

Unitary One Matrix Models: String Equation and Flows*

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Abstract

We review the Symmetric Unitary One Matrix Models. In particular we discuss the string equation in the operator formalism, the mKdV flows and the Virasoro Constraints. We focus on the τ -function formalism for the flows and we describe its connection to the (big cell of the) Sato Grassmannian $Gr^{(0)}$ via the Plucker embedding of $Gr^{(0)}$ into a fermionic Fock space. Then the space of solutions to the string equation is an explicitly computable subspace of $Gr^{(0)} \times Gr^{(0)}$ which is invariant under the flows.

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1. Introduction

One Matrix Models are quantum mechanical systems whose partition function is defined by an integral of the form:

$$Z_M = \int dM \exp\left\{-\frac{N}{\lambda} \text{tr}V(M)\right\}, \quad (1)$$

where M is an $N \times N$ matrix and the potential $V(M)$ is a polynomial in M . In the last few years, there has been tremendous progress [1–4] in matrix models through the discovery of a connection of a certain class of these models to two dimensional gravity coupled to (p, q) minimal conformal matter. This happens when M is a hermitian matrix (HMM) or when one considers generalizations to a $(q - 1)$ hermitian multi-matrix model (MHMM), where $(q - 1)$ hermitian matrices are coupled linearly to each other [5]. In the former simpler case, the Feynman graphs of the zero dimensional field theory are viewed as being dual to a discrete dynamical polygonation of an oriented two dimensional Riemann surface. Then the perturbation series can be summed in the form

$$Z_M = \sum_{h=0}^{\infty} N^\chi Z_h \quad (2)$$

where $\chi = V - E + L$ is the Euler character of the corresponding surface and h is its genus given by $\chi = 2 - 2h$. Since the number of vertices V , edges E and loops L of the Feynman graph correspond respectively to the number of faces F , edges E and vertices V of the dual graph, the above series can be shown to correspond to the discretized version of the partition function of pure two dimensional gravity

$$Z_{Gra} = \sum_h \sum_T \frac{1}{C(T)} \exp\left(-\mu_B A + \frac{1}{4\pi G_B} \chi\right). \quad (3)$$

In (3) A is the area of the surface, μ_B and G_B the bare cosmological and Newton's constant and $C(T)$ is the symmetry factor of the polygonation corresponding to dividing by the volume of the diffeomorphism group of the surface. The equality of (2) and (3) is achieved by identifying $\lambda = e^{-\mu_B}$ and $N = e^{\frac{1}{4\pi G_B}}$. The action in (3) can also be viewed as the action of a string theory embedded in zero dimensional spacetime

$$S_{str} = \log \kappa_B \int d^2\xi \sqrt{g} R + \mu_B \int d^2\xi \sqrt{g}. \quad (4)$$

Then (2) gives the genus perturbation expansion with $\kappa_B = \frac{1}{N}$, the bare string coupling. The naive continuum theory is taken by letting $N \rightarrow \infty$. In this case the area A diverges and the polygonated surface is thought to approach a smooth Riemann surface. For a critical value μ_c of μ_B the increasing entropy of large surfaces compensates the Boltzmann factor and the system undergoes a (third order) phase transition. If the critical point is approached in an arbitrary way, only the sphere Z_0 contributes to (2). The remarkable observation [1–4] was that since the singular part of $Z_h \sim (\mu_B - \mu_c)^{\chi(1+\frac{1}{2k})}$ with k a positive integer, one can obtain contributions from all genera by simultaneously taking the large N -limit and letting μ_B approach its critical value μ_c in a coordinated way. The integer k labels a series of multicritical points reached by tuning k parameters in the potential $V(M)$. Introducing a cutoff a in the theory, we define the string coupling κ_0 and renormalized cosmological constant μ_R to be

$$\kappa_0 = \frac{a^{-(2+\frac{1}{k})}}{N}, \quad \mu_R = \frac{\mu_B - \mu_c}{a^2}. \quad (5)$$

The double scaling limit is defined by taking $N \rightarrow \infty$ and $\mu_B \rightarrow \mu_c$ while keeping κ_0 and μ_R fixed. Then the continuum limit of (2) becomes

$$Z_{str} = \sum_{h=0}^{\infty} \kappa^{\chi} Z_h, \quad (6)$$

with $\kappa = \frac{\kappa_0}{\mu_R}$. The series (6) is horribly divergent. It is non-Borel summable since every term increases as $(2h)!$. This reflects our ignorance in summing the perturbation series of string theory although the fixed genus partition function Z_h can be calculated and is well defined. Happily, the theory is exactly solvable at the multicritical points and its dynamical content is given by a single differential equation, the string equation. The string equation is a differential equation in the variable x satisfied by the specific heat $-\partial^2 \log Z$, with $\kappa^2 = x^{-(2+\frac{1}{k})}$. It possesses solutions that in the weak coupling limit $\kappa \rightarrow 0$ are asymptotic to (6) and we say that the double scaling limit provides a non-perturbative definition of Z_{str} . Indeed comparison with calculations directly from the continuum theory indicates that Z_{str} corresponds to two dimensional gravity coupled to $(2k-1, 2)$ minimal conformal matter. Even more interesting is the discovery that the double scaling limit of $(q-1)$ MHMM gives two dimensional gravity coupled to (p, q) minimal conformal matter [5].

