ΕΘΝΙΚΟ ΜΕΤΣΟΒΙΟ ΠΟΛΥΤΕΧΝΕΙΟ

ΣΧΟΛΗ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ ΚΑΙ ΦΥΣΙΚΩΝ ΕΠΙΣΤΗΜΩΝ

ΣΧΟΛΗ ΜΗΧΑΝΟΛΟΓΩΝ ΜΗΧΑΝΙΚΩΝ

ΕΚΕΦΕ «ΔΗΜΟΚΡΙΤΟΣ»

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Διατμηματικό Πρόγραμμα Μεταπτυχιακών Σπουδών

«Φυσική και Τεχνολογικές Εφαρμογές»

Geometry Transition in Covariant Loop Quantum Gravity

ΜΕΤΑΠΤΥΧΙΑΚΗ ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ

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Αθήνα, Φεβρουάριος, 2023

Geometry Transition in Covariant Loop Quantum Gravity

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February 2023

Contents

Acknowledgements

Now that I am finishing my Master's in National Technical University of Athens I feel the need to express my gratitude to a handful o people.

First of all I would like to thank my parents Spyros and Yianna and my brother Panagiotis for supporting me thought my undergraduate and graduate studies on every level. Despite every talent I have in Physics I am certain that I wouldn't have achieved anything without them and I owe them everything.

Next come my teachers. I start with my supervisor Marios Christodoulou. Marios showed high interest in me from the very first time we met and made it possible for me to attend to the Loops '22 Summer School which was the beginning of my journey to Loop Quantum Gravity. Later on, he trusted me with finishing an unfinished project of his which resulted to my first paper. During writing my thesis he invited me twice to his group in IQOQI in the very beginning and in the very end of writing this thesis. This gave me the opportunity to see in person how intelligent and hard working physicist he is but also allowed me to be in a very scientifically productive environment and interact with many scientists from different backgrounds and expertise. Now, a few words about the selection committee. Kostas Anagnostopoulos is one of the most warm and kind individuals I have ever met in my life. We would have extensive discussions on General Relativity and Quantum Gravity in his office and he was always there for everything I needed during this thesis. I'm extra grateful to him because he is the one who introduced me to Marios. I've known Alex Kehagias since I was an undergrad and he always gave me the impression that he is in tune with Physics and capable of effortlessly teaching everything he wished. He would never miss the chance to squeeze some string theory in the Mathematical Methods for Physics master's class which inevitably fueled our intense disagreement but it was always super fun and enjoyable. Nick Mavromatos is an amazing person, tutor and scientist. His transfer to NTUA gave the school a huge push and I was very lucky to be taught advanced QFT by him.

I wish to thank two other young scientists. The first is Pierre Martin Dussaud with whom I overlapped during my first visit to IQOQI. Pierre is one of the brightest young researchers in LQG possessing both great physical intuition and mathematically oriented mind. I am very lucky that during my first steps in LQG he was in the nearby office and always willing to answer my questions. The second is Andrea Di Biaggio who during my second visit to IQOQI when I thought that I had mastered the basics of the theory pushed me to deeper understanding so that I could answer his sincerely queries.

I've kept a very special person for the end. Dimitra, during the last 2.5 years you've made my life a roller coaster of emotions. I don't know how things would have turned without you but I regret nothing and I thank you for everything.

During the writing of this thesis I received financial support through the ID# 62312 grant from the John Templeton Foundation, as part of the "Quantum Information Structure of Spacetime (QISS)" project (qiss.fr) and from the Blaumann Foundation. It is certain that without their aid I would not be able to perform in such a high scientific level for this period of time.

Preface

This thesis can be viewed as a brief and self-contained introduction to LQG where I used as less mathematics as possible. For someone interested in the mathematics of LQG I would recommend the excellent paper by Dussaud (1). The main source used in this thesis was the book by Rovelli and Vidotto (2). Chapter 5 is based on the paper (3).

Abstract

In Chapter 1 we briefly address the problem of quantization of Gravity along with the motivation for LQG and its qualitative features. In Chapter 2 we recast classical General Relativity in a form suitable for quantization. In Chapter 3 we construct the Kinematics of LQG and we compute the eigenvalues of the Area operator and Volume operator in a special but pedagogical case. Chapter 4 contains the main part of this thesis. We show how the fixed-spin asymptotics of the EPRL model can be used to perform the spin-sum for spin foam amplitudes defined on fixed two-complexes without interior faces and contracted with coherent spin-network states peaked on a discrete simplicial geometry with macroscopic areas. We work in the representation given in (4). We first rederive the latter in a different way suitable for our purposes. We then extend this representation to 2-complexes with a boundary and derive its relation to the coherent state representation. We give the measure providing the resolution of the identity for Thiemann's state in the twisted geometry parametrization. The above then permit us to put everything together with other results in the literature and show how the spin sum can be performed analytically for the regime of interest here. These results are relevant to analytic investigations regarding the transition of a black hole to a white hole geometry. In particular, this work gives detailed technique that was the basis of estimate for the black to white bounce appeared in (5). These results may also be relevant for applications of spinfoams to investigate the possibility of a 'big bounce'.

Chapter 1

Introduction

Arguably, the combination of General Relativity (GR) and Quantum Mechanics is the Holy Grail of Physics. There both practical and conceptual motivations to quantize Gravity. First, the singularities that arise in GR suggest that it is only an effective description of physical reality and not a fundamental theory. Second, since every other interaction we know is of quantum nature we expect the same to be true about Gravity. But why did this reasonable expectation turn out to be the hardest problem in Theoretical Physics?

The standard procedure we have for quantizing classical Theories is through Dirac's rules that have as the starting point the Hamiltonian formulation of the theory. But even this was very hard to accomplish for GR and was achieved by Dirac himself only in 1958 (6). The resulting Hamiltonian formulation was very complicated and not suitable for quantization. The situation somewhat improved when Arnowitt, Deser and Misner found a much more convenient set of variables to perform the Hamiltonian formulation (7). Immediately after that, DeWitt applied Dirac's rules to obtain what we today call the Wheeler-DeWitt equation (8). Unfortunately, this equation is ill-defined and is essentially a meaningless formality.

Another way to quantize Gravity is to utilize the standard techniques of quantum field theory (QFT) and split spacetime into a fixed background and a propagating perturbation, namely $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and try to quantize $h_{\mu\nu}$. For starters, one should be very skeptical about the split itself. Unlike the other fields, the gravitational field is not something that propagates in a fixed spacetime; it's spacetime itself. Therefore, this split is against the geist of the best understanding we have of Gravity and can be applied only in specific cases but it is totally inappropriate as a starting point of quantization. Furthermore, it is evident that the "gravitons" $h_{\mu\nu}$ propagate respecting the causal structure of the unphysical fixed background $\eta_{\mu\nu}$ to all orders in perturbation theory; the causal structure does not fluctuate as someone would hope. But even if we ignore all the red flags this scheme simply doesn't work. The reason for that is that the gravitational coupling provided by Newton's constant G has dimension of mass -2, therefore the theory is not renormalizable.

In a nutshell, this is why Gravity cannot be directly quantized as the other theories. To attack this problem many solutions have been proposed. In this manuscript we present the set of ideas that developed to the Theory that is known to the Physics community as Loop Quantum Gravity (LQG). LQG is a Quantum theory of Gravity from a Relativist's perspective; it is manifestly background-independent and non-perturbative. Following the revolution by Einstein "spacetime $\leftrightarrow$ gravitational field" the quantum object considered in LQG is spacetime itself. The spirit of LQG can be encapsulated in figure1. Given a spacetime configuration one should be able to calculate the probability amplitude between an initial and a final 3-geometry. LQG has so far made two impressive predictions about physical phenomena. The first is the replacement of the Big Bang with a Big Bounce(ref). The second is the transition from a Black Hole to a White hole in the end of the Hawking evaporation.

When promises a Quantum Theory one should provide a triplet $(\mathcal{H}, \mathcal{A}, \mathcal{W})$. The first two together are called Kinematics and are a Hilbert space and a set of operators. The third is a rule for the Dynamics. Regarding the latter one has two options; either follow the canonical quantization or the path integral quantization. Historically LQG was born in the Hamiltonian framework when Ashtekar discovered a very convenient canonical transformation for GR (9). Although mathematically more concrete, the canonical LQG program still faces difficulties regarding dynamics. Instead, in this thesis we will present the covariant version of the theory which is also known as spinfoams and turned out to be much more convenient for actual calculations.

The spirit of the theory can be captured in Fig 1.1

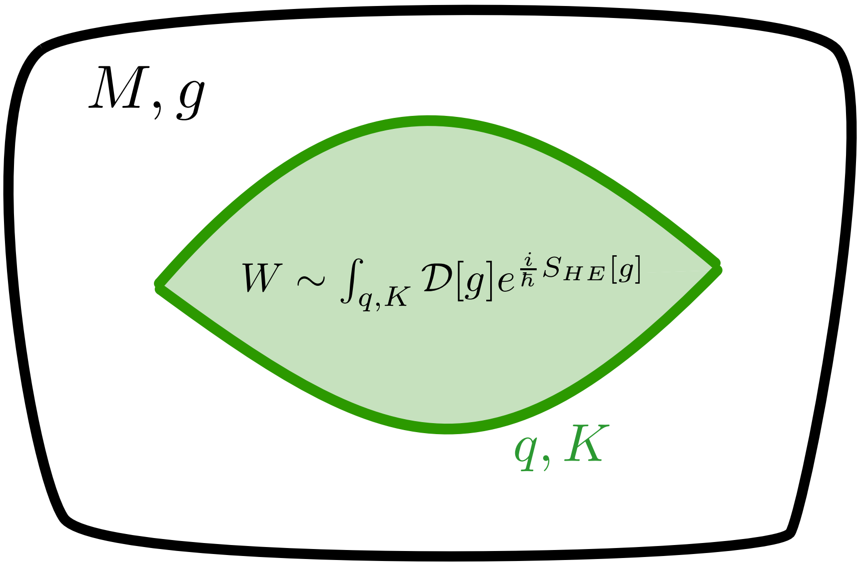


Figure 1.1: Geometry transition viewed as a path-integral over geometries. The boundary surface (dark green) separates the parts of the system treated as classical and quantum. The exterior spacetime is classical with a metric g solving Einstein's field equations. A path-integral W is performed in the interior with the metric fixed to the intrinsic metric q and extrinsic curvature K of the boundary surface. We emphasize that the interpretation of the amplitude as a path-integral over geometries is emergent in covariant LQG, in the semiclassical limit of large quantum numbers. The theory is defined in the absence of any notion of classical metric, or indeed, spacetime.

There are many models of the theory (see Perez), but the one that has met the most success in terms of calculations is the one by EPRL and it is the model presented here.

Chapter 2

Classical GR

2.1 Action

Our starting point is the well- known Einstein – Hilbert action

$$S[g]_{EH} = \int d^4x\sqrt{-g}R[g] \tag{2.1}$$

where, g is the determinant of the metric and R is the Ricci scalar of the Levi – Civita connection. To bring the action in a more suitable form for our purposes we rewrite it in terms of tetrads. We choose a set of four orthonormal vectors e_I and we expand it in terms of the coordinate basis ∂_α as

$$e_I = e_I^\alpha \partial_\alpha \tag{2.2}$$

The inverse of the matrix e_I^α is denoted by e_I^α and is defined by

$$\begin{aligned} e_I^\alpha e_\alpha^J &= \delta_j^I \\ e_I^\beta e_\alpha^I &= \delta_\alpha^\beta \end{aligned} \tag{2.3}$$

We consider $e_{\alpha I} \equiv g_{\alpha\beta} e_I^\beta$ and we rewrite the metric as

$$g = g_{\alpha\beta} dx^\alpha dx^\beta = e_{\alpha I} e_\beta^I dx^\alpha dx^\beta = e_{\alpha I} e_{\beta J} \eta^{IJ} dx^\alpha dx^\beta \tag{2.4}$$

and we immediately read

$$g = e_I e_J \eta^{IJ} \tag{2.5}$$

It is apparent that the tetrad encapsulates the same geometric information as the metric. But we notice that there is a redundancy in the description as the tetrad field is not uniquely defined; after performing a Lorentz transformation $e_I \to e_L \Lambda_I^L$ the metric remains untouched

$$g' = \Lambda_I^K e_K \Lambda_J^L e_L \eta^{IJ} = e_K e_L \eta^{KL} = g \tag{2.6}$$

where we made use of the defining property of the Lorentz transformations $\Lambda_I^K \Lambda_J^L \eta^{IJ} = \eta^{KL}$.

Returning back to General Relativity we can write the action as

$$S[e]_{EH} = \int d^4x |\det e| R[e] \tag{2.7}$$

Equation (2.7) has a peculiar feature; it involves both the coordinate basis and the tetrad. We wish to write the action entirely in terms of the tetrad field and we will proceed as follows.

First of all we know that d^4x is a shorthand notation for $dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$. The volume form vol^4 is given by

$$vol^4 = \sqrt{-g} d^4x = \det(e) d^4x \tag{2.8}$$

We compute

$$
\begin{aligned}
e^0 \wedge e^1 \wedge e^2 \wedge e^3 &= e^0_{k_0} e^1_{k_1} e^2_{k_2} e^3_{k_3} dx^{k_0} \wedge dx^{k_1} \wedge dx^{k_2} \wedge dx^{k_3} \\
&= e^0_{k_0} e^1_{k_1} e^2_{k_2} e^3_{k_3} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \epsilon^{k_0 k_1 k_2 k_3} \\
&= \det(e) d^4x
\end{aligned} \tag{2.9}
$$

and we substitute back to the E – H action to obtain

$$
S_{EH}[e] = \int e^0 \wedge e^1 \wedge e^2 \wedge e^3 R(e) \tag{2.10}
$$

Moving on, we can write the Riemann tensor using Minkowski indices as

$$
R^{IJ}{}_{KL}(e) = R^{\rho\sigma}{}_{\mu\nu}(e) e^\mu_K e^\nu_L e^I_\rho e^K_\sigma \tag{2.11}
$$

By virtue of the identity $2(\delta^K_I \delta^L_J - \delta^L_I \delta^K_J) = \epsilon_{IJCD}\epsilon^{CDKL}$ and the antisymmetry of the Riemann tensor in the last pair of indices we get

$$
\begin{aligned}
R(e) = R^{IJ}{}_{IJ}(e) = \delta^K_I \delta^L_J R^{IJ}{}_{KL}(e) = \frac{1}{2}(\delta^K_I \delta^L_J - \delta^L_I \delta^K_L) R^{IJ}{}_{KL}(e) \\
= \frac{1}{4}\epsilon_{IJCD}\epsilon^{CDKL} R^{IJ}{}_{KL}(e)
\end{aligned} \tag{2.12}
$$

We now notice that we may write

$$
\epsilon^{CDKL} vol^4 = \epsilon^{CDKL} e^0 \wedge e^1 \wedge e^2 \wedge e^3 = e^C \wedge e^D \wedge e^K \wedge e^L \tag{2.13}
$$

We can substitute (2.12) and (2.13) to the action to get

$$
S_{EH}[e] = \frac{1}{4}\int \epsilon_{IJKL} e^I \wedge e^J \wedge e^A \wedge e^B R^{KL}{}_{AB}(e) \tag{2.14}
$$

Now, we define the field strength

$$
F^{KL} := R^{KL}{}_{AB} e^A \wedge e^B \tag{2.15}
$$

which brings the action to its tetradic form

$$
S_{EH}[e] = \frac{1}{4}\int \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL}(e) \tag{2.16}
$$

We are almost done. We can use the Hodge star operator to write

$$
\star(e \wedge e)_{KL} = \frac{1}{2}\epsilon_{IJKL} e^I \wedge e^J \tag{2.17}
$$

and obtain

$$
S_{EH}[e] = \frac{1}{2}\int \star(e \wedge e)_{IJ} \wedge F^{IJ}(e) \tag{2.18}
$$

The factor of $\frac{1}{2}$ is irrelevant and we shall omitte it. To lighten the notation we will neglect the indices to get

$$
S_{EH}[e] = \int \star(e \wedge e) \wedge F[e] \tag{2.19}
$$

Now we will change our course a title bit. It is well-known that there exists an alternative and equivalent way to obtain the field equations of GR by considering the same action but treating the connection and the metric as independent objects, i.e.

$$
S_P[g, \Gamma] = \int d^4x \sqrt{-g} R[g, \Gamma] \tag{2.20}
$$

This is called the Palatini action. The variation with respect to the connection gives back the Levi-Civita connection and variation with respect to the metric gives the Einstein field equations. We can do the same thing using the tetrad and the spin connection ω. The action we consider is the following

$$S_P[e,\omega] = \int \star(e \wedge e) \wedge F[\omega] \tag{2.21}$$

Having the Palatini action in hand we can now do the last step. We can add the extra term

$$\frac{1}{\gamma} \int e \wedge e \wedge F[\omega] \tag{2.22}$$

where γ is called the Barbero-Immirzi parameter. The justification for the addition of this term comes from the Hamiltonian analysis . The extra term corresponds to the canonical transformation found by Ashtekar which corresponds to the addition of an extra term to the action that does not affect the equations of motion. Indeed, returning back to the original variables this term reads

$$\frac{1}{\gamma} \int d^4x \sqrt{-g} \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \tag{2.23}$$

which is zero when we are on-shell. The new action is called the Holst action

$$S_H[e,\omega] = \int \left(\star(e \wedge e) + \frac{1}{\gamma} e \wedge e \right) \wedge F[\omega] \tag{2.24}$$

We define the bivector

$$B[e] := \star(e \wedge e) + \frac{1}{\gamma} e \wedge e \tag{2.25}$$

And we finally write the action as

$$S_H[e,\omega] = \int B[e] \wedge F[\omega] \tag{2.26}$$

Theories of this type are called "BF" theories and are well studied. General Relativity is a special kind of BF theory in the sense that the B field is restricted by equation (2.25) to be "simple". For that reason this is also known as the simplicity constraint of General Relativity and it plays a central role in LQG.

To anticipate a bit for what comes next, it can be easily shown that F is the usual field strength of the (here $so(1,3)$ or $\underline{sl(2,\mathbb{C})}$ valued) form ω, i.e. it has the form.

$$F = d\omega + \omega \wedge \omega \tag{2.27}$$

On a $t = const$ boundary, B is the derivative of the action with respect to $\partial\omega/\partial t$, since the quadratic part of the action is $\sim B \wedge d\omega$. Thus B is the momentum canonical to the connection or equivalently the generator of Lorentz transformations.

2.1.1 Linear Simplicity Constraints

Pick a spacetime boundary surface Σ and choose coordinates $\{\sigma^i\}$ with $i = 1,2,3$. The unit vector n_I normal to the surface Σ is

$$n_I \sim \epsilon_{IJKL} e^I_\mu e^J_\nu e^K_\rho \frac{\partial x^\mu}{\partial \sigma^1} \frac{\partial x^\nu}{\partial \sigma^2} \frac{\partial x^\rho}{\partial \sigma^3} \tag{2.28}$$

Where $x^\mu(\sigma^i)$ is the embedding of the boundary Σ into spacetime. The bivector B can be decomposed into its electric $K^I = n_J B^{IJ}$ and magnetic $L^I = n_J(\star B)^{IJ}$ part. But since B is antisymmetric the electric and magnetic parts to not have components normal to Σ, that is $n_I K^I = 0$, $n_I L^I = 0$. Hence, they are three-vectors tangent to Σ that we can denote $\vec{K}$ and $\vec{L}$. Now, we exploit the fact that we can orient the Lorentz frame such in a way that locally $n_I = (1,0,0,0)$. This corresponds to gauge fixing the orientation such that the local Lorentz group $SO(3,1)$ breaks to the $SO(3)$ group of spatial rotations and this choice of n_I is called the time gauge. In this gauge we have

$$K^i = B^{i0}, \qquad L^i = \frac{1}{2} \epsilon^i{}_{jk} B^{jk} \tag{2.29}$$

Let's now compute the components. For the electric part we have

$$K^I = n_J B^{IJ} = n_J(\star e\wedge e + \frac{1}{\gamma}e\wedge e)^{IJ} = n_J(\frac{1}{2}\epsilon^{IJ}{}_{KL}e^K\wedge e^L + \frac{1}{\gamma}e^I\wedge e^J) \tag{2.30}$$

But by definition we have $n_J e^J|_\Sigma = 0$. Thus, we have

$$K^I = n_J(\star e\wedge e)^{IJ} \tag{2.31}$$

For the same reason we have for the magnetic part

$$\begin{aligned} L^I &= n_J(\star B)^{IJ} = n_J[\star(\star e\wedge e + \frac{1}{\gamma}e\wedge e)]^{IJ} \\ &= n_J(e\wedge e + \frac{1}{\gamma}\star e\wedge e)^{IJ} = \frac{1}{\gamma}n_J(\star e\wedge e)^{IJ} \end{aligned} \tag{2.32}$$

By comparing (2.30) and (2.32) we have

$$\vec{K} = \gamma\vec{L} \tag{2.33}$$

This form of the simplicity constraint is called the linear simplicity constraint and it is of utmost importance in LQG as it hints to the set of quantum state we shall use.

But what is the physical meaning of $\vec{K}$ and $\vec{L}$? As we mentioned in the last section B is the generator of Lorentz transformations. But if we go to the time gauge, there is only one option: $\vec{K}$ is the generator of the boosts and $\vec{L}$ is the generator of the rotations. The two of them are related by (2.33).

2.2 Elements of Canonical Theory

To write down a Hamiltonian Formulation of GR one needs the appropriate variables. The choice we make here is the ADM-variables. From $g_{\mu\nu}$ we define

$$q_{ab} = g_{ab} \tag{2.34}$$

$$N_a = g_{a0} \tag{2.35}$$

$$N = \sqrt{-g^{00}} \tag{2.36}$$

N is called the "Lapse" function, N_a is called the "Shift" function and q_{ab} is called the three-metric. One extremely useful and highly non-trivial fact is that the Lagrangian does not depend on $\dot{N}$ and $\dot{N}^a$, thus Lapse and Shift are Lagrange multipliers. This guarantees that the space and time split that is necessary for the Hamiltonian formulation doesn't harm diffeomorphism invariance.

In these variables the line element reads

$$ds^2 = -(N^2 - N_a N^a)dt^2 + 2N_a dx^a dt + q_{ab}dx^a dx^b \tag{2.37}$$

The extrinsic curvature of $t = const$ surfaces is

$$k_{ab} = \frac{1}{2N}(\dot{q}_{ab} - D_{(a}N_{b)}) \tag{2.38}$$

where dot is the derivative with respect to t and D_a is the covariant derivative of the Levi-Civita connection in terms of the three-metric q_ab. Using these and Theorema-Egregium the E-H action can be written as

$$S[N,\vec{N},q] = \int dt\int d^3x\sqrt{q}N(k_{ab}k^{ab} - k^2 + R[q]) \tag{2.39}$$

where $k := k_a{}^a$ and $\sqrt{q} := \sqrt{\det q}$

Substituting (2.38) to the action the Lagrangian density reads

$$\mathcal{L}[N,\vec{N},q] = \frac{\sqrt{q}(g^{ac}g^{bd} - g^{ab}g^{cd})(\dot{q}_{ab} - D_{(a}N_{b)})(\dot{q}_{cd} - D_{(c}N_{d)})}{4N} + \sqrt{q}NR[q] \tag{2.40}$$

As one can immediately observe neither $\dot{N}$ nor $\dot{N}_a$ appear in the Lagrangian, thus the corresponding canonical momenta vanish. The canonical momentum of the three metric is

$$\pi^{ab} = \frac{\partial L}{\partial \dot{q}_{ab}} = \sqrt{q} G^{abcd} k_{cd} = \sqrt{\det q}(k^{ab} - k q_{ab}) \tag{2.41}$$

where $G_{abcd} := (g_{ac}g_{bd} + g_{ad}g_{bc} - g_{ab}g_{cd})$ is called the DeWitt metric.
Returning back to the action it reads

$$S[N, \vec{N}, q] = \int dt \int d^3x (\pi^{ab}\dot{q}_{ab} - NC(\pi, q) - 2N^a C_a(\pi, q)) \tag{2.42}$$

where

$$C = G_{abcd}\pi^{ab}\pi^{cd} - \sqrt{q}R[q] \tag{2.43}$$

is called the scalar or Hamiltonian constraint and

$$C_a = D_b \pi^{ab} \tag{2.44}$$

is called the vector or diffeomorphism constraint. The Hamiltonian density is

$$H = NC + N^a C_a \tag{2.45}$$

and as a linear combination of constraints it vanishes on-shell (up to a boundary term). Following the standard quantization strategy we are led to the Wheeler-DeWitt equations which unfortunately are ill-defined.

