# New guises of $AdS_3$ and the entropy of two-dimensional CFT

N. Tetradis

University of Athens

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- AdS/CFT: duality between string theory on $AdS_5 \times S^5$ and the conformal $\mathcal{N} = 4$ SUSY with $SU(N)$ gauge symmetry in 3+1 dimensions
  $\lambda = g_{YM}^2 N = (L_{AdS}/\ell_s)^4$ and $4\pi g_{YM}^2 = g_s$
- Things simplify for large fixed $\lambda$ and $N \to \infty$
  Duality between a supergravity solution in asymptotically AdS space and a strongly coupled CFT
- The CFT "lives" on the boundary of AdS
- Many of the deduced properties of the CFT are generic for strongly coupled theories
- Relevant example: Hydrodynamic properties of CFTs on flat or Bjorken geometries $(\eta/s = 1/4\pi)$
- AdS/CFT for a FRW boundary
- Thermodynamic properties of CFT on cosmological backgrounds

## Outline

- Fefferman-Graham parametrization of $AdS_3$ and the BTZ black hole with various boundary metrics
- Stress-energy tensor
- Entropy
- Comments

P. Apostolopoulos, G. Siopsis, N. T. :
arxiv:0809.3505[hep-th], Phys. Rev. Lett. 102 (2009) 151301
N. T. : arxiv:0905.2763[hep-th], JHEP 1003 (2010) 040
N. Lamprou, S. Nonis, N. T. : arXiv:1106.1533 [gr-qc]
N. T. : arXiv:1106.2492 [hep-th]
N. T. : arXiv:1109.2335 [hep-th]

## AdS$_3$ in various coordinates ($L_{AdS} = 1$)

- Global coordinates:

$$ds^2 = -\cosh^2(\tilde{r})\, d\tilde{t}^2 + d\tilde{r}^2 + \sinh^2(\tilde{r})\, d\tilde{\phi}^2. \qquad (1)$$

$0 \le \tilde{r} < \infty$, $-\pi < \tilde{\phi} \le \pi$ (periodic)

- The boundary of AdS is approached for $\tilde{r} \to \infty$.
- Define $\tilde{\chi}$ through $\tan(\tilde{\chi}) = \sinh(\tilde{r})$. The metric becomes

$$ds^2 = \frac{1}{\cos^2(\tilde{\chi})} \left[ -d\tilde{t}^2 + d\tilde{\chi}^2 + \sin^2(\tilde{\chi})\, d\tilde{\phi}^2 \right]. \qquad (2)$$

$0 \le \tilde{\chi} < \pi/2$.

- The boundary is now approached for $\tilde{\chi} \to \pi/2$.

- Poincare coordinates:

$$ds^2 = -r^2\, dt^2 + \frac{dr^2}{r^2} + r^2 d\phi^2. \tag{3}$$

- The relation between the global and Poincare coordinates is

$$\tilde{t}(t,r,\phi) = \arctan\left[\frac{2r^2 t}{1 + r^2(1+\phi^2 - t^2)}\right] \tag{4}$$

$$\tilde{\chi}(t,r,\phi) = \arctan\sqrt{r^2\phi^2 + \frac{\left[1 - r^2(1-\phi^2+t^2)\right]^2}{4r^2}} \tag{5}$$

$$\tilde{\phi}(t,r,\phi) = \arctan\left[\frac{1 - r^2(1-\phi^2+t^2)}{2r^2\phi}\right]. \tag{6}$$

  The global coordinate $\tilde{\phi}$ is periodic with period $2\pi$. The limits $\phi \to \pm\infty$ of the Poincare coordinate $\phi$ must be identified.
- The slice $t = \tilde{t} = 0$ is covered entirely by both coordinate systems.