Unitary One Matrix Models (UMM) form another interesting class of matrix models. These are defined by (1) with M being a unitary matrix U . The interest in those models

arose a long time ago when Gross and Witten [6] showed that the partition function of two dimensional $U(N)$ QCD on a lattice is given by $Z_{QCD} = (Z_U)^{\frac{V}{a^2}}$ and that the theory undergoes a third order phase transition in the large N limit (V is the volume of the two dimensional world and a is the lattice cutoff). The theory was also shown to possess a double scaling limit $N \rightarrow \infty$ and $\lambda \rightarrow \lambda_c$ with $t = (1 - \frac{n}{N})N^{\frac{2k}{2k+1}}$ and $y = (1 - \frac{\lambda}{\lambda_c})N^{\frac{2k}{2k+1}}$ held fixed [7,8]. The string equation is a $2k^{\text{th}}$ order differential equation of the function v in the variable $x = t + y$, with $v^2 = -\partial^2 \log Z$. It has solutions that are asymptotic to (6) in the limit $x \rightarrow \infty$ with $\kappa^2 = x^{-(2+\frac{1}{k})}$. The identifications of those solutions with conformal field theories coupled to two dimensional gravity or other interesting systems is still, however, an interesting open problem. Some interesting suggestions have been made in [9]. Moreover, the surface interpretation of UMM is not as clear as in the case of HMM. In [10] Neuberger views the unitary matrix as $U = e^{iM}$ where M is hermitian and introduces $N \times N$ hermitian fermionic matrices ψ and $\bar{\psi}$ to exponentiate the Haar measure $dU \rightarrow dM \det(\frac{\delta U}{\delta M})$. The resulting surfaces contain an infinite number of types of bosonic vertices forming bosonic “webs” and fermionic loops forming their boundaries that might allow a stringy interpretation of the UMM. For another interesting suggestion see [11]. It is also interesting to note that UMM belong to the same universality class as the HMM in a different class of multicritical points, the double-cut HMM [9,12]. This is expected since the critical behaviour is governed by the scaling of the density of the eigenvalues at the edge of its support [13] and the eigenvalues of the two models scale identically there.

The continuum theory obtained in the double scaling limit has a very rich mathematical structure. When one considers perturbations by the scaling operators $\langle \sigma_k \rangle$ with sources t_k , the dependence of the specific heat (or its square root for the UMM) on the “times” t_k is given by KdV flows [5,14] for the HMM and mKdV flows for the (symmetric) UMM. The partition functions of the theory are found to be given by the corresponding τ -functions [15–17] which can be thought as sections of a line bundle over the Universal Grassmannian. Furthermore the τ -functions that solve the string equation are annihilated by constraints which for the one matrix model are the Virasoro constraints [15–18]. All of those results have counterparts in the discrete theory. The integrable flows are now with respect to the couplings in the potential $V(M)$. For the UMM these are given by Toda flows on the half line [19] and the partition function is given by the product of two Toda-chain τ -functions. The Virasoro constraints L_n have the simple interpretation of

corresponding to invariance of the partition function under specific transformations, which for the UMM are given by $\delta U = \epsilon_n(U^{n+1} - U^{1-n})$.

An interesting observation is that the string equation can be written in the form $[P, Q] = 1$ where P and Q are differential operators for the HMM [5] and 2×2 matrices of differential operators of specific order for the UMM [20]. They correspond to the continuum limits of operators acting on the space of orthonormal functions used to solve the model. One can use this form of the string equation to determine easily the points in the Universal Grassmannian that solve the string equation [21]. For the UMM these are found to correspond to a pair of points V_1 and V_2 in the (big cell of the) Sato Grassmannian satisfying certain invariance conditions. It is very important that the mKdV evolution of V_1 and V_2 gives new solutions to the string equation. The τ -functions that correspond to V_1 and V_2 are shown to satisfy the Virasoro constraints in this formalism [22] since the constraints are derived from the same invariance conditions that solutions to the string equation satisfy [23–26].

This article is organized as follows. In section 2 we review the discrete formulation of the symmetric UMM. The method of the orthogonal polynomials in the trigonometric basis is summarized and the Toda flows and Virasoro constraints are discussed. In section 3 we describe the double scaling limit and describe how the mKdV flows arise. In section 4 we give a non-rigorous approach to the connection between the Sato Grassmannian and the mKdV flows starting from the finite dimensional Grassmannians. In section 5 we describe the connection of the Sato Grassmannian to the solutions to the string equation.

2. The Symmetric Unitary Matrix Model

In this paper we will study the UMM defined by the one matrix integral

$$Z_N^U = \int DU \exp\left\{-\frac{N}{\lambda} \text{Tr} V(U + U^\dagger)\right\}, \quad (7)$$

where U is a $2N \times 2N$ or a $(2N + 1) \times (2N + 1)$ unitary matrix, DU is the Haar measure for the unitary group and the potential

$$V(U) = \sum_{k \geq 0} g_k U^k, \quad (8)$$

is a polynomial in U . As standard we first reduce the above integral to an integral over the eigenvalues [6,27] z_i of U which lie on the unit circle in the complex z plane.

$$Z_N^U = \int \left\{ \prod_j \frac{dz_j}{2\pi i z_j} \right\} |\Delta(z)|^2 \exp\left\{-\frac{N}{\lambda} \sum_i V(z_i + z_i^*)\right\}, \quad (9)$$

where $\Delta(z) = \prod_{k < j} (z_k - z_j)$ is the Vandermonde determinant. The Vandermonde determinant is conveniently expressed in terms of trigonometric orthogonal polynomials [28]

$$\begin{aligned} c_n^\pm(z) &= z^n \pm z^{-n} + \sum_{i=1}^{i_{max}} \alpha_{n,n-i}^\pm (z^{n-i} \pm z^{-n+i}) \\ &= \pm c_n^\pm(z^{-1}) \end{aligned} \quad (10)$$

where for $U(2N+1)$ n is a non-negative integer and $i_{max} = n$ and for $U(2N)$ n is a positive half-integer and $i_{max} = n - \frac{1}{2}$. The polynomials $c_n^\pm(z)$ are orthogonal with respect to the inner product

$$\begin{aligned} \langle c_n^+, c_m^+ \rangle &= \oint \frac{dz}{2\pi i z} \exp\left\{-\frac{N}{\lambda} V(z + z^*)\right\} c_n^+(z)^* c_m^+(z) \\ &= e^{\phi_n^+} \delta_{n,m}, \\ \langle c_n^-, c_m^- \rangle &= e^{\phi_n^-} \delta_{n,m}, \\ \langle c_n^+, c_m^- \rangle &= 0. \end{aligned} \quad (11)$$

