Dynamical Compactification of Extra Dimensions in the Euclidean IKKT Matrix Model via Spontaneous Symmetry Breaking

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The IKKT matrix model has been conjectured to provide a promising nonperturbative formulation of superstring theory. In this model, spacetime emerges dynamically from the microscopic matrix degrees of freedom in the large-N limit, and Monte Carlo simulations of the Lorentzian version provide evidence of an emergent (3+1)-dimensional expanding space-time. In this talk, we discuss the Euclidean version of the IKKT matrix model and provide evidence of dynamical compactification of the extra dimensions via the spontaneous symmetry breaking (SSB) of the 10D rotational symmetry. We perform numerical simulations of a system with a severe complex action problem by using the complex Langevin method (CLM). The CLM suffers from the singular-drift problem and we deform the model in order to avoid it. We study the SSB pattern as we vary the deformation parameter and we conclude that the original model has an SO(3) symmetric vacuum, in agreement with previous calculations using the Gaussian expansion method (GEM). We employ the GEM to the deformed model and we obtain results consistent with the ones obtained by CLM.

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1. Introduction

The type IIB matrix model [1], also known as the IKKT model, has been proposed as a nonperturbative formulation of superstring theory. Spacetime emerges dynamically from the eigenvalues of the bosonic matrices in the large-N limit [2] and it is possible that the extra dimensions are compactified *dynamically* via a non perturbative mechanism. Furthermore, it is possible that a unique vacuum exists in the theory, thereby solving the so-called landscape problem.

The action of the model can be formally viewed as the dimensional reduction of the $10D$, $\mathscr{N} = 1$, SU(N) super Yang-Mills (SYM) theory to zero dimensions. Numerical simulations [3, 4, 5, 6, 7, 8, 9, 10, 11] suggest that *continuum* time emerges dynamically and 3 dimensional space undergoes rapid expansion after a critical time t_c, while the remaining 6 dimensions do not expand. The cosmological time is defined by the eigenvalues of the temporal matrix A_0, and their infinite and homogeneous distribution in the large N limit is a nontrivial dynamic result, allowing one to define a continuum and infinitely extending time. Furthermore, the spatial matrices A_i have a band diagonal structure in the SU(N) basis used to diagonalize A_0 that makes possible to define space at a given time. Then one can study the structure of space, and especially its size, which for $t > t_c$, it is found to be expanding in the three dominant dimensions. This expansion is exponential at early times [5] and becomes a power law at later times [8], making possible the emergence of a realistic cosmology from the dynamics of the microscopic degrees of freedom.

Simulations of the model using Monte Carlo techniques is hard because of the complex action problem. This originates from the e^{iS_b} factor in the path integral, where S_b is the bosonic part of the action. This problem can be avoided by integrating the scale factor of the bosonic matrices [3, 4, 5, 6, 7, 8, 9]. One obtains a sharply peaked function of S_b near the origin, which can then be approximated by a Gaussian. The complex action problem vanishes, but the approximation favors singular spatial configurations [10]. In [11] the 6D model at late times was studied without this approximation, and the Complex Langevin Method (CLM) [12, 13] was used in order to confront the complex action problem. A two parameter deformation of the model was used for making the simulations possible, and provided evidence that when these parameters become small, configurations with non trivial spatial structure dominate the path integral.

In this work we study the Euclidean version of the IKKT matrix model. This model, as well as related ones, have been studied extensively [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27], because they are more tractable numerically and are finite [15, 16], making the introduction of infrared cutoffs unnecessary. There is strong evidence that dynamical compactification of extra dimensions in these models is realized via spontaneous symmetry breaking (SSB) of the SO(10) rotational symmetry of the model. The effective action of the model $S_{\text{eff}} = S_{\text{R}} + i\Gamma$, obtained after integrating out the fermionic degrees of freedom, is complex. The fluctuations of the phase $e^{i\Gamma}$ for a generic SO(d), $d > 3$ symmetric configuration, suppress $d > 3$ configurations, thereby favoring lower dimensional configurations and triggering the SSB [28]. The evidence is provided mainly by studying the model using the Gaussian Expansion Method (GEM). In [29, 30, 31, 32, 33], it was shown that the SO(3) vacuum of the IIB matrix model has the lowest free energy and the ratio between the extended three directions and the shrunken seven directions is a finite number, larger than one. Furthermore, the study of phase quenched models, show no SSB of the SO(10) rotational symmetry [14, 17, 18, 19], consistent with the expectation that the complex phase $e^{i\Gamma}$

