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Perturbations of AdS black holes

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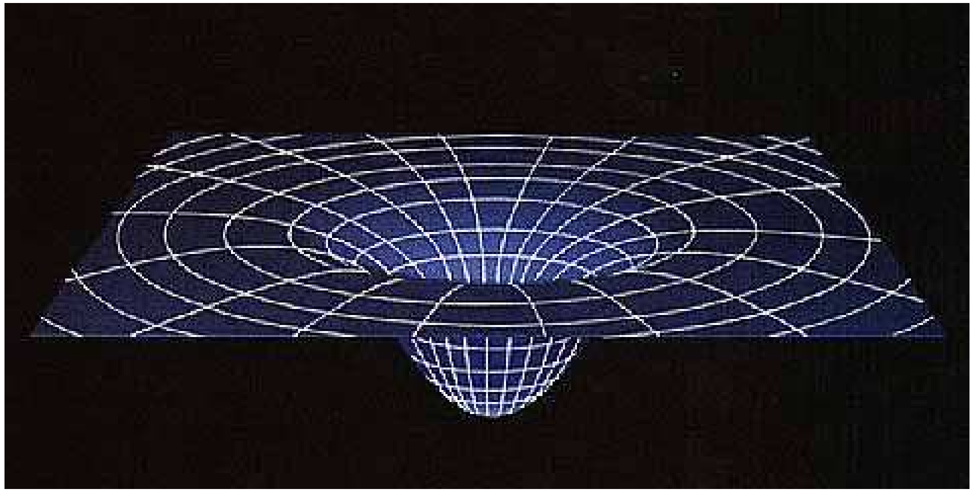


OUTLINE

- Perturbations
- Hydrodynamics
- Phase transitions
- Conclusions

Perturbations

Quasi-normal modes (QNMs) describe small perturbations of a black hole.

- A black hole is a thermodynamical system whose (Hawking) temperature and entropy are given in terms of its global characteristics (total mass, charge and angular momentum).

QNMs obtained by solving a wave equation for small fluctuations subject to the conditions that the flux be

- ingoing at the horizon and
- outgoing at asymptotic infinity.

$\Rightarrow$ discrete spectrum of complex frequencies.

- imaginary part determines the decay time of the small fluctuations

$$\Im\omega = \frac{1}{\tau}$$

AdS$_d$ Schwarzschild black holes

metric

$$ds^2 = -\left(\frac{r^2}{R^2} + K - \frac{2\mu}{r^{d-3}}\right) dt^2 + \frac{dr^2}{\frac{r^2}{R^2} + K - \frac{2\mu}{r^{d-3}}} + r^2 d\Sigma^2_{K,d-2}$$

choose units so that AdS radius $R = 1$.

horizon radius and Hawking temperature, respectively,

$$2\mu = r_+^{d-1}\left(1 + \frac{K}{r_+^2}\right) \ , \quad T_H = \frac{(d-1)r_+^2 + K(d-3)}{4\pi r_+}$$

mass and entropy, respectively,

$$M = (d-2)(K + r_+^2)\frac{r_+^{d-3}}{16\pi G} Vol(\Sigma_{K,d-2}) \ , \quad S = \frac{r_+^{d-2}}{4G} Vol(\Sigma_{K,d-2})$$

- $K = 0$: flat horizon $\mathbb{R}^{d-2}$
- $K = +1$: spherical horizon $\mathbb{S}^{d-2}$
- $K = -1$: hyperbolic horizon $\mathbb{H}^{d-2}/\Gamma$ (topological b.h.)
  Γ: discrete group of isometries

harmonics on $\Sigma_{K,d-2}$:

$$\left(\nabla^2 + k^2\right) \mathbb{T} = 0$$

- $K = 0$, k is momentum
- $K = +1$,

$$k^2 = l(l + d - 3) - \delta$$

- $K = -1$,

$$k^2 = \xi^2 + \left(\frac{d-3}{2}\right)^2 + \delta$$

  ξ is dicrete for non-trivial Γ

$\delta = 0, 1, 2$ for scalar, vector, or tensor perturbations, respectively.

AdS/CFT correspondence:

$\Rightarrow$ QNMs for AdS b.h. expected to correspond to perturbations of dual CFT.

establishment of correspondence hindered by difficulties in solving wave eq.

- In 3d: Hypergeometric equation $\therefore$ solvable

  [Cardoso, Lemos; Birmingham, Sachs, Solodukhin]

- In 5d: Heun equation $\therefore$ unsolvable.
- Numerical results in 4d, 5d and 7d

  [Horowitz, Hubeny; Starinets; Konoplya]

Asymptotic form of QNMs of AdS black holes

Approximation to the wave equation valid in the high frequency regime.

- In 3d: exact equation.
- In 5d: Heun eq. $\rightarrow$ Hypergeometric eq., as in low frequency regime.
  - analytical expression for asymptotic form of QNM frequencies
  - in agreement with numerical results.

AdS$_3$

wave equation

$$\frac{1}{R^2\, r}\partial_r\left(r^3\left(1-\frac{r_h^2}{r^2}\right)\partial_r\Phi\right)-\frac{R^2}{r^2-r_h^2}\partial_t^2\Phi+\frac{1}{r^2}\partial_x^2\Phi=m^2\Phi$$

Solution:

$$\Phi=e^{i(\omega t-px)}\Psi(y),\qquad y=\frac{r_h^2}{r^2}$$

where Ψ satisfies

$$y^2(y-1)\left((y-1)\Psi'\right)'+\hat{\omega}^2\, y\Psi+\hat{p}^2\, y(y-1)\Psi+\tfrac{1}{4}\hat{m}^2\,(y-1)\Psi=0$$

in the interval $0<y<1$, and

$$\hat{\omega}=\frac{\omega R^2}{2r_h}=\frac{\omega}{4\pi T_H},\quad \hat{p}=\frac{pR}{2r_h}=\frac{p}{4\pi R T_H}\,,\quad \hat{m}=mR$$

Two independent solutions obtained by examining the behavior near the horizon ($y \to 1$),

$$\Psi_\pm \sim (1-y)^{\pm i\widehat{\omega}}$$

Ψ_+ : outgoing; Ψ_- : ingoing.

Different set obtained by studying behavior at large r ($y \to 0$).

$$\Psi \sim y^{h_\pm} \quad , \quad h_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+\widehat{m}^2}$$

In massless case ($m = 0$): $h_+ = 1$ and $h_- = 0$

$\therefore$ one of the solutions contains logarithms.

For QNMs, we are interested in the analytic solution

$$\Psi(y) = y(1-y)^{i\widehat{\omega}}{}_2F_1(1+i(\widehat{\omega}+\widehat{p}), 1+i(\widehat{\omega}-\widehat{p}); 2; y)$$

Near the horizon ($y \to 1$): mixture of ingoing and outgoing waves
[$\because$ standard Hypergeometric function identities]

$$\Psi \sim A(1-y)^{-i\hat{\omega}} + B(1-y)^{i\hat{\omega}}$$

$$\begin{aligned} A &= \frac{\Gamma(2i\hat{\omega})}{\Gamma(1+i(\hat{\omega}+\hat{p}))\Gamma(1+i(\hat{\omega}-\hat{p}))} \\ B &= \frac{\Gamma(-2i\hat{\omega})}{\Gamma(1-i(\hat{\omega}+\hat{p}))\Gamma(1-i(\hat{\omega}-\hat{p}))} \end{aligned}$$

Ψ linear combination of Ψ_+ and Ψ_- $\therefore$

$$\Psi = A\Psi_- + B\Psi_+$$

For QNMs: Ψ purely ingoing at horizon, so set

$$B = 0$$

Solutions (QNM frequencies):

$$\boxed{\hat{\omega} = \pm\hat{p} - in} \quad , \quad n = 1, 2, \dots$$

discrete set of complex frequencies with $\Im\hat{\omega} < 0$.
NB: we obtained two sets of frequencies, with opposite $\Re\hat{\omega}$.

AdS$_5$

For a large black hole, scalar wave equation with $m = 0$

$$\frac{1}{r^3}\partial_r(r^5\, f(r)\,\partial_r\Phi) - \frac{R^4}{r^2\, f(r)}\partial_t^2\Phi - \frac{R^2}{r^2}\,\vec{\nabla}^2\Phi = 0$$

$$\hat{f}(r) = 1 - \frac{r_h^4}{r^4}$$

Solution:

$$\Phi = e^{i(\omega t - \vec{p}\cdot\vec{x})}\Psi(r)$$

change coordinate r to y,

$$y = \frac{r^2}{r_h^2}$$

Wave equation:

$$(y^2-1)\left(y(y^2-1)\Psi'\right)' + \left(\frac{\hat{\omega}^2}{4}y^2 - \frac{\hat{p}^2}{4}(y^2-1)\right)\Psi = 0$$

Two solutions by examining behavior near the horizon ($y \to 1$),

$$\Psi_\pm \sim (y-1)^{\pm i\hat{\omega}/4}$$

Different set by studying behavior at large r
($y \to \infty$)

$$\Psi \sim y^{h_\pm} \quad , \quad h_\pm = 0, -2$$

so one of the solutions contains logarithms.

For QNMs, we are interested in analytic solution

$$\Psi \sim y^{-2} \text{ as } y \to \infty$$

By considering the other (unphysical) singularity at $y = -1$,
$\Rightarrow$ another set of solutions

$$\Psi \sim (y+1)^{\pm\hat{\omega}/4} \text{ near } y = -1$$

Write wavefunction as

$$\Psi(y) = (y-1)^{-i\hat{\omega}/4}(y+1)^{\pm\hat{\omega}/4}F(y)$$

$\Rightarrow$ Two sets of modes with same $\Im\hat{\omega}$, but opposite $\Re\hat{\omega}$.

$F(y)$ satisfies the Heun equation

$$y(y^2-1)F'' + \left\{ \left(3 - \frac{i \pm 1}{2}\hat{\omega}\right) y^2 - \frac{i \pm 1}{2}\hat{\omega} y - 1 \right\} F'$$

$$+ \left\{ \frac{\hat{\omega}}{2}\left(\pm \frac{i\hat{\omega}}{4} \mp 1 - i\right) y - (i \mp 1)\frac{\hat{\omega}}{4} - \frac{\hat{p}^2}{4} \right\} F = 0$$

Solve in a region in the complex y-plane containing $|y| \geq 1$
(includes physical regime $r > r_h$)

For large $\hat{\omega}$: constant terms in Polynomial coefficients of F' and F small compared with other terms

$\therefore$ they may be dropped.

$\therefore$ wave eq. may be approximated by Hypergeometric equation

$$(y^2-1)F'' + \left\{ \left(3 - \frac{i \pm 1}{2}\hat{\omega}\right) y - \frac{i \pm 1}{2}\hat{\omega} \right\} F' + \frac{\hat{\omega}}{2}\left(\pm \frac{i\hat{\omega}}{4} \mp 1 - i\right) F = 0$$

in asymptotic limit of large frequencies $\hat{\omega}$.

Analytic solution:

$$F_0(x) = {}_2F_1(a_+, a_-; c; (y+1)/2) \ , \quad a_\pm = 1 - \tfrac{i \pm 1}{4}\hat{\omega} \pm 1 \quad , \quad c = \tfrac{3}{2} \pm \tfrac{1}{2}\hat{\omega}$$

For proper behavior at $y \to \infty$, demand that F be a *Polynomial*.

$\therefore$

$$a_+ = -n \ , \ \ n = 1, 2, \dots$$

$\therefore$ F is a Polynomial of order n, so as $y \to \infty$,

$$F \sim y^n \sim y^{-a_+}$$

$$\Psi \sim y^{-i\hat{\omega}/4} y^{\pm\hat{\omega}/4} y^{-a_+} \sim y^{-2}$$

as expected.

$\therefore$ QNM frequencies

$$\boxed{\hat{\omega} = \frac{\omega}{4\pi T_H} = 2n(\pm 1 - i)}$$

[Musiri, Siopsis]

in agreement with numerical results.