The expression for the Vandermonde determinant is

$$|\Delta(z)|^2 = \left| \det \begin{pmatrix} c_i^-(z_j) \\ c_i^+(z_j) \end{pmatrix} \right|^2, \quad (12)$$

where $j = 1, \dots, 2N$, $i = \frac{1}{2}, \frac{3}{2}, \dots, N - \frac{1}{2}$ for $U(2N)$ and $j = 1, \dots, 2N+1$, $i = 0, 1, \dots, N$ for $U(2N+1)$ (where the line $c_0^-(z) \equiv 0$ is understood to be omitted). Then the partition function of the model is given by the product of the norms of the orthogonal polynomials

$$Z_N^U = \prod_n e^{\phi_n^+} e^{\phi_n^-} = \tau_N^{(+)} \tau_N^{(-)}. \quad (13)$$

The functions $\tau_n^{(+)}$ and $\tau_n^{(-)}$ are Toda chain τ -functions on the half line [19]

$$\frac{\partial^2 \phi_n^\pm}{\partial g_1^2} = e^{\phi_{n+1}^\pm - \phi_n^\pm} - e^{\phi_n^\pm - \phi_{n-1}^\pm}, \quad (14)$$

with solutions $e^{\phi_n^\pm} = \frac{\tau_{n+1}^{(\pm)}}{\tau_n^{(\pm)}}$.

The orthogonal basis of polynomials chosen is especially useful for constructing the operator formalism of the theory. When acting on the basis of orthonormal functions

$$\pi_n^\pm(z) = e^{-\phi_n^\pm/2} e^{-\frac{N}{\lambda} V(z^+)} c_n^\pm(z) \quad (15)$$

such that

$$\begin{aligned}
\langle \pi_n^+(z), \pi_m^+(z) \rangle &= \oint \frac{dz}{2\pi iz} \pi_n^+(z)^* \pi_m^+(z) \\
&= \delta_{n,m}, \\
\langle \pi_n^-(z), \pi_m^-(z) \rangle &= \delta_{n,m}, \\
\langle \pi_n^+(z), \pi_m^-(z) \rangle &= 0,
\end{aligned} \tag{16}$$

the operators $z_{\pm} = z \pm \frac{1}{z}$ and $z\partial_z$ give finite term recursion relations

$$\begin{aligned}
z_+ \pi_n^{\pm}(z) &= \sqrt{R_{n+1}^{\pm}} \pi_{n+1}^{\pm}(z) - r_n^{\pm} \pi_n^{\pm}(z) + \sqrt{R_n^{\pm}} \pi_{n-1}^{\pm}(z), \\
z_- \pi_n^{\pm}(z) &= \sqrt{Q_{n+1}^{\mp}} \pi_{n+1}^{\mp}(z) - q_n^{\pm} \sqrt{\frac{Q_n^{\mp}}{R_n^{\pm}}} \pi_n^{\mp}(z) - \sqrt{Q_n^{\pm}} \pi_{n-1}^{\mp}(z), \\
z\partial_z \pi_n^{\pm}(z) &= -\frac{N}{2\lambda} \sum_{r=1}^k (v_z^{\pm})_{n,n+r} \pi_{n+r}^{\mp}(z) + \left\{ n \sqrt{\frac{Q_n^{\mp}}{R_n^{\pm}}} - \frac{N}{2\lambda} (v_z^{\pm})_{n,n} \right\} \pi_n^{\mp}(z) \\
&\quad + \frac{N}{2\lambda} \sum_{r=1}^k (v_z^{\pm})_{n,n-r} \pi_{n-r}^{\mp}(z),
\end{aligned} \tag{17}$$

where $R_n^{\pm} = e^{\phi_n^{\pm} - \phi_{n-1}^{\pm}}$, $Q_n^{\pm} = e^{\phi_n^{\pm} - \phi_{n-1}^{\mp}}$, $r_n^{\pm} = \frac{\partial \phi_n^{\pm}}{\partial g_1}$, $q_n^{\pm} = \frac{(Q_{n+1}^{\pm} - Q_n^{\pm}) + (R_{n+1}^{\mp} - R_n^{\pm})}{r_n^{\pm} - r_n^{\mp}}$, and

$$(v_z^{\pm})_{n,n-r} = \oint \frac{dz}{2\pi iz} \pi_{n-r}^{\mp}(z)^* \{z\partial_z V(z_+)\} \pi_n^{\pm}(z).$$

Then the discrete string equation is given by the relation $[z\partial_z, z_{\pm}] = z_{\mp}$.

Invariance of the partition function under the transformations

$$\delta_n U = \epsilon(U^{n+1} - U^{1-n}) \quad n \geq 1,$$

implies that the partition function is annihilated by the Virasoro constraints

$$L_n = \sum_{k=0}^{\infty} k g_k \frac{\partial}{\partial g_{k+n}} + \frac{1}{2} \sum_{1 \leq k \leq n} \frac{\partial^2}{\partial g_k \partial g_{n-k}}. \tag{18}$$

In [19] it was argued that the string equation can be viewed as a consistency condition of the integrable hierarchy and the Virasoro constraints.