plays a critical role in suppressing large dimensional configurations.

The Monte Carlo simulations of the Euclidean IIB matrix model confront a very strong complex action problem, which needs to be addressed by using special methods. Straightforward reweighting is not possible, since it makes the computational effort to increase exponentially with the matrix size N, which needs to be extrapolated to infinity. A density of states based method was used in [20, 34, 35, 22, 23, 24, 25], allowing one to study relatively large systems and providing evidence that SSB occurs from first principles. But it turned out to be hard to determine the pattern of SSB, and it was not until recently [26, 36, 27] that this question could be addressed by using the Complex Langevin Method (CLM)[12, 13, 37]. The CLM is applied by complexifying the degrees of freedom and defining a stochastic process where the expectation values with respect to this process are equal to the expectation values defined in the original system. This method fails in many cases, and it was not until recently that with the help of new techniques [38, 39, 40, 41, 42] it was possible to meet the conditions for the equivalence of the stochastic process defined by the CLM and the original system, and obtain correct results for an extended range of parameters [43, 44, 45, 21, 46, 47, 48]. The CLM has been applied successfully to many systems in lattice quantum field theory [49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 43, 60, 61, 62, 63, 64, 65, 48, 66, 67, 68, 69, 70], and matrix models [71, 72, 21, 73, 26, 11, 48, 74, 75, 36, 27].

In this talk we review the application of the CLM to the Euclidean type IIB matrix model discussed in [26, 36, 27]. The successful application of the method requires the deformation of the model by two parameters m_f and ε, which deform the Dirac operator and the Bosonic part of the action with mass like terms. This way, one can avoid the singular drift problem [21], which can lead to erroneous results in systems where singularities of the drift dominate in the stochastic process of the CLM [40]. In the IIB matrix model the singular drift problem occurs when the eigenvalues of the Pfaffian, when put into a canonical Youla's form, accumulate near zero. This method was applied successfully on the 6D IIB model [26], obtaining SSB to SO(3), consistent with GEM calculations in [32]. The 10D model was studied in [27]. In that case, the simulations were harder because the fermionic degrees of freedom increase by a factor of four and the finite size effects are more severe due to the increase of the dimensionality of the target space. The results are consistent with GEM calculations [33], showing SSB to SO(3). A careful extrapolation of the results, first to large N, then to small ε and finally to small m_f is necessary in order to obtain the correct large N limit of the original model. This makes the calculation tricky, and the Monte Carlo simulations were done in parallel with GEM calculations of the m_f–deformed model. By computing the free energy of vacua of different dimensionality using the GEM, it was possible to find physical solutions for the SO(d), $d = 4, 6, 7$ ansatzes, calculated up to three loops. As m_f decreases, lower dimensional vacua turn out to have lower free energies and the pattern of SSB is similar to the one obtained by the CLM.