2.2.1 Ashtekar Variables

On each $t = const$ surface we introduce triads. They are defined as

$$q_{ab}(x) = e_a^i(x) e_b^j(x) \delta_{ij} \tag{2.46}$$

We can also define the triad version of the extrinsic curvature, namely define

$$k_i^a e_b^i := k_{ab} \tag{2.47}$$

If we wish to consider these variables as a canonical conjugate pair, namely pose $\{k_i^a, e_b^j\} \sim \delta_i^j \delta_b^a$ we need to be careful; instead of $6+6$ we have $9+9$ variables, since there is no reason for k and e to be symmetrical. But extrinsic curvature k_{ab} is symmetrical, thus the antisymmetric part of the left hand side of (2.47) must vanish. Hence we have the constraint

$$G_c = \epsilon_{cab} k_i^a e_b^i = 0 \tag{2.48}$$

In this way we recover the original $6+6$ dimensional space. The next step is to introduce the connection

$$A_a^i = \Gamma_a^i[e] + \beta k_a^i \tag{2.49}$$

where $\Gamma_a^i[e]$ is the torsionless spin connection of the triad and β is an arbitrary parameter. We also introduce the Ashtekar electric field

$$E_i^a = \frac{1}{2}\epsilon_{ijk}\epsilon^{abc} e_b^j e_c^k \tag{2.50}$$

which is the inverse triad multiplied by its determinant. The connection satisfies the Poisson brackets

$$\{A_a^i(x), A_b^j(y)\} = 0 \tag{2.51}$$

and

$$\{A_a^i(x), E_j^b(y)\} = \delta_j^i \delta_a^b \delta^3(x, y) \tag{2.52}$$

therefore A_a^i and E_j^b are canonically conjugate variables.
Constraint (2.48) can be written as

$$G^i = D_a E^{ai} \tag{2.53}$$

where here D_a is the covariant derivative of the Ashtekar connection. We observe that this is the usual constraint of Yang-Mills theories.

Let's see what is the geometric interpretation of the field E_i^a. First, we go to a $t = \textit{const}$ hypersurface and chose a surface such that $x^3 = 0$. The area of this surface is given by

$$\begin{aligned} A_S &= \int_S d^2\sigma \sqrt{\det {}^{(2)}q} = \int_S d^2\sigma \sqrt{q_{11}q_{22} - q_{12}^2} = \int_S d^2\sigma \sqrt{\det q q^{33}} \\ &= \int_S d^2\sigma \sqrt{E_i^3 E_i^3} = \int_S d^2\sigma \sqrt{E_i^a n_a E_i^b n_b} \end{aligned} \tag{2.54}$$

By introducing the two-form

$$E^i = \frac{1}{2}\epsilon_{abc} E^{ai} dx^b dx^c \tag{2.55}$$

the area of the surface can be written as

$$A_S = \int_S |E| \tag{2.56}$$

thus the field E can be seen as the area element.

In the limit where the surface is small, the vector $\vec{E}_S$ defined by

$$E_S^i = \int_S E^i \tag{2.57}$$

is normal to the surface and its length is the area of the surface. I terms of triads it can be written as

$$E_S^i = \frac{1}{2}\epsilon^i{}_{jk} \int_S e^j \wedge e^k \tag{2.58}$$

Chapter 3

Kinematics

3.1 Elementary Geometry

Let there be a tetrahedron with vertices $0, 1, 2, 3$. To fully determine the tetrahedron one needs six numbers that correspond to the length of the six edges l_{ab}, $a, b = 0, 1, 2, 3$ that satisfy triangular inequalities. Given these numbers one can compute the area of the faces, the volume and the dihedral angles as functions of l_{ab}. The main disadvantage of this description is that the formulas for the geometric features of the tetrahedron are very complicated. A better description is in terms of the three vectors $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$. This corresponds to nine numbers but if we take into account the common rotations we are left again with six independent numbers. Using these vectors we can construct an even better set of variables. We define

$$\vec{L}_a = \frac{1}{2}\varepsilon_a{}^{bc}\vec{e}_b \times \vec{e}_c, \quad a, b, c = 1, 2, 3 \tag{3.1}$$

The length of the vector $|\vec{L}_a|$ corresponds to the area of the face across the vertex a. The dihedral angles are given by the dot product and the volume by

$$V^2 = \frac{2}{9}(\vec{L}_1 \times \vec{L}_2) \cdot \vec{L}_3 \tag{3.2}$$

We can use elementary geometry to construct the vector across the vertex 0, $\vec{L}_0$. It is immediate to see that the four vectors normal to the four faces satisfy

$$\vec{C} := \vec{L}_0 + \vec{L}_1 + \vec{L}_2 + \vec{L}_3 = 0 \tag{3.3}$$

In fact, we can describe directly the tetrahedron in terms of four vectors $\vec{L}_f$ that satisfy (3.3). Again, (3.3) corresponds to the invariance under common rotations. As we shall immediately see this is the key to constructing the quantum operators and the Hilbert space for quantum gravity.

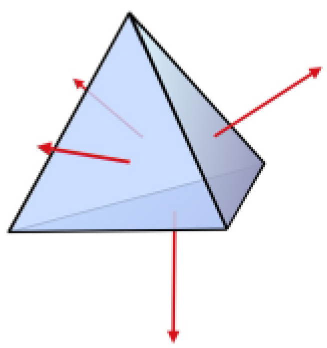


Figure 3.1: A tetrahedron and the four vectors normal to the four faces

3.2 Quantum spacetime

The main discovery of GR is that spacetime is the gravitational field. Fields are ultimately of quantum nature and the same should apply to the gravitational field. So let's examine a quantum of this field, or equivalently a chunk of spacetime. Let's focus on space in particular. The unit cell is in principle a polyhedron with the simplest version being a tetrahedron. As we saw in the previous chapter, the tetrahedron is described by four vectors that satisfy a closure condition, which is the manifestation of invariance under common rotations. To do the quantum version of the tetrahedron we promote each vector to an operator. The closure condition gives us the hint that the relevant group is the rotation group $SU(2)$, thus the operators should satisfy the $su(2)$ algebra

$$[L_f^i, L_{f'}^j] = l_0^2 i\epsilon^{ij}{}_k L_f^k \delta_{ff'} \tag{3.4}$$

where l_0^2 is a constant of dimension of an area. The only relevant dimensional constant in quantum gravity is the Planck length $L_{pl} = \sqrt{\hbar G}$ (in units where $c = 1$) so l_0^2 should be proportional to L_{pl}^2. From the canonical analysis of the theory it follows that

$$l_0^2 = 8\pi\gamma\hbar G \tag{3.5}$$

The commutation relations (3.4) have a dramatic implication in the geometry of the tetrahedron. Since the modulus of each operator corresponds to the area of a face it follows immediately that the former is quantized and its spectrum is

$$A = l_0^2\sqrt{j(j+1)}, \qquad j = 0, 1/2, 1, 3/2, ... \tag{3.6}$$

In Loop Quantum Gravity you can never go arbitrarily small; no matter what you do you can never measure a surface smaller than l_0 and this is a prediction of the theory.

What about the volume? It is a bit more tedious to find its spectrum. To begin with, the geometry of the tetrahedron is a state with area eigenvalues j_0, j_1, j_2, j_3. Hence, the Hilbert space of the quantum states of the quantum geometry is

$$\mathcal{H} = \mathcal{H}_{j_0} \otimes \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \tag{3.7}$$

But there is also the closure condition (3.3) which means that a state of the tetrahedron satisfies

$$\vec{C}\Psi = 0 \tag{3.8}$$

To find these states we have to look in the subspace $\mathcal{K}$ of $\mathcal{H}$, where

$$\mathcal{K} := \mathrm{Inv}_{SU(2)}[\mathcal{H}_{j_0} \otimes \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}] \tag{3.9}$$

The dimension of $\mathcal{K}$ is given by

$$\dim(\mathcal{K}) = \min(j_0 + j_1, j_2 + j_3) - \max(|j_0 - j_1|, |j_2 - j_3|) + 1 \tag{3.10}$$

therefore, given a spin configuration, the four spins j_0, j_1, j_2, j_3 have to satisfy the inequality

$$\dim(\mathcal{K}) = \min(j_0 + j_1, j_2 + j_3) - \max(|j_0 - j_1|, |j_2 - j_3|) + 1 > 0 \tag{3.11}$$

So that the invariant part of the tensor product of the four Hilbert spaces exist.

Now, let's see if the volume operator is well defined in $\mathcal{K}$. We need to check if it commutes with $\vec{C}$. We compute

$$
\begin{aligned}
[V^2, C^a] &= \frac{2}{9}\epsilon^{ijk}[L_1^i L_2^j L_3^k, L_0^a + L_1^a + L_2^a + L_3^a] \\
&= \frac{2}{9}\epsilon^{ijk}[L_1^i, L_1^a]L_2^j L_3^k + [L_2^j, L_2^a]L_1^i L_3^k + L_1^i L_2^j [L_3^k, L_3^a] \\
&= \frac{2}{9}\epsilon^{ijk} i l_0^2 [\epsilon^{iab} L_1^b L_j^2 L_k^3 + \epsilon^{jab} L_2^b L_1^i L_3^k + L_1^i L_2^j \epsilon^{abk} L_k^b] \\
&= \frac{2}{9} i l_0^2 [(\delta^{ja}\delta^{kb} - \delta^{jb}\delta^{ka}) L_1^b L_2^j L_3^k + (\delta^{ka}\delta^{ib} - \delta^{kb}\delta^{ja}) L_2^b L_1^i L_3^k + (\delta^{ia}\delta^{jb} - \delta^{ib}\delta^{ja}) L_1^i L_2^j L_3^b] \\
&= \frac{2}{9} i l_0^2 [L_1^b L_2^a L_3^b - L_1^b L_2^b L_3^a + L_2^b L_1^b L_3^a - L_2^b L_1^a L_3^b + L_1^a L_2^b L_3^b - L_1^b L_2^a L_3^b] \\
&= 0
\end{aligned}
\tag{3.12}
$$

where summation on repeated indices is implied. We made use of the commutation relations (3.4) and the the identity $\epsilon^{ijk}\epsilon^{nlk} = \delta^{in}\delta^{jl} - \delta^{il}\delta^{jn}$.

This little proof shows that the volume operator is well-defined. Since $\mathcal{K}$ is finite dimensional it follows that the operator has descrete spectrum.

Let's explicitly compute the spectrum in the simplest case were $j_0 = j_1 = j_2 = j_3 = 1/2$. A state in this Hilbert space is a 4-indices spinor z^{ABCD}, where on each index acts an element of the $j = \frac{1}{2}$ representation of $SU(2)$. The volume operator has the form

$$
V^2 = \frac{2}{9}\epsilon_{ijk} L_1^i L_2^j L_3^k \tag{3.13}
$$

Where $L_f^i = l_0^2 \frac{\sigma^i}{2}$. This operator acts on the f-th index thus we have

$$
(V^2\mathbf{z})^{ABCD} = \frac{2}{9}\left(\frac{l_0^2}{2}\right)^3 \epsilon^{ijk}\sigma_{i\ A'}^{A}\sigma_{j\ B'}^{B}\sigma_{k\ C'}^{C} z^{A'B'C'D} \tag{3.14}
$$

where of course nothing acts on the last index.
From elementary quantum mechanics we know that

$$
\mathcal{H}_{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}} = (o \oplus 1) \bigotimes (o \oplus 1) = 0 \oplus 1 \oplus 1 \bigoplus (0 \oplus 1 \oplus 2) \tag{3.15}
$$

Since the trivial representation 0 appears twice the dimension of $\mathcal{K}_{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}}$ is two. This means that we are looking for two $SU(2)$ invariant objects with four indices. The only such objects that exist are ε^{AB} and σ_i^{AB}. Therefore the states that span $\mathcal{K}_{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}}$ are

$$
z_1^{ABCD} = \varepsilon^{AB}\varepsilon^{CD} \quad \text{and} \quad z_2^{ABCD} = \sigma_i^{AB}\sigma_i^{CD} \tag{3.16}
$$

Now that we have a basis of $\mathcal{K}$ let's find the eigenvalues of the volume. It is a simple exercise to show that

$$
V^2\mathbf{z}_1 = -\frac{i(l_0^2)^3}{18}\mathbf{z}_2, \quad V^2\mathbf{z}_2 = \frac{i(l_0^2)^3}{6}\mathbf{z}_1 \tag{3.17}
$$

so that

$$
V^2 = -\frac{i(l_0^2)^3}{18}\begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix} \tag{3.18}
$$

The eigenvalues are

$$
V^2 = \pm\frac{(l_0^2)^3}{6\sqrt{3}} \tag{3.19}
$$

The ambiguity in sign is due to the orientation of the triple product. We can safely ignore the minus sign and obtain

$$
V = \frac{1}{\sqrt{6\sqrt{3}}}(8\pi\gamma\hbar G)^{3/2} \tag{3.20}
$$

Returning to the general spin case, it is straightforward to show that the volume operator commutes with every area operator. Thus, a state of the quantum tetrahedron is described by five numbers $|j_0, j_1, j_2, j_3, v\rangle$. Notice that it is one less than the classical tetrahedron. This is exactly the fuzziness in geometry expected when you go Quantum.

A final remark. Nothing we said about the Quantum Geometry relies on the specific form of the action of the classical theory (except for the γ). Every gravitational theory that follows the spirit of GR "spacetime→ gravitational field" should arrive at the same conclusion. In other words, this is an entirely kinematical result.

3.3 Triangulation and dual triangulation

Triangulation of spacetime is a very important procedure in LQG. Let's begin with the simplest case of a 2d surface. We truncate the surface into triangles. In the center of each triangle we place a vertex. We connect the nearby vertices with oriented links and we notice that every vertex is tri-valent, meaning that from each one emanate tree links. The graph we obtain by this procedure is called the graph dual to the triangulation or just the dual graph. Each vertex is dual to a triangle and each link is dual to a segment.

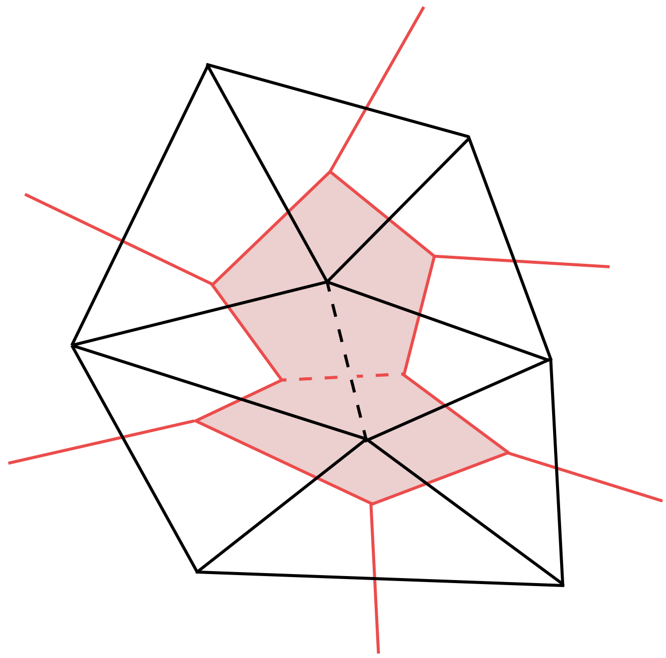
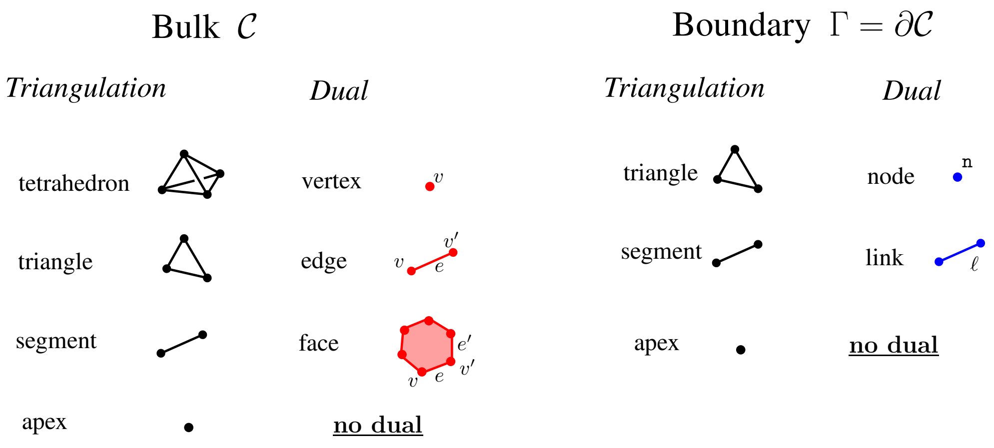


Figure 3.2: A triangulation in two dimensions. Each edge of the dual graph, shown in red, is common to two faces. As an example, the segment in dotted black is dual to the edge in dotted red, which is common to the two faces in pale red. This generalizes in d dimensions: an edge is common to exactly d faces.

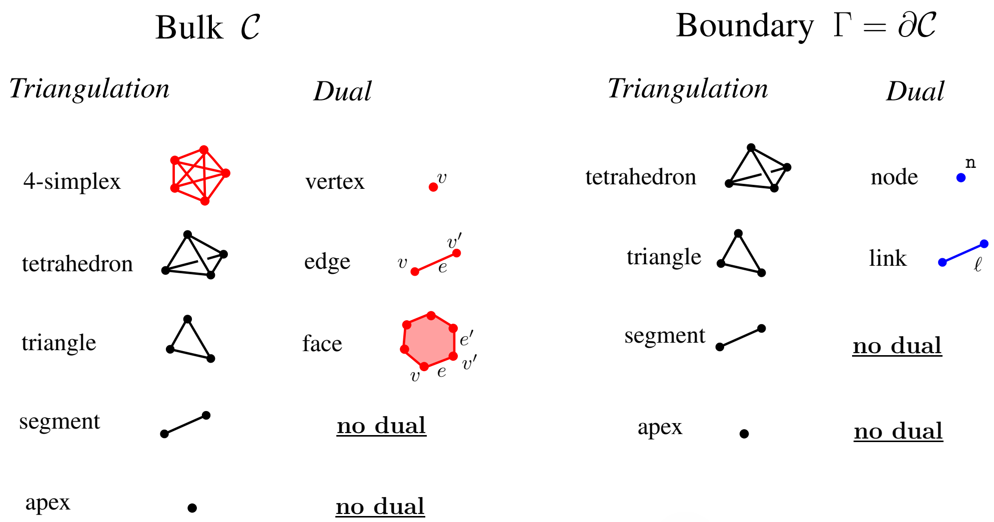
Now, let's increase the number of dimensions by going to three. We truncate 3d space intro tetrahedra and we put a vertex in each tetrahedron. We connect the vertices with edges and we get the dual graph. Vertices are four-valent and are dual to tetrahedra. Two contiguous tetrahedra share a common triangle, thus edges are dual to triangles. In the dual graph two edges that emanate from the same vertex belong to the same face. But the edges are dual to triangles thus a face is dual to the segment that belongs to both of the triangles the two edges are dual to. In four dimensions we truncate 4d spacetime with 4-simplices. In the center of each simplex we place a vertex. We connect the vertices with edges and we get the dual graph. Vertices are five-valent and are dual to 4-simplices. Using analogous reasoning as in the 3d case edges are dual to tetrahedra and faces are dual to triangles.

Let's focus on a region of 4d spacetime with a boundary. The bulk is triangulated by 4-simplices and the boundary by tetrahedra. The notation we fix for the dual graphs is vertices, edges, faces in the bulk and nodes, links on the boundary. It is very important to not confuse these. Bellow there is a comprehensive and useful list of our notation and conventions

Triangulation and 2-complex in three *dimensions*



Triangulation and 2-complex in four *dimensions*



Let's now suppose that we find ourselves somewhere in the bulk and we wish to measure the curvature. In a continuous spacetime this can be measured by the holonomy of the connection in the limit where the loop is very small. But in our case we find ourselves inside a 4-simplex in a triangulated spacetime. The best thing we can do is to pick a frame and take a tour around a triangle bounding the 4-simplex just by following the edges and return to our initial position. If the frame is rotated relatively to its initial state there is curvature. This rotation is the outcome of the individual rotations that take place every time we jump from one 4-simplex to the other by following the edges. Thus, we assign to each edge an group element $g \in SL(2, \mathbb{C})$. By the same reasoning the assign to each link of the boundary dual graph an element $h \in SU(2)$. The boundary dual graph has now the structure of a spin network.

We are ready to define our Hilbert space. As mentioned in the beginning, in Loop Quantum Gravity we study the transition from an initial to a final 3-geometry. The only variables available are the $SU(2)$

elements of the boundary graph that sit along the edges of total number L. Thus, we define the Hilbert space

$$\tilde{\mathcal{H}}_\Gamma = L_2[SU(2)^L] \tag{3.21}$$

But there is some redundancy in our description. The measure of curvature we proposed is independent of the orientation of the triad we move around. If we choose an other triad inside every tetrahedron, rotated with respect to the original, the result should be the same. Therefore, our Hilbert space is

$$\mathcal{H}_\Gamma = L_2[SU(2)^L/SU(2)^N]_\Gamma \tag{3.22}$$

Where N is the total number of the tetrahedra that truncate the boundary. The equivalence relation underlying the quotient is

$$\Psi(h_l) = \Psi(\Lambda_{s_l} h_l \Lambda_{s_t}^{-1}) \tag{3.23}$$

where $\Lambda_{s_l} h_l, \Lambda_{s_t}^{-1} \in SU(2)$ correspond to the source node and the target node of the link l. Relation (3.23) can be also written as

$$\vec{C}_n \Psi = 0 \tag{3.24}$$

where

$$\vec{C}_n = \vec{L}_{l_1} + \vec{L}_{l_2} + \vec{L}_{l_3} + \vec{L}_{l_4} \tag{3.25}$$

is the generator of the total $SU(2)$ transformation of the n node which is exactly the closure condition (3.3) we mentioned before applied to every tetrahedron (node).

There is another way to define the Hilbert space. For starters let's recall the Peter-Weyl theorem

$$L_2[SU(2)] = \bigoplus_{j=0}^{+\infty} (\mathcal{H}_j \otimes \mathcal{H}_j) \tag{3.26}$$

Hence,

$$L_2[SU(2)^L] = \bigotimes_{l=1}^{L} \left[\bigoplus_{j_l=0}^{+\infty} (\mathcal{H}_{j_l} \otimes \mathcal{H}_{j_l}) \right] = \bigoplus_{\{j_l\}} \left[\bigotimes_{l=1}^{L} (\mathcal{H}_{j_l} \otimes \mathcal{H}_{j_l}) \right] \tag{3.27}$$

Now let's recall the fact that the tensor product $\mathcal{H}_{j_l} \otimes \mathcal{H}_{j_l}$ corresponds to a link that connects two nodes. Instead of this we can focus on one node and consider the tensor product of the four Hilbert spaces that correspond to the four links that emanate (or terminate) to a node. Thus, we have

$$L_2[SU(2)^L] = \bigoplus_{\{j_l\}} \left[\bigotimes_{n=1}^{N} (\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4}) \right] \tag{3.28}$$

so that the Hilbert space $\mathcal{H}_\Gamma = L_2[SU(2)^L/SU(2)^N]_\Gamma$ can be written as

$$\mathcal{H}_\Gamma = \bigoplus_{\{j_l\}} \left[\bigotimes_{n=1}^{N} \mathrm{Inv}_{SU(2)} (\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4}) \right] \tag{3.29}$$

The actual Hilbert space is defined in an abstract way as the limit of H_Γ when the triangulation is as refined as possible, but we are not going to bother with that.