3. The Double Scaling Limit

The continuum limit of (7) is taken by letting $N \rightarrow \infty$. Then the eigenvalues α_i , where $z_i = e^{i\alpha_i}$, become continuously distributed over the unit circle $|z| = 1$ and their distribution is described by the density of eigenvalues

$$\begin{aligned} \rho(\alpha) &= \frac{ds}{d\alpha}, \quad s = \frac{i}{N} \\ \int_{-a_c}^{a_c} \rho(\alpha) d\alpha &= 1 \quad 0 < a_c \leq \pi. \end{aligned} \quad (19)$$

If $\rho(\alpha)$ is given, quantities of physical interest, like the free energy $F_{sph} = \frac{1}{N^2} \log Z$, can be calculated. For example the saddle point approximation of (7) gives

$$F_{sph} = \frac{2}{\lambda} \int_{-a_c}^{a_c} d\alpha \rho(\alpha) V(2 \cos \alpha) + P \int_{-a_c}^{a_c} d\alpha d\beta \rho(\alpha) \rho(\beta) \log \left| \sin \frac{\alpha - \beta}{2} \right| + \text{const.}, \quad (20)$$

where P denotes the principal value of the integral. Then one can think of the eigenvalues as a Dyson gas of electric charges on the unit circle subject to their mutual Coulomb repulsion and an external potential V . In the weak coupling limit $\lambda \rightarrow \infty$ the eigenvalues tend to distribute uniformly on the circle, whereas in the strong coupling limit $\lambda \rightarrow 0$ the charges are localized, say at the point $z = 1$. The system undergoes a phase transition precisely when the eigenvalue distribution develops a cut at $z = -1$ and it happens when $\lambda_c = 1$. Near the cut $\rho(\alpha)$ scales as

$$\rho_k(\alpha) \sim c_k \left(1 - \sin^2 \frac{\alpha}{2}\right)^k \quad \alpha \rightarrow \pi, \quad (21)$$

and we obtain a third order phase transition with $F \sim (\lambda - \lambda_c)^{2 + \frac{1}{k}}$ [6]. The k^{th} multicritical point is obtained by tuning k couplings in the potential $V(U)$ to their critical values.

The double scaling limit [7,8] corresponding to the k^{th} multicritical point is defined by $N \rightarrow \infty$ and $\lambda \rightarrow \lambda_c$, with $t = (1 - \frac{t}{N})N^{\frac{2k}{2k+1}}$, $y = (1 - \frac{t}{\lambda_c})N^{\frac{2k}{2k+1}}$ held fixed. It was shown in [20] that the operators z_{\pm} and $z\partial_z$ have a smooth continuum limit given by

$$\begin{aligned} z_+ &\rightarrow 2 + N^{-\frac{2}{2k+1}} \mathcal{Q}_+, \quad z_- \rightarrow -2N^{-\frac{1}{2k+1}} \mathcal{Q}_-, \\ z\partial_z &\rightarrow N^{\frac{1}{2k+1}} \mathcal{P}_k, \end{aligned} \quad (22)$$

where \mathcal{Q}_{\pm} are given by

$$\begin{aligned} \mathcal{Q}_- &= \begin{pmatrix} 0 & \partial + v \\ \partial - v & 0 \end{pmatrix}, \\ \mathcal{Q}_+ &= \begin{pmatrix} (\partial + v)(\partial - v) & 0 \\ 0 & (\partial - v)(\partial + v) \end{pmatrix} \\ &= \mathcal{Q}_-^2, \end{aligned} \quad (23)$$

and \mathcal{P}_k by

$$\mathcal{P}_k = \begin{pmatrix} 0 & \mathbf{P}_k \\ \mathbf{P}_k^\dagger & 0 \end{pmatrix}. \quad (24)$$

Here $\partial \equiv \frac{\partial}{\partial x}$ and $x = t + y$. The scaling function v^2 is proportional to the specific heat $-\partial^2 \ln Z$ of the model. The operators \mathbf{P}_k are differential operators of order $2k$. The same assertions hold if we introduce sources $t_{2k+1}(t_1 \equiv x)$ and deform the k^{th} multicritical potential V_k to $V_k(z) - \sum_l t_{2l+1} V_l(z) N^{\frac{2(k-l)}{2k+1}}$. From $[z\partial_z, z_-] = z_+$ it follows that

$$[\mathcal{P}, \mathcal{Q}_-] = 1, \quad (25)$$

where $\mathbf{P} = -\sum_{l \geq 1} (2l+1)t_{2l+1} \tilde{\mathbf{P}}_l - x$ with $\tilde{\mathbf{P}}_l = \mathbf{P}_l + x$. The function $v(x)$ becomes a function of x and the times $\{t_{2l+1}\}$ and obeys the string equation

$$\sum_{l \geq 1} (2l+1)t_{2l+1} \hat{\mathcal{D}} \mathbf{R}_l[u] = -vx. \quad (26)$$

where $\hat{\mathcal{D}} = \partial + 2v$, $u = v^2 - v'$, and $\mathbf{R}_k[u]$ are the Gel'fand-Dikii potentials defined through the recursion relation

$$\partial \mathbf{R}_{k+1}[u] = \left(\frac{1}{4} \partial^3 - \frac{1}{2} (\partial u + u \partial) \right) \mathbf{R}_k[u], \quad \mathbf{R}_0[u] = \frac{1}{2}. \quad (27)$$

The dependence of v on the times $\{t_{2l+1}\}$ is given by the mKdV flows

$$\frac{\partial v}{\partial t_{2k+1}} = -\partial \hat{\mathcal{D}} \mathbf{R}_k[u]. \quad (28)$$

It is very important that (28) is compatible with the string equation. It can be shown (see also section 5) that solutions to the string equation flow with (28) to other solutions of (26). The k^{th} multicritical point is reached when $t_{2k+1} = -\frac{a_k}{2k+1}$ and all other times are zero. In this case the string equation becomes

$$\hat{\mathcal{D}} \mathbf{R}_k[u] = a_k v x. \quad (29)$$

This is a $2k^{\text{th}}$ order differential equation which as $x \rightarrow \infty$ has asymptotic solutions of the form

$$v \sim x^{\frac{1}{2k}} \left(1 + \sum_{l=1}^{\infty} v_l x^{-l(2+\frac{1}{k})} \right), \quad (30)$$

which upon the identification $\kappa^2 = x^{-(2+\frac{1}{k})}$ gives the genus expansion of the specific heat

$$v^2 \sim x^{\frac{1}{k}} \left(1 + \sum_{h=1}^{\infty} f_h \kappa^{2g} \right), \quad (31)$$

where $f_h = 2v_h + \sum_{l_1+l_2=h} v_{l_1} v_{l_2}$.