2. The IKKT matrix model

The IKKT matrix model [1] is defined by the action:

$$S = S_b + S_f \, , \text{ where} \tag{2.1}$$

$$S_b = -\frac{1}{4} N \operatorname{tr}[A_\mu, A_\nu][A^\mu, A^\nu] \, , \tag{2.2}$$

$$S_f = -\frac{1}{2} N \operatorname{tr} \left(\psi_\alpha (\mathscr{C}\Gamma^\mu)_{\alpha\beta} [A_\mu, \psi_\beta] \right) \, . \tag{2.3}$$

For $D = 10$, the matrices A_μ ($\mu = 0, 1, 2, \ldots, D-1$) are $N \times N$ traceless Hermitian matrices which transform like vectors, and ψ_α ($\alpha = 1, 2, \ldots, 2^{D/2-1}$) are $N \times N$ traceless matrices with Grassmann entries which transform like Majorana-Weyl spinors. The $2^{D/2-1} \times 2^{D/2-1}$ matrices Γ^μ and $\mathscr{C}$ are the gamma matrices after Weyl projection and the charge conjugation matrix, respectively, in ten dimensions.

In order to obtain the Euclidean version of the IKKT matrix model, we perform the Wick rotation

$$A_0 = iA_D \, , \qquad \Gamma^0 = -i\Gamma^D \, . \tag{2.4}$$

The metric now is $\delta_{\mu\nu}$ ($\mu, \nu = 1, \ldots, D$) and the partition function becomes

$$Z = \int dA d\psi \, e^{-S} = \int dA \, e^{-S_b} \operatorname{Pf} \mathscr{M} \, , \tag{2.5}$$

where the $2^{D/2-1}(N^2-1) \times 2^{D/2-1}(N^2-1)$ anti-symmetric matrix $\mathscr{M}$ is defined by its action

$$\psi_\alpha \to (\mathscr{M}\psi)_\alpha = (\mathscr{C}\Gamma^\mu)_{\alpha\beta} [A_\mu, \psi_\beta] \tag{2.6}$$

on the linear space of traceless complex $N \times N$ matrices. The model has an SO(D) rotational symmetry acting on A_μ and ψ_α.

Dynamical compactification of extra dimensions can be realized via the SSB of the SO(D) symmetry to SO(d), with $d < D$. The order parameters of the SSB can be taken to be [21, 26]

$$\lambda_\mu = \frac{1}{N} \operatorname{tr} (A_\mu)^2 \, , \quad \mu = 1, \ldots, D \, , \tag{2.7}$$

where no sum over μ is taken. Then, we break the SO(D) symmetry explicitly by deforming the model by

$$\Delta S_b = \frac{1}{2} N \varepsilon \sum_{\mu=1}^{D} m_\mu \operatorname{tr} (A_\mu)^2 \tag{2.8}$$

using the parameters ε and $0 < m_1 \leq \ldots \leq m_D$. The SSB pattern will arise in the $\varepsilon \to 0$ limit *after* taking the large-N limit. For finite N, the choice $0 < m_1 \leq \ldots \leq m_D$ yields $\langle \lambda_1 \rangle \geq \ldots \geq \langle \lambda_D \rangle$. If there is no SSB, all $\langle \lambda_\mu \rangle$ will be equal in the $N \to \infty$ and $\varepsilon \to 0$ limits. Otherwise, SSB occurs and we will conclude that some dimensions of space are larger than others.

The SSB of the SO(D) rotational symmetry has been studied by using the GEM. This method uses a systematic expansion around a Gaussian action S_0, which contains many parameters. In order to make the calculation tractable, the number of parameters is reduced by considering SO(d)–symmetric ansatzes for $d < D$. The free energy and the expectation values of the observables is

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calculated in the expansion around S_0 as a function of these parameters, and the physical solutions are computed by finding a region in the parameter space where the results are independent of these parameters. In this way, one obtains *nonperturbative* information about the model [76]. When applied to the IKKT matrix model [29, 30, 77, 78, 79, 80, 81, 31, 82, 32, 33], the GEM yields the free energy and the average extent of spacetime in each direction. The $D = 6$ case was studied in [32], and free energy and the average extent of spacetime was calculated to fifth order. The SO(3) ansatz was found to have the lowest free energy, showing SSB to SO(3). The extended directions where found to have an extent $R^2(d)$ for $\mu = 1, \ldots, d$, whereas the shrunken directions to have and extent $r^2 = \langle \lambda_\mu \rangle \approx 0.223$, $\mu = d + 1, \ldots, D$, which is independent of d. The values of $R^2(d)$ are such that