Enough with the Hilbert space, it is time to talk about the operators. In standard quantum mechanics we have the operators $\hat{x}$ and $\hat{p}$ given by

$$\hat{x}\Psi(x) = x\Psi(x), \quad \hat{p}\Psi(x) = -i\hbar \frac{d\Psi(x)}{dx} \tag{3.30}$$

In our case the analogous to the position operator is easy and uninteresting and it is simply given by

$$\hat{h}_l \Psi(h_l) = h_l \Psi(h_l) \tag{3.31}$$

where the action is to be understood as the action of the group elements $h^a{}_b$

What about the operator that acts as a derivative? Well, we introduced already a derivative that happens to be the left-invariant vector fields on $SU(2)$ which we repeat here

$$(J^i\Psi)(h) = -i\frac{d}{dt}\Psi(he^{t\tau^i})|_{t=0} \tag{3.32}$$

The only thing we have to add is the constant of an area $8\pi\gamma\hbar G$ thus we have the operator

$$\vec{L}_l = 8\pi\gamma\hbar G\vec{J}_l \tag{3.33}$$

where of course we have one operator for every link. These operators are well-defined on $\tilde{\mathcal{H}}_\Gamma$ but since they are vector operators they have no hope of being well-defined on $\mathcal{H}_\Gamma$. But it is easy to define gauge invariant operators by

$$G_{ll'} = \vec{L}_l \cdot \vec{L}'_l \tag{3.34}$$

In the case where $l = l'$ the norm $A_l = \sqrt{G_l}$ is of course the area of the triangle punctured by the link l with spectrum

$$A_l = 8\pi\gamma\hbar G\sqrt{j_l(j_l+1)} \tag{3.35}$$

Again, we can construct the volume operator corresponding to the node n

$$V_n^2 = \frac{2}{9}(\vec{L}_{l_1} \times \vec{L}_{l_2}) \cdot \vec{L}_{l_3} \tag{3.36}$$

Finally a state is written as

$$|\Gamma, j_l, v_n\rangle \tag{3.37}$$

where Γ is there to remind us that we are referring to a specific triangulation.

Chapter 4

Dynamics

4.1 Path Integral Quantization

We are now ready to proceed to the quantization of classical GR via the path integral method. At first we assume that we have a general BF theory and we write the partition function

$$Z = \int \mathcal{D}B\mathcal{D}\omega e^{\frac{i}{\hbar}\int B\wedge F} \tag{4.1}$$

B is a two-form, therefore it's integrated on surfaces. The only surfaces available are the triangles of the truncation or equivalently the faces of the dual graph. Thus, $B \to B_f$. ω is an one-form therefore it is naturally integrated on the edges of the dual graph, thus $\omega \to \omega_e$. From ω we can easy obtain an element of the group by $U_e = \mathcal{P}e^{\int \omega_e}$. In the spirit of the previous chapter, the curvature, or equivalently the field strength F, in the truncated case can be approximated by $F = \prod_{e\in f} U_e$. Thus, we write the truncated partition function as

$$Z = \int \mathcal{D}B_f \int_G dU_e e^{\frac{i}{\hbar}\sum_f B_f \prod_{e\in f} U_e} \tag{4.2}$$

Now, by using the fact that $\int dp e^{ipx} \sim \delta(x)$ we can perform the integration in B to get

$$Z = \int_G dU_e \prod_f \delta(\prod_{e\in f} U_e) \tag{4.3}$$

The next step is to replace the group elements U_e with two other group elements (see Fig 4.1).

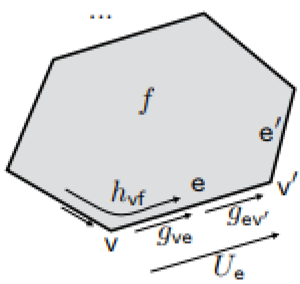


Figure 4.1: The split of the group element U_e into two group elements g_{ve} and $g_{\mathsf{ev'}}$ and the trade of g_{ev}, $g_{\mathsf{ve'}}$ for $h_{\mathsf{v}f}$

We get

$$Z = \int_G dg_{\mathsf{ev}} \prod_f \delta(g_{\mathsf{ve}} g_{\mathsf{ev'}} g_{\mathsf{v'}e'} g_{e'\mathsf{v''}} \ldots) \tag{4.4}$$

We proceed by trading two group elements $g_{e\mathsf{v}'}$ and $g_{\mathsf{v}'e'}$ for one $h_{\mathsf{v}f}$ (see again Fig 4.1). We can write

$$Z = \int_{G'} dh_{\mathsf{v}f} \int_G dg_{e\mathsf{v}} \prod_f \delta(h_{\mathsf{v}f} h_{\mathsf{v}'f} \ldots) \prod_{\mathsf{v}} \prod_{f \in \mathsf{v}} \delta(g_{e'\mathsf{v}} g_{\mathsf{v}e} h_{\mathsf{v}f}) \tag{4.5}$$

Then we can rearrange some terms and write the partition function as

$$Z = \int_{G'} dh_{\mathsf{v}f} \prod_f \delta(h_{\mathsf{v}f} h_{\mathsf{v}'f} \ldots) A_{\mathsf{v}}(h_{\mathsf{v}f}) \tag{4.6}$$

where

$$A_{\mathsf{v}}(h_{\mathsf{v}f}) = \int_G dg_{e\mathsf{v}} \prod_{f \in \mathsf{v}} \delta(g_{e'\mathsf{v}} g_{\mathsf{v}e} h_{\mathsf{v}f}) \tag{4.7}$$

is called the vertex amplitude. The δ function on a group can be expanded in terms of the unitary representations of the group as

$$\delta(U) = \sum_{j_f=0}^{+\infty} (2j_f + 1) \mathrm{Tr}_{j_f}[U] \tag{4.8}$$

Where $2j + 1$ is the dimension of the representation. The vertex amplitude can now be written as

$$A_{\mathsf{v}}(h_{\mathsf{v}f}) = \sum_{\{j_f\}} \int_G dg_{e\mathsf{v}} \prod_f (2j_f + 1) \mathrm{Tr}_{j_f}[g_{e'\mathsf{v}} g_{\mathsf{v}e} h_{\mathsf{v}f}] \tag{4.9}$$

Now, we need to remember that we are not quantizing a general BF theory but GR. GR is characterised by $SL(2,\mathbb{C})$ symmetry in the bulk, $SU(2)$ on the boundary, along with the simplicity constraint $\vec{K} = \gamma \vec{L}$ on the boundary. As we have mentioned every vertex corresponds to a 4-simplex, so the vertex amplitude is the 4-simplex amplitude. A 4-simplex is bounded by five tetraedra what correspond to the nodes on the edges that meet at the vertex (see Fig 4.2). Between these nodes there are ten links that correspond to the ten $h_{\mathsf{v}f}$ elements around the vertex. The situation is now clear; we should treat $h_{\mathsf{v}f}$ as $SU(2)$ elements that rotate the frame as we move along the boundary and $g_{e\mathsf{v}}$ as $SL(2,\mathbb{C})$ elements that rotate the frame as we move in the bulk. We can now write

$$Z = \int_{SU(2)} dh_{\mathsf{v}f} \prod_f \delta(h_{\mathsf{v}f} h_{\mathsf{v}'f} \ldots) A_{\mathsf{v}}(h_{\mathsf{v}f}) \tag{4.10}$$

and

$$A_{\mathsf{v}}(h_{\mathsf{v}f}) = \sum_{\{j_f\}} \int_{SL(2,\mathbb{C})} dg_{e\mathsf{v}} \prod_f (2j_f + 1) \mathrm{Tr}_{j_f}[g_{e'\mathsf{v}} g_{\mathsf{v}e} h_{\mathsf{v}f}] \tag{4.11}$$

There is something off with (4.11). The trace in the integrand involves both $SU(2)$ and $SL(2,\mathbb{C})$ elements. The need a map that embeds elements from the former to the later. But we have this map, it is the Y_γ map introduced in Appendix B. Hence, we should write

$$A_{\mathsf{v}}(h_{\mathsf{v}f}) = \sum_{j_f} \int_{SL(2,\mathbb{C})} dg_{e\mathsf{v}} \prod_f (2j_f + 1) \mathrm{Tr}_{j_f}[Y_\gamma^\dagger g_{e'\mathsf{v}} g_{\mathsf{v}e} Y_\gamma h_{\mathsf{v}f}] \tag{4.12}$$

where the trace is easily computed as

$$\begin{aligned} &\mathrm{Tr}_j[Y_\gamma^\dagger g Y_\gamma h] = \sum_m \langle j,m| Y_\gamma^\dagger g Y_\gamma h |j,m\rangle = \\ &\sum_m \sum_n \langle j,m| Y_\gamma^\dagger g Y_\gamma |j,n\rangle \langle j,n| h |j,m\rangle = \sum_{m,n} D^{(\gamma j, j)}_{jm,jn}(g) D^{(j)}_{nm}(h) \end{aligned} \tag{4.13}$$

The vertex amplitude is a function of the states in

$$\mathcal{H}_{\Gamma_{\mathsf{v}}} = L_2[SU(2)^{10}/SU(2)^5] \tag{4.14}$$

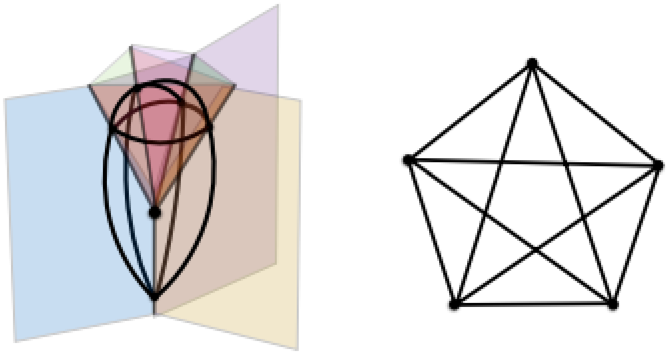


Figure 4.2: Each vertex in the bulk is connected to five edges that correspond to the five tetrahedra that bound a 4-simplex. Each face corresponds to a common triangle between two contiguous tetrahedra. In total we have 10 triangles which correspond to 10 $SU(2)$ elements. By connecting everything that can be connected to get the spin network on the right.

Finally, to obtain the transition amplitude we notice that it is a function of the h_ℓ elements that correspond to the faces of the outer boundary and should not be integrated over. We write

$$W(h_\ell) = \int_{SU(2)} dh_{\mathsf{v}f} \prod_f \delta(h_{\mathsf{v}f} h_{\mathsf{v}'f} \ldots) A_{\mathsf{v}}(h_{\mathsf{v}f}) \tag{4.15}$$

A few remarks. The final transition amplitude is abstractly defined as the limit of (4.15) in the most possible refined truncation. At first someone would worry whether this refinement results to UV divergences but this is not the case in LQG. UV divergences can arise when you are permitted to go arbitrarily small but in LQG you simply can't as we showed when we displayed the kinematics of the Theory. Nevertheless, there can be IR divergences because we are summing over all possible geometries and some of them can be arbitrarily big. But it has been proved that the version of the theory with cosmological constant is IR finite (10).

An interesting qualitative feature is that unlike other lattice gauge theories there is no parameter that should be taken to zero. This is a manifestation of the background independence of Gravity (11).

There have been a lot of persistent rumors and missinformation regarding the classical limit of LQG. We are not going to discuss this here but it has been shown that the vertex amplitudes of LQG are very strongly related to the vertex amplitudes of Regge calculus (12).

There is another form of the transition amplitude. One can expand the delta functions in representations and perform the integrals $dh_{\mathsf{v}f}$. One obtains

$$W_{\mathcal{C}}(h_\ell) = \mathcal{N} \int_{SL(2,\mathbb{C})} \left(\prod_{\mathsf{v}} \mathrm{d}\widehat{g}_{\mathsf{ve}} \right) \left(\prod_{\mathsf{f} \in \mathcal{B}} A_{\mathsf{f}} \right) \left(\prod_{\mathsf{f} \in \Gamma} A_{\mathsf{f}}(h_\ell) \right) \tag{4.16}$$

where A_f is the face amplitude for internal (bulk) faces and it's defined as

$$A_{\mathsf{f}} := \sum_{j_{\mathsf{f}}} d_{j_{\mathsf{f}}} \ \mathrm{Tr}_{j_{\mathsf{f}}} \left[\prod_{\mathsf{v} \in \mathsf{f}} Y^\dagger g_{\mathsf{ve}}^{-1} \, g_{\mathsf{ve}'} Y \right] := \sum_{j_{\mathsf{f}}} d_{j_{\mathsf{f}}} \ \mathrm{Tr}_{j_{\mathsf{f}}} \left[Y^\dagger g_{\mathsf{ev}} g_{\mathsf{ve}'} Y Y^\dagger g_{\mathsf{e}'\mathsf{v}'} g_{\mathsf{v}'\mathsf{e}''} Y \ldots Y^\dagger g_{\mathsf{e}^{(n)}\mathsf{v}^{(n)}} g_{\mathsf{v}^{(n)}\mathsf{e}} Y \right] \quad \text{for } \mathsf{f} \in \mathcal{B}, \tag{4.17}$$

and $A_f(h_\ell)$ is the face amplitude for boundary faces and it's defined as

$$\begin{aligned} A_{\mathsf{f}}(h_\ell) &:= \sum_{j_{\mathsf{f}}} d_{j_{\mathsf{f}}} \ \mathrm{Tr}_{j_{\mathsf{f}}} \left[Y^\dagger g_{\mathsf{vn}'}^{-1} g_{\mathsf{ve}'} Y \left(\prod_{\mathsf{v} \in \mathsf{f}} Y^\dagger g_{\mathsf{ve}'}^{-1} \, g_{\mathsf{ve}} Y \right) Y^\dagger g_{\mathsf{v}^{(n)}\mathsf{e}^{(n)}}^{-1} g_{\mathsf{v}^{(n)}\mathsf{n}} Y h_\ell^{-1} \right] \\ &= \sum_{j_{\mathsf{f}}} d_{j_{\mathsf{f}}} \ \mathrm{Tr}_{j_{\mathsf{f}}} \left[Y^\dagger g_{\mathsf{n}'\mathsf{v}} g_{\mathsf{ve}'} Y Y^\dagger g_{\mathsf{e}'\mathsf{v}'} g_{\mathsf{v}'\mathsf{e}''} Y \ldots Y^\dagger g_{\mathsf{e}^{(n)}\mathsf{v}^{(n)}} g_{\mathsf{v}^{(n)}\mathsf{n}} Y h_\ell^{-1} \right] \qquad \text{for } \mathsf{f} \in \Gamma. \end{aligned} \tag{4.18}$$

This form of the transition amplitude is very useful in actual calculations and it's the form that is explained further and used in the next chapter.

Chapter 5

An estimation for the transition amplitude

5.1 Semiclassical Boundary States

In this section we briefly review the boundary states used in subsequent sections. In 5.1.1 we derive the measure providing the resolution of identity in the twisted geometry parametrisation which will be needed in what follows.[1] In 5.1.2 we review the large areas limit of the states, and discuss semiclassicality conditions, the validity of which will be the central assumption for performing the spin–sum in Section 5.3.

5.1.1 Resolution of Identity in the Twisted Geometry Parametrization for the Heat Kernel States

The boundary states considered throughout this thesis are Thiemann's heat kernel states (15; 16; 17; 18), in the twisted-geometry parametrization (19; 20). When parametrized in this manner, the states are also known as coherent spin-networks (21) or extrinsic coherent states (22). They are elements of the truncated boundary Hilbert space $\mathcal{H}_\Gamma = L^2\left[SU(2)^L/SU(2)^N\right]$ and are labelled by data H_ℓ drawn from the discrete phase space $P_\Gamma = \bigtimes_\ell T^*SU(2)_\ell \simeq \bigtimes_\ell \left(\mathbb{R}^+_\ell \times S^1_\ell \times S^2_\ell \times S^2_\ell\right)$ of twisted geometries. Here, L denotes the number of links ℓ and N the number of nodes n of the boundary graph Γ. Coherent spin-networks are defined as the L-parameter family of states

$$\Psi^{t_\ell}_{\Gamma,H_\ell}(h_\ell) := \int_{SU(2)^N} \left(\prod_{\mathsf{n}} \mathrm{d}h_{\mathsf{n}(\ell)}\right) \prod_\ell K^{t_\ell}_\ell(h_\ell, h_{t(\ell)}\, H_\ell\, h^{-1}_{s(\ell)}), \tag{5.1}$$

where $s(\ell)$ and $t(\ell)$ denote source and target node of the link ℓ, $t_\ell > 0$ are the L semiclassicality parameters and $K^t(h,H)$ is the $SU(2)$ heat kernel with a complexified $SU(2)$ element as second argument. Since $SU(2)^{\mathbb{C}} \simeq SL(2,\mathbb{C})$, H is taken to be an element of $SL(2,\mathbb{C})$[2]. The Wigner D-matrices of the $SU(2)$ heat kernel in (5.1) are defined by analytical extension to the group $SL(2,\mathbb{C})$[3]. Concretely, $K^t(h,H)$ is given in the spin-representation by

$$K^t(h,H) = \sum_j d_j\, \mathrm{e}^{-j(j+1)t}\, \mathrm{Tr}\left[D^{(j)}(hH^{-1})\right]. \tag{5.2}$$

The L-parameter family of states $\Psi^{t_\ell}_{\Gamma,H_\ell}(h_\ell)$ is an overcomplete basis of the Hilbert space $\mathcal{H}_\Gamma$. The identity operator $\mathbb{1}_\Gamma$ on $\mathcal{H}_\Gamma$ is given in the holonomy representation by the delta distribution δ_Γ on

[1]This result is also crucial in order to take into account the issue raised by R. Oeckl in (13), where it is suggested that the measure must be considered in the definitions of the observables studied in (14; 5) for a black hole to white hole transition.

[2]$SL(2,\mathbb{C})$ is isomorphic to $SU(2) \times su(2) \simeq T^*SU(2)$ which corresponds to the (linkwise, not gauge invariant) classical phase space associated to the Hilbert space on a graph.

[3]The explicit defining expression for the analytically extended matrix elements D^j_{mn} can be found in (23) and (24). In fact, this provides an analytic extension to the entire $GL(2,\mathbb{C})$.

$SU(2)^L/SU(2)^N$, and we have

$$\delta_\Gamma(h_l, h'_l) = \int_{SL(2,\mathbb{C})^L} \left(\prod_l \Omega_{2t_\ell}(H_l)\,\mathrm{d}H_l \right) \Psi^{t_\ell}_{\Gamma,H_l}(h_l)\,\overline{\Psi^{t_\ell}_{\Gamma,H_l}(h'_l)}. \tag{5.3}$$

The twisted geometry parametrization relies on the Cartan decomposition of $H_\ell^{-1} \in SL(2,\mathbb{C})$, i.e.

$$H_\ell^{-1} = n_{s(\ell)}\ \mathrm{e}^{(\eta_\ell + i\gamma\zeta_\ell)\frac{\sigma_3}{2}}\ n^{-1}_{t(\ell)}. \tag{5.4}$$

The data H_ℓ are replaced by the data $(\eta_\ell, \zeta_\ell, \vec{n}_{s(\ell)}, \vec{n}_{t(\ell)})$, which at the classical level have the following geometrical interpretation: the data $\eta_\ell \in \mathbb{R}^+$ is related to the area dual to the link ℓ, $\zeta_\ell \in [0, 4\pi)$ encodes the distributional extrinsic curvature (25), γ is the Barbero-Immirzi parameter and $\vec{n}_{s(\ell)}, \vec{n}_{t(\ell)}$ are two unit vectors normal to the face dual to the link. Substituting (5.4) into (5.1) allows the construction of coherent states peaked on a prescribed boundary discrete geometry.

The measure $\Omega_{2t}(H)$ which provides the resolution of identity in terms of Thiemann's coherent states is formally given as the heat kernel on the quotient space $SL(2,\mathbb{C})/SU(2)$, i.e.

$$\Omega_{2t}(H) := \int_{SU(2)} F_{2t}(Hg)\,\mathrm{d}g, \tag{5.5}$$

where F_{2t} is the heat kernel on $SL(2,\mathbb{C})$. The integration measure in the polar decomposition of $SL(2,\mathbb{C})$,

$$H = h\ \mathrm{e}^{\vec{p}\cdot\frac{\vec{\sigma}}{2}}\ . \tag{5.6}$$

where $h \in SU(2)$ and $\vec{p}$ is a vector in $\mathbb{R}^3$, is given by (17; 21)

$$\Omega_{2t}(h\ \mathrm{e}^{\vec{p}\cdot\frac{\vec{\sigma}}{2}}) = \frac{\mathrm{e}^{\frac{t}{2}}}{(2\pi t)^{\frac{3}{2}}} \frac{|\vec{p}|}{\sinh|\vec{p}|}\ \mathrm{e}^{-\frac{|p|^2}{2t}} \qquad \mathrm{d}H = \frac{\sinh^2|\vec{p}|}{|\vec{p}|^2}\,\mathrm{d}h\,\mathrm{d}^3\vec{p} \tag{5.7}$$

With these preliminaries, in the remaining of this section we derive the measure in the twisted geometry parametrization. First, note that the $SU(2)$ element h does not appear on the right hand side of Ω_{2t} in (5.7), because it is by definition an $SU(2)$ invariant function. Similarly, for the twisted geometry parametrization (5.4) we have

$$\Omega_{2t_\ell}\left(n_{s(\ell)}\ \mathrm{e}^{\eta_\ell\frac{\sigma_3}{2}}\ \mathrm{e}^{i\gamma\zeta_\ell\frac{\sigma_3}{2}}\ n^{-1}_{t(\ell)}\right) = \Omega_{2t_\ell}\left(\mathrm{e}^{\eta_\ell\frac{\sigma_3}{2}}\right) = \frac{\mathrm{e}^{-\frac{t_\ell}{2}}}{(2\pi t_\ell)^{\frac{3}{2}}} \frac{\eta_\ell}{\sinh\eta_\ell}\ \mathrm{e}^{-\frac{\eta_\ell^2}{2t_\ell}} \tag{5.8}$$

where again the $SU(2)$ elements $n_{s(\ell)}$ and $n_{t(\ell)}$ drop out from the right hand side because of the $SU(2)$ invariance of Ω_{2t}. The measure $\mathrm{d}H_\ell$ in the Cartan decomposition reads[4]

$$\mathrm{d}H_\ell = \frac{\sinh^2\eta_\ell}{4\pi}\,\mathrm{d}\eta_\ell\,\mathrm{d}u_\ell\,\mathrm{d}v_\ell, \tag{5.9}$$

where $\mathrm{d}u_\ell$ and $\mathrm{d}v_\ell$ are $SU(2)$ Haar measures. We seem to have achieved our goal, but, the resolution of identity in the twisted geometry parametrization does not immediately follow from the above expressions because of the following subtlety. The polar decomposition of (5.6) for H_ℓ is unique and is a parametrization by six real parameters. The twisted geometry parametrization (5.4) for H_ℓ is not unique. There is a $U(1)$ gauge choice to be made, since there are seven real parameters to be integrated over in (5.9). We proceed by ansatz, choosing to drop the ζ_ℓ integration in $\mathrm{d}u_\ell$ such that the measure becomes proportional to the standard measure on the two-sphere $\mathcal{S}^2$. The measure $\mathrm{d}v_\ell$ remains the standard $SU(2)$ Haar measure. Concretely:

$$\begin{aligned}
\mathrm{d}u_\ell &:= \mathcal{N}\sin\theta_{s(\ell)}\,\mathrm{d}\phi_{s(\ell)}\,\mathrm{d}\theta_{s(\ell)} = \mathcal{N}\,\mathrm{d}^2\vec{n}_{s(\ell)} \\
\mathrm{d}v_\ell &:= \frac{1}{(4\pi)^2}\sin\theta_{t(\ell)}\,\mathrm{d}\phi_{t(\ell)}\,\mathrm{d}\theta_{t(\ell)}\,\mathrm{d}\zeta_\ell = \frac{1}{(4\pi)^2}\mathrm{d}^2\vec{n}_{t(\ell)}\,\mathrm{d}\zeta_\ell.
\end{aligned} \tag{5.10}$$

[4]See, for instance, (23).