- Fefferman-Graham coordinates:

$$ds^2 = \frac{1}{z^2}\left(dz^2 - dt^2 + d\phi^2\right). \tag{7}$$

- $r = 1/z$

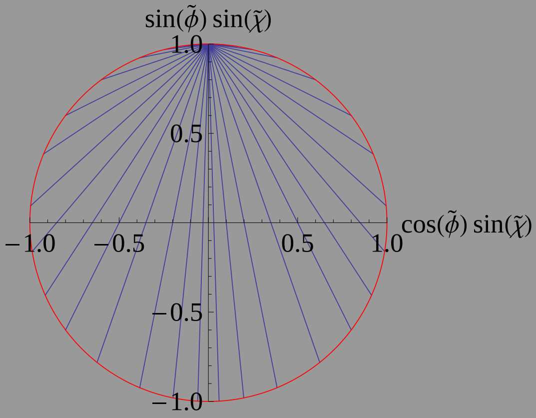


**Figure:** *Lines of constant $\phi$ for a Minkowski boundary.*

- General metric in Fefferman-Graham coordinates:

$$ds^2 = \frac{1}{z^2}\left[dz^2 + g_{\mu\nu}dx^\mu dx^\nu\right], \tag{8}$$

where

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + z^2 g^{(2)}_{\mu\nu} + z^4 g^{(4)}_{\mu\nu}. \tag{9}$$

- Holographic stress-energy tensor of the dual CFT (Skenderis 2000)

$$\langle T^{(CFT)}_{\mu\nu}\rangle = \frac{1}{8\pi G_3}\left[g^{(2)} - \mathrm{tr}\left(g^{(2)}\right) g^{(0)}\right]. \tag{10}$$

## The BTZ black hole

- Metric in Schwarzschild coordinates:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2, \qquad f(r) = r^2 - \mu. \qquad (11)$$

- $\phi$ has a period equal to $2\pi$
- Temperature, energy and entropy of the black hole ($V = 2\pi$):

$$T = \frac{1}{2\pi}\sqrt{\mu}, \qquad E = \frac{V}{16\pi G_3}\mu, \qquad S = \frac{V}{4G_3}\sqrt{\mu}. \qquad (12)$$

- Metric in Fefferman-Graham coordinates:

$$ds^2 = \frac{1}{z^2}\left[dz^2 - \left(1 - \frac{\mu}{4}z^2\right)^2 dt^2 + \left(1 + \frac{\mu}{4}z^2\right)^2 d\phi^2\right]. \qquad (13)$$

with

$$z = \frac{2}{\mu}\left(r \mp \sqrt{r^2 - \mu}\right), \qquad r = \frac{1}{z} + \frac{\mu}{4}z. \qquad (14)$$

- $z$ takes values $0 < z \le z_e = 2/\sqrt{\mu}$ and $z_e \le z < \infty$, covering twice the region outside the event horizon.
  $r$ takes values $r_e = \sqrt{\mu} \le r < \infty$. Throat at $z = 2/\sqrt{\mu}$.
- Holographic stress-energy tensor of the dual CFT
  Energy density and pressure:

$$\rho = \frac{E}{V} = -\langle T^t_{\ t}\rangle = \frac{\mu}{16\pi G_3}, \qquad (15)$$

$$p = \langle T^{\phi}_{\ \phi}\rangle = \frac{\mu}{16\pi G_3}. \qquad (16)$$

## AdS$_3$ with a Rindler boundary

- AdS$_3$ can be put in the form

$$ds^2 = \frac{1}{z^2}\left[dz^2 - a^2x^2\left(1+\frac{z^2}{4x^2}\right)^2 dt^2 + \left(1-\frac{z^2}{4x^2}\right)^2 dx^2\right], \tag{17}$$

with a boundary corresponding to the Rindler wedge ($x > 0$)

$$ds_0^2 = g^{(0)}_{\mu\nu}dx^\mu dx^\nu = -a^2x^2dt^2 + dx^2. \tag{18}$$

- Transformation from Poincare to Fefferman-Graham coordinates:

$$r(z,x) = a\left(\frac{x}{z}+\frac{z}{4x}\right) \tag{19}$$

$$\phi(z,x) = \frac{1}{a}\log[ax] - \frac{8}{a(4+z^2/x^2)}. \tag{20}$$

- The region near negative infinity for $\phi$ is mapped to the neighborhood of zero for $x$ .
- The limits $x \to 0$ and $x \to \infty$ must be identified.