$$R^d(d)\, r^{D-d} \approx l^D \,, \tag{2.9}$$

where $v \equiv l^D$ is the spacetime volume. It turns out that $l^2 \approx 0.627$ and $R^2(3) \approx 1.76$. The $D = 10$ case was studied in [33], where a systematic expansion to third order was performed for SO(d) ansatzes with $2 \leq d \leq 7$. The SO(3) ansatz turns out to have the lowest free energy, also showing SSB to SO(3). In that case

$$R^2(3) \approx 3.27 \,, \qquad r^2 \approx 0.155 \,, \qquad l^2 \approx 0.383 \,. \tag{2.10}$$

These results are consistent with the ones obtained by Monte Carlo simulations [25]. They are finite, in contrast to the case of the Lorentzian model, where $R^2(d)$ expands indefinitely.

3. The Complex Langevin Method

In this section we review how to apply the CLM to the Euclidean IKKT matrix model. The partition function can be written as

$$Z = \int dA\, \mathrm{e}^{-S_{\mathrm{eff}}} \,, \tag{3.1}$$

where the effective action $S_{\mathrm{eff}} = S_{\mathrm{b}} - \log \mathrm{Pf}\,\mathcal{M}$ is complex. In the CLM the matrices A_μ become general complex traceless matrices, which amounts to the complexification of the degrees of freedom in the CLM. The time evolution in the fictitious Langevin time is given by

$$\frac{d\left(A_\mu(t)\right)_{ij}}{dt} = -\frac{\partial S_{\mathrm{eff}}[A_\mu(t)]}{\partial \left(A_\mu\right)_{ji}} + \left(\eta_\mu\right)_{ij}(t) \,. \tag{3.2}$$

The noise $\eta_\mu(t)$ is made from traceless Hermitian matrices whose elements are random variables obeying the Gaussian distribution $\propto \exp\left(-\frac{1}{4}\int \mathrm{tr}\left\{\eta_\mu(t)\right\}^2 dt\right)$. The first term on the right-hand side is the drift term

$$\frac{\partial S_{\mathrm{eff}}}{\partial \left(A_\mu\right)_{ji}} = \frac{\partial S_{\mathrm{b}}}{\partial \left(A_\mu\right)_{ji}} - \frac{1}{2}\mathrm{Tr}\left(\mathcal{M}^{-1}\frac{\partial \mathcal{M}}{\partial \left(A_\mu\right)_{ji}}\right) \,, \tag{3.3}$$

where Tr represents the trace of a $16(N^2 - 1) \times 16(N^2 - 1)$ matrix. The expectation value of an observable $\mathcal{O}[A_\mu]$ can be calculated from

$$\langle \mathcal{O}[A_\mu] \rangle = \frac{1}{T}\int_{t_0}^{t_0+T} \mathcal{O}[A_\mu(t)]dt \,, \tag{3.4}$$

where $A_\mu(t)$ is a general complex matrix solution of (3.2), t_0 is the thermalization time, and T is large enough in order to obtain good statistics. Upon complexification of the matrices $A_\mu(t)$, the observable $\mathscr{O}[A_\mu(t)]$ depends on general complex matrices. The analyticity of the function $\mathscr{O}[A_\mu]$ plays a crucial role in the proof of the validity of (3.4)[38, 39, 41]. The discretization of (3.2) is given by

$$\left(A_\mu\right)_{ij}(t+\Delta t) = \left(A_\mu\right)_{ij}(t) - \Delta t \frac{\partial S[A_\mu(t)]}{\partial \left(A_\mu\right)_{ji}} + \sqrt{\Delta t}\,\left(\eta_\mu\right)_{ij}(t)\,, \qquad (3.5)$$

which we have used in order to solve (3.2) numerically.