The full ansatz for the resolution of identity measure then reads

$$\Omega_{2t_\ell}(H_\ell)\,\mathrm{d}H_\ell = \frac{\mathcal{N}}{(4\pi)^2}\,\Omega_{2t_\ell}\left(\mathrm{e}^{\eta_\ell\frac{\sigma_3}{2}}\right)\,\sinh^2\eta_\ell\,\mathrm{d}\eta_\ell\,\mathrm{d}\zeta_\ell\,\mathrm{d}^2\vec{n}_{s(\ell)}\,\mathrm{d}^2\vec{n}_{t(\ell)}. \tag{5.11}$$

We fix the normalization $\mathcal{N}$ by requiring that the "volume" of this measure over $SL(2,\mathbb{C})$ is the same in the polar decomposition and the Cartan decomposition

$$\int_{SL(2,\mathbb{C})}\mathcal{N}\,\Omega_{2t_\ell}\left(\mathrm{e}^{\eta_\ell\frac{\sigma_3}{2}}\right)\,\sinh^2\eta_\ell\,\mathrm{d}\eta_\ell\,\mathrm{d}\zeta_\ell\,\mathrm{d}^2\vec{n}_{s(\ell)}\,\mathrm{d}^2\vec{n}_{t(\ell)} = \int_{SL(2,\mathbb{C})}\Omega_{2t_\ell}(\mathrm{e}^{\vec{p}\cdot\frac{\vec{\sigma}}{2}})\,\frac{\sinh^2|\vec{p}|}{|\vec{p}|^2}\,\mathrm{d}h\,\mathrm{d}^3\vec{p}. \tag{5.12}$$

On the left hand side the two integrations over $\mathcal{S}^2$ and the integration over ζ_ℓ give an overall factor of $(4\pi)^3$ and on the right hand side the $SU(2)$ integration gives unit. There remain the integrations over η_ℓ and $\vec{p}$. From (5.7), the integrand on the right hand side only depends on the norm $p \equiv |\vec{p}|$. We can therefore change to polar coordinates for the vector $\vec{p}$ which gives another factor of 4π from the angular integration. The remaining integrals over η_ℓ and p are of the same form and they are both non zero since the integrand is positive definite. Hence, collecting all terms and solving for $\mathcal{N}$ we find $\mathcal{N} = 1$.

We now proceed to show by direct calculation (that is, that the above ansatz indeed works out) that the measure (5.12) indeed gives the resolution of identity for the states (5.1) in the twisted geometry parametrization. To simplify the notation and render the computations more readable, we drop the gauge-averaging integrations over $SU(2)$ and only consider a single link. The full proof proceeds similarly. We wish to prove that

$$\delta(hh'^\dagger) = \int_{\mathbb{R}^+}\mathrm{d}\eta\,\nu_t(\eta)\int_0^{4\pi}\mathrm{d}\zeta\int_{\mathcal{S}^2}\mathrm{d}^2\vec{n}_s\int_{\mathcal{S}^2}\mathrm{d}^2\vec{n}_t\,\Psi^t_H(h)\,\overline{\Psi^t_H(h')}, \tag{5.13}$$

where

$$\delta(hh'^\dagger) = \sum_j d_j\,\mathrm{Tr}_j[hh'^\dagger] = \sum_j d_j\sum_{|m|\le j}\sum_{|n|\le j}D^j_{mn}(h)D^j_{nm}(h'^\dagger) \tag{5.14}$$

is the Dirac distribution on $SU(2)$ and $\nu_t(\eta)$ is given by

$$\nu_t(\eta) := \frac{\mathrm{e}^{-\frac{t}{2}}}{(4\pi)^2(2\pi t)^{3/2}}\eta\sinh\eta\;\mathrm{e}^{-\frac{\eta^2}{2t}}\,. \tag{5.15}$$

The states (5.1) in the twisted geometry parametrization (5.4) are explicitly given by

$$\begin{aligned}\Psi^t_{\Gamma,H}(h) &= \sum_j d_j\;\mathrm{e}^{-j(j+1)t}\sum_{m,n,k}D^j_{mn}(h)D^j_{nk}(n_t)D^j_{km}(\mathrm{e}^{i\gamma\zeta\frac{\sigma_3}{2}}\,n_s^\dagger)\;\mathrm{e}^{-\eta k}\\ \overline{\Psi^t_{\Gamma,H}(h')} &= \sum_{j'} d_{j'}\;\mathrm{e}^{-j'(j'+1)t}\sum_{m',n',k'}D^{j'}_{n'm'}(h'^\dagger)\,D^{j'}_{k'n'}(n_t^\dagger)D^{j'}_{m'k'}(n_s\,\mathrm{e}^{-i\gamma\zeta\frac{\sigma_3}{2}})\;\mathrm{e}^{-\eta k'}\,.\end{aligned} \tag{5.16}$$

By noticing that $n_s = \mathrm{e}^{-i\phi_s\frac{\sigma_3}{2}}\,\mathrm{e}^{-i\theta_s\frac{\sigma_3}{2}}$ lives in a subspace of $SU(2)$ we can introduce the auxiliary variable $g := n_s\,\mathrm{e}^{-i\gamma\zeta\frac{\sigma_3}{2}}$, which is a genuine $SU(2)$ element. This allows us to perform the n_s and the ζ integration simultaneously by virtue of the Peter-Weyl theorem

$$\begin{aligned}A &:= \int_{\mathcal{S}^2}\mathrm{d}^2\vec{n}_s\int_0^{4\pi}\mathrm{d}\zeta\,\Psi^t_H(h)\,\overline{\Psi^t_H(h')} = (4\pi)^2\int_{SU(2)}\mathrm{d}g\,\Psi^t_H(h)\,\overline{\Psi^t_H(h')}\\ &= (4\pi)^2\delta^{jj'}\delta_{mm'}\delta_{kk'}\sum_j d_j\,\mathrm{e}^{-2j(j+1)t}\sum_{m,n,k,n'}D^j_{mn}(h)D^j_{nk}(n_t)D^j_{n'm}(h'^\dagger)D^j_{kn'}(n_t^\dagger)\,\mathrm{e}^{-2\eta k}\,.\end{aligned} \tag{5.17}$$

To perform the next integration we notice that n_t is also parametrized as $n_t = \mathrm{e}^{-i\phi_t\frac{\sigma_3}{2}}\;\mathrm{e}^{-i\theta_t\frac{\sigma_2}{2}}$ and that therefore we have

$$D^j_{nk}(n_t) = \mathrm{e}^{-i\phi_t n}\,d^j_{nk}(\theta_t),\qquad D^j_{kn'}(n_t^\dagger) = \mathrm{e}^{i\phi_t n'}\,d^j_{kn'}(-\theta_t), \tag{5.18}$$

which implies that the n_t integration is zero unless $n = n'$. This allows us to do the following step:

$$\int_{\mathcal{S}^2} \mathrm{d}^2\vec{n}_t\, D^j_{nk}(n_t) D^j_{kn'}(n_t^\dagger) = \delta_{nn'} \int_{\mathcal{S}^2} \mathrm{d}^2\vec{n}_t\, D^j_{nk}(n_t) D^j_{kn}(n_t^\dagger)\, \frac{1}{4\pi} \int_0^{4\pi} \mathrm{d}\xi\ \mathrm{e}^{i\xi k}\ \mathrm{e}^{-i\xi k}$$
$$= (4\pi)\delta_{nn'} \int_{SU(2)} \mathrm{d}g'\, D^j_{nk}(g') D^j_{kn}(g'^\dagger) = \frac{4\pi\delta_{nn'}}{d_j}. \tag{5.19}$$

Above, we inserted the identity in the first equality and defined the auxiliary variable $g' := n_t\ \mathrm{e}^{-i\xi\frac{\sigma_3}{2}}$, which allowed us to use again the Peter-Weyl theorem. Hence we find:

$$B := \int_{\mathcal{S}^2} \mathrm{d}^2\vec{n}_t\, A = (4\pi)^3 \sum_j \mathrm{e}^{-2j(j+1)t} \sum_{m,n} D^j_{mn}(h) D_{nm}(h'^\dagger) \sum_k \mathrm{e}^{-2\eta k}\,. \tag{5.20}$$

The last sum is easily performed by recognizing that it can be split into two geometric sums and yields

$$\sum_{|k|\le j} \mathrm{e}^{-2\eta k} = \frac{\sinh\left((2j+1)\eta\right)}{\sinh\eta}, \tag{5.21}$$

which holds for both, integer and half-integer values of j. What is left is the integral over η which gives

$$(4\pi)^3 \int_{\mathbb{R}^+} \mathrm{d}\eta\, \nu_t(\eta) \frac{\sinh\left((2j+1)\eta\right)}{\sinh\eta} = d_j\ \mathrm{e}^{2j(j+1)t}\,. \tag{5.22}$$

Putting everything together we obtain

$$\int_{\mathbb{R}^+} \mathrm{d}\eta\, \nu_t(\eta) B = \sum_j d_j \sum_{m,n} D^j_{mn}(h) D^j_{nm}(h'^\dagger) = \delta(hh'^\dagger). \tag{5.23}$$

This completes the proof. The above steps extend straightforwardly to gauge-invariant states (5.1) on a general graph Γ, and one can then also prove the identity

$$\delta_\Gamma(h, h'^\dagger) = \int_{SL(2,\mathbb{C})^L} \left(\prod_\ell \nu_{t_\ell}(\eta_\ell) \mathrm{d}\eta_\ell\, \mathrm{d}\zeta_\ell\, \mathrm{d}\vec{n}_{s(\ell)}\, \mathrm{d}\vec{n}_{t(\ell)} \right) \Psi^{t_\ell}_{\Gamma, H_\ell}(h) \overline{\Psi^{t_\ell}_{\Gamma, H_\ell}(h')} \tag{5.24}$$

with the Dirac distribution δ_Γ on $SU(2)^L/SU(2)^N$ explicitly given by

$$\delta_\Gamma(h, h'^\dagger) = \int_{SU(2)^N} \left(\prod_{\mathsf{n}} \mathrm{d}h_{\mathsf{n}(\ell)} \right) \int_{SU(2)^N} \left(\prod_{\mathsf{n}} \mathrm{d}\tilde{h}_{\mathsf{n}(\ell)} \right) \prod_\ell \delta\left(h^\dagger_{t(\ell)} h h_{s(\ell)} \left(\tilde{h}^\dagger_{t(\ell)} h' \tilde{h}_{s(\ell)} \right)^\dagger \right). \tag{5.25}$$

In summary, the integration measure giving the resolution of the identity for Thiemann's coherent states in the twisted geometry parametrization reads

$$\Omega_{2t_\ell}(\mathrm{e}^{\eta_\ell\frac{\sigma_3}{2}})\,\mathrm{d}H_\ell = \frac{\mathrm{e}^{-\frac{t_\ell}{2}}}{(4\pi)^2(2\pi t_\ell)^{3/2}} \eta_\ell \sinh\eta_\ell\ \mathrm{e}^{-\frac{\eta_\ell^2}{2t_\ell}}\ \mathrm{d}\eta_\ell\, \mathrm{d}\zeta_\ell\, \mathrm{d}^2\vec{n}_{s(\ell)}\, \mathrm{d}^2\vec{n}_{t(\ell)}$$
$$\eta_\ell \in \mathbb{R}^+, \quad \zeta_\ell \in [0, 4\pi), \quad \vec{n}_{s(\ell)} \in \mathcal{S}^2, \quad \vec{n}_{t(\ell)} \in \mathcal{S}^2 \tag{5.26}$$

5.1.2 The large Areas Limit

In this subsection we briefly review the large area (large η) limit of the Thiemann's states in the twisted geometry parametrization and the interpretation of the data $(\eta_\ell, \zeta_\ell, \vec{n}_{s(\ell)}, \vec{n}_{t(\ell)})$, as appeared in (21). This discussion also provides the kinematical setup and assumptions under which we perform the spin–sum in Section 5.3.

We henceforth drop the gauge-averaging $SU(2)$ integrals in (5.1) because in the following sections we will consider the boundary states in contraction with a spinfoam amplitude and so the $SL(2,\mathbb{C})$ integrals in the vertex amplitude will automatically implement gauge invariance at the nodes.

The first simplification we make is to consider states where all semiclassicality parameters t_ℓ are set equal

$$t_\ell = t \; \forall \ell \in \Gamma \tag{5.27}$$

The semiclassicality parameter t controls the spread of the gaussians over the spins and is thus set to be inversly proportional to a typical macroscopic area A of the triangulation

$$t = \left(\frac{l_p^2}{A}\right)^n \quad \text{with } n \in [0, 2], \tag{5.28}$$

The multiplication by the Planck area l_P^2 is so that t is indeed dimensionless. The reasoning for the restriction of the values of n to $0, 1, 2$ is explained below. Since by assumption

$$A \gg l_p^2, \tag{5.29}$$

we have that

$$t \ll 1. \tag{5.30}$$

When

$$\eta_\ell \gg 1 \; \forall \ell \in \Gamma, \tag{5.31}$$

the states (5.1) are proportional to (21)

$$\Psi^t_{\Gamma, H_\ell}(h_\ell) \propto \sum_{\{j_\ell\}} \prod_\ell d_{j_\ell} \, \mathrm{e}^{-(j_\ell - \omega_\ell)^2 t + i\gamma \zeta_\ell j_\ell} \; \psi_{\Gamma, j_\ell, \vec{n}_{s(\ell)}, \vec{n}_{t(\ell)}}(h_\ell), \tag{5.32}$$

where we dropped a multiplicative factor $\prod_\ell \exp\big((\eta_\ell - t)^2/4t\big)$ and defined the area data

$$\omega_\ell := \frac{\eta_\ell - t}{2t} \approx \frac{\eta_\ell}{2t}. \tag{5.33}$$

The states $\psi_{\Gamma, j_\ell, \vec{n}_{s(\ell)}, \vec{n}_{t(\ell)}}(h_\ell)$ are given by

$$\psi_{\Gamma, j_\ell, \vec{n}_{s(\ell)}, \vec{n}_{t(\ell)}}(h_\ell) = \sum_{m_s, m_t} D^{j_\ell}_{j_\ell m_t}(n^\dagger_{t(\ell)}) \; D^{j_\ell}_{m_t m_s}(h_\ell) \; D^{j_\ell}_{m_s j_\ell}(n_{s(\ell)}). \tag{5.34}$$

The gauged averaged version of the above are the Livine-Speziale coherent states (26) also known as intrinsic coherent states (22).

From the above we see that the large η limit of Thiemann's states parametrized in the twisted geometry parametrization indeed corresponds to a large area limit: large η implies large ω from (5.30) and (5.33), with omega admitting a direct interpretation as the macroscopic area on which spins are peaked. The states (5.32) are peaked on $j_\ell = \omega_\ell$ due to the Gaussian weight factors, which have a spread

$$\sigma := \frac{1}{\sqrt{2t}} \gg 1. \tag{5.35}$$

In particular, the expectation values A_ℓ of the area operator on these states are given by

$$A_\ell \approx \gamma \, l_p^2 \, \omega_\ell, \tag{5.36}$$

the parameters ω_ℓ (and consequently the parameters η_ℓ) are directly related to physical areas A_ℓ. We are assuming the Immirzi parameter to be of order unit

$$\gamma \sim 1 \tag{5.37}$$

By tuning the parameter t, it is possible to peak the states on a prescribed intrinsic and extrinsic semiclassical geometry. For this to be the case, t has to be chosen such that the spreads in the areas ΔA_ℓ and the spreads in the holonomies Δh_ℓ are much smaller than the expectation values of the corresponding operators. This requirement translates into

$$\Delta A_\ell \sim \frac{l_p^2}{\sqrt{t}} \ll A_\ell \quad \text{and} \quad \Delta h_\ell \sim \sqrt{t} \ll 1 \quad \forall \ell \in \Gamma. \tag{5.38}$$

Combining the two, we obtain the semiclassicality condition that will be used in what follows:

$$l_p^2 \ll \sqrt{t}\, A_\ell \ll A_\ell$$

$$\Leftrightarrow \;\; 1 \ll \sqrt{t}\, \omega_\ell \ll \omega_\ell. \tag{5.39}$$

The above translate to the condition $n \in (0, 2)$ for the exponent in (5.28).

In summary, the above reflect a physical setup where there is a single typical area scale. This is for instance the case in the transition of a black to a white hole in spherical symmetry, where the relevant area scale is given by $\frac{m^2}{m_P} l_P^2$, where m is the mass of the hole and m_P is the Planck mass, as in for instance (5). In a homogeneous cosmological setup, the relevant area scale would be given by the squared of the area factor. Recall that the bounce in Loop Quantum Cosmology(27; 28) occurs when the typical area is still macroscopic; i.e. while we are still in the large areas regime of equation (5.29).

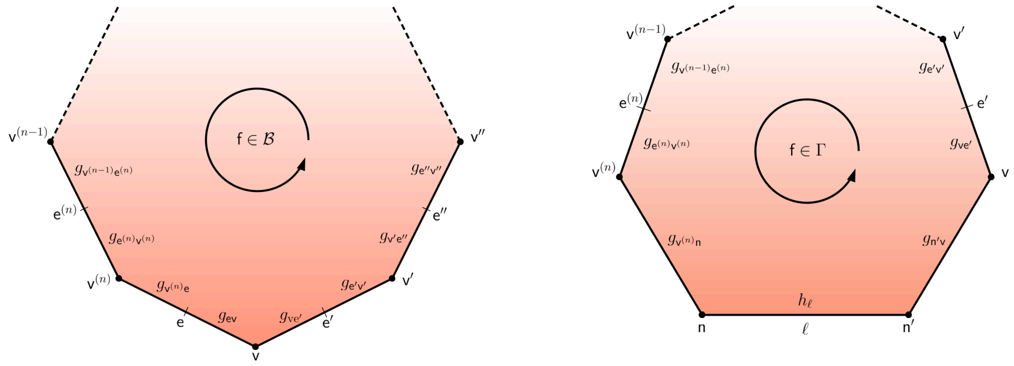
5.2 The Path Integral Representation of The Lorentzian EPRL Amplitude

In this section we give a different derivation of the path integral representation of the EPRL amplitude discovered in (4). In that work, the authors employed group theoretical methods which allowed them to give a path-integral representation for the case of a 2-complex without boundary that precisely captures the number of degrees of freedom, without the need of introducing auxiliary spinorial variables such as those appearing in the coherent state representation. That has the advantage of rendering the critical point equations in the asymptotic analysis particularly transparent. However, because the method used in (4) for the derivation differs significantly from the coherent state representation techniques it becomes difficult to combine it with the analysis carried out in (29), which is what we want to do here. The derivation of the representation in (4) that we give here uses similar techniques as in (29). This then allows us to extend the representation of (4) to two-complexes with boundary. The contraction of the EPRL amplitude with boundary coherent states (that is, performing the spin sum) then becomes straightforward by combining the results of (4) and (29). This is done in Section 5.3.2.

First we fix notation and terminology. We will consider a topological two-complex $\mathcal{C}$ with a non-empty boundary $\Gamma := \partial\mathcal{C} \neq \emptyset$ and bulk $\mathcal{B} := \mathcal{C}\backslash\Gamma$. The two-complex is assumed to be dual to a four dimensional simplicial triangulation and consists of a collection of five-valent vertices v connected by edges e which in turn bound faces f. All vertices belong to the bulk $\mathcal{B}$, but some of the edges which emanate from a vertex intersect the boundary and therefore terminate at a node n. Nodes are four-valent and connected by links ℓ. Vertices and edges are considered to be part of the bulk structure (also referred to as one-skeleton) while nodes and links constitute the boundary graph. Faces are said to be bulk faces when they are bounded by vertices and edges and we also write $\mathsf{f} \in \mathcal{B}$. If a face is bounded by vertices, edges, nodes and links it is said to be a boundary face and we write $\mathsf{f} \in \Gamma$ with a slight abuse of notation (Γ is a graph, it has no faces, it is a boundary link of the face f that belongs to Γ).

The EPRL amplitude is a map $W_\mathcal{C} : \mathcal{H}_\Gamma \to \mathbb{C}$ defined on the two-complex $\mathcal{C}$, which associates complex numbers to states from the boundary Hilbert space $\mathcal{H}_\Gamma = L^2\left[SU(2)^L/SU(2)^N\right]$. A group element $g_{\mathsf{ve}} \in SL(2,\mathbb{C})$ is associated to every half edge in the bulk (see Figure 5.1) and by convention we set $g_{\mathsf{ve}} = g_{\mathsf{ev}}^{-1}$. If an edge originating from v terminates at a node n, then it is not split in two and the single group element is associated to it is denoted as $g_{\mathsf{vn}} \in SL(2,\mathbb{C})$. Links carry $SU(2)$ group elements h_ℓ and all faces, whether they are boundary or bulk faces, are colored by a half-integer spin $j_\mathsf{f} > 0$. Moreover, all faces carry an orientation which in turn induces an orientation on the edges and links (see Figure 5.1). In particular, the face orientation induces the notion of ingoing and outgoing group elements. An element of the form g_{ev} sits on the half edge e which enters the vertex v and is called ingoing while the element $g_{\mathsf{ve}'}$ sits on the half edge e' which exits the vertex v and is called outgoing.