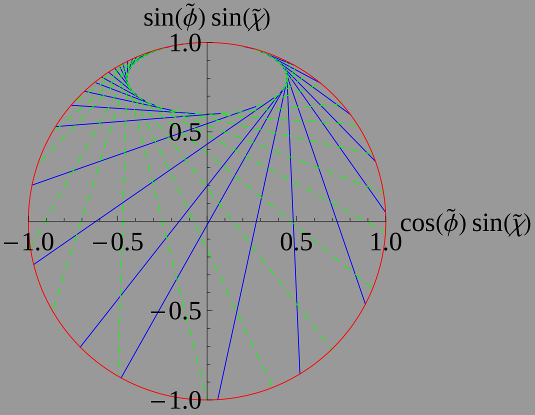


**Figure:** *Lines of constant $x$ for a Rindler boundary with $a = 0.5$.*

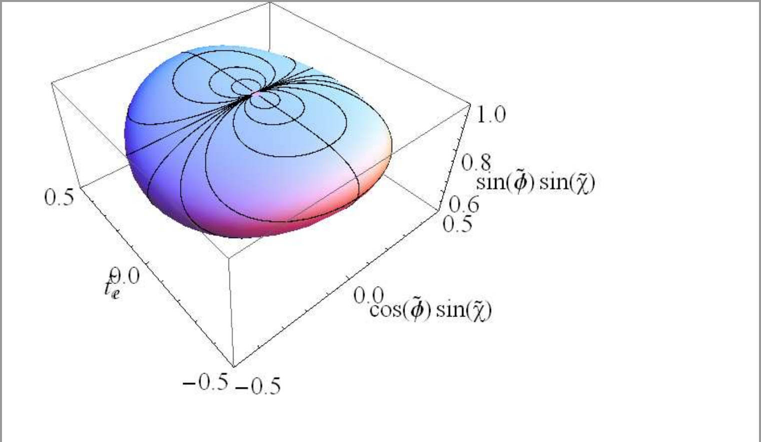


**Figure:** *The region not covered by Fefferman-Graham coordinates for a Rindler boundary with* $a = 0.5$.

- For fixed $x$, there is a minimal value for $r(z,x)$ as a function of $z$. It is obtained for

$$z_m(x) = 2x \tag{21}$$

and is equal to

$$r_m(x) = a. \tag{22}$$

The corresponding value of $\phi$ is

$$\phi_m(x) \equiv \phi_m(z_m(x), x) = (\log[ax] - 1)/a. \tag{23}$$

- Bridge connecting the two asymptotic regions at $z \to 0$ and $z \to \infty$.
- The holographic stress-energy tensor of the CFT at $z = 0$ is

$$\rho = -\langle T^t_{\ t}\rangle = -\frac{1}{16\pi G_3}\frac{1}{x^2}, \tag{24}$$

$$p = \langle T^x_{\ x}\rangle = -\frac{1}{16\pi G_3}\frac{1}{x^2}. \tag{25}$$

It displays the expected singularity at $x = 0$. The conformal anomaly vanishes.

## A different state

- The AdS$_3$ metric can also be put in the form

$$ds^2 = \frac{1}{z^2}\left[dz^2 - a^2x^2d\eta^2 + dx^2\right], \tag{26}$$

with a Rindler boundary. The coordinate transformation that achieves this is given by

$$t(\eta, x) = x\sinh(a\eta) \tag{27}$$

$$r(z) = \frac{1}{z} \tag{28}$$

$$\phi(z, x) = x\cosh(a\eta). \tag{29}$$

- The corresponding stress-energy tensor vanishes.

## AdS$_3$ with de Sitter boundary

- $$ds^2 = \frac{1}{z^2}\left[dz^2 - (1-H^2\rho^2)\left(1+\frac{1}{4}\left[\frac{H^2}{1-H^2\rho^2}-H^2\right]z^2\right)^2 dt^2 + \left(1-\frac{1}{4}\left[\frac{H^2}{1-H^2\rho^2}+H^2\right]z^2\right)^2 \frac{d\rho^2}{1-H^2\rho^2}\right], \quad (30)$$

  with a de Sitter boundary at $z=0$.
- The coordinate transformation is $(-1/H < \rho < 1/H)$

$$r(z,\rho) = \frac{\sqrt{1-H^2\rho^2}}{z} + \frac{H^4\rho^2}{4\sqrt{1-H^2\rho^2}}z \quad (31)$$

$$\phi(z,\rho) = \frac{1}{2H}\log\left[\frac{1+H\rho}{1-H\rho}\right] - \frac{H^2\rho z^2}{2(1-H^2\rho^2+H^4\rho^2z^2/4)} \quad (32)$$

- The transformation maps the region near negative infinity for $\phi$ to the vicinity of $-1/H$ for $\rho > -1/H$, and the region near positive infinity for $\phi$ to the vicinity of $1/H$ for $\rho < 1/H$.