   The solutions of the stochastic equation (3.2) are random variables with distribution $P(A_\mu^{(R)}, A_\mu^{(I)}; t)$, where $A_\mu^{(R)}(t) = (A_\mu(t) + A_\mu^\dagger(t))/2$, $A_\mu^{(I)}(t) = (A_\mu(t) - A_\mu^\dagger(t))/2i$, and in order that in the $t \to \infty$ limit an observable $\mathscr{O}[A_\mu]$ has an expectation value in this distribution equal to $\langle \mathscr{O}[A_\mu] \rangle$, given by the path integral (2.5), certain conditions need to be met. In [41] it was shown that if the magnitude of the drift falls off exponentially or faster, then these conditions are met. For this, the $A_\mu(t)$ should not make long trips in the anti-Hermitian direction. Gauge cooling has been applied in order to restrict those trips and meet this condition[26, 27]. Adaptive stepsize techniques have also been applied in order to keep the stability of the time evolution. Furthermore, one has to avoid the singular drift problem. This problem occurs in the term with $\mathscr{M}^{-1}$ in Eq. (3.3), when the eigenvalues of $\mathscr{M}$ accumulate densely near zero. In order to avoid this problem, we deform the fermionic action by adding a term

$$\Delta S_{\mathrm{f}} = -i m_{\mathrm{f}} \frac{N}{2} \operatorname{tr}\left(\psi_\alpha (\mathscr{C}\Gamma_8\Gamma_9^\dagger\Gamma_{10})_{\alpha\beta}\psi_\beta\right)\,, \qquad (3.6)$$

where m_{f} is the deformation parameter. This term shifts the eigenvalue distribution of $\mathscr{M}$ away from the origin [21], but it also breaks the SO(10) symmetry down to SO(7) × SO(3) explicitly. Therefore, we examine if the remaining SO(7) symmetry breaks down to smaller subgroups as m_{f} is varied and discuss what occurs at $m_{\mathrm{f}} = 0$. As $m_{\mathrm{f}} \to \infty$, the fermionic degrees of freedom decouple and we obtained the so–called bosonic model. This model is known to be SO(10) symmetric and no SSB occurs [25].

   The severeness of the singular drift problem depends on the parameters m_{f} and ε. For large enough m_{f} the problem disappears, but as m_{f} is lowered the problem reappears for small enough ε. In our simulations we monitor the fall off of the drift and make sure that it falls of faster than exponentially. The singular drift problem is expected to vanish for large enough N [27].

4. Results

   In our numerical investigation we consider whether the remaining SO(7) symmetry is broken down to a smaller group. We study the model deformed by (2.8) and (3.6). We use $m_{\mathrm{f}} = 3.0$, 1.4, 1.0, 0.9, 0.7, in order to extrapolate to the undeformed IKKT model at $m_{\mathrm{f}} = 0$. The m_μ in Eq. (2.8) are chosen so that this term does not break SO(10) completely, because otherwise the spectrum of m_μ becomes too wide to make the $\varepsilon \to 0$ extrapolation reliably. For $m_{\mathrm{f}} = 3.0$, we choose $m_\mu = (0.5, 0.5, 0.5, 1, 2, 4, 8, 8, 8, 8)$, which enables us to distinguish SO(d) vacua with $d = 3, 4, 5, 6, 7$. For smaller values of m_{f}, we choose $m_\mu = (0.5, 0.5, 1, 2, 4, 8, 8, 8, 8, 8)$, which enables us to distinguish SO(d) vacua with $d = 2, 3, 4, 7$. In particular, we may confirm that the SO(3) symmetry remains unbroken by seeing that $\langle \lambda_1 \rangle = \langle \lambda_2 \rangle$ and $\langle \lambda_3 \rangle$ agree in the $N \to \infty$ and