The amplitude $W_\mathcal{C}$ can be written as a product of face amplitudes, associated to every face of the two-complex. For bulk faces, the face amplitude A_f is constructed as follows: At every vertex v we build the product of ingoing group element g_{ev} and outgoing group element $g_{\mathsf{ve}'}$, i.e. $g_{\mathsf{ev}}g_{\mathsf{ve}'}$. Every such product is multiplied from the left by $Y^\dagger$ and from the right by Y, yielding a product of the form $Y^\dagger g_{\mathsf{ev}}g_{\mathsf{ve}'}Y$ at every vertex. The unitary injection Y (Y-map) will be defined precisely below. These terms combine as



(a) Bulk

(b) Boundary

Figure 5.1: The notation and conventions used. All faces are assumed to have an (arbitrarily chosen) orientation.

we go around the face, and the face amplitude is defined as

$$A_{\mathsf{f}} := \sum_{j_{\mathsf{f}}} d_{j_{\mathsf{f}}} \operatorname{Tr}_{j_{\mathsf{f}}} \left[\prod_{\mathsf{v} \in \mathsf{f}} Y^\dagger g_{\mathsf{ve}}^{-1} \, g_{\mathsf{ve}'} Y \right] := \sum_{j_{\mathsf{f}}} d_{j_{\mathsf{f}}} \operatorname{Tr}_{j_{\mathsf{f}}} \left[Y^\dagger g_{\mathsf{ev}} g_{\mathsf{ve}'} Y Y^\dagger g_{\mathsf{e}'\mathsf{v}'} g_{\mathsf{v}'\mathsf{e}''} Y \dots Y^\dagger g_{\mathsf{e}^{(n)}\mathsf{v}^{(n)}} g_{\mathsf{v}^{(n)}\mathsf{e}} Y \right] \quad \text{for } \mathsf{f} \in \mathcal{B}, \tag{5.40}$$

where the summation in j_{f} runs over $\frac{1}{2}\mathbb{N}\backslash\{0\}$ in half-integer steps. The trace is explicitly defined by

$$\operatorname{Tr}_{j_{\mathsf{f}}} \left[\prod_{\mathsf{v} \in \mathsf{f}} Y^\dagger g_{\mathsf{ve}}^{-1} \, g_{\mathsf{ve}'} Y \right] = \sum_{\{m_{\mathsf{e}}\}} D^{(\gamma j_{\mathsf{f}}, j_{\mathsf{f}})}_{j_{\mathsf{f}} m_{\mathsf{e}} j_{\mathsf{f}} m'_{\mathsf{e}}}(g_{\mathsf{ev}} g_{\mathsf{ve}'}) D^{(\gamma j_{\mathsf{f}}, j_{\mathsf{f}})}_{j_{\mathsf{f}} m_{\mathsf{e}'} j_{\mathsf{f}} m_{\mathsf{e}''}}(g_{\mathsf{e}'\mathsf{v}'} g_{\mathsf{v}'\mathsf{e}''}) \dots D^{(\gamma j_{\mathsf{f}}, j_{\mathsf{f}})}_{j_{\mathsf{f}} m_{\mathsf{e}^{(n)}} j_{\mathsf{f}} m_{\mathsf{e}}}(g_{\mathsf{e}^{(n)}\mathsf{v}^{(n)}} g_{\mathsf{v}^{(n)}\mathsf{e}}), \tag{5.41}$$

where $D^{(\gamma j, j)}_{jmjm'}(g)$ are representation matrices of the principal series of unitary irreducible representations of $SL(2,\mathbb{C})$, and $\sum_{\{m_{\mathsf{e}}\}}$ is a short hand notation for multiple sums (in this case, over all magnetic indices m_{e} appearing in (5.41)). The face amplitude $A_{\mathsf{f}}(h_\ell)$ for boundary faces is defined analogously, the difference being that edges terminating in nodes are not split into half edges and therefore carry only one $SL(2,\mathbb{C})$ group element, and there is an $SU(2)$ group element h_ℓ on the link:

$$\begin{aligned} A_{\mathsf{f}}(h_\ell) &:= \sum_{j_{\mathsf{f}}} d_{j_{\mathsf{f}}} \operatorname{Tr}_{j_{\mathsf{f}}} \left[Y^\dagger g_{\mathsf{vn}'}^{-1} g_{\mathsf{ve}'} Y \left(\prod_{\mathsf{v} \in \mathsf{f}} Y^\dagger g_{\mathsf{ve}'}^{-1} \, g_{\mathsf{ve}} Y \right) Y^\dagger g_{\mathsf{v}^{(n)}\mathsf{e}^{(n)}}^{-1} g_{\mathsf{v}^{(n)}\mathsf{n}} Y h_\ell^{-1} \right] \\ &= \sum_{j_{\mathsf{f}}} d_{j_{\mathsf{f}}} \operatorname{Tr}_{j_{\mathsf{f}}} \left[Y^\dagger g_{\mathsf{n}'\mathsf{v}} g_{\mathsf{ve}'} Y Y^\dagger g_{\mathsf{e}'\mathsf{v}'} g_{\mathsf{v}'\mathsf{e}''} Y \dots Y^\dagger g_{\mathsf{e}^{(n)}\mathsf{v}^{(n)}} g_{\mathsf{v}^{(n)}\mathsf{n}} Y h_\ell^{-1} \right] \qquad \text{for } \mathsf{f} \in \Gamma. \end{aligned} \tag{5.42}$$

In the above definition we used the fact that for a 2-complex dual to a simplicial triangulation there is only one link per boundary face. The amplitude $W_{\mathcal{C}}(h_\ell)$ associated to the two-complex $\mathcal{C}$ is finally defined as

$$W_{\mathcal{C}}(h_\ell) := \mathcal{N} \int_{SL(2,\mathbb{C})} \left(\prod_{\mathsf{v}} \mathrm{d}\widehat{g}_{\mathsf{ve}} \right) \left(\prod_{\mathsf{f} \in \mathcal{B}} A_{\mathsf{f}} \right) \left(\prod_{\mathsf{f} \in \Gamma} A_{\mathsf{f}}(h_\ell) \right), \tag{5.43}$$

where $\mathcal{N}$ is an arbitrary normalization constant. This constant is finite when any one of the five $SL(2,\mathbb{C})$ integrations at each vertex is dropped (30). This is indicated by the notation $\mathrm{d}\widehat{g}_{\mathsf{ve}}$, which is the product of four $SL(2,\mathbb{C})$ Haar measures, explicitly defined as

$$\mathrm{d}g = \frac{\mathrm{d}\beta \, \mathrm{d}\overline{\beta} \, \mathrm{d}\gamma \, \mathrm{d}\overline{\gamma} \, \mathrm{d}\delta \, \mathrm{d}\overline{\delta}}{|\delta|^2} \quad \text{for} \quad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{C}). \tag{5.44}$$

5.2.1 A different derivation of the Krajewski-Han representation for a Two-Complex without Boundary

To recast the EPRL amplitude (5.43) in a path integral form, we work in a representation of the principal series of $SL(2,\mathbb{C})$ and its subgroup $SU(2)$ on the space of homogeneous functions $\mathcal{H}^{(k,p)}$ in two complex variables $\mathbf{z} = (z^0, z^1)^\intercal \in \mathbb{C}^2$. A self-contained review of the $SL(2,\mathbb{C})$ and $SU(2)$ representation theory on this space is given in Appendix B.1. Here, we recall only what is necessary for the calculations that follow. The unitary, irreducible, infinite dimensional representations of the principal series of $SL(2,\mathbb{C})$ on $\mathcal{H}^{(k,p)}$ are labeled by two parameters, $(k,p) \in \mathbb{R}\times\frac{1}{2}\mathbb{Z}$. In terms of these two parameters, the functions $F \in \mathcal{H}^{(k,p)}$ satisfy the homogeneity property

$$F(\lambda\mathbf{z}) = \lambda^{ik+p-1}\,\overline{\lambda}^{ik-p-1}F(\mathbf{z}) \quad \forall\lambda\in\mathbb{C}\backslash\{0\}. \tag{5.45}$$

The space $\mathcal{H}^{(k,p)}$ decomposes as $\mathcal{H}^{(k,p)} \simeq \bigoplus_{j=|p|}^{\infty}\mathcal{V}^j$, where $\mathcal{V}^j$ is the space of homogeneous polynomials of degree $2j$ in two complex variables. The Y-map provides us with a unitary injection

$$Y:\mathcal{V}^j \to \mathcal{H}^{(\gamma j,j)} \quad , \quad f(\mathbf{z}) \mapsto F(\mathbf{z}) = \langle\mathbf{z}|\mathbf{z}\rangle^{i\gamma j-j-1}\,f(\mathbf{z}) \quad \forall f\in\mathcal{V}^j, \tag{5.46}$$

where $\langle\mathbf{x}|\mathbf{y}\rangle = \overline{x}^0y^0 + \overline{x}^1y^1$ is the $SU(2)$ (but not $SL(2,\mathbb{C})$) invariant inner product on $\mathbb{C}^2$. The Y-map allows us to easily determine a basis of $\mathcal{H}^{(k,p)}$. As shown in Appendix B.1, the space $\mathcal{V}^j$ is spanned by the basis polynomials

$$P^j_m(\mathbf{z}) = \left[\frac{(2j)!}{(j+m)!(j-m)!}\right]^{\frac{1}{2}} z_0^{j+m}\,z_1^{j-m}, \quad m\in\{-j,...,j\} \tag{5.47}$$

and acting with the Y-map on these basis elements yields

$$\phi^{(\gamma j,j)}_m(\mathbf{z}) := Y \triangleright P^j_m(\mathbf{z}) = \sqrt{\frac{d_j}{\pi}}\,\langle\mathbf{z}|\mathbf{z}\rangle^{i\gamma j-j-1}\,P^j_m(\mathbf{z}), \tag{5.48}$$

which is a basis for $\mathcal{H}^{(k,p)}$. The basis $\phi^{(\gamma j,j)}_m$ is orthonormal with respect to the inner product

$$\langle f,g\rangle := \int_{\mathbb{C}\mathrm{P}^1}\mathrm{d}\Omega\,\overline{f(\mathbf{z})}\,g(\mathbf{z}), \quad \forall f,g\in\mathcal{H}^{(\gamma j,j)}, \tag{5.49}$$

where $\mathrm{d}\Omega = \frac{i}{2}(z^0\mathrm{d}z^1 - z^1\mathrm{d}z^0)\wedge(\overline{z}^0\mathrm{d}\overline{z}^1 - \overline{z}^1\mathrm{d}\overline{z}^0)$ is a homogeneous and $SL(2,\mathbb{C})$ invariant measure on $\mathbb{C}^2\backslash\{0\} \simeq \mathbb{C}\mathrm{P}^1$. By virtue of this inner product, the $SL(2,\mathbb{C})$ representation matrices can be written as

$$D^{(\gamma j,j)}_{j\,m\,j\,m'}(g) \equiv \langle jm|\,Y^\dagger gY\,|jm'\rangle = \int_{\mathbb{C}\mathrm{P}^1}\mathrm{d}\Omega\,\overline{\phi^{(\gamma j,j)}_m(\mathbf{z})}\,\phi^{(\gamma j,j)}_{m'}(g^\intercal\mathbf{z}). \tag{5.50}$$

If g lies in the $SU(2)$ subgroup, the usual Wigner D-matrices are recovered, see (23). Equation (5.50) is crucial in what follows since it is the key to rewrite the trace of the bulk face amplitude (5.40) in the representation found in (4). From the definition of the trace (5.41) together with (5.50) it follows that

$$\begin{aligned}
\mathrm{Tr}_{j_\mathsf{f}}\left[\prod_{\mathsf{v}\in\mathsf{f}}Y^\dagger g_{\mathsf{ve}}^{-1}\,g_{\mathsf{ve}'}Y\right] &= \sum_{\{m_\mathsf{e}\}}\prod_{\mathsf{v}\in\mathsf{f}}D^{(\gamma j_\mathsf{f},j_\mathsf{f})}_{j_\mathsf{f}\,m_\mathsf{e}\,j_\mathsf{f}\,m_{\mathsf{e}'}}(g_{\mathsf{ve}}^{-1}g_{\mathsf{ve}'})\\
&= \sum_{\{m_\mathsf{e}\}}\prod_{\mathsf{v}\in\mathsf{f}}\int_{\mathbb{C}\mathrm{P}^1}\mathrm{d}\Omega_{\mathsf{vf}}\,\overline{\phi^{(\gamma j_\mathsf{f},j_\mathsf{f})}_{m_\mathsf{e}}(\mathbf{z}_{\mathsf{vf}})}\,\phi^{(\gamma j_\mathsf{f},j_\mathsf{f})}_{m_{\mathsf{e}'}}(g^\intercal_{\mathsf{ve}'}\,(g_{\mathsf{ve}}^{-1})^\intercal\,\mathbf{z}_{\mathsf{vf}})\\
&= \sum_{\{m_\mathsf{e}\}}\prod_{\mathsf{v}\in\mathsf{f}}\int_{\mathbb{C}\mathrm{P}^1}\mathrm{d}\Omega_{\mathsf{vf}}\,\overline{\phi^{(\gamma j_\mathsf{f},j_\mathsf{f})}_{m_\mathsf{e}}(g^\intercal_{\mathsf{ve}}\,\mathbf{z}_{\mathsf{vf}})}\,\phi^{(\gamma j_\mathsf{f},j_\mathsf{f})}_{m_{\mathsf{e}'}}(g^\intercal_{\mathsf{ve}'}\,\mathbf{z}_{\mathsf{vf}}).
\end{aligned} \tag{5.51}$$

To get the last line we performed the change of integration variables $\mathbf{z}_{\mathsf{vf}} \to g^\intercal_{\mathsf{ve}}\,\mathbf{z}_{\mathsf{vf}}$ and used the $SL(2,\mathbb{C})$ invariance of the measure $\mathrm{d}\Omega_{\mathsf{vf}}$. Exploiting the fact that the trace (5.51) appears under an integral with an $SL(2,\mathbb{C})$ Haar measure in (5.43), we perform the replacement $g_{\mathsf{ve}} \to \overline{g}_{\mathsf{ve}}$ on all group variables. We define spinorial variables associated to vertices and half edges of a given face:

$$\mathbf{Z}_{\mathsf{vef}} := g^\dagger_{\mathsf{ve}}\,\mathbf{z}_{\mathsf{vf}} \quad , \quad \mathbf{Z}_{\mathsf{ve'f}} := g^\dagger_{\mathsf{ve}'}\,\mathbf{z}_{\mathsf{vf}}. \tag{5.52}$$

Using the explicit expression (5.48) for the basis functions $\phi_m^{(\gamma j,j)}$, the trace is brought to the form

$$
\begin{aligned}
\mathrm{Tr}_{j_{\mathsf{f}}}\left[\prod_{\mathsf{v}\in\mathsf{f}} Y^\dagger g_{\mathsf{ve}}^{-1}\, g_{\mathsf{ve}'} Y\right] &= \sum_{\{m_{\mathsf{e}}\}} \prod_{\mathsf{v}\in\mathsf{f}} \int_{\mathbb{CP}^1} \mathrm{d}\Omega_{\mathsf{vf}}\, \overline{\phi_{m_{\mathsf{e}}}^{(\gamma j_{\mathsf{f}},j_{\mathsf{f}})}(\mathbf{Z}_{\mathsf{vef}})}\, \phi_{m_{\mathsf{e}'}}^{(\gamma j_{\mathsf{f}},j_{\mathsf{f}})}(\mathbf{Z}_{\mathsf{ve'f}}) \\
&= \sum_{\{m_{\mathsf{e}}\}} \prod_{\mathsf{v}\in\mathsf{f}} \frac{d_{j_{\mathsf{f}}}}{\pi} \int_{\mathbb{CP}^1} \mathrm{d}\Omega_{\mathsf{vf}}\, \langle \mathbf{Z}_{\mathsf{vef}} | \mathbf{Z}_{\mathsf{vef}} \rangle^{-i\gamma j_{\mathsf{f}} - j_{\mathsf{f}} - 1} \langle \mathbf{Z}_{\mathsf{ve'f}} | \mathbf{Z}_{\mathsf{ve'f}} \rangle^{i\gamma j_{\mathsf{f}} - j_{\mathsf{f}} - 1} P_{m_{\mathsf{e}'}}^{j_{\mathsf{f}}}(\mathbf{Z}_{\mathsf{ve'f}})\, P_{m_{\mathsf{e}}}^{j_{\mathsf{f}}}\left(\overline{\mathbf{Z}}_{\mathsf{vef}}\right). \qquad (5.53)
\end{aligned}
$$

In the above expression, the spinorial inner products do not depend on any magnetic indices m_{e}. Hence, the sums only extend over the $SU(2)$ basis polynomials P_m^j. There are two such polynomials per edge e which carry the same magnetic index m_{e} (as there are two $\mathbf{Z}$ spinors per edge, but pertaining to different vertices). This follows from the contraction pattern in (5.41). Consequently, the sum $\sum_{\{m_{\mathsf{e}}\}}$ decomposes into a certain number[5] of single, independent sums of the form

$$
\begin{aligned}
&\sum_{|m_{\mathsf{e}'}|\le j_{\mathsf{f}}} P_{m_{\mathsf{e}'}}^{j_{\mathsf{f}}}(\mathbf{Z}_{\mathsf{ve'f}})\, P_{m_{\mathsf{e}'}}^{j_{\mathsf{f}}}\left(\overline{\mathbf{Z}}_{\mathsf{v'e'f}}\right) && (5.54) \\
&= \sum_{|m_{\mathsf{e}'}|\le j_{\mathsf{f}}} \frac{(2j_{\mathsf{f}})!}{(j_{\mathsf{f}}+m_{\mathsf{e}'})!(j_{\mathsf{f}}-m_{\mathsf{e}'})!} \left(\overline{Z}^0_{\mathsf{v'e'f}} Z^0_{\mathsf{ve'f}}\right)^{j_{\mathsf{f}}+m_{\mathsf{e}'}} \left(\overline{Z}^1_{\mathsf{v'e'f}} Z^1_{\mathsf{ve'f}}\right)^{j_{\mathsf{f}}-m_{\mathsf{e}'}} \\
&= \sum_{s=0}^{2j_{\mathsf{f}}} \binom{2j_{\mathsf{f}}}{s} \left(\overline{Z}^0_{\mathsf{v'e'f}} Z^0_{\mathsf{ve'f}}\right)^{s} \left(\overline{Z}^1_{\mathsf{v'e'f}} Z^1_{\mathsf{ve'f}}\right)^{2j_{\mathsf{f}}-s} \\
&= \left(\overline{Z}^0_{\mathsf{v'e'f}} Z^0_{\mathsf{ve'f}} + \overline{Z}^1_{\mathsf{v'e'f}} Z^1_{\mathsf{ve'f}}\right)^{2j_{\mathsf{f}}} = \langle \mathbf{Z}_{\mathsf{v'e'f}} | \mathbf{Z}_{\mathsf{ve'f}} \rangle^{2j_{\mathsf{f}}}. && (5.55)
\end{aligned}
$$

In the first line we use the definition (5.48) of $P_{m_{\mathsf{e}'}}^{j_{\mathsf{f}}}$ and in the second line we performed the change of summation variable $s = j_{\mathsf{f}} + m_{\mathsf{e}'}$. The resulting binomial sum is trivial and yields the result on the third line. Plugging (5.54) into (5.53) and changing from a product over vertices $\mathsf{v}\in\mathsf{f}$ to an equivalent product over edges $\mathsf{e}\in\mathsf{f}$ brings the bulk face amplitude into the form

$$
\begin{aligned}
A_{\mathsf{f}} &= \sum_{j_{\mathsf{f}}} d_{j_{\mathsf{f}}} \prod_{\mathsf{e}\in\mathsf{f}} \frac{d_{j_{\mathsf{f}}}}{\pi} \int_{\mathbb{CP}^1} \mathrm{d}\tilde{\Omega}_{\mathsf{vef}}\, \frac{\langle \mathbf{Z}_{\mathsf{v'e'f}} | \mathbf{Z}_{\mathsf{ve'f}} \rangle^{2j_{\mathsf{f}}}}{\langle \mathbf{Z}_{\mathsf{v'e'f}} | \mathbf{Z}_{\mathsf{v'e'f}} \rangle^{i\gamma j_{\mathsf{f}} + j_{\mathsf{f}}} \langle \mathbf{Z}_{\mathsf{ve'f}} | \mathbf{Z}_{\mathsf{v'e'f}} \rangle^{-i\gamma j_{\mathsf{f}} + j_{\mathsf{f}}}} \\
&= \sum_{j_{\mathsf{f}}} d_{j_{\mathsf{f}}} \prod_{\mathsf{e}\in\mathsf{f}} \frac{d_{j_{\mathsf{f}}}}{\pi} \int_{\mathbb{CP}^1} \mathrm{d}\tilde{\Omega}_{\mathsf{vef}}\, \mathrm{e}^{j_{\mathsf{f}} S_{\mathsf{f}}[g_{\mathsf{ve}}, \mathbf{z}_{\mathsf{vf}}]} \qquad \forall \mathsf{f}\in\mathcal{B}. \qquad (5.56)
\end{aligned}
$$

As in (4) and (29) we introduced the rescaled measure

$$
\mathrm{d}\tilde{\Omega}_{\mathsf{vef}} := \frac{\mathrm{d}\Omega_{\mathsf{vf}}}{\langle \mathbf{Z}_{\mathsf{vef}} | \mathbf{Z}_{\mathsf{vef}} \rangle \langle \mathbf{Z}_{\mathsf{ve'f}} | \mathbf{Z}_{\mathsf{ve'f}} \rangle} \qquad (5.57)
$$

and an "action" $S_{\mathsf{f}}[g_{\mathsf{ve}}, \mathbf{z}_{\mathsf{vf}}]$ associated to bulk faces:

$$
S_{\mathsf{f}}[g_{\mathsf{ve}}, \mathbf{z}_{\mathsf{vf}}] := \log \frac{\langle \mathbf{Z}_{\mathsf{v'e'f}} | \mathbf{Z}_{\mathsf{ve'f}} \rangle^2}{\langle \mathbf{Z}_{\mathsf{vef}} | \mathbf{Z}_{\mathsf{vef}} \rangle \langle \mathbf{Z}_{\mathsf{ve'f}} | \mathbf{Z}_{\mathsf{ve'f}} \rangle} + i\gamma \log \frac{\langle \mathbf{Z}_{\mathsf{ve'f}} | \mathbf{Z}_{\mathsf{ve'f}} \rangle}{\langle \mathbf{Z}_{\mathsf{vef}} | \mathbf{Z}_{\mathsf{vef}} \rangle}. \qquad (5.58)
$$

When the two-complex has no boundary, i.e. when $\Gamma = \emptyset$, then the EPRL transition amplitude in its path integral form would read

$$
\begin{aligned}
W_{\mathcal{C}} &= \mathcal{N} \int_{SL(2,\mathbb{C})} \left(\prod_{\mathsf{v}} \mathrm{d}\widehat{g}_{\mathsf{ve}}\right) \prod_{\mathsf{f}\in\mathcal{B}} \left(\sum_{j_{\mathsf{f}}} d_{j_{\mathsf{f}}} \prod_{\mathsf{e}\in\mathsf{f}} \frac{d_{j_{\mathsf{f}}}}{\pi} \int_{\mathbb{CP}^1} \mathrm{d}\tilde{\Omega}_{\mathsf{vef}}\, \mathrm{e}^{j_{\mathsf{f}} S_{\mathsf{f}}[g_{\mathsf{ve}}, \mathbf{z}_{\mathsf{vf}}]}\right) \\
&= \mathcal{N} \sum_{\{j_{\mathsf{f}}\}} \int_{SL(2,\mathbb{C})} \left(\prod_{\mathsf{v}} \mathrm{d}\widehat{g}_{\mathsf{ve}}\right) \left(\prod_{\mathsf{f}\in\mathcal{B}} d_{j_{\mathsf{f}}} \prod_{\mathsf{e}\in\mathsf{f}} \frac{d_{j_{\mathsf{f}}}}{\pi} \int_{\mathbb{CP}^1} \mathrm{d}\tilde{\Omega}_{\mathsf{vef}}\right) \mathrm{e}^{\sum_{\mathsf{f}\in\mathcal{B}} j_{\mathsf{f}} S_{\mathsf{f}}[g_{\mathsf{ve}}, \mathbf{z}_{\mathsf{vf}}]}. \qquad (5.59)
\end{aligned}
$$

This is precisely the result first obtained in (4) by different means.

[5] The number of sums is equal to the number of edges which constitute the face.