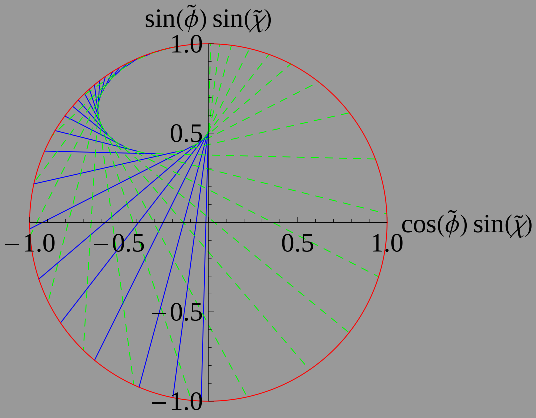


**Figure:** *Lines of constant $\rho < 0$ for a static de Sitter boundary with $H = 0.8$.*

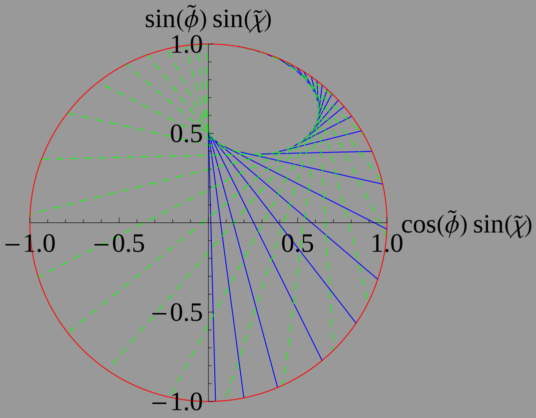


**Figure:** *Lines of constant $\rho > 0$ for a static de Sitter boundary with $H = 0.8$.*

- The coordinates starts covering the AdS space for a second time when $\partial r(z,\rho)/\partial\rho = 0$. For fixed $\rho$, the turning point is

$$z_t(\rho) = \frac{2}{H}\sqrt{\frac{1-H^2\rho^2}{2-H^2\rho^2}}. \tag{33}$$

It corresponds to

$$r_t(\rho) \equiv r(z_t(\rho),\rho) = \frac{H}{\sqrt{2-H^2\rho^2}}, \tag{34}$$

$$\phi_t(\rho) = \frac{1}{2H}\log\left[\frac{1+H\rho}{1-H\rho}\right] - \rho. \tag{35}$$

- Bridge connecting the asymptotic regions at $z \to 0$ and $z \to \infty$.

- Stress-energy tensor of the CFT at $z = 0$:

$$\rho = -\langle T^{t}_{\ t}\rangle = -\frac{1}{16\pi G_3}\left(\frac{H^2}{1-H^2\rho^2} + H^2\right) \quad (36)$$

$$p = \langle T^{\rho}_{\ \rho}\rangle = -\frac{1}{16\pi G_3}\left(\frac{H^2}{1-H^2\rho^2} - H^2\right). \quad (37)$$

- The conformal anomaly is $\langle T^{\mu\,(CFT)}_{\ \mu}\rangle = H^2/(8\pi G_3)$.

## A different state

- The metric

$$ds^2 = \frac{1}{z^2}\left[dz^2 + \left(1 - \frac{1}{4}H^2z^2\right)^2 \left(-(1 - H^2\rho^2)d\eta^2 + \frac{d\rho^2}{1 - H^2\rho^2}\right)\right], \tag{38}$$

also has a de Sitter boundary.

- The stress-energy tensor is

$$\rho = -\langle T^t{}_t\rangle = -\frac{H^2}{16\pi G_3} \tag{39}$$

$$p = \langle T^\rho{}_\rho\rangle = \frac{H^2}{16\pi G_3}. \tag{40}$$

- The conformal anomaly is the same as before.