The connection to the τ -function formalism of the mKdV hierarchy is shown by noting that the specific heat v^2 can be written in the form [17]

$$v^2 = -\partial^2 \log(\tau_1 \tau_2) \quad (32)$$

with τ_1 and τ_2 the τ -functions of the mKdV hierarchy (28). These are simply connected to the Miura transformed functions $u_1 = v^2 + v'$ and $u_2 = v^2 - v'$ by $u_i = -2\partial^2 \log \tau_i$, $i = 1, 2$. Then the partition function is given by

$$Z = \tau_1 \cdot \tau_2, \quad (33)$$

which is the continuum analog of (13).

The Virasoro constraints [17] are obtained by first substituting (28) into (26) and then using (32). The result is

$$L_0 \tau_i = \mu \tau_i, \quad (34)$$

with $L_0 = \sum_{k=0}^{\infty} (k + \frac{1}{2}) t_{2k+1} \frac{\partial}{\partial t_{2k+1}} + \frac{1}{16}$ and μ an arbitrary constant. The flows and the recursion relations relate L_{n+1} to L_n and one obtains

$$L_n \tau_i = 0 \quad \text{with} \quad n \geq 1, \quad (35)$$

where $L_n = \sum_{k=0}^{\infty} (k + \frac{1}{2}) t_{2k+1} \frac{\partial}{\partial t_{2(k+n)+1}} + \frac{1}{2} \sum_{k=1}^n \frac{\partial^2}{\partial t_{2k-1} \partial t_{2(n-k)+1}}$. We will further discuss the Virasoro constraints in section 5.

4. The mKdV Hierarchy and the Sato Grassmannian

As we already mentioned in the introduction, the analysis of the solutions of the string equation in the Sato Grassmannian Gr depends crucially on the association of the mKdV τ -functions τ_1 and τ_2 to points V_1 and V_2 in the big cell of the Sato Grassmannian $Gr^{(0)}$. In this section we take a pedestrian approach to explaining this association and the reader familiar with the subject might want to skip to the next section. For more rigorous treatments on the subject see [22] and the references therein.

Since the Sato Grassmannian is an infinite dimensional generalization of finite dimensional Grassmannians, we start by reviewing the relevant concepts in the finite dimensional case. For a nice review along these lines see [29]. The Grassmannian $Gr(k, N)$ consists of all k -dimensional linear subspaces of C^N . A point $V \in Gr(k, N)$ is described by a basis $\{v_i\}$ with $i = 1, \dots, k$ and a basis of the orthogonal complement of V $\{w_i\}$ with $i = k + 1, \dots, N$. Then the pair (v, w) specifies a point in $Gr(k, N)$. A pair (v', w') , however, gives the same point if

$$(v', w') = (v, w) \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$

Then

$$Gr(k, N) \simeq GL(N)/P$$

with $P = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \right\}$. The relation between $Gr(k, N)$ and fermions is established by considering the $GL(N)$ representation on a fermionic Fock space F defined by the vacua

$$|k\rangle = e_1 \wedge \dots \wedge e_k \quad \langle k| = i_{e_k} \dots i_{e_1} \quad \langle i|k\rangle = \delta_{ik}, \quad (36)$$

where $\{e_i\}$ is a basis of C^N and $i_{e_i}(e_j) = \delta_{ij}$ is the inner product operator. The fermionic operators are defined by

$$\psi_i^\dagger = e_i \wedge |\chi\rangle \quad \psi_i^\dagger = i_{e_i} |\chi\rangle, \quad (37)$$

and satisfy canonical anticommutation relations

$$\{\psi_i, \psi_j^\dagger\} = \delta_{ij}, \quad \{\psi_i, \psi_j\} = \{\psi_i^\dagger, \psi_j^\dagger\} = 0. \quad (38)$$

The vacua $|k\rangle$ carry charge k and $\psi_i^\dagger(\psi_i)$ create a charge $+1(-1)$. Then

$$\begin{aligned} \psi_i^\dagger |k\rangle = 0 \quad i = 1, \dots, k & \quad \psi_i |k\rangle = 0 \quad i = k + 1, \dots, N \\ \langle k| \psi_i^\dagger = 0 \quad i = k + 1, \dots, N & \quad \langle k| \psi_i = 0 \quad i = 1, \dots, k, \end{aligned} \quad (39)$$

The Plucker embedding is defined by assigning to every point $V \in Gr(k, N)$ a state

$$|v\rangle = c v_1 \wedge \dots \wedge v_k \quad \text{with} \quad v_i = \sum v_{ij} e_j, \quad (40)$$

where $\{v_i\}$ is a basis of V and c is an arbitrary constant. A change of basis $v_i \rightarrow a_{ij} v_j$ corresponds to $c \rightarrow (\det a) c$ and the state $|v\rangle$ is well defined. The condition

$$\psi^\dagger[v_i] |v\rangle = 0 \quad \forall i, \quad (41)$$

with $\psi^\dagger[v_i] = \sum v_{ij} \psi_i^\dagger$ defines equivalently the state $|v\rangle$ up to the constant c .