Monodromy argument

If the function has no singularities other than $y = \pm 1$, the contour around $y = +1$ may be unobstructedly deformed into the contour around $y = -1$,

$$\mathcal{M}(1)\mathcal{M}(-1) = 1$$

Since

$$\mathcal{M}(1) = e^{\pi\widehat{\omega}/2} \quad , \quad \mathcal{M}(-1) = e^{\mp i\pi\widehat{\omega}/2}$$

and using $\Im\widehat{\omega} < 0$, we deduce

$$\widehat{\omega} = 2n(\pm 1 - i)$$

same as before.

Gravitational perturbations

$\boxed{K = +1}$

▶ derive analytical expressions including first-order corrections.

▶ results in good agreement with results of numerical analysis.

radial wave equation

$$-\frac{d^2\Psi}{dr_*^2} + V[r(r_*)]\Psi = \omega^2\Psi \ ,$$

in terms of the tortoise coordinate defined by

$$\frac{dr_*}{dr} = \frac{1}{f(r)} \ .$$

potential V from Master Equation *[Ishibashi and Kodama]*

For tensor, vector and scalar perturbations, we obtain, respectively,

[Natário and Schiappa]

$$V_{\mathsf{T}}(r) = f(r)\left\{\frac{\ell(\ell+d-3)}{r^2} + \frac{(d-2)(d-4)f(r)}{4r^2} + \frac{(d-2)f'(r)}{2r}\right\}$$

$$V_{\mathsf{V}}(r) = f(r)\left\{\frac{\ell(\ell+d-3)}{r^2} + \frac{(d-2)(d-4)f(r)}{4r^2} - \frac{rf'''(r)}{2(d-3)}\right\}$$

$$\begin{aligned} V_{\mathsf{S}}(r) \;=\; & \frac{f(r)}{4r^2}\left[\ell(\ell+d-3)-(d-2)+\frac{(d-1)(d-2)\mu}{r^{d-3}}\right]^{-2} \\ \times\; & \left\{\frac{d(d-1)^2(d-2)^3\mu^2}{R^2r^{2d-8}} - \frac{6(d-1)(d-2)^2(d-4)[\ell(\ell+d-3)-(d-2)]\mu}{R^2r^{d-5}}\right. \\ & + \frac{(d-4)(d-6)[\ell(\ell+d-3)-(d-2)]^2r^2}{R^2} + \frac{2(d-1)^2(d-2)^4\mu^3}{r^{3d-9}} \\ & + \frac{4(d-1)(d-2)(2d^2-11d+18)[\ell(\ell+d-3)-(d-2)]\mu^2}{r^{2d-6}} \\ & + \frac{(d-1)^2(d-2)^2(d-4)(d-6)\mu^2}{r^{2d-6}} - \frac{6(d-2)(d-6)[\ell(\ell+d-3)-(d-2)]^2\mu}{r^{d-3}} \\ & - \frac{6(d-1)(d-2)^2(d-4)[\ell(\ell+d-3)-(d-2)]\mu}{r^{d-3}} \\ & \left. +4[\ell(\ell+d-3)-(d-2)]^3 + d(d-2)[\ell(\ell+d-3)-(d-2)]^2\right\} \end{aligned}$$

Near the black hole singularity ($r \sim 0$),

$$V_{\mathsf{T}} = -\frac{1}{4r_*^2} + \frac{\mathcal{A}_{\mathsf{T}}}{[-2(d-2)\mu]^{\frac{1}{d-2}}} r_*^{-\frac{d-1}{d-2}} + \dots \,, \quad \mathcal{A}_{\mathsf{T}} = \frac{(d-3)^2}{2(2d-5)} + \frac{\ell(\ell+d-3)}{d-2},$$

$$V_{\mathsf{V}} = \frac{3}{4r_*^2} + \frac{\mathcal{A}_{\mathsf{V}}}{[-2(d-2)\mu]^{\frac{1}{d-2}}} r_*^{-\frac{d-1}{d-2}} + \dots \,, \quad \mathcal{A}_{\mathsf{V}} = \frac{d^2-8d+13}{2(2d-15)} + \frac{\ell(\ell+d-3)}{d-2}$$

and

$$V_{\mathsf{S}} = -\frac{1}{4r_*^2} + \frac{\mathcal{A}_{\mathsf{S}}}{[-2(d-2)\mu]^{\frac{1}{d-2}}} r_*^{-\frac{d-1}{d-2}} + \dots \,,$$

where

$$\mathcal{A}_{\mathsf{S}} = \frac{(2d^3-24d^2+94d-116)}{4(2d-5)(d-2)} + \frac{(d^2-7d+14)[\ell(\ell+d-3)-(d-2)]}{(d-1)(d-2)^2}$$

We may summarize the behavior of the potential near the origin by

$$V = \frac{j^2-1}{4r_*^2} + \mathcal{A}\, r_*^{-\frac{d-1}{d-2}} + \dots$$

where $j = 0$ (2) for scalar and tensor (vector) perturbations.

for large r,

$$V = \frac{j_\infty^2 - 1}{4(r_* - \bar{r}_*)^2} + \dots \quad , \quad \bar{r}_* = \int_0^\infty \frac{dr}{f(r)}$$

where $j_\infty = d-1$, $d-3$ and $d-5$ for tensor, vector and scalar perturbations, respectively.

After rescaling the tortoise coordinate $(z = \omega r_*)$, wave equation

$$\left(\mathcal{H}_0 + \omega^{-\frac{d-3}{d-2}} \mathcal{H}_1 \right) \Psi = 0,$$

where

$$\mathcal{H}_0 = \frac{d^2}{dz^2} - \left[\frac{j^2 - 1}{4z^2} - 1 \right] \quad , \quad \mathcal{H}_1 = -\mathcal{A}\, z^{-\frac{d-1}{d-2}}.$$

By treating $\mathcal{H}_1$ as a perturbation, we may expand the wave function

$$\Psi(z) = \Psi_0(z) + \omega^{-\frac{d-3}{d-2}} \Psi_1(z) + \dots$$

and solve wave eq. perturbatively.

The zeroth-order wave equation,

$$\mathcal{H}_0 \Psi_0(z) = 0,$$

may be solved in terms of Bessel functions,

$$\Psi_0(z) = A_1 \sqrt{z}\, J_{\frac{j}{2}}(z) + A_2 \sqrt{z}\, N_{\frac{j}{2}}(z).$$

For large z, it behaves as

$$\begin{aligned} \Psi_0(z) &\sim \sqrt{\frac{2}{\pi}} \left[A_1 \cos(z - \alpha_+) + A_2 \sin(z - \alpha_+) \right], \\ &= \frac{1}{\sqrt{2\pi}} (A_1 - iA_2) e^{-i\alpha_+} e^{iz} + \frac{1}{\sqrt{2\pi}} (A_1 + iA_2) e^{+i\alpha_+} e^{-iz} \end{aligned}$$

where $\alpha_\pm = \frac{\pi}{4}(1 \pm j)$.

large z ($r \longrightarrow \infty$)

wavefunction ought to vanish $\therefore$ acceptable solution

$$\Psi(r_*) = B\sqrt{\omega(r_* - \bar{r}_*)}\; J_{\frac{j_\infty}{2}}(\omega(r_* - \bar{r}_*))$$

NB: $\Psi \longrightarrow 0$ as $r_* \longrightarrow \bar{r}_*$, as desired.

Asymptotically, it behaves as

$$\Psi(r_*) \sim \sqrt{\frac{2}{\pi}}\, B \cos\left[\omega(r_* - \bar{r}_*) + \beta\right] \;, \quad \beta = \frac{\pi}{4}(1 + j_\infty)$$

match this to asymptotic behavior in the vicinity of the black-hole singularity along the Stokes line $\Im z = \Im(\omega r_*) = 0$

$\Rightarrow$ constraint on the coefficients $A_1,\ A_2$,

$$A_1 \tan(\omega\bar{r}_* - \beta - \alpha_+) - A_2 = 0.$$

impose boundary condition at the horizon

$$\Psi(z) \sim e^{iz} \;, \quad z \to -\infty \;,$$

$\Rightarrow$ second constraint

analytically continue wavefunction near the origin to negative values of z.

- rotation of z by $-\pi$ corresponds to a rotation by $-\frac{\pi}{d-2}$ near the origin in the complex r-plane.

using

$$J_\nu(e^{-i\pi}z) = e^{-i\pi\nu}J_\nu(z)\ , \quad N_\nu(e^{-i\pi}z) = e^{i\pi\nu}N_\nu - 2i\cos\pi\nu\, J_\nu(z)$$

for $z < 0$, the wavefunction changes to

$$\Psi_0(z) = e^{-i\pi(j+1)/2}\sqrt{-z}\left\{\left[A_1 - i(1+e^{i\pi j})A_2\right] J_{\frac{j}{2}}(-z) + A_2 e^{i\pi j}\, N_{\frac{j}{2}}(-z)\right\}$$

whose asymptotic behavior is given by

$$\Psi \sim \frac{e^{-i\pi(j+1)/2}}{\sqrt{2\pi}}\left[A_1 - i(1+2e^{j\pi i})A_2\right] e^{-iz} + \frac{e^{-i\pi(j+1)/2}}{\sqrt{2\pi}}\left[A_1 - iA_2\right] e^{iz}$$

$\Rightarrow$ second constraint

$$A_1 - i(1+2e^{j\pi i})A_2 = 0$$

constraints compatible provided

$$\begin{vmatrix} 1 & -i(1+2e^{j\pi i}) \\ \tan(\omega\bar{r}_* - \beta - \alpha_+) & -1 \end{vmatrix} = 0$$

∴ quasi-normal frequencies

$$\boxed{\omega\bar{r}_* = \frac{\pi}{4}(2 + j + j_\infty) - \tan^{-1}\frac{i}{1 + 2e^{j\pi i}} + n\pi}$$

[Natário and Schiappa]

First-order corrections

[Musiri, Ness and Siopsis]

To first order, the wave equation becomes

$$\mathcal{H}_0\Psi_1 + \mathcal{H}_1\Psi_0 = 0$$

The solution is

$$\Psi_1(z) = \sqrt{z}\, N_{\frac{j}{2}}(z) \int_0^z dz' \frac{\sqrt{z'}\, J_{\frac{j}{2}}(z')\mathcal{H}_1\Psi_0(z')}{\mathcal{W}} - \sqrt{z}\, J_{\frac{j}{2}}(z) \int_0^z dz' \frac{\sqrt{z'}\, N_{\frac{j}{2}}(z')\mathcal{H}_1\Psi_0(z')}{\mathcal{W}}$$

$\mathcal{W} = 2/\pi$ is the Wronskian.

