sum_{\substack{\phi,\ \text{flat connections} \\ \text{up to gauge}}} \mathrm{tor}[\phi]$$

  where tor is a topological invariant, the torsion, associated to the manifold and flat connection.
  (actually simple homotopy invariant).

Amplitudes

- Insert functions over set of flat connections

$$\langle\psi|\chi\rangle = \sum_{\substack{\phi,\ \text{flat connections} \\ \text{up to gauge}}} \overline{\psi[\phi]}\ \mathsf{tor}[\phi]\ \chi[\phi]$$

- Spin network quantization ?

Lattice construction of the model

Discretize the fields

$$\begin{aligned} \text{1-form } A &\rightarrow A_e \text{ on edges} \\ (d-2)\text{-form } B &\rightarrow B_f \text{ on faces} \end{aligned}$$

Action of gauge symmetry 1, $A + d_A\omega$

$$\text{function } \omega \rightarrow \omega_v \text{ on vertices}$$

Action of gauge symmetry II, $B + d_A\eta$

$$(d-3)\text{-form } \eta \rightarrow \eta_{3c} \text{ on 3-cells}$$

Covariant derivative

For a flat connection ϕ on the lattice

$$d_\phi^2 = F = 0 \quad \longrightarrow \quad \delta_\phi : k\text{-cells} \mapsto (k+1)\text{-cells}$$

$\delta_\phi^2 = 0$

Gauge-Fixing

- Freidel-Louapre gauge fixing, trivial FP determinant on $\Sigma_g \times [0,1]$
- Generically non-trivial $\Delta_{FP}(\phi) \neq 1$

2-complex not enough, need to known the full cellular structure to get the correct path integral