Then $a \in \mathfrak{gl}(N)$ acts on F by

$$\hat{a}|\chi\rangle = \sum \psi_i^\dagger a_{ij} \psi_j |\chi\rangle \quad |\chi\rangle \in F, \quad (42)$$

and on the space of operators on F by

$$[\psi_i, \hat{a}] = \sum_k a_{ik} \psi_k, \quad [\hat{a}, \psi_i^\dagger] = \sum_k \psi_k^\dagger a_{ki}. \quad (43)$$

The action of $g \in \mathrm{GL}(N)$ is defined by exponentiation of (42). For example

$$\hat{g} \psi_{i_1}^\dagger \psi_{i_2}^\dagger \dots \psi_{i_1} \psi_{i_2} \dots |0\rangle = (\psi^\dagger g)_{i_1} (\psi^\dagger g)_{i_2} \dots (g\psi)_{i_1} (g\psi)_{i_2} \dots |0\rangle \quad (44)$$

with $(\psi^\dagger g)_i \equiv \psi_j^\dagger g_{ji}$ and $(g\psi)_i \equiv g_{ij} \psi_j$. Then a $\mathfrak{gl}(N)$ operator a acting on $V \in \mathrm{Gr}(k, N)$ by $av = \sum (a_{ij} v_j) e_i$ corresponds to a fermionic operator $\hat{a} = \sum \psi_i^\dagger a_{ij} \psi_j$. Then if $\hat{a}_1 \leftrightarrow a_1$ and $\hat{a}_2 \leftrightarrow a_2$, equations (43) give

$$[\hat{a}_1, \hat{a}_2] \leftrightarrow [a_1, a_2]. \quad (45)$$

Moreover note that if

$$\hat{a}|v\rangle = \text{const.}|v\rangle \Leftrightarrow aV \subset V. \quad (46)$$

The state $|v\rangle$ belongs to the $\mathrm{GL}(N)$ orbit of the state $|k\rangle$. Since for $|v\rangle = v_1 \wedge \dots \wedge v_k$ every vector v_i can be written in the form $v_i = g e_i$ for some fixed $g \in \mathrm{GL}(N)$, we have that $|v\rangle = \hat{g}|k\rangle$ as defined in (44). Therefore the image of $\mathrm{Gr}(k, N)$ under the Plucker embedding can be identified with the orbit $\mathrm{GL}(N)|k\rangle$.

The τ -functions are given by fermion correlators

$$\tau_V^{\mathcal{O}} = \langle \mathcal{O} \rangle_V = \langle k | \mathcal{O} | v \rangle, \quad (47)$$

with \mathcal{O} a zero charge operator. Since the topology of $\mathrm{Gr}(k, N)$ is non-trivial, we divide it into cells $(\mathcal{U}_a, a \in I)$. A point $V \in \mathcal{U}_a$ is represented by a basis $\{v_i^{(a)}\}$ and the state $|v\rangle^{(a)} = v_1^{(a)} \wedge \dots \wedge v_k^{(a)}$. Then if $V \in \mathcal{U}_a \cap \mathcal{U}_b$ we have $v_k^{(a)} = a_{ki}^{(ab)} v_i^{(b)}$ and

$$\tau_V^{\mathcal{O}(a)} = \det a^{(ab)} \tau_V^{\mathcal{O}(b)}$$

Therefore the τ -functions are really sections of a determinant line bundle over $\mathrm{Gr}(k, N)$ whose transition functions are given by $\det a^{(ab)}$.

Most of the results carry over almost unchanged to the infinite dimensional case. For the infinite dimensional vector space we consider the space of formal Laurent series

$$H = \left\{ \sum_n a_n z^n, \quad a_n = 0 \quad \text{for } n \gg 0 \right\}$$

and its decomposition

$$H = H_+ \oplus H_-,$$

where $H_+ = \left\{ \sum_{n \geq 0} a_n z^n, \quad a_n = 0 \quad \text{for } n \gg 0 \right\}$. Then the big cell of the Sato Grassmannian $Gr^{(0)}$ consists of all subspaces $V \subset H$ comparable to H_+ , in the sense that the natural projection $\pi_+ : V \rightarrow H_+$ is an isomorphism. Then V admits a basis of the form $\{\phi_i(z)\}_{i \geq 0}$ where $\phi_i(z) = z^i + \text{lower order terms}$. The Plucker embedding (40) is defined by the semi-infinite wedge product

$$|v\rangle = c \phi_1(z) \wedge \phi_2(z) \wedge \dots \quad (48)$$

Care has to be taken so that a $GL(\infty)$ change of basis $\phi_i(z) \rightarrow a_{ij} \phi_j(z)$ does not introduce infinities, since $\det a$ can be infinite. We choose a set of admissible bases for $V \in Gr$ to be those whose matrix relating $\{\pi_+(\phi_i(z))\}_{i \geq 0}$ to $\{z^i\}_{i \geq 0}$ differs from the identity by an operator of trace class. Then the fermionic representation is defined on the Fock space built on the vacuum state of zero charge

$$|0\rangle = 1 \wedge z \wedge z^2 \wedge \dots, \quad (49)$$

by fermions ψ_i^\dagger and ψ_i defined as in (37). The states ($m > 0$)

$$|m\rangle = \psi_m^\dagger \dots \psi_1^\dagger |0\rangle, \quad |-m\rangle = \psi_{-m+1} \dots \psi_0 |0\rangle \quad (50)$$

are the filled states with charge m and $-m$ respectively. The generalization of $\mathfrak{gl}(N)$ is given by $\mathfrak{gl}(\infty)$ and is represented on F by its central extension $\mathfrak{gl}^*(\infty)$ with

$$\hat{a} = \sum_{i,j} : \psi_i^\dagger a_{ij} \psi_j : \quad (51)$$

where

$$: \psi_i^\dagger \psi_j : = \psi_i^\dagger \psi_j - \langle \psi_i^\dagger \psi_j \rangle = \begin{cases} \psi_i^\dagger \psi_j & i > 0 \\ -\psi_j \psi_i^\dagger & i \leq 0 \end{cases} \quad (52)$$

is the normal ordering. The reason for introducing normal ordering is that the naive operator $\sum_{i,j} \psi_i^\dagger a_{ij} \psi_j$ maps an admissible basis to a non-admissible one.