5.2.2 Extension of the Krajewski-Han Path Integral representation to Two-Complexes with Boundary

We now proceed to generalize the calculation of the previous section to a two-complex with a boundary. To this end, it is necessary to also rewrite the trace in the boundary face amplitude (5.42) in terms of functions on $\mathcal{H}^{(\gamma j,j)}$. From the first line of the definition (5.42) it follows that the product over vertices can be treated in the same way as for the bulk face amplitude as none of the group elements lives on an edge which terminates in a node. The only group elements we need to consider here are the first two and the last three in the trace of (5.42) (see also Figure 5.1). For conciseness, we write $(\star)$ as placeholder for the product over vertices. We have

$$
\begin{aligned}
&\mathrm{Tr}_{j_\mathsf{f}}\left[Y^\dagger g_{\mathsf{vn}'}^{-1} g_{\mathsf{ve}'} Y \left(\prod_{\mathsf{v}\in\mathsf{f}} Y^\dagger g_{\mathsf{ve}'}^{-1}\, g_{\mathsf{ve}} Y\right) Y^\dagger g_{\mathsf{v}^{(n)}\mathsf{e}^{(n)}}^{-1} g_{\mathsf{v}^{(n)}\mathsf{n}} Y h_\ell^{-1}\right] \\
&\quad = \sum_{\{m_\mathsf{e}\}} D^{(\gamma j_\mathsf{f}, j_\mathsf{f})}_{j_\mathsf{f} m_{\mathsf{n}'} j_\mathsf{f} m_{\mathsf{e}'}}(g_{\mathsf{vn}'}^{-1} g_{\mathsf{ve}'})\,(\star)\, D^{(\gamma j_\mathsf{f}, j_\mathsf{f})}_{j_\mathsf{f} m_{\mathsf{e}^{(n)}} j_\mathsf{f} m_\mathsf{n}}(g_{\mathsf{v}^{(n)}\mathsf{e}^{(n)}}^{-1} g_{\mathsf{v}^{(n)}\mathsf{n}}) D^{j_\mathsf{f}}_{m_\mathsf{n} m_{\mathsf{n}'}}(h_\ell^{-1}) \\
&\quad = \sum_{\{m_\mathsf{e}\}} \int_{\mathbb{CP}^1} \mathrm{d}\Omega_{\mathsf{vf}}\, \overline{\phi^{(\gamma j_\mathsf{f}, j_\mathsf{f})}_{m_{\mathsf{n}'}}(\mathbf{z}_{\mathsf{vf}})} \phi^{(\gamma j_\mathsf{f}, j_\mathsf{f})}_{m_{\mathsf{e}'}}(g_{\mathsf{ve}'}^\intercal (g_{\mathsf{vn}'}^{-1})^\intercal \mathbf{z}_{\mathsf{vf}})\,(\star) \int_{\mathbb{CP}^1} \mathrm{d}\Omega_{\mathsf{v}^{(n)}\mathsf{f}}\, \overline{\phi^{(\gamma j_\mathsf{f}, j_\mathsf{f})}_{m_{\mathsf{e}^{(n)}}}(\mathbf{z}_{\mathsf{v}^{(n)}\mathsf{f}})} \phi^{(\gamma j_\mathsf{f}, j_\mathsf{f})}_{m_\mathsf{n}}(g_{\mathsf{v}^{(n)}\mathsf{n}}^\intercal (g_{\mathsf{v}^{(n)}\mathsf{e}^{(n)}}^{-1})^\intercal \mathbf{z}_{\mathsf{v}^{(n)}\mathsf{f}}) \times \\
&\qquad \times \int_{\mathbb{CP}^1} \mathrm{d}\Omega_\ell\, \overline{\phi^{(\gamma j_\mathsf{f}, j_\mathsf{f})}_{m_\mathsf{n}}(\mathbf{z}_\ell)} \phi^{(\gamma j_\mathsf{f}, j_\mathsf{f})}_{m_{\mathsf{n}'}}((h_\ell^{-1})^\intercal \mathbf{z}_\ell) \\
&\quad = \sum_{\{m_\mathsf{e}\}} \int_{\mathbb{CP}^1} \mathrm{d}\Omega_{\mathsf{vf}}\, \overline{\phi^{(\gamma j_\mathsf{f}, j_\mathsf{f})}_{m_{\mathsf{n}'}}(\mathbf{Z}_{\mathsf{vn}'\mathsf{f}})} \phi^{(\gamma j_\mathsf{f}, j_\mathsf{f})}_{m_{\mathsf{e}'}}(\mathbf{Z}_{\mathsf{ve}'\mathsf{f}})\,(\star) \int_{\mathbb{CP}^1} \mathrm{d}\Omega_{\mathsf{v}^{(n)}\mathsf{f}}\, \overline{\phi^{(\gamma j_\mathsf{f}, j_\mathsf{f})}_{m_{\mathsf{e}^{(n)}}}(\mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{e}^{(n)}\mathsf{f}})} \phi^{(\gamma j_\mathsf{f}, j_\mathsf{f})}_{m_\mathsf{n}}(\mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{nf}}) \times \\
&\qquad \times \int_{\mathbb{CP}^1} \mathrm{d}\Omega_\ell\, \overline{\phi^{(\gamma j_\mathsf{f}, j_\mathsf{f})}_{m_\mathsf{n}}(h_\ell^\intercal \mathbf{z}_\ell)} \phi^{(\gamma j_\mathsf{f}, j_\mathsf{f})}_{m_{\mathsf{n}'}}(\mathbf{z}_\ell).
\end{aligned}
\tag{5.60}
$$

To get the last line we exploited again the $SL(2,\mathbb{C})$ invariance of the measures and performed the same change of integration variables as before. That is, we introduce the following spinorial variables associated to the two edges which terminate in the nodes n, n':

$$
\mathbf{Z}_{\mathsf{vn}'\mathsf{f}} := g_{\mathsf{vn}'}^\dagger \mathbf{z}_{\mathsf{vf}} \quad , \quad \mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{nf}} := g_{\mathsf{v}^{(n)}\mathsf{n}}^\dagger \mathbf{z}_{\mathsf{v}^{(n)}\mathsf{f}},
\tag{5.61}
$$

Next, we collect only the relevant terms in (5.60) and compute

$$
\begin{aligned}
&\sum_{m_\mathsf{n}, m_{\mathsf{n}'}} \overline{\phi^{(\gamma j_\mathsf{f}, j_\mathsf{f})}_{m_{\mathsf{n}'}}(\mathbf{Z}_{\mathsf{vn}'\mathsf{f}})}\, \phi^{(\gamma j_\mathsf{f}, j_\mathsf{f})}_{m_\mathsf{n}}(\mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{nf}})\, \overline{\phi^{(\gamma j_\mathsf{f}, j_\mathsf{f})}_{m_\mathsf{n}}(h_\ell^\intercal \mathbf{z}_\ell)}\, \phi^{(\gamma j_\mathsf{f}, j_\mathsf{f})}_{m_{\mathsf{n}'}}(\mathbf{z}_\ell) \\
&\quad = \langle \mathbf{Z}_{\mathsf{vn}'\mathsf{f}} | \mathbf{Z}_{\mathsf{vn}'\mathsf{f}} \rangle^{-i\gamma j_\mathsf{f} - j_\mathsf{f} - 1} \langle \mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{nf}} | \mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{nf}} \rangle^{i\gamma j_\mathsf{f} - j_\mathsf{f} - 1} \langle \mathbf{z}_\ell | \mathbf{z}_\ell \rangle^{-2(j_\mathsf{f}+1)} \sum_{m_\mathsf{n}} P^{j_\mathsf{f}}_{m_\mathsf{n}}(h_\ell^\dagger \overline{\mathbf{z}}_\ell) P^{j_\mathsf{f}}_{m_\mathsf{n}}(\mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{nf}}) \sum_{m_{\mathsf{n}'}} P^{j_\mathsf{f}}_{m'_\mathsf{n}}(\overline{\mathbf{Z}}_{\mathsf{vn}'\mathsf{f}}) P^{j_\mathsf{f}}_{m'_\mathsf{n}}(\mathbf{z}_\ell) \\
&\quad = \langle \mathbf{Z}_{\mathsf{vn}'\mathsf{f}} | \mathbf{Z}_{\mathsf{vn}'\mathsf{f}} \rangle^{-i\gamma j_\mathsf{f} - j_\mathsf{f} - 1} \langle \mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{nf}} | \mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{nf}} \rangle^{i\gamma j_\mathsf{f} - j_\mathsf{f} - 1} \langle \mathbf{z}_\ell | \mathbf{z}_\ell \rangle^{-2(j_\mathsf{f}+1)} \langle h_\ell^\intercal \mathbf{z}_\ell | \mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{nf}} \rangle^{2j_\mathsf{f}} \langle \mathbf{Z}_{\mathsf{vn}'\mathsf{f}} | \mathbf{z}_\ell \rangle^{2j_\mathsf{f}}.
\end{aligned}
\tag{5.62}
$$

Plugging the above result back into (5.60) yields

$$
\begin{aligned}
&\mathrm{Tr}_{j_\mathsf{f}}\left[Y^\dagger g_{\mathsf{vn}'}^{-1} g_{\mathsf{ve}'} Y \left(\prod_{\mathsf{v}\in\mathsf{f}} Y^\dagger g_{\mathsf{ve}'}^{-1}\, g_{\mathsf{ve}} Y\right) Y^\dagger g_{\mathsf{v}^{(n)}\mathsf{e}^{(n)}}^{-1} g_{\mathsf{v}^{(n)}\mathsf{n}} Y h_\ell^{-1}\right] \\
&\quad = \left(\prod_{\mathsf{e}\in\mathsf{f}} \frac{d_{j_\mathsf{f}}}{\pi} \int_{\mathbb{CP}^1} \mathrm{d}\tilde{\Omega}_{\mathsf{vef}}\right) \left(\frac{d_{j_\mathsf{f}}^3}{\pi^3} \int_{(\mathbb{CP}^1)^3} \mathrm{d}\tilde{\Omega}_{\mathsf{n}\ell\mathsf{n}'}\right) \mathrm{e}^{j_\mathsf{f} S_\mathsf{f}[g_{\mathsf{ve}}, \mathbf{z}_{\mathsf{vf}}] + j_\mathsf{f} B_\ell[g_{\mathsf{vn}}, h_\ell, \mathbf{z}_\ell]},
\end{aligned}
\tag{5.63}
$$

where we have introduced the rescaled $(\mathbb{CP}^1)^3$ measures

$$
\mathrm{d}\tilde{\Omega}_{\mathsf{n}\ell\mathsf{n}'} := \frac{\mathrm{d}\Omega_{\mathsf{v}^{(n)}\mathsf{f}}}{\langle \mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{nf}} | \mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{nf}} \rangle} \frac{\mathrm{d}\Omega_\ell}{\langle \mathbf{z}_\ell | \mathbf{z}_\ell \rangle^2} \frac{\mathrm{d}\Omega_{\mathsf{vf}}}{\langle \mathbf{Z}_{\mathsf{vn}'\mathsf{f}} | \mathbf{Z}_{\mathsf{vn}'\mathsf{f}} \rangle}
\tag{5.64}
$$

associated to the vertices attached to the nodes n, n' and to the link ℓ. Moreover, the action $S_\mathsf{f}[g_{\mathsf{ve}}, \mathbf{z}_{\mathsf{vf}}]$ is defined as in (5.58), except that the sum over edges excludes the two edges attached to the nodes.

These edges are accounted for in the newly defined boundary face action

$$B_\ell[g_{\mathsf{vn}}, h_\ell, \mathbf{z}_\ell] := \log \frac{\langle \mathbf{Z}_{\mathsf{vn'f}}|\mathbf{z}_\ell\rangle^2}{\langle \mathbf{Z}_{\mathsf{vn'f}}|\mathbf{Z}_{\mathsf{vn'f}}\rangle \langle \mathbf{z}_\ell|\mathbf{z}_\ell\rangle} + \log \frac{\langle h_\ell^\intercal \mathbf{z}_\ell|\mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{nf}}\rangle^2}{\langle \mathbf{z}_\ell|\mathbf{z}_\ell\rangle \langle \mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{nf}}|\mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{nf}}\rangle} + i\gamma \log \frac{\langle \mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{nf}}|\mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{nf}}\rangle}{\langle \mathbf{Z}_{\mathsf{vn'f}}|\mathbf{Z}_{\mathsf{vn'f}}\rangle}. \tag{5.65}$$

The full EPRL amplitude on a two-complex with boundary in its path integral form is finally given by

$$W_{\mathcal{C}}(h_\ell) = \mathcal{N} \sum_{\{j_\mathsf{f}\}} \int_{SL(2,\mathbb{C})} \left(\prod_{\mathsf{v}} \mathrm{d}\widehat{g}_{\mathsf{ve}}\right) \left(\prod_{\mathsf{f}\in\mathcal{C}} d_{j_\mathsf{f}} \prod_{\mathsf{e}\in\mathsf{f}} \frac{d_{j_\mathsf{f}}}{\pi} \int_{\mathbb{C}\mathrm{P}^1} \mathrm{d}\tilde{\Omega}_{\mathsf{vef}}\right) \left(\prod_{\ell\in\Gamma} \frac{d_{j_\mathsf{f}}^3}{\pi^3} \int_{(\mathbb{C}\mathrm{P}^1)^3} \mathrm{d}\tilde{\Omega}_{\mathsf{n}\ell\mathsf{n}'}\right) \mathrm{e}^{\sum_{\mathsf{f}\in\mathcal{C}} j_\mathsf{f} S_\mathsf{f} + \sum_{\ell\in\Gamma} j_\mathsf{f} B_\ell}. \tag{5.66}$$

The action term B_ℓ seems oddly asymmetric due to the presence of the $h_\ell^\intercal$ element. This can in principle be remedied by arbitrarily splitting h_ℓ in a product of two $SU(2)$ elements. This amounts to splitting the link into half links and makes (5.65) appear more symmetric. As we see below, this splitting happens naturally when the amplitude (5.66) is contracted with coherent boundary states.

5.2.3 The Holomorphic Amplitude

The spinfoam amplitude (5.66) depends on L arbitrary $SU(2)$ elements and is therefore really a map from $L_2[SU(2)^L/SU(2)^N]$ to the reals, defined on a two-complex. The number associated to a two-complex is called the transition amplitude, obtained from contracting the amplitude (5.66) with a boundary state. Below, we use the coherent states (5.1) discussed in subsection 5.1.1 and consider the contraction

$$W_{\mathcal{C}}^{t_\ell}(H_\ell) := \left\langle W_{\mathcal{C}} \middle| \Psi_{\Gamma,H_\ell}^{t_\ell}\right\rangle := \int_{SU(2)^L} \left(\prod_{\ell\in\Gamma} \mathrm{d}h_\ell\right) W_{\mathcal{C}}(h_\ell)\, \Psi_{\Gamma,H_\ell}^{t_\ell}(h_\ell), \tag{5.67}$$

which is known in the literature as the holomorphic amplitude. We have also dropped the gauge averaging $SU(2)$ integrals from $\Psi_{\Gamma,H_\ell}^{t_\ell}$ as the $SL(2,\mathbb{C})$ integrals of $W_{\mathcal{C}}$ take care of gauge-invariance.

To compute this transition amplitude we need consider only the boundary face amplitude $A_\mathsf{f}(h_\ell)$ times the state $\Psi_{\Gamma,H_\ell}^{t_\ell}$ and employ the Peter-Weyl theorem. The relevant part of the computation yields

$$\begin{aligned}\int_{SU(2)^L} &\left(\prod_{\ell\in\Gamma} \mathrm{d}h_\ell\right) A_\mathsf{f}(h_\ell) \Psi_{\Gamma,H_\ell}^{t_\ell}(h_\ell)\\ &= \sum_{\{j_\ell\}} \prod_{\ell\in\Gamma} d_{j_\ell}\, \mathrm{e}^{-j_\ell(j_\ell+1)t_\ell}\, \mathrm{Tr}_{j_\ell} \left[Y^\dagger g_{\mathsf{vn'}}^{-1} g_{\mathsf{ve'}} Y \left(\prod_{\mathsf{v}\in\mathsf{f}} Y^\dagger g_{\mathsf{ve'}}^{-1}\, g_{\mathsf{ve}} Y\right) Y^\dagger g_{\mathsf{v}^{(n)}\mathsf{e}^{(n)}}^{-1} g_{\mathsf{v}^{(n)}\mathsf{n}} Y H_\ell^{-1}\right].\end{aligned} \tag{5.68}$$

The integration exchanged the arbitrary $h_\ell^{-1} \in SU(2)$ with the group elements $H_\ell^{-1} \in SL(2,\mathbb{C})$ given by (5.4) and completely determined by the boundary data[6]. Moreover, the integration gives rise to a $\delta^{j_\mathsf{f} j_\ell}$, which forces the spins j_f colouring the boundary face to be the same as the spins j_ℓ appearing in the boundary states and which live on the links.

Rewriting this trace in terms of functions on $\mathcal{H}^{(\gamma j_\ell, j_\ell)}$ involves the same steps as in the previous subsection. However, before continuing we will make use of an approximation that is pertinent for the physical applications we have in mind. We are interested in boundary states peaked on geometries with large areas, that is, states with $\eta_\ell \gg 1$, for which the highest weight approximation is appropriate (31)

$$D_{ab}^{j_\ell}(H_\ell^{-1}) = D^{j_\ell}(n_{s(\ell)}\, \mathrm{e}^{(\eta_\ell + i\gamma\zeta_\ell)\frac{\sigma_3}{2}}\, n_{t(\ell)}^{-1}) = D_{aj_\ell}^{j_\ell}(n_{s(\ell)}) D_{j_\ell b}^{j_\ell}(n_{t(\ell)}^{-1})\, \mathrm{e}^{(\eta_\ell + i\gamma\zeta_\ell) j_\ell} \left(1 + \mathcal{O}(\mathrm{e}^{-\eta_\ell})\right). \tag{5.69}$$

To rewrite the two Wigner matrices of the $SU(2)$ matrices, which essentially split the link into two parts, we make use of (5.48) to obtain

$$\phi_j^{(\gamma j, j)}(n^\intercal \mathbf{z}) = \sqrt{\frac{d_j}{\pi}}\, \langle \mathbf{z}|\mathbf{z}\rangle^{i\gamma j - j - 1} \langle \overline{\mathbf{n}}|\mathbf{z}\rangle^{2j}, \tag{5.70}$$

[6]This is achieved by analytical continuation of the Wigner D-matrix $D_{ab}^j(H_\ell^{-1})$ to $SL(2,\mathbb{C})$, see (24; 23).

where $\mathbf{n}$ is the spinor corresponding to the $SU(2)$ element n. This yields

$$D^{j_\ell}_{aj_\ell}(n_{s(\ell)}) = \frac{d_{j_\ell}}{\pi}\int_{\mathbb{CP}^1} \mathrm{d}\Omega_{\mathsf{n}\ell}\ \langle \mathbf{z}_{\mathsf{n}\ell}|\mathbf{z}_{\mathsf{n}\ell}\rangle^{-2(j_\ell+1)}\ \langle \overline{\mathbf{n}}_{s(\ell)}|\mathbf{z}_{\mathsf{n}\ell}\rangle^{2j_\ell}\ \overline{P^{j_\ell}_a(\mathbf{z}_{\mathsf{n}\ell})}$$

$$D^{j_\ell}_{j_\ell b}(n^{-1}_{t(\ell)}) = \frac{d_{j_\ell}}{\pi}\int_{\mathbb{CP}^1} \mathrm{d}\Omega_{\mathsf{n}'\ell}\ \langle \mathbf{z}_{\mathsf{n}'\ell}|\mathbf{z}_{\mathsf{n}'\ell}\rangle^{-2(j_\ell+1)}\ \langle \mathbf{z}_{\mathsf{n}'\ell}|\overline{\mathbf{n}}_{t(\ell)}\rangle^{2j_\ell}\ P^{j_\ell}_b(\mathbf{z}_{\mathsf{n}'\ell}). \tag{5.71}$$

Repeating the same steps as in the previous section one arrives without much effort at

$$\begin{aligned}
&\mathrm{e}^{-j_\ell(j_\ell+1)t_\ell}\,\mathrm{Tr}_{j_\ell}\left[Y^\dagger g^{-1}_{\mathsf{vn}'} g_{\mathsf{ve}'} Y\left(\prod_{\mathsf{v}\in\mathsf{f}} Y^\dagger g^{-1}_{\mathsf{ve}'}\, g_{\mathsf{ve}} Y\right) Y^\dagger g^{-1}_{\mathsf{v}^{(n)}\mathsf{e}^{(n)}} g_{\mathsf{v}^{(n)}\mathsf{n}} Y H^{-1}_\ell\right]\\
&\qquad = \mathrm{e}^{\frac{(\eta_\ell - t_\ell)^2}{4t_\ell}}\left(\prod_{\mathsf{e}\in\mathsf{f}}\frac{d_{j_\ell}}{\pi}\int_{\mathbb{CP}^1}\mathrm{d}\tilde{\Omega}_{\mathsf{vef}}\right)\left(\frac{d^4_{j_\ell}}{\pi^4}\int_{(\mathbb{CP}^1)^4}\mathrm{d}\tilde{\Omega}_{s\ell t}\right)\,\mathrm{e}^{j_\ell F_{\mathsf{f}}[g_{\mathsf{ve}},\mathbf{z}_{\mathsf{vf}}]+B_\ell[j_\ell;H_\ell]}\left(1+\mathcal{O}(\mathrm{e}^{-\eta_\ell})\right),
\end{aligned} \tag{5.72}$$

where we defined

$$\mathrm{d}\tilde{\Omega}_{s\ell t} := \frac{\mathrm{d}\Omega_{\mathsf{v}^{(n)}\mathsf{f}}}{\langle \mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{nf}}|\mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{nf}}\rangle}\ \frac{\mathrm{d}\Omega_{\mathsf{n}\ell}}{\langle \mathbf{z}_{\mathsf{n}\ell}|\mathbf{z}_{\mathsf{n}\ell}\rangle^2}\ \frac{\mathrm{d}\Omega_{\mathsf{n}'\ell}}{\langle \mathbf{z}_{\mathsf{n}'\ell}|\mathbf{z}_{\mathsf{n}'\ell}\rangle^2}\ \frac{\mathrm{d}\Omega_{\mathsf{vf}}}{\langle \mathbf{Z}_{\mathsf{vn}'\mathsf{f}}|\mathbf{Z}_{\mathsf{vn}'\mathsf{f}}\rangle} \tag{5.73}$$

and

$$\begin{aligned}
F_\ell[g_{\mathsf{ve}},\mathbf{z}_{\mathsf{n}\ell};\mathbf{n}_{\mathsf{n}(\ell)}] &:= S_\ell[g_{\mathsf{ve}},\mathbf{z}_{\mathsf{n}\ell}] + \log\frac{\langle \overline{\mathbf{n}}_{s(\ell)}|\mathbf{z}_{\mathsf{n}\ell}\rangle^2\langle \mathbf{z}_{\mathsf{n}'\ell}|\overline{\mathbf{n}}_{t(\ell)}\rangle^2}{\langle \mathbf{z}_{\mathsf{n}\ell}|\mathbf{z}_{\mathsf{n}\ell}\rangle^2\langle \mathbf{z}_{\mathsf{n}'\ell}|\mathbf{z}_{\mathsf{n}'\ell}\rangle^2} + \log\frac{\langle \mathbf{Z}_{\mathsf{vn}'\ell}|\mathbf{z}_{\mathsf{n}'\ell}\rangle^2\langle \mathbf{z}_{\mathsf{n}\ell}|\mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{n}\ell}\rangle^2}{\langle \mathbf{Z}_{\mathsf{vn}'\ell}|\mathbf{Z}_{\mathsf{vn}'\ell}\rangle\langle \mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{n}\ell}|\mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{n}\ell}\rangle} + i\gamma\log\frac{\langle \mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{n}\ell}|\mathbf{Z}_{\mathsf{v}^{(n)}\mathsf{n}\ell}\rangle}{\langle \mathbf{Z}_{\mathsf{vn}'\ell}|\mathbf{Z}_{\mathsf{vn}'\ell}\rangle}\\
G_\ell[j_\ell;H_\ell] &:= i\gamma j_\ell\zeta_\ell - \left(j_\ell - \omega_\ell(\eta_\ell,t_\ell)\right)^2 t_\ell.
\end{aligned} \tag{5.74}$$

The definition of $\mathrm{d}\tilde{\Omega}_{\mathsf{vef}}$ and $S_\ell[g_{\mathsf{ve}},\mathbf{z}_{\mathsf{vf}}]$ remain the same as in the previous sections and the exponential pre-factor $\exp\big((\eta_\ell - t_\ell)^2/4t_\ell\big)$, which only depends on the data η_ℓ and the parameter t_ℓ, arises from completing the square such that the Gaussian weight $\exp\big(-(j_\ell-\omega_\ell)^2 t_\ell\big)$ appears in (5.74). Absorbing the pre-factor into the normalization $\mathcal{N}$ of the EPRL amplitude we finally arrive at the holomorphic amplitude in its path integral form:

$$W^{t_\ell}_{\mathcal{C}}(H_\ell) = \mathcal{N}\sum_{\{j_{\mathsf{f}},j_\ell\}}\int_{SL(2,\mathbb{C})}\left(\prod_{\mathsf{v}}\mathrm{d}\widehat{g}_{\mathsf{ve}}\right)\left(\prod_{\mathsf{f}\in\mathcal{C}} d_{j_{\mathsf{f}}}\prod_{\mathsf{e}\in\mathsf{f}}\frac{d_{j_{\mathsf{f}}}}{\pi}\int_{\mathbb{CP}^1}\mathrm{d}\tilde{\Omega}_{\mathsf{vef}}\right)\left(\prod_{\ell\in\Gamma}\frac{d^4_{j_\ell}}{\pi^4}\int_{(\mathbb{CP}^1)^4}\mathrm{d}\tilde{\Omega}_{s\ell t}\right)\mathrm{e}^{\sum_{\mathsf{f}\in\mathcal{B}} j_{\mathsf{f}} S_{\mathsf{f}}+\sum_{\ell\in\Gamma}(j_\ell F_\ell+G_\ell)}. \tag{5.75}$$

This amplitude is the object of main interest in this thesis. In the next section, we will give an approximate expression for this amplitude when defined two-complexes without interior faces.