$\varepsilon \to 0$ limits. On the other hand, this choice of m_μ has a drawback that λ_6 and λ_7 are mixed up because of $m_6 = m_7$, and hence one cannot distinguish SO(5) and SO(6) vacua. This does not cause any harm, however, as far as we find that $\langle \lambda_4 \rangle$ and $\langle \lambda_5 \rangle$ do not agree in the $N \to \infty$ and $\varepsilon \to 0$ limits, which implies that the SO(7) symmetry is broken to SO(4) or lower symmetries.
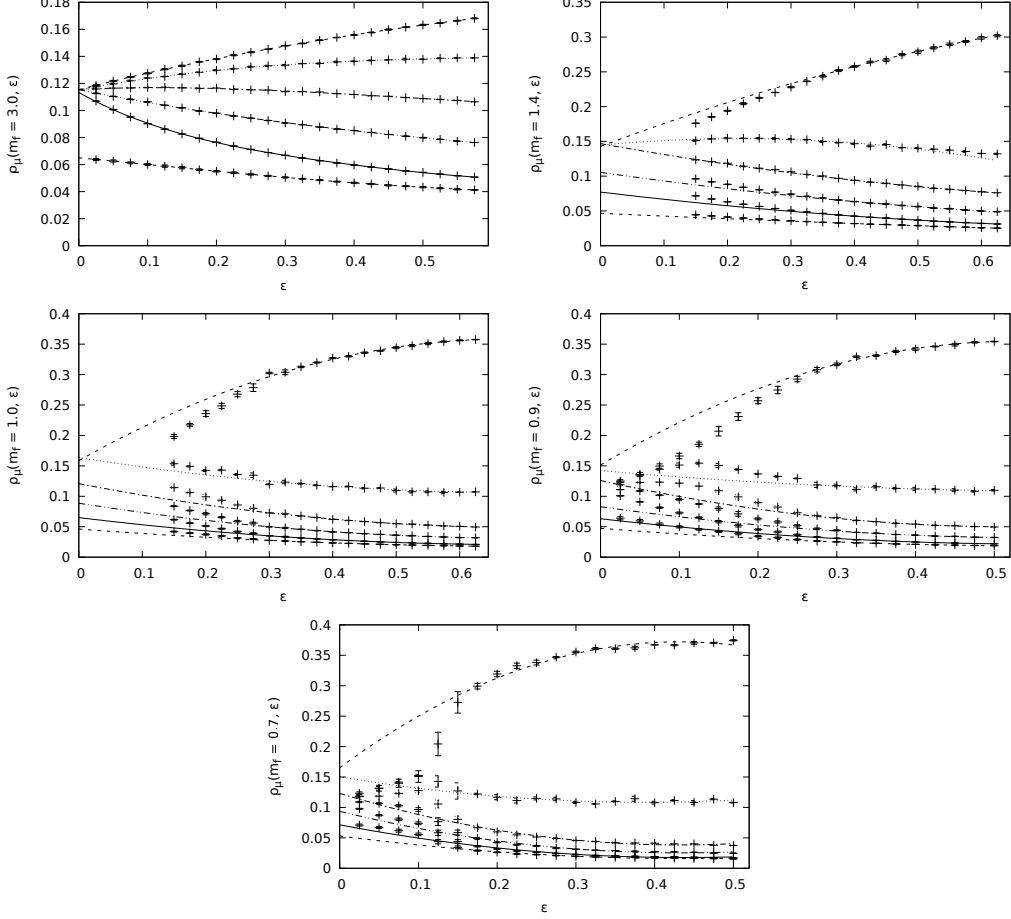

