Matter in Inhomogeneous Loop Quantum Cosmology: The Gowdy Model

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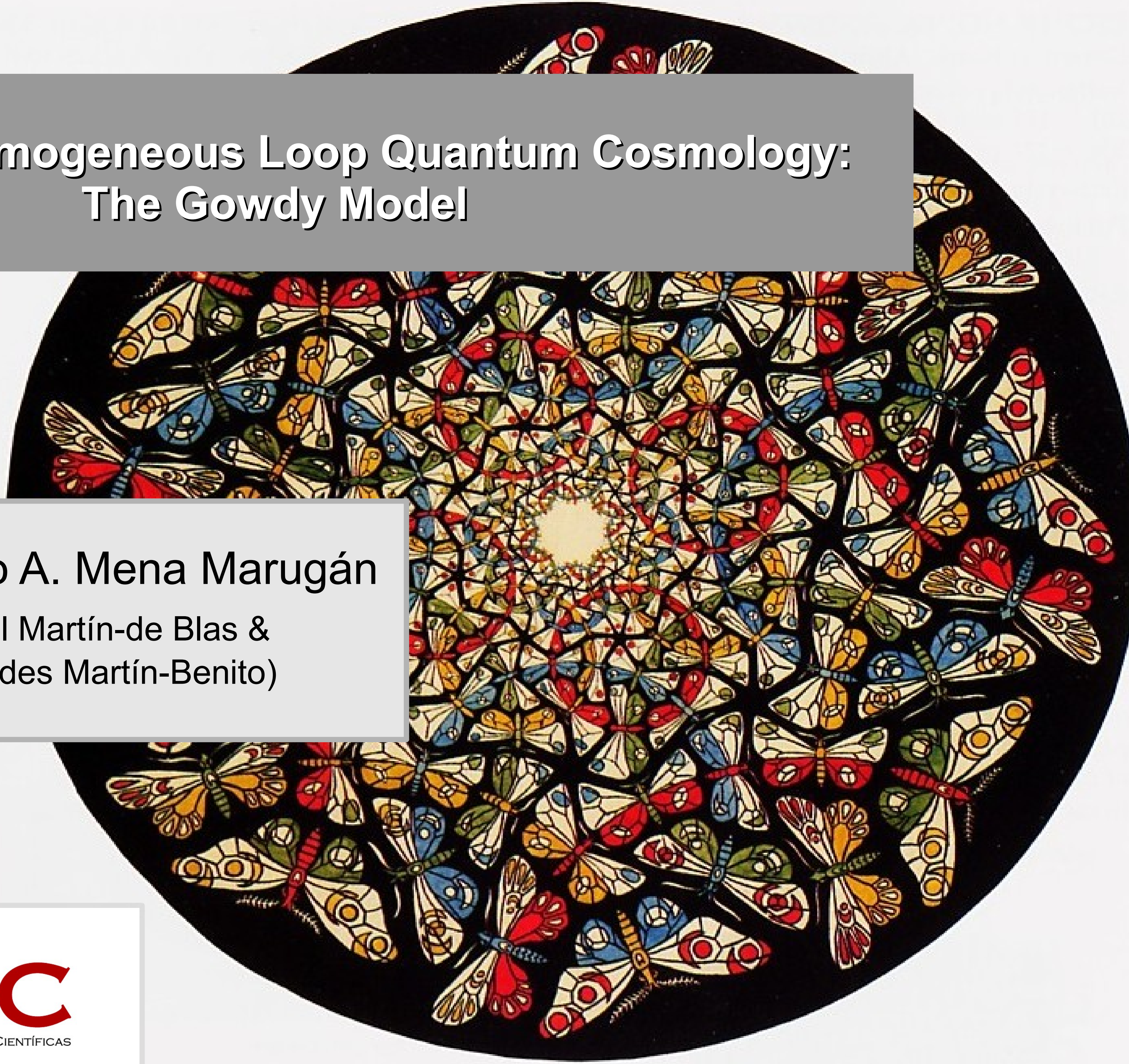