$\therefore$ wavefunction up to first order

$$\Psi(z) = \{A_1[1 - b(z)] - A_2 a_2(z)\}\sqrt{z} J_{\frac{j}{2}}(z) + \{A_2[1 + b(z)] + A_1 a_1(z)\}\sqrt{z} N_{\frac{j}{2}}(z)$$

where

$$\begin{aligned}
a_1(z) &= \frac{\pi\mathcal{A}}{2}\omega^{-\frac{d-3}{d-2}} \int_0^z dz'\, z'^{-\frac{1}{d-2}} J_{\frac{j}{2}}(z') J_{\frac{j}{2}}(z') \\
a_2(z) &= \frac{\pi\mathcal{A}}{2}\omega^{-\frac{d-3}{d-2}} \int_0^z dz'\, z'^{-\frac{1}{d-2}} N_{\frac{j}{2}}(z') N_{\frac{j}{2}}(z') \\
b(z) &= \frac{\pi\mathcal{A}}{2}\omega^{-\frac{d-3}{d-2}} \int_0^z dz'\, z'^{-\frac{1}{d-2}} J_{\frac{j}{2}}(z') N_{\frac{j}{2}}(z')
\end{aligned}$$

$\mathcal{A}$ depends on the type of perturbation.

asymptotically

$$\Psi(z) \sim \sqrt{\frac{2}{\pi}} [A'_1 \cos(z - \alpha_+) + A'_2 \sin(z - \alpha_+)] \ ,$$

where

$$A'_1 = [1 - \bar{b}]A_1 - \bar{a}_2 A_2 \ \ , \quad A'_2 = [1 + \bar{b}]A_2 + \bar{a}_1 A_1$$

and we introduced the notation

$$\bar{a}_1 = a_1(\infty) \ \ , \quad \bar{a}_2 = a_2(\infty) \ \ , \quad \bar{b} = b(\infty) \ .$$

First constraint modified to

$$A'_1 \tan(\omega \bar{r}_* - \beta - \alpha_+) - A'_2 = 0$$

$\therefore$

$$[(1-\bar{b}) \tan(\omega \bar{r}_* - \beta - \alpha_+) - \bar{a}_1] A_1 - [1 + \bar{b} + \bar{a}_2 \tan(\omega \bar{r}_* - \beta - \alpha_+)] A_2 = 0$$

For second constraint,

$\hookrightarrow$ approach the horizon

$\hookrightarrow$ rotate by $-\pi$ in the z-plane

$$
\begin{aligned}
a_1(e^{-i\pi}z) &= e^{-i\pi\frac{d-3}{d-2}} e^{-i\pi j} a_1(z) \,, \\
a_2(e^{-i\pi}z) &= e^{-i\pi\frac{d-3}{d-2}} \left[e^{i\pi j} a_2(z) - 4\cos^2\frac{\pi j}{2} a_1(z) - 2i(1+e^{i\pi j}) b(z) \right] \,, \\
b(e^{-i\pi}z) &= e^{-i\pi\frac{d-3}{d-2}} \left[b(z) - i(1+e^{-i\pi j}) a_1(z) \right]
\end{aligned}
$$

$\therefore$ in the limit $z \to -\infty$,

$$
\Psi(z) \sim -ie^{-ij\pi/2} B_1 \cos(-z-\alpha_+) - ie^{ij\pi/2} B_2 \sin(-z-\alpha_+)
$$

where

$$
\begin{aligned}
B_1 &= A_1 - A_1 e^{-i\pi\frac{d-3}{d-2}} [\bar{b} - i(1+e^{-i\pi j})\bar{a}_1] \\
&\quad - A_2 e^{-i\pi\frac{d-3}{d-2}} \left[e^{+i\pi j}\bar{a}_2 - 4\cos^2\frac{\pi j}{2}\bar{a}_1 - 2i(1+e^{+i\pi j})\bar{b} \right] \\
&\quad - i(1+e^{i\pi j}) \left[A_2 + A_2 e^{-i\pi\frac{d-3}{d-2}} [\bar{b} - i(1+e^{-i\pi j})\bar{a}_1] + A_1 e^{-i\pi\frac{d-3}{d-2}} e^{-i\pi j}\bar{a}_1 \right] \\
B_2 &= A_2 + A_2 e^{-i\pi\frac{d-3}{d-2}} [\bar{b} - i(1+e^{-i\pi j})\bar{a}_1] + A_1 e^{-i\pi\frac{d-3}{d-2}} e^{-i\pi j}\bar{a}_1
\end{aligned}
$$

$\therefore$ second constraint

$$
[1 - e^{-i\pi\frac{d-3}{d-2}}(i\bar{a}_1 + \bar{b})] A_1 - [i(1+2e^{i\pi j}) + e^{-i\pi\frac{d-3}{d-2}}((1+e^{i\pi j})\bar{a}_1 + e^{i\pi j}\bar{a}_2 - i\bar{b})] A_2 = 0
$$

compatibility of the two first-order constraints,

$$\begin{vmatrix} 1+\bar{b}+\bar{a}_2\tan(\omega\bar{r}_*-\beta-\alpha_+) & i(1+2e^{i\pi j})+e^{-i\pi\frac{d-3}{d-2}}((1+e^{i\pi j})\bar{a}_1+e^{i\pi j}\bar{a}_2-i\bar{b}) \\ (1-\bar{b})\tan(\omega\bar{r}_*-\beta-\alpha_+)-\bar{a}_1 & 1-e^{-i\pi\frac{d-3}{d-2}}(i\bar{a}_1+\bar{b}) \end{vmatrix}=0$$

$\Rightarrow$ first-order expression for quasi-normal frequencies,

$$\begin{aligned} \omega\bar{r}_* &= \frac{\pi}{4}(2+j+j_\infty)+\frac{1}{2i}\ln 2+n\pi \\ &\quad -\frac{1}{8}\left\{6i\bar{b}-2ie^{-i\pi\frac{d-3}{d-2}}\bar{b}-9\bar{a}_1+e^{-i\pi\frac{d-3}{d-2}}\bar{a}_1+\bar{a}_2-e^{-i\pi\frac{d-3}{d-2}}\bar{a}_2\right\} \end{aligned}$$

where

$$\begin{aligned} \bar{a}_1 &= \frac{\pi\mathcal{A}}{4}\left(\frac{n\pi}{2\bar{r}_*}\right)^{-\frac{d-3}{d-2}}\frac{\Gamma(\frac{1}{d-2})\Gamma(\frac{j}{2}+\frac{d-3}{2(d-2)})}{\Gamma^2(\frac{d-1}{2(d-2)})\Gamma(\frac{j}{2}+\frac{d-1}{2(d-2)})} \\ \bar{a}_2 &= \left[1+2\cot\frac{\pi(d-3)}{2(d-2)}\cot\frac{\pi}{2}\left(-j+\frac{d-3}{d-2}\right)\right]\bar{a}_1 \\ \bar{b} &= -\cot\frac{\pi(d-3)}{2(d-2)}\,\bar{a}_1 \end{aligned}$$

▶ first-order correction is $\sim O(n^{-\frac{d-3}{d-2}})$.

4d

compare with numerical results *[Cardoso, Konoplya and Lemos]*

set the AdS radius $R = 1$: radius of horizon r_H related to black hole mass μ by

$$2\mu = r_H^3 + r_H$$

$f(r)$ has two more (complex) roots, r_- and its complex conjugate, where

$$r_- = e^{i\pi/3}\left(\sqrt{\mu^2 + \frac{1}{27}} - \mu\right)^{1/3} - e^{-i\pi/3}\left(\sqrt{\mu^2 + \frac{1}{27}} + \mu\right)^{1/3}$$

The integration constant in the tortoise coordinate is

$$\bar{r}_* = \int_0^\infty \frac{dr}{f(r)} = -\frac{r_-}{3r_-^2 + 1} \ln \frac{r_-}{r_H} - \frac{r_-^*}{3r_-^{*2} + 1} \ln \frac{r_-^*}{r_H}$$

Scalar perturbations

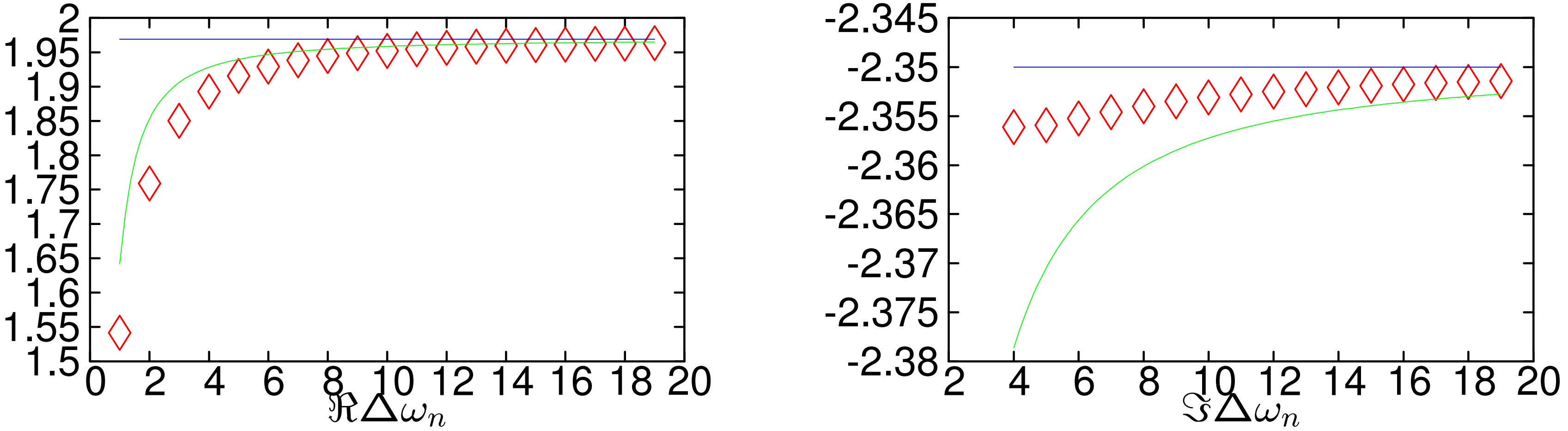


Fig. 1: $r_H = 1$ and $\ell = 2$: zeroth (horizontal line) and first order (curved line) analytical compared with numerical data (diamonds).

$$\omega_n \bar{r}_* = \left(n + \frac{1}{4}\right)\pi + \frac{i}{2}\ln 2 + e^{i\pi/4}\frac{\mathcal{A}_{\mathsf{S}}\Gamma^4(\frac{1}{4})}{16\pi^2}\sqrt{\frac{\bar{r}_*}{2\mu n}}\,, \quad \mathcal{A}_{\mathsf{S}} = \frac{\ell(\ell+1)-1}{6}$$

only the first-order correction is ℓ-dependent.

In the limit of large horizon radius ($r_H \approx (2\mu)^{1/3} \gg 1$),

$$\bar{r}_* \approx \frac{\pi(1+i\sqrt{3})}{3\sqrt{3}r_H}$$

Numerically for $\ell = 2$,

$$\frac{\omega_n}{r_H} = (1.299 - 2.250i)n + 0.573 - 0.419i + \frac{0.508 + 0.293i}{r_H^2\sqrt{n}}$$

which compares well with the result of numerical analysis,

$$\left(\frac{\omega_n}{r_H}\right)_{\text{numerical}} \approx (1.299 - 2.25i)n + 0.581 - 0.41i$$

including both leading order and offset.

For an intermediate black hole, $r_H = 1$, we obtain

$$\omega_n = (1.969 - 2.350i)n + 0.752 - 0.370i + \frac{0.654 + 0.458i}{\sqrt{n}}$$

In Fig. 1 we compare with data from numerical analysis. We plot the gap

$$\Delta\omega_n = \omega_n - \omega_{n-1}$$

because the offset does not always agree with numerical results.

► numerical estimates of the offset ought to be improved.

For a small black hole, $r_H = 0.2$, we obtain

$$\omega_n = (1.695 - 0.571i)n + 0.487 - 0.0441i + \frac{1.093 + 0.561i}{\sqrt{n}}$$

to be compared with the result of numerical analysis,

$$(\omega_n)_{\text{numerical}} \approx (1.61 - 0.6i)n + 2.7 - 0.37i$$

The two estimates of the offset disagree with each other.