The connection of the fermion representation of $Gr^{(0)}$ and the KP and mKP hierarchies is made explicit by making use of the boson-fermion equivalence in two dimensions. The fermionic currents

$$J_n = \sum_{r \in \mathbb{Z}} : \psi_{n-r}^\dagger \psi_r : \quad n \in \mathbb{Z} \quad (53)$$

satisfy the bosonic commutation relations

$$[J_m, J_n] = m\delta_{m,-n}. \quad (54)$$

By representing the bosonic Fock space by $B \cong \mathbb{C}[t_1, t_2, \dots; u, u^{-1}]$, the space of polynomials in $t_1, t_2, \dots; u, u^{-1}$, $\frac{\partial}{\partial t_n}$ and $-nt_{-n}$ (with $n \geq 0$) act as creation and annihilation operators on B satisfying the algebra (54). Then fermionic operators can be mapped to operators acting on B and states in F to states in B by mapping the state $|m\rangle$ of F to u^m . Then the k^{th} modified KP hierarchy τ -functions correspond to correlators (47) where $\mathcal{O} = e^{\sum_{p \geq 1} t_p J_p}$ and the states $|v\rangle$ correspond to the $GL(\infty)$ orbit of $|i\rangle$ with $i = 0, \dots, k-1$. In particular the solutions to the second mKP hierarchy is given by two τ -functions

$$\tau_i(t) = \langle i-1 | \exp\left\{ \sum_{p \geq 1} t_p J_p \right\} g | i-1 \rangle \quad (i = 1, 2), \quad (55)$$

where $g \in GL(\infty)$. The modified KdV hierarchy that arises in UMM is the second reduced mKP hierarchy of the above equation and it corresponds to eliminating from (55) the dependence on the even times $\{t_{2n}\}$. Therefore every solution $\tau_1(t)$ and $\tau_2(t)$ of the mKdV hierarchy corresponds to points $V_1(t)$ and $V_2(t)$ in $Gr^{(0)}$ given by the states $|v_i(t)\rangle = \exp\left\{ \sum_{p \geq 1} t_p J_p \right\} g | i-1 \rangle$. Then the time dependence of $V_i(t)$ is given by

$$\frac{\partial}{\partial t_{2k+1}} |v_i(t)\rangle = J_{2k+1} |v_i(t)\rangle \quad \text{and} \quad J_{2k} |v_i(t)\rangle = 0, \quad (56)$$

or by using the correspondence (45)

$$\frac{\partial}{\partial t_{2k+1}} V_i(t) = z^{2k+1} V_i(t) \quad \text{and} \quad z^{2k} V_i(t) \subset V_i(t). \quad (57)$$

Then $V_i(t) = \exp\left\{ \sum_k t_{2k+1} z^{2k+1} \right\} V_i \equiv \gamma(t, z) V_i$.

5. The Solutions to the String Equation

Since to every solution of the mKdV hierarchy correspond points $V_1(t)$ and $V_2(t)$ in $Gr^{(0)}$ satisfying (57), one would like to determine those that are solutions to the string equation (25). This is particularly easy because the commutator $[\mathcal{P}, \mathcal{Q}_-]$ is equal to a constant [22].

Consider the space Ψ of pseudodifferential operators $W = \sum_{i \leq k} w_i(x) \partial^i$ where the functions $w_i(x)$ are taken to be formal power series (i.e. $w_i(x) = \sum_{k \geq 0} w_{ik} x^k$, $w_{ik} = 0$, $k \gg 0$). W is then a pseudodifferential operator of order k . It is called monic if $w_k(x) = 1$ and normalized if $w_{k-1}(x) = 0$. The space Ψ forms an algebra. The space of monic, zeroth-order pseudodifferential operators forms a group \mathcal{G} .

There is a natural action of Ψ on H defined by

$$\begin{aligned} x^m \partial^n : H &\rightarrow H \\ \phi &\rightarrow \left(-\frac{d}{dz}\right)^m (z)^n \phi. \end{aligned}$$

Then it is well known [30] that every point $V \in Gr^{(0)}$ can be uniquely represented in the form $V = SH_+$ with $S \in \mathcal{G}$. This will imply that for every operator \mathcal{Q}_- we can uniquely associate a pair of points $V_1, V_2 \in Gr^{(0)}$.

Indeed, consider S_1 and $S_2 \in \mathcal{G}$ such that

$$\hat{S} \mathcal{Q}_- \hat{S}^{-1} = \tilde{\mathcal{Q}}_- \tag{58}$$

where

$$\hat{S} = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \quad \tilde{\mathcal{Q}}_- = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}. \tag{59}$$

Then

$$\begin{aligned} S_1(\partial + v)S_2^{-1} &= \partial, \\ S_2(\partial - v)S_1^{-1} &= \partial. \end{aligned} \tag{60}$$

S_1 and S_2 can be shown to exist and are unique up to a redefinition $S_i \rightarrow S_i R$ with $R = 1 + \sum_{i \geq 0} r_i \partial^{-i}$ and r_i constants.

Since $V \subset Gr^{(0)}$ is given uniquely by $V = SH_+$, the operator \mathcal{Q}_- determines two spaces $V_1 = S_1 H_+$ and $V_2 = S_2 H_+$. Conversely given spaces V_1 and V_2 determine \mathcal{Q}_-

uniquely. The operator \mathcal{Q}_- , however, is a differential operator and V_1, V_2 cannot be arbitrary. Indeed, since every differential operator leaves H_+ invariant, we obtain

$$\begin{aligned} (\partial + v) H_+ \subset H_+ &\Leftrightarrow S_1^{-1} \partial S_2 H_+ \subset H_+ \\ &\Leftrightarrow \partial V_2 \subset V_1 \\ &\Leftrightarrow z V_2 \subset V_1 \end{aligned} \quad (61)$$

Similarly, $z V_1 \subset V_2$. Notice that these conditions are consistent with the second equation in (57).