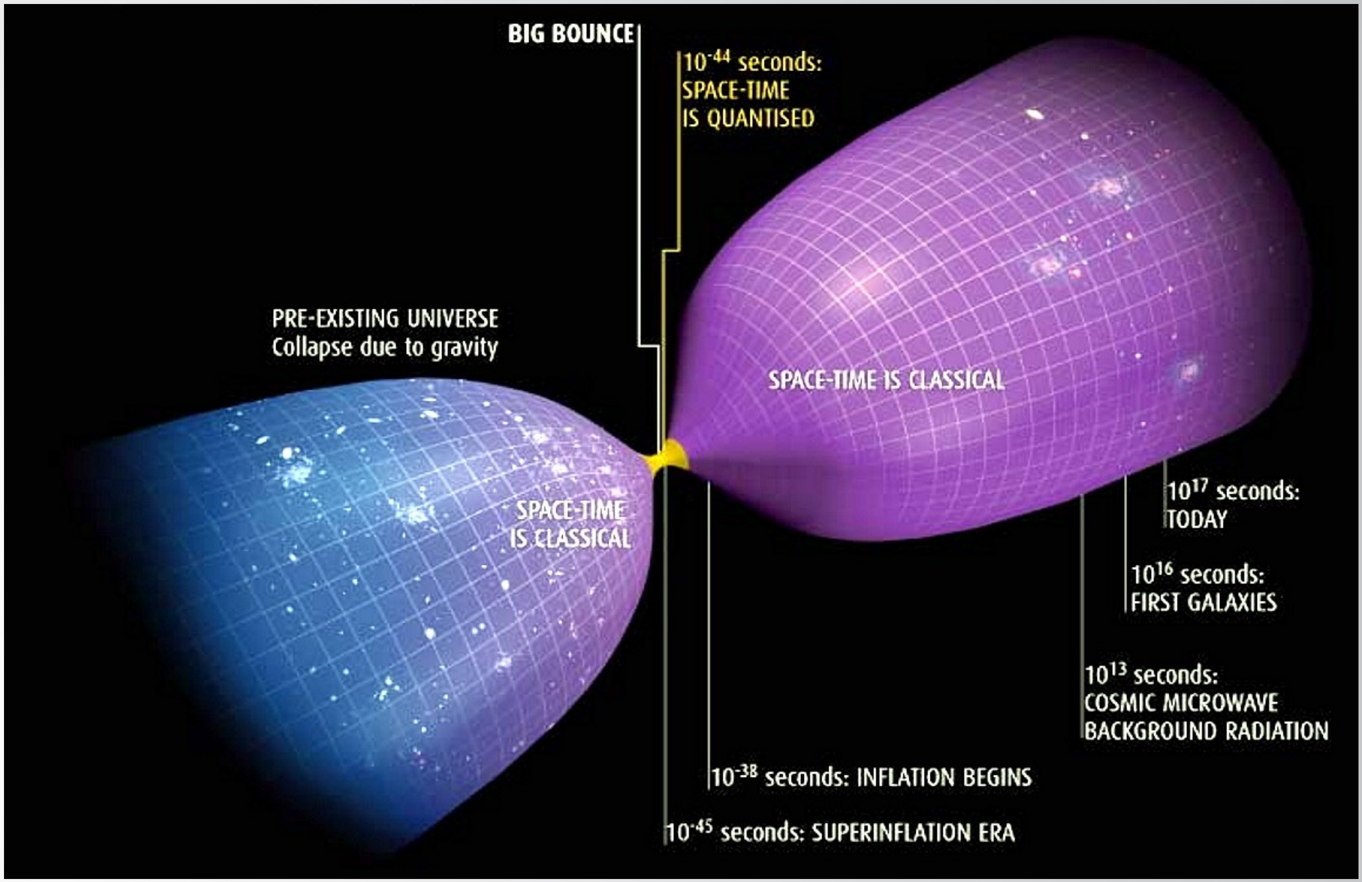
CSIC

Consejo Superior de Investigaciones Científicas

Introduction

- **Loop Quantum Cosmology** (LQC) is a quantum approach for cosmological systems inspired by the ideas and methods of Loop Quantum Gravity (LQG).