Tensor perturbations

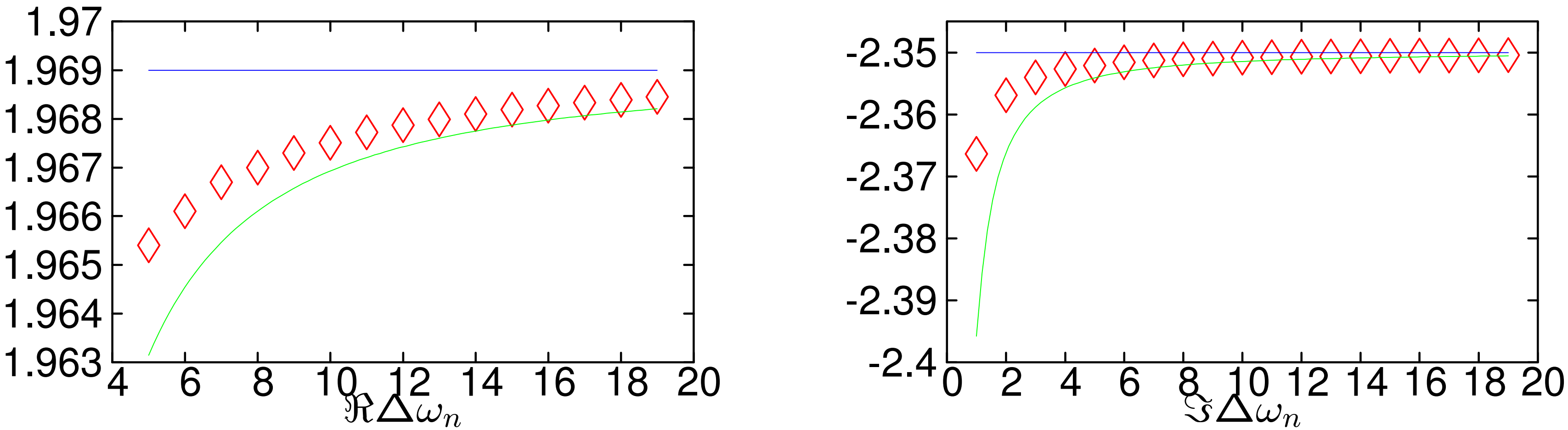


Fig. 2: $r_H = 1$ and $\ell = 0$: zeroth (horizontal line) and first order (curved line) analytical compared with numerical data (diamonds).

$$\omega_n \bar{r}_* = \left(n + \frac{1}{4}\right)\pi + \frac{i}{2}\ln 2 + e^{i\pi/4}\frac{\mathcal{A}_{\mathsf{T}}\Gamma^4(\frac{1}{4})}{16\pi^2}\sqrt{\frac{\bar{r}_*}{2\mu n}}\,, \quad \mathcal{A}_{\mathsf{T}} = \frac{3\ell(\ell+1)+1}{6}$$

Numerically for large r_H and $\ell = 0$,

$$\frac{\omega_n}{r_H} = (1.299 - 2.250i)n + 0.573 - 0.419i + \frac{0.102 + 0.0586i}{r_H^2\sqrt{n}}$$

For an intermediate black hole, $r_H = 1$, we obtain

$$\omega_n = (1.969 - 2.350i)n + 0.752 - 0.370i + \frac{0.131 + 0.0916i}{\sqrt{n}}$$

in good agreement with the result of numerical analysis (Fig. 2), including the offset.

For a small black hole, $r_H = 0.2$, we obtain

$$\omega_n = (1.695 - 0.571i)n + 2.182 - 0.615i + \frac{0.489 + 0.251i}{\sqrt{n}}$$

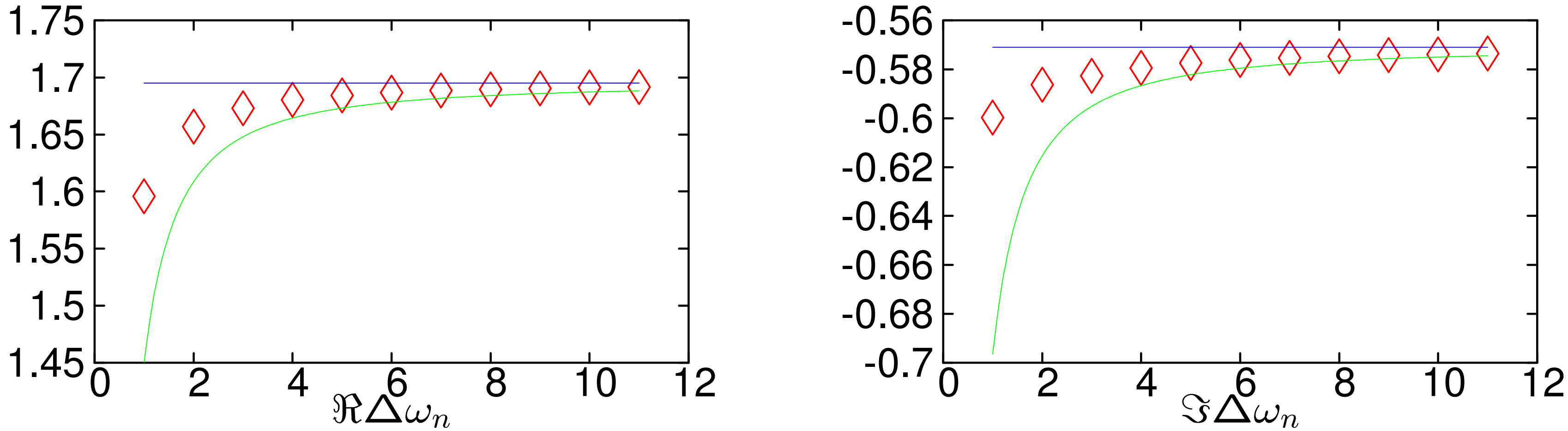


Fig. 3: $r_H = 0.2$ and $\ell = 0$: zeroth (horizontal line) and first order (curved line) analytical compared with numerical data (diamonds).

Vector perturbations

$$\omega_n \bar{r}_* = \left(n + \frac{1}{4}\right)\pi + \frac{i}{2}\ln 2 + e^{i\pi/4}\frac{\mathcal{A}_{\rm V}\Gamma^4(\frac{1}{4})}{48\pi^2}\sqrt{\frac{\bar{r}_*}{2\mu n}}\,, \quad \mathcal{A}_{\rm V} = \frac{\ell(\ell+1)}{2} + \frac{3}{14}$$

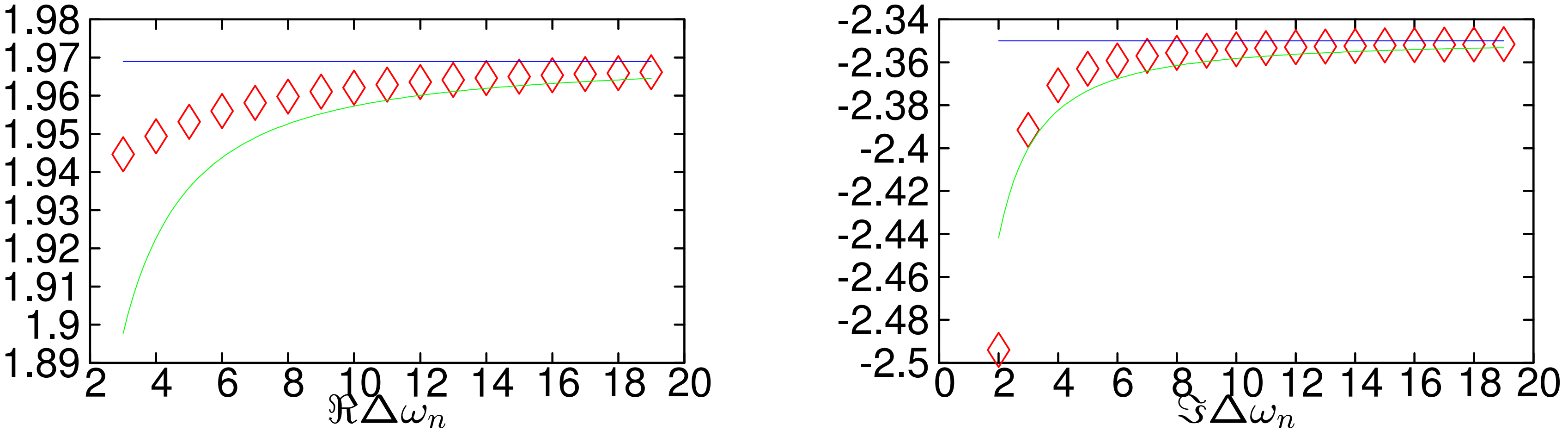


Fig. 4: $r_H = 1$ and $\ell = 2$: zeroth (horizontal line) and first order (curved line) analytical (eq. ()) compared with numerical data (diamonds).

Numerically for large r_H and $\ell = 2$,

$$\frac{\omega_n}{r_H} = (1.299 - 2.250i)n + 0.573 - 0.419i + \frac{8.19 + 6.29i}{r_H^2\sqrt{n}}$$

to be compared with the result of numerical analysis,

$$\left(\frac{\omega_n}{r_H}\right)_{\text{numerical}} \approx (1.299 - 2.25i)n + 0.58 - 0.42i$$

For an intermediate black hole, $r_H = 1$, we obtain

$$\omega_n = (1.969 - 2.350i)n + 0.752 - 0.370i + \frac{0.741 + 0.519i}{\sqrt{n}}$$

and for a small black hole, $r_H = 0.2$, we obtain

$$\omega_n = (1.695 - 0.571i)n + 0.487 - 0.0441i + \frac{1.239 + 0.6357i}{\sqrt{n}}$$

estimates of the offset agree for large r_H but diverge as $r_H \to 0$.

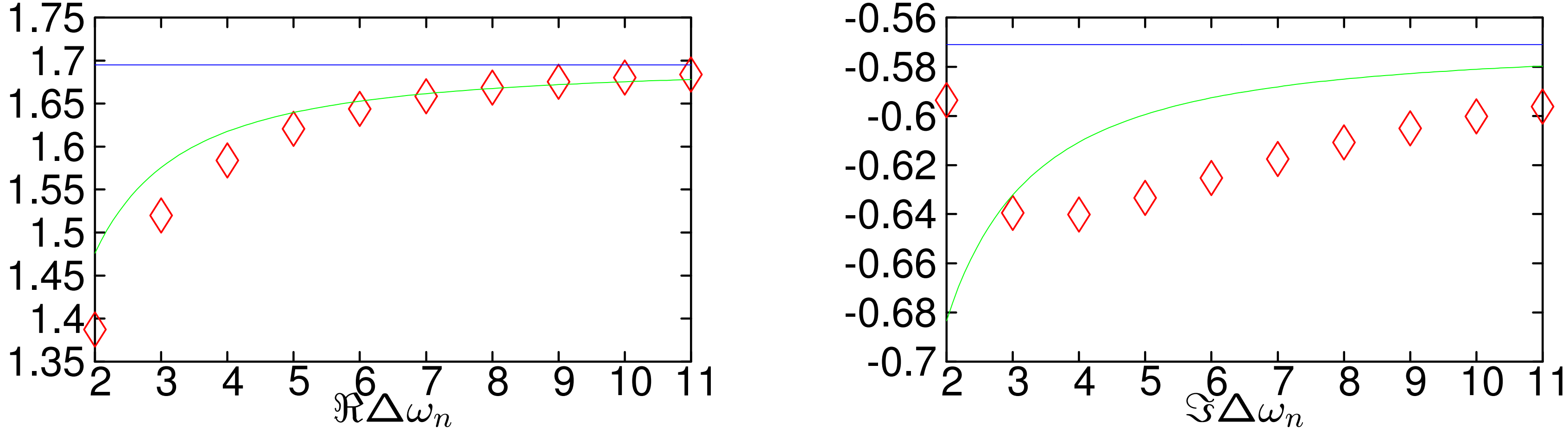


Fig. 5: $r_H = 0.2$ and $\ell = 2$: zeroth (horizontal line) and first order (curved line) analytical (eq. ()) compared with numerical data (diamonds).

Electromagnetic perturbations

electromagnetic potential

$$V_{\text{EM}} = \frac{\ell(\ell+1)}{r^2} f(r).$$

Near the origin,

$$V_{\text{EM}} = \frac{j^2 - 1}{4r_*^2} + \frac{\ell(\ell+1) r_*^{-3/2}}{2\sqrt{-4\mu}} + \dots \, ,$$

where $j = 1$ - vanishing potential to zeroth order!

► need to include first-order corrections for QNMs.

QNMs

$$\omega \bar{r}_* = n\pi - \frac{i}{4} \ln n + \frac{1}{2i} \ln \left(2(1+i)\mathcal{A}\sqrt{\bar{r}_*} \right) \, , \quad \mathcal{A} = \frac{\ell(\ell+1)}{2\sqrt{-4\mu}}$$

► correction behaves as $\ln n$.