Figure 1: The $\rho_\mu(m_\mathrm{f}, \varepsilon)$ in Eq. (4.2) are plotted against ε for $m_\mathrm{f} = 3.0$ (Top-Left), $m_\mathrm{f} = 1.4$ (Top-Right), $m_\mathrm{f} = 1.0$ (Middle-Left), $m_\mathrm{f} = 0.9$ (Middle-Right) and $m_\mathrm{f} = 0.7$ (Bottom). We use $m_\mu = (0.5, 0.5, 0.5, 1, 2, 4, 8, 8, 8, 8)$ for $m_\mathrm{f} = 3.0$ and $m_\mu = (0.5, 0.5, 1, 2, 4, 8, 8, 8, 8, 8)$ for the other values of m_f. The continuous lines are polynomial fits in ε. For $m_\mathrm{f} = 3.0$ a quartic fit is performed, whereas for the other values of m_f the fits are quadratic in ε. In the $m_\mathrm{f} = 3.0$ plot, the curves from top to bottom are $(\rho_1 + \rho_2 + \rho_3)/3$, ρ_4, ρ_5, ρ_6, ρ_7 and $(\rho_8 + \rho_9 + \rho_{10})/3$. For the other plots, the curves from top to bottom are $(\rho_1 + \rho_2)/2$, ρ_3, ρ_4, ρ_5, $(\rho_6 + \rho_7)/2$ and $(\rho_8 + \rho_9 + \rho_{10})/3$.

In order to probe the SSB, one has to take the $N \to \infty$ limit first and then the $\varepsilon \to 0$ limit. For given m_μ in Eq. (2.8), the large-N limit is obtained by first computing the ratio

$$\rho_\mu(m_\mathrm{f}, \varepsilon, N) = \frac{\langle \lambda_\mu \rangle_{m_\mathrm{f}, \varepsilon, N}}{\sum_{v=1}^{10} \langle \lambda_v \rangle_{m_\mathrm{f}, \varepsilon, N}} \ , \tag{4.1}$$

and then by making a large-N extrapolation

$$\rho_\mu(m_\mathrm{f}, \varepsilon) = \lim_{N \to \infty} \rho_\mu(m_\mathrm{f}, \varepsilon, N) \ . \tag{4.2}$$

The large-N extrapolation is performed by plotting $\rho_\mu(m_{\mathrm{f}}, \varepsilon, N)$ against $1/N$ and making a quadratic fit with respect to $1/N$. Then we make the $\varepsilon \to 0$ extrapolation

$$\rho_\mu(m_{\mathrm{f}}) = \lim_{\varepsilon \to 0} \rho_\mu(m_{\mathrm{f}}, \varepsilon) \tag{4.3}$$

by fitting $\rho_\mu(m_{\mathrm{f}}, \varepsilon)$ to a polynomial in ε. In Fig. 1 we plot the large-N extrapolated values $\rho_\mu(m_{\mathrm{f}}, \varepsilon)$ as a function of ε for $m_{\mathrm{f}} = 3.0, 1.4, 1.0, 0.9$ and 0.7 together with the performed fits. In those fits, the data that is affected by finite size effects must be excluded. As it was argued in [27], these become quite severe in the $\varepsilon \to 0$ limit. These effects are apparent in Fig. 1, where a crossover to a symmetric phase is observed for $m_{\mathrm{f}} \leq 1.4$ as $\varepsilon \to 0$. This crossover is expected to vanish in the large–N limit.

From the extrapolated values $\rho_\mu(m_{\mathrm{f}})$, we find that the SO(7) symmetry of the deformed model is not spontaneously broken at $m_{\mathrm{f}} = 3.0$, but it is actually broken to SO(4) for $m_{\mathrm{f}} = 1.4$ and to SO(3) for $m_{\mathrm{f}} = 1.0, 0.9, 0.7$. Thus as m_{f} is decreased, the SO(7) symmetry seems to be spontaneously broken to smaller subgroups gradually in the same way as it was observed in the $D = 6$ case [26]. However, we consider that the symmetry is not going to be broken further down to SO(2) at smaller m_{f}. This is based on the fact that the Pfaffian vanishes identically for strictly 2D configurations [28, 83], which implies that the mechanism of SSB due to the phase of the Pfaffian no longer works there[1]. Hence our results are consistent with the results obtained by the GEM for the undeformed model, which show that the SO(3) vacuum has the smallest free energy.