- It has been successfully applied mainly to homogeneous and isotropic (FRW) models, predicting e.g. the **Big Bounce** mechanism, which eludes the initial singularity.



Introduction

- To deal with more complicated systems, and test the robustness of the results, LQC has been applied also to **anisotropic** and to **inhomogeneous** models.

- The best studied inhomogeneous system is the **vacuum Gowdy mode**l with three-torus topology. A **hybrid** approach has been adopted, which combines techniques of LQC with a Fock quantization of the inhomogeneities.

- More realistic analyses call for the **introduction of matter**.

- In particular, the subfamily of homogeneous solutions in vacuo does not contain **FRW spacetimes**. Then, one cannot study the behavior around them quantum mechanically.

- The goal is to discuss the effect of LQC phenomena in the (matter and gravitational) inhomogeneities, and viceversa. Developing **perturbative approaches** seems important.

- With this aim, we are going to introduce a **massless scalar field** in the Gowdy model (with the same symmetries as those spacetimes) and study its hybrid quantization.

The Model

- We consider **linearly polarized** Gowdy T^3 cosmologies with a minimally coupled massles scalar field, Φ.

- These cosmologies are globally hyperbolic spacetimes with three-torus topology and two axial and hypersurface orthogonal **Killing vectors**, which are also matter symmetries.

- We can choose coordinates so that the metric depends only on a cyclic one, $\theta \in S^1$, and on time t.

- The **geometry** is described by three scale factors (a_i, one for each direction i) and a **field** without zero mode in its **Fourier expansion** in θ.

- This field describes linearly polarized gravitational waves, which propagate in a **Bianchi I** background determined by the scale factors a_i.

- The content of this background is the **zero mode** ϕ of the **matter** scalar field. In particular, there exist **FRW solutions**.

The Model: Variables

- We describe the **background geometry** with Ashtekar variables --densitized triads and su(2) connections-- as in **LQC**:

$$(\tilde{E}^{BI})_i^a = \frac{p_i}{4\pi^2}\delta_i^a, \quad (A^{BI})_a^i = \frac{c^i}{2\pi}\delta_a^i,$$

where we have used a diagonal gauge and (with γ being the Immirzi parameter):

$$p_i^2 = (2\pi)^4 a_j^2 a_k^2 \ (i \neq j \neq k), \qquad \{c^i, p_j\} = 8\pi G\gamma\delta_j^i.$$

- The **background matter** content is described with the canonical pair (ϕ, p_ϕ).

- Matter field (M) and gravitational waves (W): we rescale both fields, **without zero modes**, by a factor $\sqrt{|p_\theta|}/(2\pi)$ and expand in **Fourier modes**.

- We then define **creation and annihilation variables** as if they were ***massless free*** fields: $(a_m^{(A)*}, a_m^{(A)})$, $\quad (m \in \mathbb{Z} - \{0\}, \quad A = M, W)$.

The Model: Metric and Constraints

- The Gowdy **metric** can be written in the form:

$$ds^2 = \frac{|p_\theta p_\sigma p_\delta|}{4\pi^2}\left[e^{\Gamma[\xi, p_\theta]}\left(-\frac{N^2}{(2\pi)^4}dt^2 + \frac{d\theta^2}{p_\theta^2}\right) + e^{-\frac{2\pi\xi}{\sqrt{|p_\theta|}}}\frac{d\sigma^2}{p_\sigma^2} + e^{\frac{2\pi\xi}{\sqrt{|p_\theta|}}}\frac{d\delta^2}{p_\delta^2}\right].$$

where Γ is determined in terms of ξ and p_θ, and has **no zero mode**.