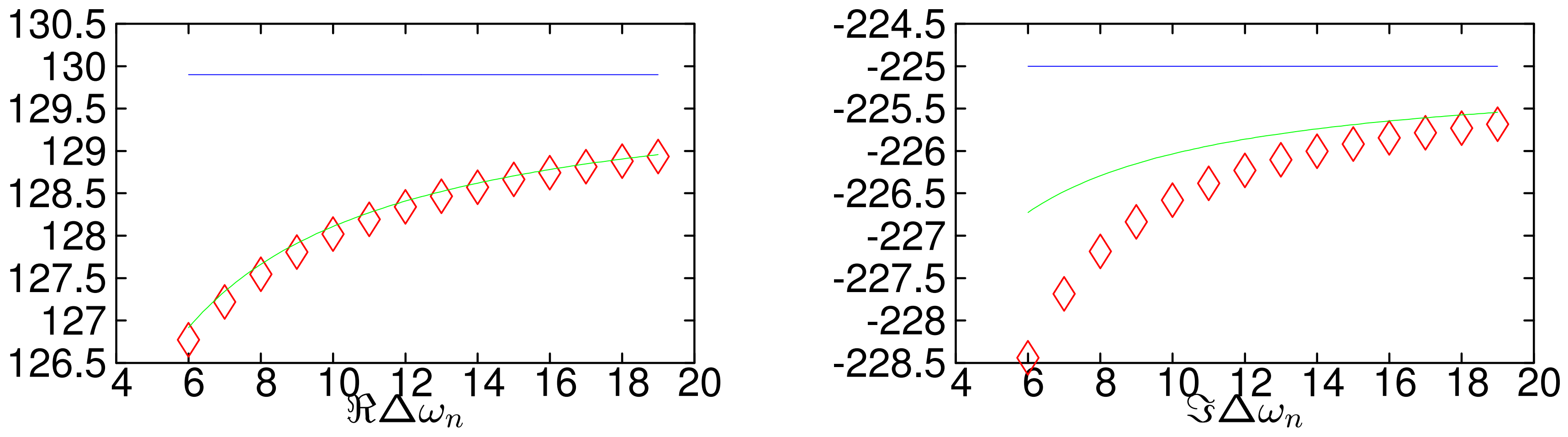
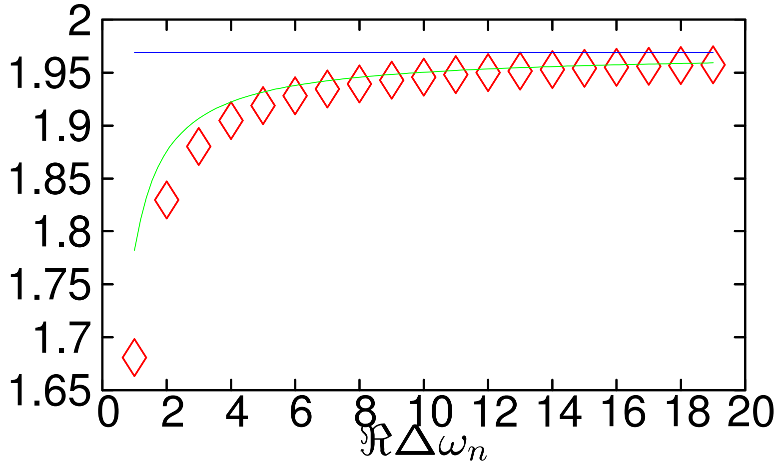


Fig. 6: $r_H = 100$ and $\ell = 1$: zeroth (horizontal line) and first order (curved line) analytical compared with numerical data (diamonds).

For a large black hole, we obtain the spectrum

$$\frac{\Delta\omega_n}{r_H} \approx \frac{3\sqrt{3}(1 - i\sqrt{3})}{4}\left(1 - \frac{i}{4\pi n} + \dots\right) = 1.299 - 2.25i - \frac{0.179 + 0.103i}{n} + \dots$$



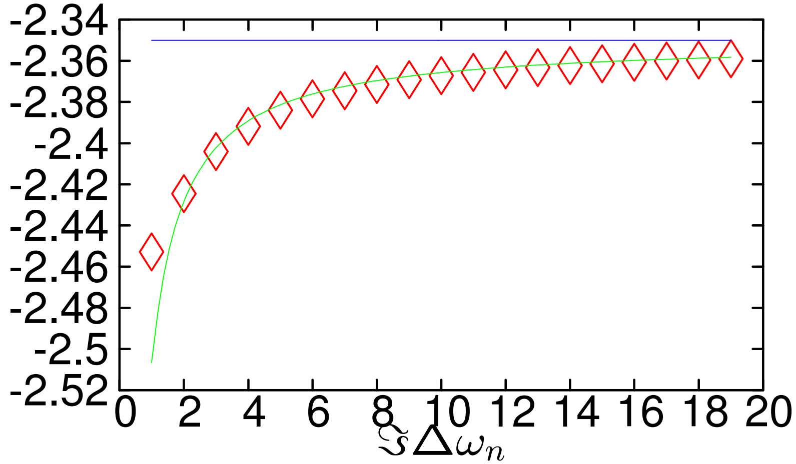


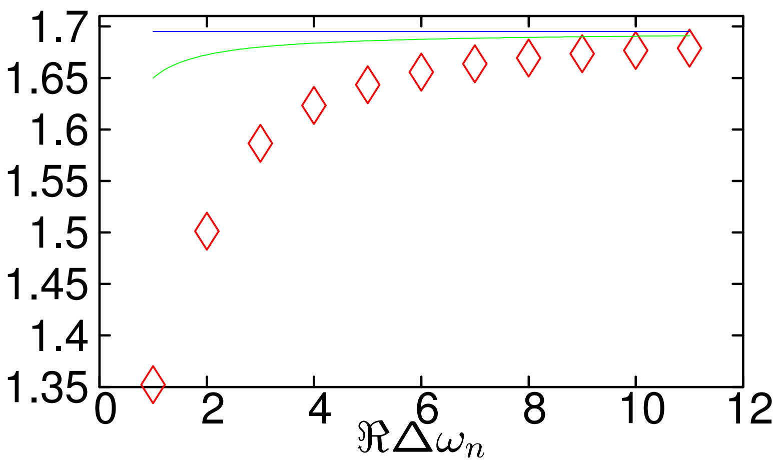
Fig. 7: $r_H = 1$ and $\ell = 1$: zeroth (horizontal line) and first order (curved line) analytical compared with numerical data (diamonds).

For an intermediate black hole, $r_H = 1$,

$$\omega_n = (1.969 - 2.350i)n - (0.187 + 0.1567i)\ln n + \dots$$

and for a small black hole, $r_H = 0.2$,

$$\omega_n = (1.695 - 0.571i)n - (0.045 + 0.135i)\ln n + \dots$$



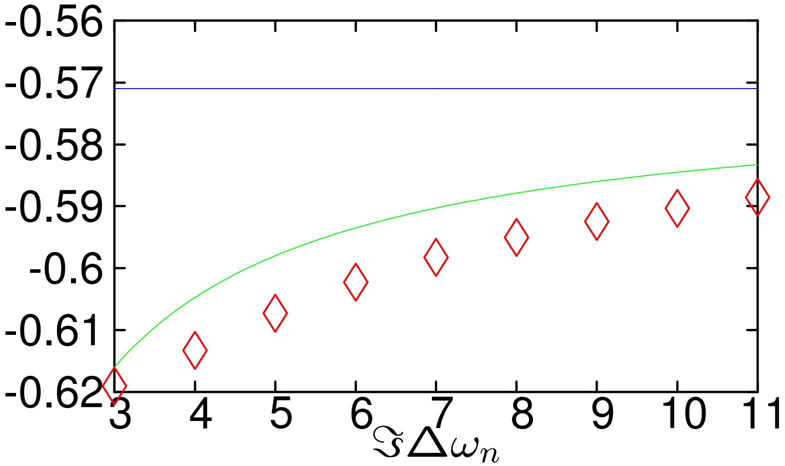


Fig. 8: $r_H = 0.2$ and $\ell = 1$: zeroth (horizontal line) and first order (curved line) analytical compared with numerical data (diamonds).

All first-order analytical results are in good agreement with numerical results.

Hydrodynamics

AdS/CFT correspondence and hydrodynamics

[Policastro, Son and Starinets]

correspondence between $\mathcal{N} = 4$ SYM in the large N limit and type-IIB string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$.

- in strong coupling limit of field theory, string theory is reduced to classical supergravity, which allows one to calculate all field-theory correlation functions.
  $\hookrightarrow$ nontrivial prediction of gauge theory/gravity correspondence

entropy of $\mathcal{N} = 4$ SYM theory in the limit of large 't Hooft coupling is precisely 3/4 the value in zero coupling limit.

long-distance, low-frequency behavior of any interacting theory at finite temperature must be described by fluid mechanics (hydrodynamics).

universality: hydrodynamics implies very precise constraints on correlation functions of conserved currents and stress-energy tensor:

- correlators fixed once a few transport coefficients are known.

Vector perturbations

[G. S., hep-th/0702079]

introduce the coordinate

$$u = \left(\frac{r_H}{r}\right)^{d-3}$$

wave equation

$$-(d-3)^2 u^{\frac{d-4}{d-3}} \hat{f}(u) \left(u^{\frac{d-4}{d-3}} \hat{f}(u) \Psi' \right)' + \hat{V}_{\mathsf{V}}(u)\Psi = \hat{\omega}^2 \Psi \ , \quad \hat{\omega} = \frac{\omega}{r_H}$$

where prime denotes differentiation with respect to u and

$$\hat{f}(u) \equiv \frac{f(r)}{r^2} = 1 - u^{\frac{2}{d-3}} \left(u - \frac{1-u}{r_H^2} \right)$$

$$\hat{V}_{\mathsf{V}}(u) \equiv \frac{V_{\mathsf{V}}}{r_H^2} = \hat{f}(u) \left\{ \hat{L}^2 + \frac{(d-2)(d-4)}{4} u^{-\frac{2}{d-3}} \hat{f}(u) - \frac{(d-1)(d-2)\left(1+\frac{1}{r_H^2}\right)}{2} u \right\}$$

where $\hat{L}^2 = \frac{\ell(\ell+d-3)}{r_H^2}$

First consider large black hole limit $r_H \to \infty$ keeping $\hat{\omega}$ and $\hat{L}$ fixed (small).

Factoring out the behavior at the horizon ($u = 1$)

$$\Psi = (1-u)^{-i\frac{\widehat{\omega}}{d-1}} F(u)$$

the wave equation simplifies to

$$\mathcal{A}F'' + \mathcal{B}_{\widehat{\omega}}F' + \mathcal{C}_{\widehat{\omega},\hat{L}}F = 0$$

where

$$\begin{aligned}
\mathcal{A} &= -(d-3)^2 u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}}) \\
\mathcal{B}_{\widehat{\omega}} &= -(d-3)[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}} - 2(d-3)^2\frac{i\widehat{\omega}}{d-1}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{1-u} \\
\mathcal{C}_{\widehat{\omega},\hat{L}} &= \hat{L}^2 + \frac{(d-2)[d-4-3(d-2)u^{\frac{d-1}{d-3}}]}{4}u^{-\frac{2}{d-3}} \\
&\quad -\frac{\widehat{\omega}^2}{1-u^{\frac{d-1}{d-3}}} + (d-3)^2\frac{\widehat{\omega}^2}{(d-1)^2}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2} \\
&\quad -(d-3)\frac{i\widehat{\omega}}{d-1}\frac{[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}}}{1-u} - (d-3)^2\frac{i\widehat{\omega}}{d-1}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2}
\end{aligned}$$

solve perturbatively:

$$(\mathcal{H}_0 + \mathcal{H}_1)F = 0$$

where

$$\begin{aligned} \mathcal{H}_0 F &\equiv \mathcal{A}F'' + \mathcal{B}_0 F' + \mathcal{C}_{0,0} F \\ \mathcal{H}_1 F &\equiv (\mathcal{B}_{\hat{\omega}} - \mathcal{B}_0)F' + (\mathcal{C}_{\hat{\omega},\hat{L}} - \mathcal{C}_{0,0})F \end{aligned}$$

Expanding the wavefunction perturbatively,

$$F = F_0 + F_1 + \dots$$

at zeroth order we have

$$\mathcal{H}_0 F_0 = 0$$

whose acceptable solution is

$$F_0 = u^{\frac{d-2}{2(d-3)}}$$

regular at horizon ($u = 1$) and boundary ($u = 0$, or $\Psi \sim r^{-\frac{d-2}{2}} \to 0$ as $r \to \infty$).