5.3 Approximation of the EPRL Amplitude on Tree-Level Two-Complexes

In the previous section we derived the Lorentzian EPRL amplitude in the Krajewski-Han path integral representation (4) in a formalism suitable for our purposes, and extended it to two-complexes with boundary. The arising boundary terms and the extended amplitude are summarized in (5.64), (5.65), (5.66).
The large spin asymptotic of the bulk partial amplitude in (32) have been studied in detail in (4) using of the coherent state representation of the EPRL amplitude. The results corroborate the ones derived in (29). In (29), the authors also gave a detailed analysis of the boundary partial amplitude, again using the coherent state representation. With the analysis of the previous section, we can now combine these results to proceed to the coherent state representation of the transition amplitude. This simply amounts to inserting resolutions of the identity in terms of $SU(2)$ coherent states at each bulk face and does not affect the asymptotics of the boundary partial amplitude as performed in (29). Therefore, all results obtained in the coherent state representation will carry over to the Krajewski-Han path integral representation.

The above will be used in this section to develop an approximation of the holomorphic EPRL amplitude (5.75) defined on a special class of two-complexes. On a general two-complex, it is a difficult task to perform the spin-sum analytically while keeping the approximation scheme under control. Attempts in this direction can be found in (33; 34; 35). Another option that has been explored is to use symmetry reduced models (36; 37; 38). Here, we only consider what we call tree-level two-complexes $\mathcal{T}$: these are dual to a four-dimensional simplicial triangulation of spacetime as before, but with the additional restriction that they only have boundary faces $\mathsf{f} \in \Gamma := \partial\mathcal{T}$. That is, there are no faces which lie completely in the bulk. Considering only these two-complexes allows us to carry the out the calculation to the end. This comes from the observation that the subset of extrinsic boundary states (5.32) which are tuned to satisfy the semiclassicality condition (5.39) are sharply peaked on spin values ω_ℓ, which are taken to correspond to macroscopic classical areas of a discrete boundary geometry. This peakedness manifests itself in the Gaussian weight factors $\exp\big(-(j_\ell - \omega_\ell)^2 t_\ell\big)$ present in (5.72). These weight factors provide a strong regulator for the boundary face amplitude and they allow to truncate the spin-sums over boundary spins while keeping the approximation under control.

The class of is quite restrictive, but, it is relevant for existing studies of possible physical applications for spinfoams. A concrete example of such a two-complex can be found in (14; 5; 39), where it has been used to model the transition of a black hole into a white hole. It is of course desirable to consider also bulk faces. This which is beyond the scope of the present work and is left for future analysis.

5.3.1 Truncated Spin-Sums, Triangle Inequalities and Semiclassicality

The holomorphic amplitude in the highest weight approximation defined on a tree-level two-complex is formally given by

$$W^{t_\ell}_{\mathcal{T}}(H_\ell) = \mathcal{N} \sum_{\{j_\ell\}} \int_{SL(2,\mathbb{C})} \left(\prod_{\mathsf{v}} \mathrm{d}\widehat{g}_{\mathsf{ve}}\right) \left(\prod_{\mathsf{f}\in\Gamma} d_{j_\ell} \prod_{\mathsf{e}\in\mathsf{f}} \frac{d_{j_\ell}}{\pi} \int_{\mathbb{CP}^1} \mathrm{d}\tilde{\Omega}_{\mathsf{vef}}\right) \left(\prod_{\ell\in\Gamma} \frac{d^4_{j_\ell}}{\pi^4} \int_{(\mathbb{CP}^1)^4} \mathrm{d}\tilde{\Omega}_{s\ell t}\right) \mathrm{e}^{\sum_{\ell\in\Gamma} j_\ell F_\ell + \sum_{\ell\in\Gamma} G_\ell}\,. \tag{5.76}$$

In what follows it is not necessary to keep track of all the details given in the precise definitions of the previous sections. In order to make the discussion in this section more concise we drop most of the indices referring to the structure of the two-complex and rewrite the amplitude as

$$W^{t}_{\mathcal{T}}(H_\ell) = \mathcal{N} \sum_{\{j_\ell\}\in D^k_\omega} \mu_j\, \mathrm{e}^{-t\sum_\ell (j_\ell - \omega_\ell)^2}\, \mathrm{e}^{i\gamma \sum_\ell \zeta_\ell j_\ell} \int_{D_{g,\mathbf{z}}} \mathrm{d}\mu_{g,\Omega}\, \mathrm{e}^{\sum_\ell j_\ell F_\ell(g,\mathbf{z};\mathbf{n}_{\ell(\mathsf{n})})}\,. \tag{5.77}$$

The notation

$$\int_{D_{g,\mathbf{z}}} \mathrm{d}\mu_{g,\Omega} := \int_{SL(2,\mathbb{C})} \left(\prod_{\mathsf{v}} \mathrm{d}\widehat{g}_{\mathsf{ve}}\right) \left(\prod_{\mathsf{f}\in\Gamma}\prod_{\mathsf{e}\in\mathsf{f}} \int_{\mathbb{CP}^1} \mathrm{d}\tilde{\Omega}_{\mathsf{vef}}\right) \left(\prod_{\ell\in\Gamma} \int_{(\mathbb{CP}^1)^4} \mathrm{d}\tilde{\Omega}_{s\ell t}\right) \tag{5.78}$$

has been introduced to summarize all $SL(2,\mathbb{C})$ and $\mathbb{CP}^1$ integrals while the notation

$$\mu_j := \left(\prod_{\mathsf{f}\in\Gamma}\prod_{\mathsf{e}\in\mathsf{f}} d_{j_\ell}\right) \left(\prod_{\ell\in\Gamma} d^4_{j_\ell}\right), \tag{5.79}$$

represents the summation "measure", and where irrelevant factors of π have been absorbed into the normalization $\mathcal{N}$. Moreover, the summation over boundary spins is only performed over the domain

$$D^k_\omega := \bigtimes_\ell \left\{ \left\lfloor \omega_\ell - \frac{k}{\sqrt{2t}} \right\rfloor, \left\lfloor \omega_\ell + \frac{k}{\sqrt{2t}} \right\rfloor \right\} \quad \text{with} \quad 0 < k \in \mathbb{N}. \tag{5.80}$$

The symbol $\lfloor x \rfloor$ denotes the floor function which, in this article, is defined to be the largest *half integer* number equal to or less than x. The restriction to the summation domain D^k_ω implements the truncation of the spin-sum discussed in the introduction of this section.[7] The Gaussian weight factors (5.74)

[7] That this is a good approximation of the actual sum follows from the fact that the partial amplitude is an oscillating and finite function of the spins. The Gaussian weights therefore strongly dominate. Further justification is provided by the procedure performed in (33), where the author introduced a regulator $\sim \mathrm{e}^{-j}$ to study phase transitions in large spin foams. Here, the coherent states naturally provide us with the stronger regulator $\sim \mathrm{e}^{-j^2}$.

regulate the spin-sums while the parameter k acts as cut-off. It measures how many standard deviations $\sigma = 1/\sqrt{2t}$ the summation moves away from the peak ω_ℓ and is of order unit.

The main subtlety that needs to be addressed in what follows is that the sums in (5.77) cannot immediately be treated as independent. The summand vanishes when the triangle inequalities among the spins are not satisfied. More precisely, the summand in (5.77) vanishes whenever any one of the intertwiner spaces associated to the nodes of the two-complex is of dimension zero. Therefore, in order to treat the sums as independent and exchange them with the integrals, the spin-sums need to be restricted to spin-configurations for which the intertwiner space is always non-trivial. Let us now see that this is not an issue for the set up of this work.

To implement this requirement and since by assumption the nodes of the two-complex are four-valent we define the set

$$\begin{aligned} D_\Gamma &:= \left\{ \{j_\ell\} \middle| \dim \mathrm{Inv}_{SU(2)} \left[\bigotimes_{\ell=1}^{4} \mathcal{H}_{j_\ell} \right] > 0 \quad \forall \mathsf{n} \in \Gamma \right\} \\ &= \left\{ \{j_\ell\} \middle| \min\left(j_1+j_2, j_3+j_4\right) - \max\left(|j_1-j_2|, |j_3-j_4|\right) + 1 > 0 \quad \forall \mathsf{n} \in \Gamma \right\}. \end{aligned} \tag{5.81}$$

This is the set of all spin configurations $\{j_\ell\}$ for which the intertwiner spaces over the whole boundary graph Γ are non-trivial. To adequately truncate the spin-sums we must now choose the cut-off parameter k such that

$$\{j_\ell\} \in D^k_\omega \subseteq D_\Gamma. \tag{5.82}$$

To rewrite this condition, it is convenient to split the boundary spins j_ℓ into fixed background contributions λa_ℓ and *fluctuations* s_ℓ, i.e.

$$j_\ell = \lambda a_\ell + s_\ell \quad \text{with} \quad \omega_\ell \equiv \lambda a_\ell \quad \text{and} \quad s_\ell \in \left\{ -\left\lfloor \frac{k}{\sqrt{2t}} \right\rfloor, \left\lfloor \frac{k}{\sqrt{2t}} \right\rfloor \right\} \quad \forall \ell \in \Gamma. \tag{5.83}$$

In this decomposition the a_ℓ's are assumed to be of order unit in λ and $\lambda \gg 1$. Combining (5.83) with (5.82) leads to

$$\begin{aligned} \lambda\, a_{\mathrm{sum}} - \frac{2k}{\sqrt{2t}} + 1 &> \lambda\, a_{\mathrm{diff}} + \frac{2k}{\sqrt{2t}} \\ a_{\mathrm{sum}} := \min\left(a_1+a_2, a_3+a_4\right) &\qquad a_{\mathrm{diff}} := \max\left(|a_1-a_2|, |a_3-a_4|\right) \end{aligned} \tag{5.84}$$

which can be rearranged to

$$\lambda\, \left(a_{\mathrm{sum}} - a_{\mathrm{diff}}\right) \sqrt{t} > \frac{4k}{\sqrt{2}} - \sqrt{t} \approx \frac{4k}{\sqrt{2}} \tag{5.85}$$

since t was assumed to be much smaller than unit (5.30). By the assumptions of this section, the difference $\lambda\, a_{\mathrm{diff}}$ is negligible compared to the sum $\lambda\, a_{\mathrm{sum}}$ and hence (5.85) is satisfied when the semiclassicality condition (5.39) holds. The semiclassicality condition can also be read as a geometricity condition on the coherent states. It imposes that the intrinsic states (5.34) have spins which are well within the triangle inequalities. This in turn means that the coherent states are composed of a superposition of intrinsic coherent states (5.34) each peaked on a triangulation of a spacelike hypersurface.

Next we turn to the dimension factors d_{j_ℓ}. From (5.85) and applying the decomposition (5.83) we get from (5.79)

$$\begin{aligned} \mu_j &= \left(\prod_{\mathsf{f}\in\Gamma} \prod_{\mathsf{e}\in\mathsf{f}} (2j_\ell+1) \right) \left(\prod_{\ell\in\Gamma} (2j_\ell+1)^4 \right) \approx \left(\prod_{\mathsf{f}\in\Gamma} \prod_{\mathsf{e}\in\mathsf{f}} 2j_\ell \right) \left(\prod_{\ell\in\Gamma} (2j_\ell)^4 \right) \\ &= 2^{N_c} \left(\prod_{\mathsf{f}\in\Gamma} \prod_{\mathsf{e}\in\mathsf{f}} (\lambda a_\ell + s_\ell) \right) \left(\prod_{\ell\in\Gamma} (\lambda a_\ell + s_\ell)^4 \right) \\ &= (2\lambda a_\ell)^{N_c} \left(1 + \mathcal{O}\left(\frac{s_\ell}{\lambda a_\ell} \right) \right) \end{aligned} \tag{5.86}$$

Dropping $\mathcal{O}(s_\ell/\lambda\, a_\ell)$ is justified when $|s_\ell| \ll \lambda\, a_\ell$ which is equivalent to $\frac{k}{\sqrt{2}} \ll \lambda\, a_\ell \sqrt{t}$. But this again follows from by the semiclassicality condition (5.39), since k is of order unit and hence we can safely drop the $\mathcal{O}(s_\ell/\lambda\, a_\ell)$ term.

5.3.2 Performing the Spin-Sum

Due to the semiclassicality condition, the spin-sums over the finite summation domain D^k_ω (with k chosen appropriately) can be treated as independent. After applying the decomposition (5.83) to the holomorphic amplitude (5.77), it can be rewritten as

$$W^t_\Gamma(H_\ell) = \mathcal{N}\int_{D_{g,\mathbf{z}}} \mu_j \mathrm{d}\mu_{g,\Omega}\mathcal{U}(g,\mathbf{z};t,H_\ell)\,\mathrm{e}^{\lambda\Sigma(a_\ell,g,\mathbf{z};\mathbf{n}_{\ell(\mathsf{n})})} \tag{5.87}$$

where

$$\mathcal{U}(g,\mathbf{z};t,H_\ell) := \prod_\ell\left(\sum_{s_\ell\in D^k_\omega} \mathrm{e}^{-s_\ell^2 t+(i\gamma\zeta_\ell+F_\ell(g,\mathbf{z};\mathbf{n}_{\ell(\mathsf{n})}))s_\ell}\right), \qquad \Sigma(a_\ell,g,\mathbf{z};\mathbf{n}_{\ell(\mathsf{n})}) := \sum_\ell(a_\ell F_\ell(g,\mathbf{z};\mathbf{n}_{\ell(\mathsf{n})})+i\gamma\zeta_\ell a_\ell) \tag{5.88}$$

The large parameter λ only appears linearly in the exponent and the newly defined function $\mathcal{U}$ is continuous in the variables g and $\mathbf{z}$. Hence, the generalized stationary phase theorem (40) may be applied. The critical point equations

$$\mathrm{Re}\Sigma(a_\ell,g,\mathbf{z};\mathbf{n}_{\ell(\mathsf{n})}) = \delta_g\Sigma(a_\ell,g,\mathbf{z};\mathbf{n}_{\ell(\mathsf{n})}) = \delta_{\mathbf{z}}\Sigma(a_\ell,g,\mathbf{z};\mathbf{n}_{\ell(\mathsf{n})}) \tag{5.89}$$

are exactly those of the fixed-spin asymptotics of (29) and hence their results can directly be used here. The data H_ℓ provided by the semiclassical states is either Regge-like, in which case there will be a geometrical critical point corresponding to one of three possible types of simplicial geometries, or there will be no critical point. We may assume the data $(\omega_\ell, n_{\ell(\mathsf{n})})$ to be Regge-like and moreover we may choose it such that vector geometries are excluded.

By virtue of the stationary phase theorem we have the following estimation for the amplitude

$$W^t_{\mathcal{T}}(H_\ell) = N\sum_c \mu_j\lambda^{M^c_{\mathcal{C}}}\mathcal{H}_c(a_\ell,\mathbf{n}_{\ell(\mathsf{n})})\mathcal{U}(g_c,z_c;t,H_l)\,\mathrm{e}^{\lambda\Sigma(a_\ell,g,\mathbf{z};\mathbf{n}_{\ell(\mathsf{n})})}\left(1+\mathcal{O}(\lambda^{-1})\right), \tag{5.90}$$

Here, the summation over c denotes a summation over the 2^N critical points. Each critical point comes with a 2^N degeneracy, corresponding to the different configurations for the orientation $s(\mathsf{v})$ where $s(\mathsf{v})$ takes the values ± 1 on each vertex of $\mathcal{C}$ see (? ? ? ?) . Note that $\mathcal{H}_c$ contains the determinant of the Hessian of Σ. The important point to keep for physical applications is that in the first order approximation, the scale λ appears only as an overall scaling factor $\lambda^{M^c_{\mathcal{C}}}$ and as a linear term in the exponential. In particular, $\mathcal{H}_c$ does not depend on λ.

We proceed to evaluate $\mathcal{U}$ at the critical point by using

$$F_\ell(g,\mathbf{z};\mathbf{n}_{\ell(\mathsf{n})}) = -i\gamma\,\phi_\ell(s_{c(\mathsf{v})},a_\ell,\mathbf{n}_{\ell(\mathsf{n})}), \tag{5.91}$$

where $\phi_\ell(s_{c(\mathsf{v})},a_\ell,\mathbf{n}_{\ell(\mathsf{n})})$ is the Palatini deficit angle. Thus, $\mathcal{U}$ evaluated at c reads

$$\mathcal{U}(g_c,\mathbf{z}_c;t,H_\ell) = \prod_\ell\left(\sum_{s_\ell\in D^k_\omega}\mathrm{e}^{-s_\ell^2 t+i\gamma(\zeta_\ell-\phi_\ell(g,\mathbf{z};\mathbf{n}_{\ell(\mathsf{n})}))s_\ell}\right) \tag{5.92}$$

Since the phase $i\gamma(\zeta_\ell-\phi_\ell)$ is purely imaginary and independent of s_ℓ, the sum is dominated by the exponential damping factor $\exp(-s_\ell^2 t)$. It can reasonably be expected that due to this exponential damping the sum converges very fast and that it is therefore a good approximation to remove the cut-off k and sum s_ℓ from $-\infty$ to ∞ for all $\ell\in\Gamma$. This allows us to get a closed analytic expression for the spin-sums, which approximates them well:

$$\sum_{s_\ell=-\infty}^{\infty}\mathrm{e}^{-s_\ell^2 t+i\gamma(\zeta_\ell-\phi_\ell)s_\ell} = 2\sqrt{\frac{\pi}{t}}\,\mathrm{e}^{-\frac{\gamma^2}{4t}(\zeta_\ell-\phi_\ell)^2}\,\vartheta_3\left(-\frac{i\pi\gamma(\zeta_\ell-\phi_\ell)}{t},\mathrm{e}^{-\frac{4\pi^2}{t}}\right), \tag{5.93}$$

where

$$\vartheta_3(u,q) := 1+2\sum_{n=1}^{\infty}q^{n^2}\cos(2nu) \tag{5.94}$$

is the third Jacobi theta function. Hence,

$$\mathcal{U}(g_c, \mathbf{z}_c; t, H_\ell) \approx \prod_\ell 2\sqrt{\frac{\pi}{t}}\, \mathrm{e}^{-\frac{\gamma^2}{4t}(\zeta_\ell-\phi_\ell)^2}\, \vartheta_3\left(-\frac{i\pi\gamma(\zeta_\ell-\phi_\ell)}{t}, \mathrm{e}^{-\frac{4\pi^2}{t}}\right). \tag{5.95}$$

Substituting everything to (5.90) we obtain

$$W^t_{\mathcal{T}}(H_\ell) = \mathcal{N} \sum_{\{s(\mathsf{v})\}} \lambda^N \mu(a) \prod_\ell \left(\mathrm{e}^{-\frac{\gamma^2}{4t}(\zeta_\ell-\phi_\ell)^2}\, \vartheta_3\left(-\frac{i\pi\gamma(\zeta_\ell-\phi_\ell)}{t}, \mathrm{e}^{-\frac{4\pi^2}{t}}\right)\right) e^{\sum_\ell(-\lambda i\gamma a_\ell \phi_\ell(s_{c(\mathsf{v})}, a_\ell, \mathbf{n}_{\ell(\mathsf{n})}) + i\lambda\gamma\zeta_\ell a_\ell)} \tag{5.96}$$

The power N is in general a half integer that depends on the rank of the hessian at the critical point and the combinatorics of the 2-complex C. The function $\mu(a)$ includes the summation measure over the spins and the Hessian evaluated at the critical point.

In Appendix A it is explained that when $\gamma \leq \frac{1}{2}$ θ_3 can be approximated by unit. Note that this is consistent with the fixing of the value of γ that comes from calculating Black Hole entropy using LQG (41; 42; 43). Thus, we obtain

$$W^t_{\mathcal{T}}(H_\ell) \approx \mathcal{N} \sum_{\{s(\mathsf{v})\}} \lambda^N \mu(a) \prod_\ell e^{\frac{-\gamma^2}{4t}(\zeta_\ell-\phi_\ell)^2 + i\gamma(\zeta_\ell-\phi_\ell)\omega_\ell} \left(1 + \mathcal{O}(\lambda^{-1})\right) \tag{5.97}$$

The above result can be generalized to include all geometric cases of critical points. Following the same procedure we arrive at

$$W^t_{\mathcal{T}}(H_\ell) \approx \mathcal{N} \sum_{\{s(\mathsf{v})\}} \lambda^N \mu(a) \prod_\ell e^{\frac{-\Delta_\ell{}^2}{4t} + i\Delta_\ell\omega_\ell} \left(1 + \mathcal{O}(\lambda^{-1})\right) \tag{5.98}$$

where, $\Delta_\ell := \gamma\zeta_\ell - \beta\phi_\ell(a_\ell) + \Pi_\ell$.

We take a moment to go through the various quantities appearing in this formula as we have introduced a few important subtleties regarding the different kinds of geometrical critical points that we neglected in the derivation above. The Π_ℓ contribution accounts for an extra phase in the Lorentzian intertwiners, see (44; 24). The power N is in general a half integer that depends on the rank of the hessian at the critical point and the combinatorics of the two-complex $\mathcal{C}$. The function $\mu(a)$ includes the summation measure over the spins and the Hessian evaluated at the critical point. The important point here is that neither the summation measure nor the Hessian scale with λ.

The estimation (5.98) is valid for all three types of possible geometrical critical points. If ω_ℓ and $\mathbf{n}_{\ell(\mathsf{n})}$ specify a Lorentzian geometry, then

$$\beta = \gamma \tag{5.99}$$

and

$$\Pi_\ell = \begin{cases} 0 & \text{thick wedge} \\ \pi & \text{thin wedge} \end{cases} \tag{5.100}$$

If ω_ℓ and $\mathbf{n}_{\ell(\mathsf{n})}$ specify a degenerate geometry, then the dihedral angles $\phi_\ell(\omega_\ell, \mathbf{n}_{\ell(\mathsf{n})})$ either vanish or are equal to π, according to whether we are in a thick or thick wedge. By abuse of notation, we express this simply by setting $\beta = 0$ in this case and keeping Π_ℓ defined as above.

If ω_ℓ and $\mathbf{n}_{\ell(\mathsf{n})}$ specify a Euclidean geometry, then we have

$$\beta = 1 \tag{5.101}$$

and

$$\Pi_\ell = 0 \tag{5.102}$$

The function $\phi_\ell(s_c(v); \delta_\ell, \mathbf{n}_{\ell(\mathsf{n})})$ denotes the Palatini deficit angle.

This completes the analysis. The above results is the technique underlying the calculation presented in (5) to give an estimation of the bounce time for the black to white hole transition from spinfoams, which was based on a 2-complex without bulk faces. We expect future work to extend these results to also treat 2-complexes that include bulk faces.