- Two constraints remain in this partially gauge-fixed model:

i) C_θ, which generates **rigid rotations** in θ.
This constraint gets contributions only from **inhomogeneous** (i.e., non-zero) modes.

ii) The zero mode of the densitized **Hamiltonian** constraint, C.
It has a **homogenous** part (due to zero modes) and an **inhomogeneous** contribution:

$$C = C_{hom} + C_{inh}.$$

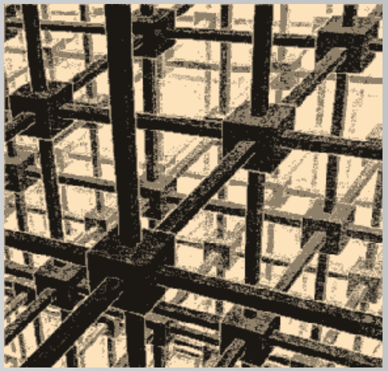
- The inhomogeneous parts of the constraints are the **sum** of two **identical contributions**, one for each field (matter and gravitational waves).

Hybrid Quantization. Loop Variables

- We adopt a **hybrid quantization**, assuming that the most relevant quantum geometry effects are those affecting the zero modes.

- For the background geometry, the elementary variables are p_i (**triads**) and **holonomies** of the connections, along edges of coordinate length $2\pi\mu_i$ for each direction.

The holonomy elements are linear combinations of $N_{\mu_j}(c_j)=e^{i c_j \mu_j/2}$.



- For each direction we consider the **basis of states** $\{|\mu_i\rangle, \quad \mu_i \in \mathbb{R}\}$.

- $$\hat{p}_i|\mu_i\rangle=4\pi\gamma G\hbar\mu_i|\mu_i\rangle, \quad \hat{N}_{\mu_i'}|\mu_i\rangle=|\mu_i+\mu_i'\rangle.$$

- The Hilbert space is the completion *wrt* the **discrete** product $\langle\mu_i|\mu_i'\rangle=\delta_{\mu_i\mu_i'}$.

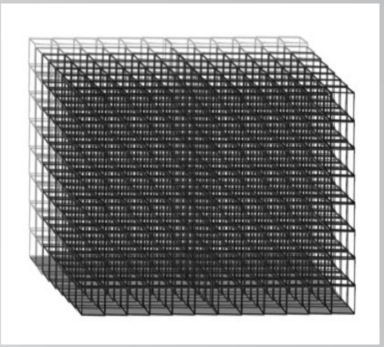
- The Bianchi I Hilbert space is the **tensor product** of those for the three directions.

- The **inverse volume** is regularized by Thiemann's trick (commutators with holonomies).
Curvature is defined in terms of holonomies along closed circuits.

Improved Dynamics & Fock Space

- The holonomy edges, of coordinate length $2\pi\bar{\mu}_i$, form rectangles whose **physical area** is set equal to the minimum area Δ allowed by the LQG spectrum (*improved dynamics*).

- It is then convenient to change the **labels** μ_i of the states to:



$$\left(v, \lambda_\theta, \chi = \lambda_\sigma / \lambda_\delta\right) \leftarrow \qquad \lambda_i^2 = \frac{(4\pi\gamma G\hbar)^{1/3}}{\Delta^{1/3}} |\mu_i|, \qquad v = 2\lambda_\theta\lambda_\sigma\lambda_\delta.$$

- $\hat{N}_{\bar{\mu}_i}$ produces a **unit shift** in v, but it also scales λ_i in a complicated v-dependent way.

- For the **zero mode of the matter** field we use a standard quantization.

- For the **inhomogeneities**, creation and annihilation variables become operators and we construct the **Fock space**.

- This Fock quantization is **privileged** (when the background is classical) under symmetry and unitarity requirements (*Cortez, Mena-Marugán, Velhinho*).