The transformation $\hat{S} \mathcal{Q}_- \hat{S}^{-1} = \tilde{\mathcal{Q}}_-$ is a similarity transformation and the string equation will be left invariant if we define $\tilde{\mathcal{P}}_{(k)} = \hat{S} \mathcal{P}_{(k)} \hat{S}^{-1}$. Then the solution to $[\tilde{\mathcal{P}}_{(k)}, \tilde{\mathcal{Q}}_-] = 1$ is easily found to be given by

$$\tilde{\mathcal{P}}_{(k)} = \begin{pmatrix} 0 & A_k \\ A_k & 0 \end{pmatrix}, \text{ where } A_k = \frac{d}{dz} + \sum_{i=0}^k \alpha_i z^{2i} \quad \text{and } \alpha_i = \text{const.} \quad (62)$$

The requirement that \mathcal{P} be a differential operator is equivalent to the conditions $A_k V_1 \subset V_2$ and $A_k V_2 \subset V_1$. The space of solutions to the string equation is the space of operators \mathcal{Q}_- such that there exists $\mathcal{P}_{(k)}$ with $[\mathcal{P}_{(k)}, \mathcal{Q}_-] = 1$. We conclude that this space is isomorphic to the set of elements $V_1, V_2 \subset Gr^{(0)}$ that satisfy the conditions:

$$\begin{aligned} z V_1 \subset V_2 \quad z V_2 \subset V_1 \\ A_k V_1 \subset V_2 \quad A_k V_2 \subset V_1 \end{aligned} \quad (63)$$

for some $A_k = \frac{d}{dz} + \sum_{i=0}^k \alpha_i z^{2i}$.

The string equation is left invariant by the flows (57). Indeed

$$\begin{aligned} z \gamma(z, t) V_1 \subset \gamma(t, z) V_2 &\Rightarrow z V_1(t) \subset V_2(t) \\ A_k(t) \gamma(z, t) V_1 \subset \gamma(t, z) V_2 &\Rightarrow A_k(t) V_1(t) \subset V_2(t), \end{aligned} \quad (64)$$

where

$$A_k(t) \equiv \gamma A_k \gamma^{-1} = A_k - \sum_k (2k+1) t_{2k+1} z^{2k} \quad (65)$$

and analogous equations with V_1 and V_2 interchanged.

It is now easy to see that (63) implies the Virasoro constraints for the τ -functions. Without going into the details (see [23]), we first notice that the operators $l_n = z^{2n+1} A$ leave V_i invariant

$$z^{2n+1} A V_i \subset V_i. \quad (66)$$

Then using the correspondence (45), one can construct the corresponding fermion operators \hat{l}_n and from them their bosonic counterparts L_n [23]. These have the exact form as equations (34)–(35). We can immediately see that they form a Virasoro algebra by noting that $l_n \sim z^{2n+1} \frac{d}{dz}$ are the generators of the Virasoro algebra and by using lemma (45). Since l_n leave V_i invariant, then using (46) we conclude that the operators L_n annihilate the τ -functions τ_1 and τ_2 and obtain equations (34) and (35).

We conclude this section by showing how conditions (63) can be used to calculate the space of solutions to the string equation [22]. We will start by describing the spaces V_1, V_2 .

First choose vectors $\phi_1(z), \phi_2(z) \in V_1$, such that

$$\phi_1(z) = 1 + \text{lower order terms}, \quad \phi_2(z) = z + \text{lower order terms}.$$

Then the condition $z^2 V_1 \subset V_1$ and $\pi_+(V_1) \cong H_+$ shows that we can choose a basis for V_1

$$\phi_1, \phi_2, z^2 \phi_1, z^2 \phi_2, \dots$$

Since $z V_1 \subset V_2$ and $\pi_+(V_2) \cong H_+$ we can choose a basis for V_2 to be

$$\psi, z\phi_1, z\phi_2, z^3 \phi_1, z^3 \phi_2, \dots$$

where $\psi(z) = 1 + \text{lower order terms}$. Using $z V_2 \subset V_1$ we have $z\psi = \alpha\phi_1 + \beta\phi_2$. Choose ϕ_1, ϕ_2 such that $z\psi = \phi_2$. Then we obtain the following basis for V_1, V_2 ($\phi \equiv \phi_1$):

$$\begin{aligned} V_1 : & \quad \phi, z\psi, z^2 \phi, z^3 \psi, \dots \\ V_2 : & \quad \psi, z\phi, z^2 \psi, z^3 \phi, \dots \end{aligned} \tag{67}$$

Then it is clear that ϕ, ψ specify the spaces V_1, V_2 . Using the conditions $AV_1 \subset V_2$ and $AV_2 \subset V_1$ we obtain

$$\begin{aligned} \left(\frac{d}{dz} + f_k(z^2)\right)\phi &= P_{00}(z)\phi + P_{01}(z)\psi \\ \left(\frac{d}{dz} + f_k(z^2)\right)\psi &= P_{10}(z)\phi + P_{11}(z)\psi. \end{aligned} \tag{68}$$

Since a generic system of the form (68) will lead to exponential evolution of the functions ϕ and ψ , the requirement that they keep their polynomial form puts severe conditions on $P_{ij}(z)$. A detailed calculation shows that the space of solutions to the string equation (25) is the two fold covering of the space of matrices $\begin{pmatrix} P_{ij}(z) \end{pmatrix}$ with polynomial entries in z such that $P_{01}(z)$ and $P_{10}(z)$ are even polynomials having equal degree and

leading terms and $P_{00}(z)$ and $P_{11}(z)$ are odd polynomials satisfying the conditions $P_{00}(z) + P_{11}(z) = 0$ and $\deg P_{00}(z) < \deg P_{01}(z)$.

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