Wronskian

$$\mathcal{W} = \frac{1}{u^{\frac{d-4}{d-3}}(1 - u^{\frac{d-1}{d-3}})}$$

Another linearly independent solution

$$\check{F}_0 = F_0 \int \frac{\mathcal{W}}{F_0^2}$$

unacceptable $\because$ diverges at both horizon ($\check{F}_0 \sim \ln(1-u)$ for $u \approx 1$) and boundary ($\check{F}_0 \sim u^{-\frac{d-4}{2(d-3)}}$ for $u \approx 0$, or $\Psi \sim r^{\frac{d-4}{2}} \to \infty$ as $r \to \infty$).

At first order we have

$$\mathcal{H}_0 F_1 = -\mathcal{H}_1 F_0$$

whose solution may be written as

$$F_1 = F_0 \int \frac{\mathcal{W}}{F_0^2} \int \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A}\mathcal{W}}$$

The limits of the inner integral may be adjusted at will

$\because$ this amounts to adding an arbitrary amount of the unacceptable solution.

To ensure regularity at the horizon, choose one of the limits at $u = 1$

- ▶ integrand is regular at the horizon, by design.

at the boundary ($u = 0$),

$$F_1 = \breve{F}_0 \int_0^1 \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A}\mathcal{W}} + \text{regular terms}$$

The coefficient of the singularity ought to vanish,

$$\int_0^1 \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A}\mathcal{W}} = 0$$

$\Rightarrow$ constraint on the parameters (dispersion relation)

$$\mathbf{a}_0 \hat{L}^2 - i\mathbf{a}_1 \hat{\omega} - \mathbf{a}_2 \hat{\omega}^2 = 0$$

After some algebra, we arrive at

$$\mathbf{a}_0 = \frac{d-3}{d-1} \ , \quad \mathbf{a}_1 = d - 3$$

The coefficient $\mathbf{a}_2$

- may also be found explicitly for each dimension d,

- it cannot be written as a function of d in closed form.
- it does not contribute to the dispersion relation at lowest order.
- E.g., for $d = 4, 5$, we obtain, respectively

$$\mathbf{a}_2 = \frac{65}{108} - \frac{1}{3}\ln 3 \ , \quad \frac{5}{6} - \frac{1}{2}\ln 2$$

quadratic in $\hat{\omega}$ eq. has two solutions,

$$\hat{\omega}_0 \approx -i\frac{\hat{L}^2}{d-1} \ , \quad \hat{\omega}_1 \approx -i\frac{d-3}{\mathbf{a}_2} + i\frac{\hat{L}^2}{d-1}$$

In terms of frequency ω and quantum number ℓ,

$$\omega_0 \approx -i\frac{\ell(\ell+d-3)}{(d-1)r_H} \ , \quad \frac{\omega_1}{r_H} \approx -i\frac{d-3}{\mathbf{a}_2} + i\frac{\ell(\ell+d-3)}{(d-1)r_H^2}$$

The smaller of the two, ω_0,

- is inversely proportional to the radius of the horizon,
- is not included in the asymptotic spectrum.

The other solution, ω_1,

- is a crude estimate of the first overtone in the asymptotic spectrum.
- shares important features with asymptotic spectrum:
  - it is proportional to r_H
  - dependence on ℓ is $O(1/r_H^2)$.

The approximation may be improved by including higher-order terms

- ▶ Inclusion of higher orders also increases the degree of the polynomial in the dispersion relation whose roots then yield approximate values of more QNMs.
- ▶ this method reproduces the asymptotic spectrum albeit not in an efficient way.

Include finite size effects:

$\hookrightarrow$ use perturbation (assuming $1/r_H$ is small) and replace $\mathcal{H}_1$ by

$$\mathcal{H}_1' = \mathcal{H}_1 + \frac{1}{r_H^2}\mathcal{H}_H$$

where

$$\mathcal{H}_H F \equiv \mathcal{A}_H F'' + \mathcal{B}_H F' + \mathcal{C}_H F$$

$$\begin{aligned}
\mathcal{A}_H &= -2(d-3)^2 u^2 (1-u) \\
\mathcal{B}_H &= -(d-3)u \left[(d-3)(2-3u) - (d-1) \frac{1-u}{1-u^{\frac{d-1}{d-3}}} u^{\frac{d-1}{d-3}} \right] \\
\mathcal{C}_H &= \frac{d-2}{2} \left[d-4-(2d-5)u-(d-1) \frac{1-u}{1-u^{\frac{d-1}{d-3}}} u^{\frac{d-1}{d-3}} \right]
\end{aligned}$$

Interestingly, zeroth order wavefunction F_0 is eigenfunction of $\mathcal{H}_H$,

$$\mathcal{H}_H F_0 = -(d-2) F_0$$

$\therefore$ first-order finite-size effect is simple shift of angular momentum

$$\hat{L}^2 \to \hat{L}^2 - \frac{d-2}{r_H^2}$$

$\therefore$ QNMs of lowest frequency are modified to

$$\omega_0 = -i \frac{\ell(\ell+d-3)-(d-2)}{(d-1) r_H} + O(1/r_H^2)$$

For $d = 4, 5$, we have respectively,

$$\omega_0 = -i\frac{(\ell-1)(\ell+2)}{3r_H} \ , \quad -i\frac{(\ell+1)^2-4}{4r_H}$$

in agreement with numerical results

[Cardoso, Konoplya and Lemos; Friess, Gubser, Michalogiorgakis and Pufu]

$\Rightarrow$ maximum lifetime

$$\tau_{\max} = \frac{4\pi}{d} T_H$$

- Flat horizon ($K = 0$):

$$\omega_0 = -i\frac{k^2}{(d-1)r_+} \Rightarrow \text{ diffusion constant } D = \frac{1}{4\pi T_H}$$

- Hyperbolic horizon ($K = -1$):

$$\omega_0 = -i\frac{\xi^2 + \frac{(d-1)^2}{4}}{(d-1)r_+} \ , \ \tau = \frac{1}{|\omega_0|} < \frac{16\pi}{(d-1)^2} T_H$$

NB: For $d = 5$, these modes live longer (important for plasma behavior).

[Alsup and GS]

Scalar perturbations

$\widehat{V}_{\mathsf{V}}$ replaced by

$$\begin{aligned}
\widehat{V}_{\mathsf{S}}(u) &= \frac{\widehat{f}(u)}{4}\left[\widehat{m}+\left(1+\frac{1}{r_H^2}\right)u\right]^{-2} \\
&\times \left\{d(d-2)\left(1+\frac{1}{r_H^2}\right)^2 u^{\frac{2d-8}{d-3}} - 6(d-2)(d-4)\widehat{m}\left(1+\frac{1}{r_H^2}\right)u^{\frac{d-5}{d-3}}\right. \\
&+(d-4)(d-6)\widehat{m}^2 u^{-\frac{2}{d-3}} + (d-2)^2\left(1+\frac{1}{r_H^2}\right)^3 u^3 \\
&+2(2d^2-11d+18)\widehat{m}\left(1+\frac{1}{r_H^2}\right)^2 u^2 \\
&+\frac{(d-4)(d-6)\left(1+\frac{1}{r_H^2}\right)^2}{r_H^2}u^2 - 3(d-2)(d-6)\widehat{m}^2\left(1+\frac{1}{r_H^2}\right)u \\
&\left.-\frac{6(d-2)(d-4)\widehat{m}\left(1+\frac{1}{r_H^2}\right)}{r_H^2}u + 2(d-1)(d-2)\widehat{m}^3 + d(d-2)\frac{\widehat{m}^2}{r_H^2}\right\}
\end{aligned}$$

where $\widehat{m} = 2\frac{\ell(\ell+d-3)-(d-2)}{(d-1)(d-2)r_H^2} = \frac{2(\ell+d-2)(\ell-1)}{(d-1)(d-2)r_H^2}$

In the large black hole limit $r_H \to \infty$ with $\widehat{m}$ fixed, potential simplifies

$$\widehat{V}_S^{(0)}(u) = \frac{1-u^{\frac{d-1}{d-3}}}{4(\widehat{m}+u)^2}\Big\{ d(d-2)u^{\frac{2d-8}{d-3}} - 6(d-2)(d-4)\widehat{m}u^{\frac{d-5}{d-3}} + (d-4)(d-6)\widehat{m}^2 u^{-\frac{2}{d-3}} + (d-2)^2 u^3 + 2(2d^2-11d+18)\widehat{m}u^2 - 3(d-2)(d-6)\widehat{m}^2 u + 2(d-1)(d-2)\widehat{m}^3 \Big\}$$

- additional singularity due to double pole of scalar potential at $u = -\widehat{m}$.
- desirable to factor out the behavior not only at the horizon, but also at the boundary and the pole of the scalar potential,

$$\Psi = (1-u)^{-i\frac{\widehat{\omega}}{d-1}} \frac{u^{\frac{d-4}{2(d-3)}}}{\widehat{m}+u} F(u)$$

$\therefore$ wave equation

$$\mathcal{A}F'' + \mathcal{B}_{\widehat{\omega}}F' + \mathcal{C}_{\widehat{\omega}}F = 0$$

where

$$\begin{aligned}
\mathcal{A} &= -(d-3)^2 u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}}) \\
\mathcal{B}_{\hat{\omega}} &= -(d-3)u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})\left[\frac{d-4}{u}-\frac{2(d-3)}{\hat{m}+u}\right] \\
&\quad -(d-3)[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}} - 2(d-3)^2\frac{i\hat{\omega}}{d-1}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{1-u} \\
\mathcal{C}_{\hat{\omega}} &= -u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})\left[-\frac{(d-2)(d-4)}{4u^2}-\frac{(d-3)(d-4)}{u(\hat{m}+u)}+\frac{2(d-3)^2}{(\hat{m}+u)^2}\right] \\
&\quad -\left[\left\{d-4-(2d-5)u^{\frac{d-1}{d-3}}\right\}u^{\frac{d-5}{d-3}}+2(d-3)\frac{i\hat{\omega}}{d-1}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{1-u}\right]\left[\frac{d-4}{2u}-\frac{d-3}{\hat{m}+u}\right] \\
&\quad -(d-3)\frac{i\hat{\omega}}{d-1}\frac{[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}}}{1-u} - (d-3)^2\frac{i\hat{\omega}}{d-1}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2} \\
&\quad +\frac{\hat{V}_S^{(0)}(u)-\hat{\omega}^2}{1-u^{\frac{d-1}{d-3}}}+(d-3)^2\frac{\hat{\omega}^2}{(d-1)^2}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2}
\end{aligned}$$

Define zeroth-order wave equation $\mathcal{H}_0 F_0 = 0$, where

$$\mathcal{H}_0 F \equiv \mathcal{A}F'' + \mathcal{B}_0 F'$$

Acceptable zeroth-order solution

$$F_0(u) = 1$$

- plainly regular at all singular points ($u = 0, 1, -\hat{m}$).
- corresponds to a wavefunction vanishing at the boundary ($\Psi \sim r^{-\frac{d-4}{2}}$ as $r \to \infty$).

Wronskian

$$\mathcal{W} = \frac{(\hat{m} + u)^2}{u^{\frac{2d-8}{d-3}}(1 - u^{\frac{d-1}{d-3}})}$$

Unacceptable solution: $\check{F}_0 = \int \mathcal{W}$

- can be written in terms of hypergeometric functions.
- for $d \geq 6$, has a singularity at the boundary, $\check{F}_0 \sim u^{-\frac{d-5}{d-3}}$ for $u \approx 0$, or $\Psi \sim r^{\frac{d-6}{2}} \to \infty$ as $r \to \infty$.
- for $d = 5$, acceptable wavefunction $\sim r^{-1/2}$; unacceptable $\sim r^{-1/2} \ln r$
- for $d = 4$, roles of F_0 and $\check{F}_0$ reversed; results still valid.
- $\check{F}_0$ is also singular (logarithmically) at the horizon ($u = 1$).