Quantum Constraints

- Global momentum constraint: $\hat{C}_{\theta}=\sum_{A}\sum_{m=1}^{\infty} m\,\hat{X}_{m}^{(A)},\qquad \hat{X}_{m}^{(A)}=\hat{a}_{m}^{(A)\dagger}\hat{a}_{m}^{(A)}-\hat{a}_{-m}^{(A)\dagger}\hat{a}_{-m}^{(A)}.$

- Global Hamiltonian constraint $\hat{C}=\hat{C}_{hom}+\sum_{A}\hat{C}_{inh}^{(A)}$ *(up to a global constant factor)*:

$$\hat{C}_{hom}=\frac{4\,\hat{p}_{\phi}^{2}}{\pi G\hbar^{2}}-\sum_{i,\,j\neq i}\frac{\hat{\Theta}_{i}\hat{\Theta}_{j}}{2\kappa^{2}},$$

$$\hat{C}_{inh}^{(A)}=\frac{16}{\beta}\widehat{|\lambda_{\theta}|^{2}}\hat{H}_{0}^{(A)}+\frac{\beta}{2\kappa^{2}}\widehat{\left[\frac{1}{\sqrt{|\lambda_{\theta}|}}\right]^{2}}\left(\hat{\Theta}_{\sigma}+\hat{\Theta}_{\delta}\right)^{2}\widehat{\left[\frac{1}{\sqrt{|\lambda_{\theta}|}}\right]^{2}}\hat{H}_{I}^{(A)}.$$

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$$\hat{C}_{inh}^{(A)} = \frac{16}{\beta} \widehat{|\lambda_\theta|^2} \hat{H}_0^{(A)} + \frac{\beta}{2\kappa^2} \left[\widehat{\frac{1}{\sqrt{|\lambda_\theta|}}}\right]^2 \left(\hat{\Theta}_\sigma + \hat{\Theta}_\delta\right)^2 \left[\widehat{\frac{1}{\sqrt{|\lambda_\theta|}}}\right]^2 \hat{H}_I^{(A)}.$$

Here, $\hat{p}_\phi = -i\hbar \partial_\phi, \quad \kappa = \pi \gamma G \hbar, \quad \beta = (4\kappa\sqrt{\Delta})^{2/3} / (G\hbar).$

$$\hat{\Theta}_j = i\kappa \widehat{\sqrt{|v|}} \left[\left(\hat{N}_{-\bar{\mu}_j} - \hat{N}_{\bar{\mu}_j}\right) \widehat{sign(\lambda_j)} + \widehat{sign(\lambda_j)} \left(\hat{N}_{-\bar{\mu}_j} - \hat{N}_{\bar{\mu}_j}\right)\right] \widehat{\sqrt{|v|}}$$

represents $c_j p_j$,

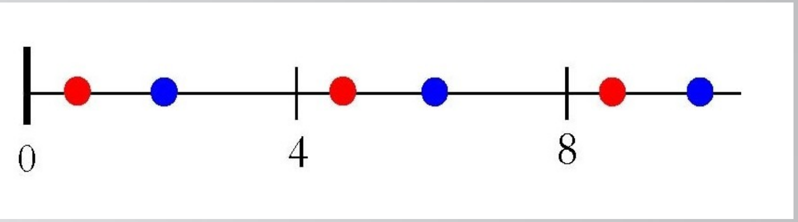
$$\hat{H}_0^{(A)} = \sum_{m=1}^{\infty} m \hat{N}_m^{(A)}, \quad \hat{H}_I^{(A)} = \sum_{m=1}^{\infty} \frac{1}{m} \left(\hat{N}_m^{(A)} + \hat{a}_m^{(A)\dagger} \hat{a}_{-m}^{(A)\dagger} + \hat{a}_m^{(A)} \hat{a}_{-m}^{(A)}\right), \quad \hat{N}_m^{(A)} = \hat{a}_m^{(A)\dagger} \hat{a}_m^{(A)} + \hat{a}_{-m}^{(A)\dagger} \hat{a}_{-m}^{(A)}.$$

Recall that the inverse of $\sqrt{|\lambda_\theta|}$ is regularized by Thiemann's trick.

Superselection. LRS Model

- **Superselection sectors:** Not all values of v and λ_i are related by the constraints.

- The v sectors are (e.g.) positive **semilattices** of **4 units** step. The minimum volume characterizes the sector: $\epsilon \in (0,4]$.