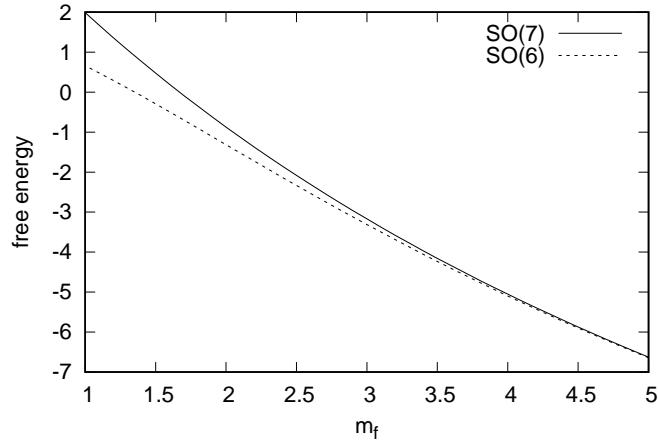**Figure 2:** The free energy calculated up to three loops for the solutions found with the SO(7) and SO(6) ansatzes are plotted against the fermion mass m_{f}. We observe a clear tendency that the SO(6) symmetric vacuum is more favored as m_{f} is decreased, whereas the free energy for the two ansatzes tends to be degenerate as m_{f} is increased.

The above mentioned results have also been checked against a calculation using the GEM on the deformed model (3.6). The GEM has been applied to the Euclidean IKKT matrix model in [33], where SO(10) was found to be broken down to SO(3). The GEM has the advantage that the

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[1]This is also reflected in the GEM results [32, 33] for the free energy of the SO(d) vacuum, which becomes much larger for $d = 2$ than for $d \geq 3$.

large-N limit is easily taken by considering only planar graphs and that the small-ε extrapolation is not necessary. The systematic errors of the GEM are due to the truncation of the expansion and the errors in determining the parameters that give the physical solutions. As such, the two methods can be considered to be completely independent. In [27], we performed a three-loop calculation using SO(d) symmetric ansatzes for $d = 6, 7$, and calculated the free energy. We observed that by decreasing m_f, the free energy of the SO(6) vacuum becomes smaller than the free energy of the SO(7) vacuum. In Fig. 2 we plot the free energy calculated up to three loops for the solutions found with the SO(7) and SO(6) ansatzes against the fermion mass m_f. We observe a clear tendency that the SO(6) symmetric vacuum is more favored as m_f is decreased. However, the free energy for the two ansatzes tends to become degenerate as m_f is increased. In this situation it is difficult to identify the critical point, given the accuracy of the GEM results.

At $m_f = 3.0$, the extent of space was found to agree very well between the two methods. In [27], it is found that

$$\rho_1 = \cdots = \rho_7 = 0.116 \,, \quad \rho_8 = \rho_9 = \rho_{10} = 0.064 \,, \tag{4.4}$$

whereas the CLM results of Fig. 1 give

$$\rho_1 = \cdots = \rho_7 = 0.115 \,, \quad (\rho_8 + \rho_9 + \rho_{10})/3 = 0.065 \,. \tag{4.5}$$

Therefore, for the first time, we have a first principle study of the Euclidean IKKT matrix model that produced clear results on the question of dynamical compactification of extra dimensions via SSB of the SO(10) rotational symmetry of the model. The SO(10) rotational symmetry breaking of the Euclidean IKKT matrix model down to SO(3) due to the phase of the Pfaffian is interesting, but it makes the model somewhat difficult to interpret. Given the promising properties of the Lorentzian model [3, 4, 5, 6, 7, 8, 9, 10, 11], we consider that the naive Wick rotation to the Euclidean model is not the right direction to pursue. On the other hand, the fact that the CLM enabled us to obtain a clear SSB pattern for the deformed model, which suffers from a severe sign problem, is encouraging. We hope that the CLM is equally useful in investigating the Lorentzian IKKT model, in particular in the presence of fermionic matrices, which are not included yet in Ref. [11].

Acknowledgements

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