## BTZ black hole with FRW boundary

- The BTZ metric can be expressed as

$$ds^2 = \frac{1}{z^2}\left[dz^2 - \mathcal{N}^2(\tau,z)d\tau^2 + \mathcal{A}^2(\tau,z)d\phi^2\right], \tag{41}$$

with

$$\mathcal{A}(\tau,z) = a(\tau)\left(1 + \frac{\mu - \dot{a}^2(\tau)}{4a(\tau)^2}z^2\right) \tag{42}$$

$$\mathcal{N}(\tau,z) = 1 - \frac{\mu - \dot{a}^2 + 2a\ddot{a}}{4a^2}z^2 = \frac{\dot{\mathcal{A}}(\tau,z)}{\dot{a}}. \tag{43}$$

- $\phi$ is now periodic with period $2\pi$
- The boundary has the form

$$ds_0^2 = g^{(0)}_{\mu\nu}dx^\mu dx^\nu = -d\tau^2 + a^2(\tau)d\phi^2, \tag{44}$$

with $a(\tau)$ an arbitrary function.

- The coordinate transformation is

$$r(\tau, z) = \frac{\mathcal{A}(\tau, z)}{z} = \frac{a}{z} + \frac{\mu - \dot{a}^2}{4}\frac{z}{a}. \tag{45}$$

- The coordinates $(\tau, z)$ do not span the full BTZ geometry. They cover the two regions outside the event horizons, located at

$$z_{e1} = \frac{2a}{\sqrt{\mu} + \dot{a}}, \qquad z_{e2} = \frac{2a}{\sqrt{\mu} - \dot{a}}. \tag{46}$$

The quantities $z_{e1}$, $z_{e2}$ are the two roots of the equation $r(\tau, z) = r_e = \sqrt{\mu}$.

- The coordinates also cover part of the regions behind the horizons. For constant $\tau$, the minimal value of $r(\tau, z)$ is obtained for

$$z_m(\tau) = \frac{2a}{\sqrt{\mu - \dot{a}^2}}, \tag{47}$$

corresponding to

$$r_m(\tau) = \sqrt{\mu - \dot{a}^2}. \tag{48}$$

Clearly, $r_m \leq r_e$. Time-dependent throat.

- The transformation of the time coordinate for $z < z_{e1}$ or for $z > z_{e2}$ is

$$t(\tau, z) = \frac{\epsilon}{2\sqrt{\mu}} \log \left[ \frac{4a^2 - \left(\sqrt{\mu} + \dot{a}\right)^2 z^2}{4a^2 - \left(\sqrt{\mu} - \dot{a}\right)^2 z^2} \right] + \epsilon\, c(\tau), \qquad (49)$$

where the function $c(\tau)$ satisfies $\dot{c} = 1/a(\tau)$ and $\epsilon = \pm 1$.

- For $z_{e1} < z < z_{e2}$ the transformation is

$$t(\tau, z) = \frac{\epsilon}{2\sqrt{\mu}} \log \left[ \frac{-4a^2 + \left(\sqrt{\mu} + \dot{a}\right)^2 z^2}{4a^2 - \left(\sqrt{\mu} - \dot{a}\right)^2 z^2} \right] + \epsilon\, c(\tau). \qquad (50)$$

- The transformation is singular on the event horizons.

- The coordinate $\phi$ remains unaffected by the transformation. It is periodic, with periodicity $2\pi$.
- Dual picture: thermalized CFT on an expanding background, with a scale factor $a(\tau)$.
- Stress-energy tensor:

$$\rho = \frac{E}{V} = -\langle T^{\tau}_{\ \tau} \rangle = \frac{1}{16\pi G_3} \frac{\mu - \dot{a}^2}{a^2} \tag{51}$$

$$P = \langle T^{\phi}_{\ \phi} \rangle = \frac{1}{16\pi G_3} \frac{\mu - \dot{a}^2 + 2a\ddot{a}}{a^2}, \tag{52}$$

- Casimir energy $\sim \dot{a}^2/a^2$.
- Conformal anomaly:

$$\langle T^{\mu\,(CFT)}_{\mu} \rangle = \frac{1}{8\pi G_3} \frac{\ddot{a}}{a}. \tag{53}$$

- Conjecture: The entropy is proportional to the narrowest part of the throat or bridge.
- This line defines the boundary of the part of the bulk geometry that is not covered by the Fefferman-Graham parametrization. In a sense, it determines the part of the bulk that is not included in the construction of the dual theory.
- In quantitative terms:

$$S = \frac{1}{4G_3} A, \tag{54}$$

with $A$ the length of the narrowest part of the throat or bridge at a given time.