- The cosmological **singularities are resolved**: their quantum analogs are removed.

- The **physical** Hilbert space can be obtained from the data at the minimum volume.

- The λ_i sectors are **dense** (e.g.) in the positive semiaxis, and depend on ϵ. In particular, we will work with the **real label** $\Lambda_\theta = \ln(\lambda_\theta)$.

- The model is symmetric under the interchange of the directions σ and δ. For **simplicity**, we consider the **LRS submodel**. Quantum mechanically, this can be achieved by the **map:**

$$|\Psi(v, \Lambda_\theta, \lambda_\sigma/\lambda_\delta)\rangle \rightarrow |\psi(v, \Lambda_\theta)\rangle = \sum_{\lambda_\sigma/\lambda_\delta} |\Psi(v, \Lambda_\theta, \lambda_\sigma/\lambda_\delta)\rangle.$$

LRS Hamiltonian Constraint

- Let us define $\hat{\Theta}_{\Lambda}=\hat{\Theta}_{\theta}-\hat{\Theta}_{\sigma}$ and eliminate the coordinate subindices.

Then, the **LRS Hamiltonian constraint**, $\underline{\hat{C}}=\underline{\hat{C}}_{hom}+\sum_{A}\underline{\hat{C}}_{inh}^{(A)}$, is :

$$\underline{\hat{C}}_{hom}=\frac{4\hat{p}_{\phi}^{2}}{\pi G\hbar^{2}}-\frac{\left(3\hat{\Theta}^{2}+\hat{\Theta}_{\Lambda}\hat{\Theta}+\hat{\Theta}\hat{\Theta}_{\Lambda}\right)}{\kappa^{2}}=\hat{C}_{FRW}-\frac{\left(\hat{\Theta}_{\Lambda}\hat{\Theta}+\hat{\Theta}\hat{\Theta}_{\Lambda}\right)}{\kappa^{2}},$$

$$\underline{\hat{C}}_{inh}^{(A)}=\frac{16}{\beta}\widehat{e^{2\Lambda}}\hat{H}_{0}^{(A)}+\frac{2\beta}{\kappa^{2}}\widehat{e^{-2\Lambda}}\widehat{D_{v}}\hat{\Theta}^{2}\widehat{D_{v}}\hat{H}_{I}^{(A)}=\hat{C}_{inh}^{0\,(A)}+\hat{C}_{inh}^{I\,(A)},\qquad D_{v}=v^{2}\left(\sqrt{1+\frac{1}{v}}-\sqrt{\left|1-\frac{1}{v}\right|}\right)^{2}.$$

- $\hat{\Theta}$ and $\hat{\Lambda}$ commute.

- It can be interpreted as the constraint of a **FRW model** with the contributions of two massless free fields, and two types of corrections:

i) an **anisotropy** contribution, which **does not commute** with the FRW constraint,

ii) **interaction terms** between field modes.

Approximations

- The **spectrum** of the geometry operator for **FRW**, $\hat{\Theta}^2$, is absolutely continuous, positive and nondegenerate.

- Let $|e_\omega\rangle$ be the generalized **eigenstates**, where ω^2 is the eigenvalue $(\omega \geq 0)$.

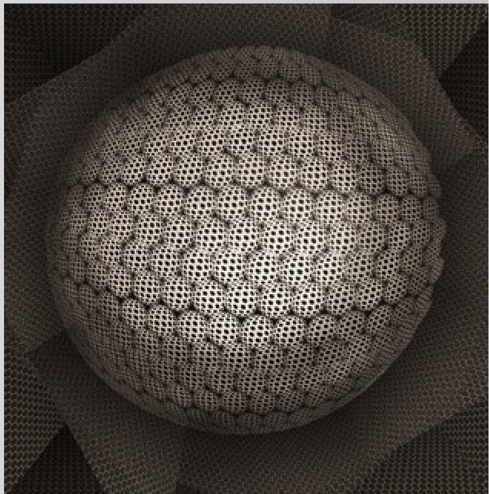
- The eigenfunctions are **exponentially suppressed** for small $\kappa v / \omega$.