Working as in the case of vector modes, we arrive at the first-order constraint

$$\int_0^1 \frac{\mathcal{C}_{\hat{\omega}}}{\mathcal{A}\mathcal{W}} = 0$$

$\because \quad \mathcal{H}_1 F_0 \equiv (\mathcal{B}_{\hat{\omega}} - \mathcal{B}_0) F_0' + \mathcal{C}_{\hat{\omega}} F_0 = \mathcal{C}_{\hat{\omega}}$

$\therefore$ dispersion relation

$$\mathbf{a}_0 - \mathbf{a}_1 i\hat{\omega} - \mathbf{a}_2 \hat{\omega}^2 = 0$$

After some algebra, we obtain

$$\mathbf{a}_0 = \frac{d-1}{2} \frac{1+(d-2)\hat{m}}{(1+\hat{m})^2} \; , \quad \mathbf{a}_1 = \frac{d-3}{(1+\hat{m})^2} \; , \quad \mathbf{a}_2 = \frac{1}{\hat{m}} \{1 + O(\hat{m})\}$$

For small $\hat{m}$, the quadratic equation has solutions

$$\hat{\omega}_0^{\pm} \approx -i\frac{d-3}{2}\,\hat{m} \pm \sqrt{\frac{d-1}{2}\,\hat{m}}$$

related to each other by $\hat{\omega}_0^{+} = -\hat{\omega}_0^{-*}$

► general symmetry of the spectrum.

Finite size effects at first order amount to a shift of the coefficient $\mathbf{a}_0$ in the dispersion relation

$$\mathbf{a}_0 \to \mathbf{a}_0 + \frac{1}{r_H^2}\mathbf{a}_H$$

after some tedious but straightforward algebra, we obtain

$$\mathbf{a}_H = \frac{1}{\widehat{m}}\{1 + O(\widehat{m})\}$$

The modified dispersion relation yields the modes

$$\widehat{\omega}_0^{\pm} \approx -i\frac{d-3}{2}\,\widehat{m} \pm \sqrt{\frac{d-1}{2}\,\widehat{m} + 1}$$

in terms of the quantum number ℓ,

$$\omega_0^{\pm} \approx -i(d-3)\,\frac{\ell(\ell+d-3)-(d-2)}{(d-1)(d-2)r_H} \pm \sqrt{\frac{\ell(\ell+d-3)}{d-2}}$$

in agreement with numerical results

[Friess, Gubser, Michalogiorgakis and Pufu]

- imaginary part inversely proportional to r_H, as in vector case
- finite real part independent of r_H

$\Rightarrow$ maximum lifetime

$$\tau_{\max} = \frac{d-2}{(d-3)d} 4\pi T_H$$

- $K = 0$,

$$\omega = \pm \frac{k}{\sqrt{d-2}} - i \frac{d-3}{(d-1)(d-2)r_+} k^2$$

$\Rightarrow$ speed of sound $v = \frac{1}{\sqrt{d-2}}$ (CFT!) and diffusion constant $D = \frac{d-3}{d-2} \frac{1}{4\pi T_H}$.

- $K = -1$,

$$\omega = \pm \sqrt{\frac{\xi^2 + (\frac{d-3}{2})^2}{d-2}} - i \frac{(d-3)[\xi^2 + \frac{(d-1)^2}{4}]}{(d-1)(d-2)r_+} \;, \quad \tau < \frac{4(d-2)}{(d-3)(d-1)^2} 4\pi T_H$$

NB: For $d = 5$, $K = -1$ scalar modes live longer than any other modes (important for plasma behavior).

[Alsup and GS]

Tensor perturbations

Unlike the other two cases, asymptotic spectrum is entire spectrum.
In large bh limit, wave equation

$$-(d-3)^2(u^{\frac{2d-8}{d-3}} - u^3)\Psi'' - (d-3)[(d-4)u^{\frac{d-5}{d-3}} - (2d-5)u^2]\Psi' + \left\{\hat{L}^2 + \frac{d(d-2)}{4}u^{-\frac{2}{d-3}} + \frac{(d-2)^2}{4}u - \frac{\hat{\omega}^2}{1-u^{\frac{d-1}{d-3}}}\right\}\Psi = 0$$

For zeroth-order eq., set $\hat{L} = 0 = \hat{\omega}$
$\hookrightarrow$ two solutions are ($\Psi = F_0$ at zeroth order)

$$F_0(u) = u^{\frac{d-2}{2(d-3)}} \quad , \quad \check{F}_0(u) = u^{-\frac{d-2}{2(d-3)}} \ln\left(1 - u^{\frac{d-1}{d-3}}\right)$$

Neither behaves nicely at both ends ($u = 0, 1$)

$\therefore$ both are unacceptable.

$\therefore$ impossible to build a perturbation theory to calculate small frequencies.

in agreement with numerical results and in accordance with the

AdS/CFT correspondence

- there is no ansatz that can be built from tensor spherical harmonics $\mathbb{T}_{ij}$ satisfying the linearized hydrodynamic eqs because of the conservation and tracelessness properties of $\mathbb{T}_{ij}$.

Hydrodynamics on the AdS boundary

- calculate the hydrodynamics in the linearized regime of a $d-1$ dimensional fluid with dissipative effects.

metric

$$ds_\partial^2 = -dt^2 + d\Sigma_{K,d-2}^2$$

hydrodynamic equations

$$\nabla_\mu T^{\mu\nu} = 0$$

$$\text{CFT} \quad \Rightarrow \quad T^\mu_\mu = 0\,, \quad \epsilon = (d-2)p \quad , \quad \zeta = 0$$

In rest frame $u^\mu = (1,0,0,0)$, const. pressure p_0; with perturbations

$$u^\mu = (1, u^i) \quad , \quad p = p_0 + \delta p$$

apply hydrodynamic equations

$$\begin{aligned} (d-2)\partial_t \delta p + (d-1) p_0 \nabla_i u^i &= 0 \\ (d-1) p_0 \partial_t u^i + \partial^i \delta p - \eta \left[\nabla^j \nabla_j u^i + K(d-3) u^i + \frac{d-4}{d-2} \partial^i (\nabla_j u^j) \right] &= 0 \end{aligned}$$

where we used $R_{ij} = K(d-3) g_{ij}$

Vector perturbations – ansatz

$$\delta p = 0\ , \quad u^i = \mathcal{C}_V e^{-i\omega t}\mathbb{V}^i$$

$\mathbb{V}^i$: vector harmonic

hydrodynamic equations $\Rightarrow$

$$-i\omega(d-1)p_0 + \eta\left[k_V^2 - K(d-3)\right] = 0$$

Using

$$\frac{\eta}{p_0} = (d-2)\frac{\eta}{s}\frac{S}{M} = \frac{4\pi\eta}{s}\frac{r_+}{K + r_+^2}$$

with ω from gravity dual, we obtain for large r_+,

$$\boxed{\frac{\eta}{s} = \frac{1}{4\pi}}$$

[Policastro, Son and Starinets]

Scalar perturbations – ansatz

$$u^i = \mathcal{A}_S e^{-i\omega t} \partial^i \mathbb{S} \ , \quad \delta p = \mathcal{B}_S e^{-i\omega t} \mathbb{S}$$

$\mathbb{S}$: scalar harmonic

hydrodynamic equations $\Rightarrow$

$$(d-2) i\omega \mathcal{B}_S + (d-1) p_0 k_S^2 \mathcal{A}_S = 0$$
$$\mathcal{B}_S + \mathcal{A}_S \left[-i\omega (d-1) p_0 - 2(d-3) K\eta + 2\eta k_S^2 \frac{d-3}{d-2} \right] = 0$$

$\therefore$ determinant must vanish

$$\begin{vmatrix} (d-2) i\omega & (d-1) p_0 k_S^2 \\ 1 & -i\omega (d-1) p_0 - 2(d-3) K\eta + 2\eta k_S^2 \frac{d-3}{d-2} \end{vmatrix} = 0$$

along the same lines as for vector perturbations, we arrive at

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

► same as vector QNMs!

Conformal soliton flow

$K = +1$

the holographic image on Minkowski space of the global AdS_5-Schwarzschild black hole is a spherical shell of plasma first contracting and then expanding.

▶ conformal map from $S^{d-2} \times \mathbb{R}$ to $(d-1)$-dim Minkowski space

[Friess, Gubser, Michalogiorgakis, Pufu]

$d = 5$ QNMs $\Rightarrow$ properties of plasma

-

$$\frac{v_2}{\delta} = \frac{1}{6\pi} \mathrm{Re} \frac{\omega^4 - 40\omega^2 + 72}{\omega^3 - 4\omega} \sin \frac{\pi\omega}{2}$$

  - $v_2 = \langle \cos 2\phi \rangle$ at $\theta = \frac{\pi}{2}$ (mid-rapidity), average with respect to energy density at late times
  - $\delta = \frac{\langle y^2 - x^2 \rangle}{\langle y^2 + x^2 \rangle}$ (eccentricity at time $t = 0$).

Numerically, $\frac{v_2}{\delta} = 0.37$, *cf.* with result from RHIC data, $\frac{v_2}{\delta} \approx 0.323$

[PHENIX Collaboration, arXiv:nucl-ex/0608033]

- thermalization time

$$\tau = \frac{1}{2|\mathrm{Im}\,\omega|} \approx \frac{1}{8.6 T_{\mathrm{peak}}} \approx 0.08 \text{ fm/c} \quad , \qquad T_{\mathrm{peak}} = 300 \text{ MeV}$$

cf. with RHIC result $\tau \sim 0.6$ fm/c

[Arnold, Lenaghan, Moore, Yaffe, Phys. Rev. Lett. **94** *(2005) 072302]*

Not in agreement, but encouragingly small

► perturbative QCD yields $\tau \gtrsim 2.5$ fm/c.

[Baier, Mueller, Schiff, Son; Molnar, Gyulassy]

$\boxed{K = -1}$

- needs work for conformal map $\mathbb{H}^{d-2}/\Gamma \times \mathbb{R} \mapsto (d-1)$–dim Minkowski space.
- important case $\because$ these modes live the longest.

[Alsup and Siopsis]

Phase transitions

Black Holes with Scalar Hair

$K = 0$, $d = 4$

scalar Ψ of mass $m^2 = -2$ (above Breitenlohner-Freedman (BF) bound) and charge q (large - probe limit - set $q = 1$) and electrostatic potential Φ in black hole background

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\vec{x}^2 \,, \quad f(r) = r^2 - \frac{2\mu}{r}$$

Horizon and Hawking temperature

$$r_+ = (2\mu)^{1/3} \,, \quad T = \frac{3r_+}{4\pi}$$

assuming spherical symmetry, Einstein-Maxwell eqs. $\Rightarrow$

$$\Psi'' + \left(\frac{f'}{f} + \frac{2}{r}\right)\Psi' + \left(\frac{\Phi}{f}\right)^2 \Psi + \frac{2}{f}\Psi = 0$$

$$\Phi'' + \frac{2}{r}\Phi' - \frac{2\Psi^2}{f}\Phi = 0$$

[Hartnoll, Herzog and Horowitz]

As $r \to \infty$,

$$\Psi = \frac{\Psi^{(1)}}{r} + \frac{\Psi^{(2)}}{r^2} + \dots \, , \quad \Phi = \Phi^{(0)} + \frac{\Phi^{(1)}}{r} + \dots$$

where one of the $\Psi^{(i)} = 0$ ($i = 1, 2$) for stability, $\Phi^{(0)}$ is the chemical potential and $\Phi^{(1)} = -\rho$ (charge density).

Below a critical temperature T_0 a condensate forms,

$$\langle \mathcal{O}_i \rangle = \sqrt{2} \Psi^{(i)}$$

of an operator of dimension $\Delta = i$.

At $T = T_0$, we may set $\Psi = 0$ in eq. for Φ and deduce ($\Phi(r_+) = 0$)

$$\Phi = \rho \left(\frac{1}{r_+} - \frac{1}{r} \right)$$

Eq. for Ψ turns into an eigenvalue problem $\Rightarrow$

$$T_0 \approx 0.226 \sqrt{\rho} \, , \quad 0.118 \sqrt{\rho}$$

depending on B.C.