- BTZ black hole with a flat boundary:

$$A = 2\pi\sqrt{\mu}, \qquad S_{th} = \frac{\pi}{2G_3}\sqrt{\mu}. \tag{55}$$

## AdS$_3$ with Rindler boundary

- The throat is located at:

$$z_m(x) = 2x \tag{56}$$

in Fefferman-Graham coordinates.

- Equivalently, it is located at

$$r_m(x) = a, \qquad \phi_m(x) \equiv \phi_m(z_m(x), x) = (\log[ax] - 1)/a \tag{57}$$

in Poincare coordinates.

- The entropy is

$$S = \frac{1}{4G_3}\int_{-\infty}^{\infty} a\,d\phi = \frac{1}{4G_3}\int_0^{\infty}\frac{dx}{x} = \frac{1}{4G_3}\int_0^{\infty}\frac{dz}{z}. \tag{58}$$

- The infinities come from the endpoints, where the line approaches the boundary.
- Regulate !

$$S = \frac{2}{4G_3}\int_{\epsilon}\frac{dz}{z}, \tag{59}$$

- For the theory at $z = 0$, the regulated effective Newton's constant $G_2$ is

$$\frac{1}{G_2} = \frac{1}{G_3} \int_\epsilon \frac{dz}{z}. \tag{60}$$

- For $\epsilon \to 0$ we have $G_2 \to 0$.
- We obtain

$$S = \frac{2}{4G_2}. \tag{61}$$

- Our construction provides a holographic description of the Rindler wedge ($0 \le x < \infty$), with an identification of the limits $x \to 0$ and $x \to \infty$.
- The region near $x = \infty$ mimicks a horizon.
- The Rindler entropy is 1/2 of the above

$$S_R = \frac{1}{4G_2}. \tag{62}$$

## AdS$_3$ with de Sitter boundary

- The bridge approaches the boundary twice, so that

$$S = \frac{2}{4G_3} \int_{\epsilon} \frac{dz}{z}. \tag{63}$$

- This gives

$$S_{dS} = \frac{1}{2G_2}. \tag{64}$$

## BTZ black hole with time-dependent boundary

- The throat has $r_m(\tau) = \sqrt{\mu - \dot{a}^2}$, so that

$$S = \frac{\pi}{2G_3}\sqrt{\mu - \dot{a}^2}. \quad (65)$$

- The asymptotic symmetries of (2+1)-dimensional Einstein gravity with a negative cosmological constant correspond to a pair of Virasoro algebras, with central charges $c = \tilde{c} = 3/(2G_3)$ (Brown, Henneaux 1986). For the BTZ black hole, the eigenvalues $\Delta$, $\tilde{\Delta}$ of the generators $L_0$, $\tilde{L}_0$ are

$$\Delta = \tilde{\Delta} = \frac{\mu}{16G_3}. \quad (66)$$

- The Cardy formula gives

$$S = 2\pi\sqrt{\frac{c}{6}\left(\Delta - \frac{c}{24}\right)} + 2\pi\sqrt{\frac{\tilde{c}}{6}\left(\tilde{\Delta} - \frac{\tilde{c}}{24}\right)} = \frac{\pi}{2G_3}\sqrt{\mu - 1}. \quad (67)$$

For $\mu \gg 1$, it reproduces correctly the entropy of the thermalized CFT.

- The shift of the mass term by 1 is a result of the influence of the Casimir energy on the entropy.
- Our result gives a generalization of the Cardy formula for a time-dependent background, with Casimir energy $\sim \dot{a}^2/a^2$.

## Comments

- The de Sitter entropy was calculated through holographic means in:
  Hawking, Maldacena, Strominger 2001
  Iwashita, Kobayashi, Shiromizu, Yoshiho 2006
  The Randall-Sundrum construction was used.
- A general framework for the calculation of entanglement entropy for a flat boundary through holography was given in:
  Ryu,Takayanagi 2006 Hubeny,Rangamani,Takayanagi 2007

- Connection with minimal surfaces ?
- Entanglement entropy for a time-dependent boundary ?
- Higher dimensions ?
- $AdS_5$ with a static de Sitter boundary ?