- For $\omega \gg 8\kappa$, one can approximate them by their **Wheeler-DeWitt** (WDW) limit in the matrix elements of the anisotropy and interaction terms. This limit is **well known** (*Martín-Benito, Mena Marugán, Olmedo, Pawlowski*).

-Besides, when $\omega \gg 8\kappa$, the sums in those elements (coming from the discrete inner product) can be approximated by **integral expressions**.

► In particular, one can prove that $\widehat{D_v}\hat{\Theta}^2\widehat{D_v}|e_\omega\rangle \approx \omega^2|e_\omega\rangle.$

This was expected, since $D_v \approx 1$ for $v \gg 1$.
Actually, the diference is a state of **finite (kinematical) norm**.



Approximations: Born-Oppenheimer

- The **anisotropy** operator $\hat{\Theta}_\Lambda$ does not commute with $\hat{\Theta}$ in the **improved dynamics.**

Otherwise, the constraint (without inverse volume corrections) would act **diagonally** on $|e_\omega\rangle$.

- One can prove that the "**WDW limit**" of the operator $\hat{\Theta}_\Lambda \hat{\Theta} + \hat{\Theta}\hat{\Theta}_\Lambda$ is $-i 8\kappa |\hat{\Theta}| \partial_\Lambda$.

- This limit is a **good approximation** for states with $\omega \gg 8\kappa$ and which *do not vary much* on Λ regions of size $\ln(1+4\kappa/\omega)$.

► Proceeding as for the inverse volume corrections, one can then show:

$$\hat{\Theta}_\Lambda \hat{\Theta} + \hat{\Theta}\hat{\Theta}_\Lambda \approx 2\hat{P}_\Lambda^{(\Theta)} |\hat{\Theta}|, \qquad \hat{P}_\Lambda^{(\Theta)}|\Lambda\rangle = i\frac{8\kappa}{y_\epsilon^{(\Theta)}}\left(|\Lambda + y_\epsilon^{(\Theta)}\rangle - |\Lambda - y_\epsilon^{(\Theta)}\rangle\right)$$

$$y_\epsilon^{(\Theta)} = \ln\left(1 + \frac{2}{\epsilon + 2n_\epsilon^{(\Theta)}}\right), \qquad n_\epsilon^{(\Theta=\omega)} = max\left\{\left[\frac{\omega}{4\kappa} - 2\right]_{ent}, 0\right\}.$$

- $\hat{P}_\Lambda^{(\Theta)}$ and $|\hat{\Theta}|$ **COMMUTE**. This allows a true **Born-Oppenheimer** approximation.

Approximations: Interaction Terms

$$\hat{P}_{\Lambda}^{(\Theta)}|\Lambda\rangle = \frac{i 8\kappa}{y_{\epsilon}^{(\Theta)}}\left(\left|\Lambda + y_{\epsilon}^{(\Theta)}\right\rangle - \left|\Lambda - y_{\epsilon}^{(\Theta)}\right\rangle\right)$$

- The operator $\hat{P}_{\Lambda}^{(\Theta)}$ is well defined on the superselection sector for Λ, on sublattices of step $y_{\epsilon}^{(\Theta)}$.

- It has an absolutely continuous and doubly degenerated **spectrum**: the real line (the eigenvalue problem is a difference equation relating three points in the **sublattices**).

- For each real eigenvalue, $p_{\Lambda}^{(\Theta)}$, one can find a generalized eigenstate $\left|p_{\Lambda}^{+(\Theta)}\right\rangle$ which tends to **zero at minus infinity.**

- We call $\left|p_{\Lambda}^{-(\Theta)}\right\rangle$ the **orthogonal** eigenstate, and $H_{\Lambda}^{\pm}$ the spaces with bases $\left\{\left|p_{\Lambda}^{\pm(\Theta)}\right\rangle\right\}$ (for all sublattices).