EM perturbation:

$$A'' + \frac{f'}{f}A' + \left(\frac{\omega^2}{f^2} - \frac{2\Psi^2}{f}\right)A = 0$$

B.C.: ingoing at horizon, $A \sim f^{-i\omega/(4\pi T)}$, and at boundary ($r \to \infty$),

$$A = A^{(0)} + \frac{A^{(1)}}{r} + \dots$$

Ohm's law $\Rightarrow$ conductivity

$$\sigma(\omega) = \frac{A^{(1)}}{i\omega A^{(0)}}$$

For $T \geq T_0$, $\Psi = 0$, $\therefore A \sim e^{i\omega r_*}$ (r_*: tortoise coordinate) $\therefore$

$$\sigma(\omega) = 1$$

At low T, for $\langle\mathcal{O}_1\rangle \neq 0$, we have

$$\Psi \approx \frac{\langle\mathcal{O}_1\rangle}{\sqrt{2}\, r}$$

Since $r_+ \to 0$, we obtain $A \sim e^{i\omega' r_*}$, where $\omega' = \sqrt{\omega^2 - \langle\mathcal{O}_1\rangle^2}$.

$\therefore$ for $\omega < \langle\mathcal{O}_1\rangle$, $\mathrm{Re}\,\sigma = 0 \Rightarrow$ superconductor with a gap!

$K = -1\ ,\ d = 4$

[Koutsoumbas, Papantonopoulos and GS]

scalar Ψ of mass $m^2 = -2$ (above Breitenlohner-Freedman (BF) bound) and charge q conformally coupled in potential

$$V(\Psi) = \frac{8\pi G}{3}|\Psi|^4$$

Exact solution (MTZ black hole)

$$ds^2 = -f_{MTZ}(r)dt^2 + \frac{dr^2}{f_{MTZ}(r)} + r^2 d\sigma^2 \quad , \quad f_{MTZ} = r^2 - \left(1 + \frac{r_0}{r}\right)^2 ,$$

$$\Psi(r) \equiv -\sqrt{\frac{3}{4\pi G}}\frac{r_0}{r + r_0} \quad , \quad \Phi = 0$$

[Martinez, Troncoso and Zanelli]

temperature, entropy and mass, respectively

$$T = \frac{1}{\pi}\left(r_+ - \frac{1}{2}\right) \ , \quad S_{MTZ} = \frac{\sigma}{4G}\left(2r_+ - 1\right) \ , \quad M_{MTZ} = \frac{\sigma r_+}{4\pi G}\left(r_+ - 1\right) .$$

▶ law of thermodynamics $dM = TdS$ holds.

At $M = 0$, MTZ coincides with TBH,

$$ds^2_{\mathrm{AdS}} = -(r^2 - 1)dt^2 + \frac{dr^2}{r^2 - 1} + r^2 d\sigma^2$$

enhanced scaling symmetry (pure AdS) at critical temperature

$$T_0 = \frac{1}{2\pi}$$

phase transition

$$\Delta F = F_{TBH} - F_{MTZ} = -\frac{\sigma l}{8\pi G}\pi^3 l^3 (T - T_0)^3 + \dots ,$$

$\therefore$ 3rd order phase transition between MTZ and TBH at T_0.

- Checked perturbative stability of MTZ for $T < T_0$ ($M < 0$).

[Koutsoumbas, Papantonopoulos and GS]

The Dual Superconductor

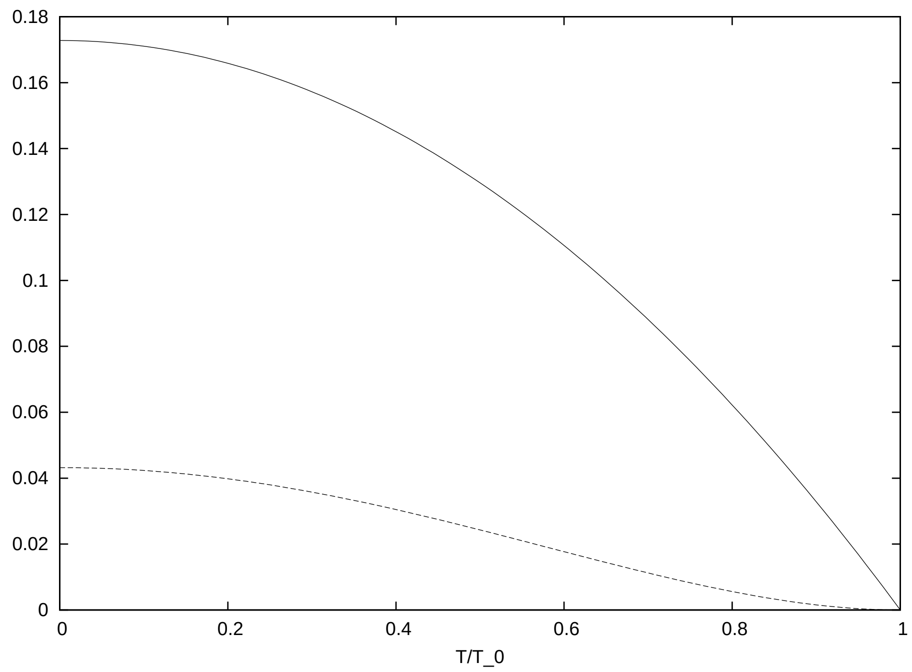
- the condensation of the scalar field has a geometrical origin and is due entirely to its coupling to gravity.

heat capacities in normal and superconducting phases, respectively, as $T \to 0$

$$C_n \approx \frac{\pi\sigma}{3\sqrt{3}G} T \ , \quad C_s \approx \frac{\pi\sigma}{2G} T \ ,$$

Both condensates are present,

$$\langle \mathcal{O}_1 \rangle = \sqrt{\frac{3\pi^3}{2G}} (T_0^2 - T^2) \ ,$$
$$\langle \mathcal{O}_2 \rangle = \sqrt{\frac{3\pi^7}{2G}} (T_0^2 - T^2)^2$$



EM perturbations

1st-order perturbation theory $\Rightarrow$

$$A = e^{-i\omega r_*} + \frac{q^2}{2i\omega} e^{i\omega r_*} \int_{r_+}^{r} dr' \Psi^2(r') e^{-2i\omega r_*} - \frac{q^2}{2i\omega} e^{-i\omega r_*} \int_{r_+}^{r} dr' \Psi^2(r') \; .$$

conductivity to 1st-order in q^2

$$\sigma(\omega) = \frac{A^{(1)}}{i\omega A^{(0)}} = 1 - \frac{q^2}{i\omega} \int_{r_+}^{\infty} dr \Psi^2(r) e^{-2i\omega r_*} \; .$$

superfluid density from

$$\mathrm{Re}\,[\sigma(\omega)] \sim \pi n_s \delta(\omega) \;\; , \quad \mathrm{Im}\,[\sigma(\omega)] \sim \frac{n_s}{\omega} \; , \quad \omega \to 0 \; .$$

$\therefore$

$$n_s = q^2 \int_{r_+}^{\infty} dr \Psi^2(r) = \frac{3q^2}{4\pi G} \frac{r_0^2}{r_+ + r_0} = \alpha\,(T_0 - T)^2 \;\; , \quad \alpha = \frac{3\pi q^2}{4G} \; .$$

near $T = 0$,

$$n_s(0) - n_s(T) \approx \frac{\alpha}{\pi} T^{\delta} \,, \quad \delta = 1$$

$q/\sqrt{G}$	1	3	5
δ	1.025 ± 0.007	1.52 ± 0.03	1.78 ± 0.03

normal, non-superconducting, component of DC conductivity

$$n_n = \lim_{\omega \to 0} \mathrm{Re}\,[\sigma(\omega)] \,.$$

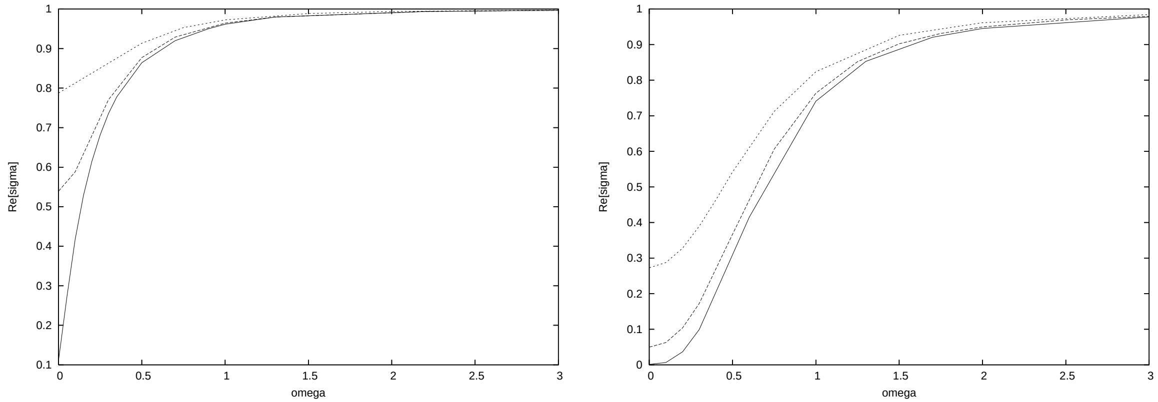
$\therefore$

$$\ln n_n = 2q^2 \int_{r_+}^{\infty} dr \Psi^2(r) r_* \,.$$

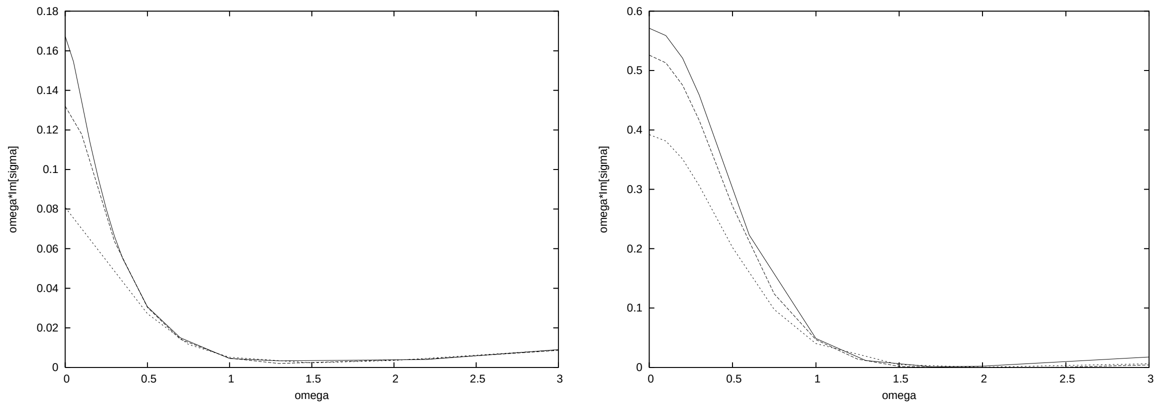
At low T,

$$n_n \sim T^{\gamma} \,, \quad \gamma = \frac{3q^2}{4\pi G} \,.$$

$q/\sqrt{G}$	$\gamma_{\text{numerical}}$	$\gamma_{\text{analytical}}$	$\alpha_{\text{numerical}}$	$\alpha_{\text{analytical}}$
0.1	0.0020	0.0024	0.0225	0.024
0.5	0.0538	0.0597	0.552	0.589
1.0	0.187	0.239	2.196	2.356
2.0	0.684	0.955	8.678	9.425
3.0	1.325	2.15	20.35	21.21
5.0	2.522	5.97	52.90	58.90



The real part of the conductivity *vs* ω for $q/\sqrt{G} = 2$ (left) and $q/\sqrt{G} = 5$ (right) and $T = 0.0032, 0.032, 0.064$. The lowest curve corresponds to the lowest temperature.



The imaginary part of the conductivity multiplied by ω *vs* ω for $q/\sqrt{G} = 2$ (left) and $q/\sqrt{G} = 5$ (right) and $T = 0.0032, 0.032, 0.064$. The uppermost curve corresponds to the lowest temperature.

CONCLUSIONS

- Quasi-normal modes are a powerful tool in understanding hydrodynamic behavior of gauge theory fluid at strong coupling
- Quark-gluon plasma understood in terms of gravitational perturbations of a dual black hole
- Superconductors understood in terms of electromagnetic perturbations of dual hairy black holes
- Physical role of high overtones not clear