- On H_{Λ}^{+}, the interactions of the inhomogeneous modes, proportional to $e^{-2\Lambda}$, should be *small*. In this sense,

$$\hat{C}_{FRW} - \frac{2}{\kappa^2}\hat{P}_{\Lambda}^{(\Theta)}|\hat{\Theta}| + \frac{16}{\beta}\widehat{e^{2\Lambda}}\hat{H}_0^{(A)} = \hat{C}_{pert}, \qquad \hat{C}_{pert} = \frac{\left(\hat{\Theta}_{\Lambda}\hat{\Theta} + \hat{\Theta}\hat{\Theta}_{\Lambda} - 2\hat{P}_{\Lambda}^{(\Theta)}|\hat{\Theta}|\right)}{\kappa^2} + \hat{C}_{inh}^{I\,(A)}.$$

Unperturbed Constraint

- We can regard $\hat{C}_{pert}$ as a "*perturbation*".

- Let us focus on the *unperturbed* constraint. This constraint is **solvable**.

Using the spectral decomposition associated with the FRW constraint and the n-particle states of the free field, we obtain a (family of) **one-dimensional problem**(s):

$$\hat{Q}_{\Lambda}^{(\omega,\{n\})}|\varphi(\Lambda)\rangle = \left(\frac{2\omega}{\kappa^2}\hat{P}_{\Lambda}^{(\Theta=\omega)} - \frac{16}{\beta}\widehat{e^{2\Lambda}}H_0(\{n\})\right)|\varphi(\Lambda)\rangle = \left(-\frac{3\omega^2}{\kappa^2} + \frac{4p_{\phi}^2}{\pi G\hbar^2}\right)|\varphi(\Lambda)\rangle.$$

We call $\delta_{FRW} = \left(-\frac{3\omega^2}{\kappa^2} + \frac{4p_{\phi}^2}{\pi G\hbar^2}\right)$.

- We expect $\hat{Q}_{\Lambda}^{(\omega,\{n\})}$ to have a doubly degenerated, absolutely continuous spectrum.

- $\hat{Q}_{\Lambda}^{(\omega,\{n\})}$ is **well defined** on H_{Λ}^{+}. Let $|q_{\Lambda}^{+(\omega,\{n\})}\rangle$ be the corresponding eigenstates, and $|q_{\Lambda}^{-(\omega,\{n\})}\rangle$ the orthogonal ones.

Solutions

- The **solutions** to the *unperturbed* constraint can be expressed in the form:

$$|\Xi\rangle = \int_{\mathbb{R}^+} d\omega \int_{\mathbb{R}} dp_\phi \sum_{\{n\}} \sum_{s=\pm} \Psi_s(\omega, p_\phi, \{n\}) \; |e_\omega\rangle \otimes |p_\phi\rangle \otimes |\{n\}\rangle \otimes |q_\Lambda^{s(\omega,\{n\})} = \delta_{FRW}\rangle .$$

- We are only interested in those for which $\hat{C}_{pert}$ should be a ***perturbation***:

▶ Small δ_{FRW} .

▶ $\omega \gg 8\kappa$.

▶ $s = +$.

- It is not difficult to provide the space of solutions with a Hilbert structure (e.g., by means of reality conditions) to obtain the space of **physical states**.

Conclusions

► We have **completed the quantization** of the linearly polarized Gowdy T^3 model with an inhomogeneous scalar field using hybrid techniques in LQC.

► The analogs of the cosmological singularities are eliminated quantum mechanically.

►We have **approximated** the **Hamiltonian constraint** by a **solvable** one and discussed in detail under which conditions the perturbations are expected to be small.

► We have found the **solutions** to this approximated constraint. They can be regarded as solutions of the Born-Oppenheimer type, constructed **in terms of FRW states**.

► Lines for future research:

- **Perturbative treatments** in interaction picture.

- Effects of the bounce in the anisotropy and the inhomogenities. **Numerical simulations**:

i) With a truncated number of modes.

ii) In the effective dynamics.