Appendix A

The approximation $\vartheta_3 \approx 1$

Omitting details not relevant here and focusing only on one link for notational simplicity, the amplitude we would like to compute is given by

$$W(A,\zeta) \simeq \sum_{j=0}^{\infty} \mathrm{e}^{-t(A-j)^2+i\gamma\zeta j} \int_{\Omega} \mathrm{d}\mu(g)\, \mathrm{d}\nu(z)\, \mathrm{e}^{jF(g,z)} \quad , \quad g \in \mathrm{SL}(2,\mathbb{C}), z \in \mathbb{C}^2. \tag{A.1}$$

Using the splitting $j = A + s$ into fixed background geometry and fluctuations we get

$$W(A,\zeta) \simeq \mathrm{e}^{i\gamma\zeta A} \sum_{s=-\infty}^{\infty} \mathrm{e}^{-ts^2+i\gamma\zeta s} \int_{\Omega} \mathrm{d}\mu(g)\, \mathrm{d}\nu(z)\, \mathrm{e}^{(A+s)F(g,z)}. \tag{A.2}$$

Note that $\mathrm{e}^{i\gamma\zeta A}$ is a pure phase (also in the general case when several links are present) and can therefore be neglected in what follows. Moreover, we used the approximation that $s \in (-\infty, \infty)$. The usual spin foam asymptotic analysis tells us that

$$\int_{\Omega} \mathrm{d}\mu(g)\, \mathrm{d}\nu(z)\, \mathrm{e}^{(A+s)F(g,z)} \sim \mathrm{e}^{-i\gamma\phi(g,z)A}\, \mathrm{e}^{-i\gamma\phi(g,z)s}. \tag{A.3}$$

We neglected here the Hessian and some numerical factors. Also, the phase $\mathrm{e}^{-i\gamma\phi(g,z)A}$ can be neglected in what follows. We are hence left with

$$W(A,\zeta) \sim \sum_{s=-\infty}^{\infty} \mathrm{e}^{-ts^2+i\gamma(\zeta-\phi)s}. \tag{A.4}$$

Now, the sum (A.4) can be written down in closed form in terms of known functions as

$$\sum_{s=-\infty}^{\infty} \mathrm{e}^{-ts^2+i\gamma(\zeta-\phi)s} = \sqrt{\frac{\pi}{t}}\mathrm{e}^{-\frac{\gamma^2}{4t}(\zeta-\phi)^2}\vartheta_3\left(-\frac{i\pi\gamma(\zeta-\phi)}{2t}, \mathrm{e}^{-\frac{\pi^2}{t}}\right), \tag{A.5}$$

where

$$\vartheta_3\,(u,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nu) \tag{A.6}$$

is one of Jacobi's Theta functions. Hence, we have for the amplitude

$$W(A,\zeta) \sim \sqrt{\frac{\pi}{t}}\mathrm{e}^{-\frac{\gamma^2}{4t}(\zeta-\phi)^2}\vartheta_3\left(-\frac{i\pi\gamma(\zeta-\phi)}{2t}, \mathrm{e}^{-\frac{\pi^2}{t}}\right) \tag{A.7}$$

If we approximate ϑ_3 with 1 we obtain

$$W(A,\zeta) \sim \sqrt{\frac{\pi}{t}}\mathrm{e}^{-\frac{\gamma^2}{4t}(\zeta-\phi)^2} \tag{A.8}$$

As we will see (A.8) is good an approximation to (A.7) in our setting. First, note that the two expressions have a significant qualitative difference: the former is periodic in ζ while the latter is not. We can read off the periodicity directly from the left hand side of (A.5):

$$\mathsf{e}^{i\gamma(\zeta-\phi)s} \Rightarrow \text{The period is } \frac{2\pi}{\gamma}. \tag{A.9}$$

Let's examine carefully this periodicity. Since e^{-ts^2} is always positive, we see that the maxima of the sum are located at

$$\zeta_k = \phi + \frac{2\pi k}{\gamma} \quad , \quad k \in \mathbb{N}_0, \;\; \phi \in [0, 2\pi). \tag{A.10}$$

The value of the maxima is then given by

$$\sqrt{\frac{\pi}{t}}\mathsf{e}^{\frac{-k^2\pi^2}{t}}\vartheta_3\left(-\frac{ik\pi^2}{t}, \mathsf{e}^{-\frac{\pi^2}{t}}\right) \equiv \sqrt{\frac{\pi}{t}}\vartheta_3\left(0, \mathsf{e}^{-\frac{\pi^2}{t}}\right). \tag{A.11}$$

In general we will have $K = \lfloor 2\gamma \rfloor$ full periods in the interval $\zeta \in [0, 4\pi)$ and $M = 1 + \left\lfloor \gamma\left(2 - \frac{\phi}{2\pi}\right)\right\rfloor$ maxima. Now, we can exploit the freedom in restricting the value of the parameter γ. Since $\zeta \in [0, 4\pi)$ we find from $\frac{2\pi}{\gamma} \geq 4\pi$ that for $\gamma \leq \frac{1}{2}$ the periodicity of $W(A, \zeta)$ is not at all a problem. There will be less than one period and exactly one maximum in the interval $[0, 4\pi)$.

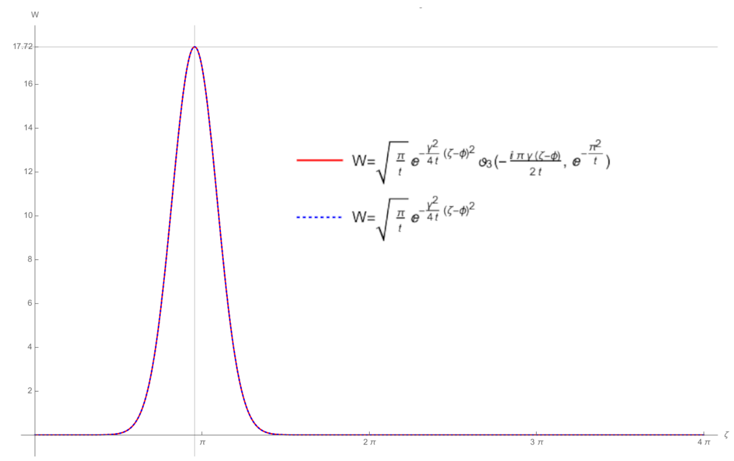
For completeness, we also note that ϕ essentially just moves around the maxima along the ζ-axis, while t determines their height and the spread of the Gaussians, as can be seen from (A.11). It is also easy to see that the imaginary part of (A.5) is exactly zero and that the real part is larger or equal to zero for all values of ζ, ϕ and t.

In summary, when $\gamma \leq \frac{1}{2}$ we can safely use (A.8) instead of the more complicated result (A.7). This is consistent with the fixing of the value of γ that comes from calculating Black Hole entropy using LQG (41; 42; 43).

Below we report graphical comparisons of the two expressions for the amplitude to illustrate the above reasoning.

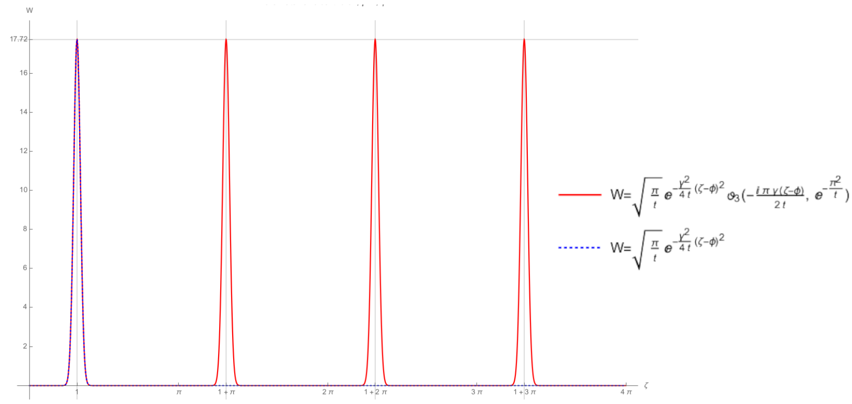
Example 1: The choice of parameters is $t = 0.01, \gamma = \frac{1}{3}$ and $\phi = 3$. This means:

- Period: $\frac{2\pi}{\gamma} = 6\pi$
- Number of full periods: $\lfloor 2\gamma \rfloor = 0$
- Number of maxima: $1 + \left\lfloor \gamma\left(2 - \frac{\phi}{2\pi}\right)\right\rfloor = 1$
- Location of maximum: $\phi + \frac{2\pi k}{\gamma} = 3$
- Height of maximum: $\sqrt{\frac{\pi}{t}}\vartheta_3\left(0, \mathsf{e}^{-\frac{\pi^2}{t}}\right) = 17.72$



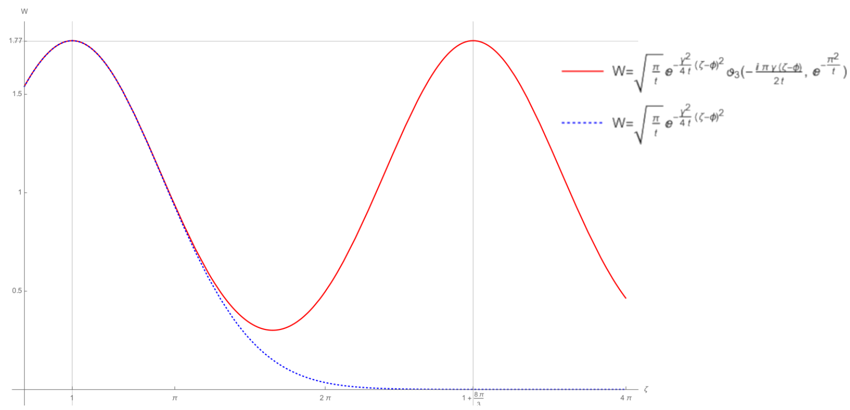
Example 2: The choice of parameters is $t = 0.01, \gamma = 2$ and $\phi = 1$.

- Period: $\frac{2\pi}{\gamma} = \pi$
- Number of full periods: $\lfloor 2\gamma \rfloor = 4$
- Number of maxima: $1 + \left\lfloor \gamma \left(2 - \frac{\phi}{2\pi} \right) \right\rfloor = 1 + 3 = 4$
- Location of maxima: $\{1, 1+\pi, 1+2\pi, 1+3\pi\}$
- Height of maxima: $\sqrt{\frac{\pi}{t}}\vartheta_3 \left(0, \mathrm{e}^{-\frac{\pi^2}{t}}\right) = 17.72$



Example 3: The choice of parameters is $t = 1, \gamma = \frac{3}{4}$ and $\phi = 1$.

- Period: $\frac{2\pi}{\gamma} = \frac{8\pi}{3}$
- Number of full periods: $\lfloor 2\gamma \rfloor = 1$
- Number of maxima: $1 + \left\lfloor \gamma \left(2 - \frac{\phi}{2\pi} \right) \right\rfloor = 1 + 1 = 2$
- Location of maxima: $\{1, 1 + \frac{8\pi}{3}\}$
- Height of maxima: $\sqrt{\frac{\pi}{t}}\vartheta_3 \left(0, \mathrm{e}^{-\frac{\pi^2}{t}}\right) = 1.77$



Appendix B

$SU(2)$, $SL(2,\mathbb{C})$ and the Y_Γ map

$SU(2)$ is the group of 2×2 unitary matrices with determinant equal to one. The underlying manifold is S^3 and an element of the group can be written as

$$h^A{}_B = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, \quad A,B = 0,1 \tag{B.1}$$

where $|a|^2 + |b|^2 = 1$.

The tangent space to the identity, ie the lie algebra $su(2)$ is a three dimensional vector space. A very convenient basis is given in terms of the Pauli matrices

$$\sigma_i{}^A{}_B = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \quad i = 1,2,3 \tag{B.2}$$

The Pauli matrices form a closed algebra, namely

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ij}{}^k \sigma_k \tag{B.3}$$

Sometimes it is better to work with the tau matrices

$$\tau_i := -\frac{i}{2}\sigma_i \tag{B.4}$$

Any element $h = h^A{}_B$ can be obtained by the standard exponentiation method for Lie Groups

$$h = e^{i\alpha \hat{n}\cdot\vec{\sigma}}, \quad \alpha \in [0, 2\pi] \tag{B.5}$$

An other parametrization is given in terms of the Euler angles in terms of which a group element is written as

$$h(\theta,\psi,\phi) = e^{\psi\tau_3} e^{\theta\tau_2} e^{\phi\tau_3} \tag{B.6}$$

in terms of these coordinates the Haar measure reads

$$dh = \frac{1}{16\pi^2}\int_0^{2\pi} d\psi \int_0^{\pi} sin\theta d\theta \int_0^{4\pi} d\phi \tag{B.7}$$

We can define derivative operators that act on functions $\Psi(h)$ by

$$(J^i\Psi)(h) = -i\frac{d}{dt}\Psi(he^{t\tau^i})|_{t=0} \tag{B.8}$$

These are also the left-invariant vector fields on the manifold (one could also define the right-invariant vector fields by changing the order).

The operator $J^2 := J^i J^i$ will also be very useful.

The matrix elements of the j irrep of an $SU(2)$ element are given by the Wigner matrices $D^i_{mn}(h)$. Let's focus for a moment in the fundamental $j = 1$ representation. The space that carries this representation is the space of spinors $\mathbb{C}^2$. A spinor $\mathbf{z} \in \mathbb{C}^2$ has the form

$$\mathbf{z} = \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} \tag{B.9}$$

We can also use the abstract index notation $z^A = \mathbf{z}$ with $A = 0, 1$.
Consider now the two antisymmetric tensors

$$\epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon_{AB} \tag{B.10}$$

These can be use to raise and lower indices. For example we have

$$z_A = \epsilon_{AB} z^B, \quad z^A = z_B \epsilon^{BA} \tag{B.11}$$

The order here matters and we follow the bottom-left-top-right rule. By this rule we have

$$z^A{}_A = \epsilon_{AB} z^{AB} = -\epsilon_{BA} z^{AB} = -z_A{}^A \tag{B.12}$$

An extremely useful property of ϵ is that it is invariant under the action of $SU(2)$. It is straightforward to show that

$$h^A{}_B h^C{}_D \epsilon^{BD} = \epsilon^{AC} \tag{B.13}$$

We are interested in the Hilbert space of square integrable functions on $SU(2)$, namely $L_2[SU(2)]$. According to the Peter-Weyl theorem the Wigner matrix elements in the j irrep $D^j_{mn}(h)$, seen as functions, form an orthogonal basis on $L_2[SU(2)]$, ie

$$\int dh \overline{D^j_{mn}(h)} D^{j'}_{m'n'}(h) = \delta_{jj'} \delta_{mm'} \delta_{nn'} \frac{1}{2j+1} \tag{B.14}$$

where dh is the Haar measure on $SU(2)$. Thus, a function Ψ on $L_2[SU(2)]$ can be written as

$$\Psi(h) = \sum_{j=0}^{+\infty} \sum_{|m| \leq j} \sum_{|n| \leq j} c_j{}^{mn} D^j_{mn}(h) \tag{B.15}$$

This allows for a very convenient decomposition of $L_2[SU(2)]$ as

$$L_2[SU(2)] = \bigoplus_{j=0}^{+\infty} (\mathcal{H}_j \otimes \mathcal{H}_j) \tag{B.16}$$

The direct sum is easy to understand. To understand the tensor product think that the first $\mathcal{H}_j$ contains elements V_m, the second $\mathcal{H}_j$ contains elements V_n and the tensor product between the two contains elements W_{mn} which is exactly what the Wigner matrices are.

The operator C acting on the basis D^j_{mn} gives

$$J^2 D^j_{mn}(h) = j(j+1) D^j_{mn}(h) \tag{B.17}$$

The group $SL(2, \mathbb{C})$ can be thought as a complexificaton of $SU(2)$. It is the double cover of $SO(1,3)$ and has 6 generators (3 rotations and 3 boosts) and two Casimir operators

$$C_1 = |\vec{K}|^2 - |\vec{L}|^2 \tag{B.18}$$

and

$$C_2 = \vec{K} \cdot \vec{L} \tag{B.19}$$

Since we are doing quantum gravity we are interested in the unitary representations of this group. It turns out that these are labelled by a positive real number p and a non-negative half-integer k. The Hilbert space $V^{(p,k)}$ of the (p,k) representation can be decomposed as

$$V^{(p,k)} = \bigoplus_{j=k}^{+\infty} \mathcal{H}^j \tag{B.20}$$

where $\mathcal{H}^j$ is the usual $2j+1$ dimensional irreducible representation space that carries the usual j spin representation of $SU(2)$. We notice that the unitary representation space for $SL(2,\mathbb{C})$ is infinite dimensional as expected due to the fact that the group is non-compact.

A state in $V^{(p,k)}$ can be written as

$$|p,k;j,m\rangle \tag{B.21}$$

where the first two numbers reflect to the two Casimir operators and the last two to the decomposition (B.20). The eigenvalues of the two Casimir are

$$C_1\,|p,k;j,m\rangle = (p^2-k^2+1)\,|p,k;j,m\rangle \tag{B.22}$$

and

$$C_2\,|p,k;j,m\rangle = pk\,|p,k;j,m\rangle \tag{B.23}$$

Now, we keep in mind that the linear simplicity constraint $\vec{K}=\gamma\vec{L}$ that characterizes GR needs to somehow be imposed on the quantum level in the large quantum numbers regime. By using this, the operators take the form

$$C_1 = (\gamma^2-1)|\vec{L}|^2 \tag{B.24}$$

and

$$C_2 = \gamma|\vec{L}|^2 \tag{B.25}$$

which in terms of eigenvalues corresponds to

$$p^2-k^2+1 = (\gamma^2-1)j(j+1) \tag{B.26}$$

and

$$pk = \gamma j(j+1) \tag{B.27}$$

As mentioned before we are interested in the large quantum numbers limit where the last two equations give

$$p^2-k^2 = (\gamma^2-1)j^2 \tag{B.28}$$

and

$$pk = \gamma j^2 \tag{B.29}$$

which is solved by the set of equations

$$p = \gamma k \tag{B.30}$$

$$k = j \tag{B.31}$$

Thus, the states of our interest are not the most general $|p,k;j,m\rangle$ but

$$|\gamma j,j;j,m\rangle \tag{B.32}$$

Notice, that a state of type (B.32) is fully determined by the quantum numbers j and m precisely as a state of SU(2). Therefore, there is one to one correspondence between the two representation spaces. This is encoded to the map Y_γ

$$\begin{aligned} Y_\gamma:\ \mathcal{H}^j &\longrightarrow\ V^{(p=\gamma j,k=j)} \\ |j,m\rangle &\longmapsto |\gamma j,j;j,m\rangle \end{aligned} \tag{B.33}$$

It can be proved that every vector in the image of Y_γ satisfies the linear simplicity constraint in the large j limit, namely

$$\langle K^i - \gamma L^i \rangle \approx 0 \tag{B.34}$$

This is the central idea of the EPRL model (32). To satisfy the linear simplicity constraints on the quantum level one should chose special states of $SL(2,\mathbb{C})$ rather than the more general.

B.1 The principal series representation

Let $\mathcal{V}^j$ with $j \in \frac{1}{2}\mathbb{N}$ be the vector space of homogeneous polynomials of degree $2j$ in two complex variables $\mathbf{z} = (z_0, z_1)^\intercal \in \mathbb{C}^2$. More precisely, there exist coefficients $(a_0, \dots, a_{2j}) \in \mathbb{C}^{2j+1}$ such that

$$P(\mathbf{z}) = \sum_{k=0}^{2j} a_k \, z_0^k \, z_1^{2j-k}, \tag{B.35}$$

which has the obvious property $P(\lambda \mathbf{z}) = \lambda^{2j} P(\mathbf{z}) \; \forall \lambda \in \mathbb{C} \backslash \{0\}$. In order to obtain a representation of $SU(2)$ on the vector space $\mathcal{V}^j$ we define the action of $h \in SU(2)$ as

$$h \triangleright P(\mathbf{z}) = P(h^\intercal \mathbf{z}) \quad \forall P \in \mathcal{V}^j. \tag{B.36}$$

and it is easy to verify that the two defining properties of a representation, i.e.

$$\mathbb{1} \triangleright P = P \quad \text{and} \quad (g_1 g_2) \triangleright P = g_1 \triangleright (g_2 \triangleright P) \tag{B.37}$$

are satisfied. The so defined representation is finite-dimensional with $\dim \mathcal{V}^j = 2j+1$ and one shows without much effort that it is also irreducible. Since all finite-dimensional irreducible representations of $SU(2)$ are isomorphic to one another we can relate the representation over $\mathcal{V}^j$ to the more familiar representation in terms of vectors $|jm\rangle \in \mathcal{H}^j$ by defining a linear map $\mathcal{I} : \mathcal{V}^j \to \mathcal{H}^j$ with the properties

$$\mathcal{I}(P^j_m) = |jm\rangle \quad \text{and} \quad \mathcal{I}(h \triangleright P) = h \triangleright \mathcal{I}(P). \tag{B.38}$$

This map allows us to determine a basis P^j_m of $\mathcal{V}^j$. All we need is to do is to compare the action of $h \in SU(2)$ on P^j_m and $|jm\rangle$. We therefore consider both sides of the equation

$$h \triangleright P^j_m = \mathcal{I}^{-1}(h \triangleright |jm\rangle) \tag{B.39}$$

separately and compare them in the end. Using the ansatz (B.35) for P^j_m and the group action (B.36) we get after some lengthy algebra

$$h \triangleright P^j_m = \sum_{|l|\leq j} \sum_{|q|\leq j} a_{j+q} \left[\frac{(j+q)!(j-q)!}{(j+l)!(j-l)!} \right]^{\frac{1}{2}} D^j_{lq}(h) z_0^{j+l} z_1^{j-l}. \tag{B.40}$$

for the left hand side of (B.39). The evaluation of the right hand side is straightforward and we obtain

$$\mathcal{I}^{-1}(h \triangleright |jm\rangle) = \sum_{|r|\leq j} \sum_{|s|\leq j} a_{j+s} \, D^j_{rm}(h) \, z_0^{j+s} \, z_1^{j-s}. \tag{B.41}$$

Comparing these expressions term by term, i.e. by setting $l = s$ we obtain the condition

$$a_{j+q} \left[\frac{(j+q)!(j-q)!}{(j+s)!(j-s)!} \right]^{\frac{1}{2}} D^j_{sq}(h) \stackrel{!}{=} a_{j+s} \, D^j_{rm}(h). \tag{B.42}$$

This equation can only be satisfied for $s = r$ and $q = m$ from which it follows that

$$a_{j+m} \sqrt{(j+m)!(j-m)!} = a_{j+s} \sqrt{(j+s)!(j-s)!}. \tag{B.43}$$

Since m is a fixed label we deduce that a_{j+s} has to be of the form

$$a_{j+s} = \frac{C \delta_{sm}}{\sqrt{(j+m)!(j-m)!}} \quad \Longrightarrow \quad P^j_m(\mathbf{z}) = \sum_{|s|\leq j} a_{j+s} \, z_0^{j+s} \, z_1^{j-s} = C \frac{z_0^{j+m} \, z_1^{j-m}}{\sqrt{(j+m)!(j-m!)}} \tag{B.44}$$

for some constant $C \in \mathbb{C}\backslash\{0\}$. This constant can easily be fixed by requiring that the basis P^j_m be orthonormal with respect to an appropriate inner product on $\mathcal{V}^j$. When defining such an inner product we need to keep in mind convergence issues arising from integrating complex polynomials over $\mathbb{C}^2$. This excludes the Lebesgue measure and suggests the use of the measure

$$\frac{\mathsf{e}^{-\langle\mathbf{z}|\mathbf{z}\rangle}}{\pi^2}\mathrm{d}^4\mathbf{z} := \frac{\mathsf{e}^{-\langle\mathbf{z}|\mathbf{z}\rangle}}{\pi^2}\mathrm{dRe}\,(z_0)\ \mathrm{dIm}\,(z_0)\ \mathrm{dRe}\,(z_1)\ \mathrm{dIm}\,(z_1)\,, \tag{B.45}$$

where the exponential damping factor ensures convergence. Hence, we can devise a well-defined and $SU(2)$ invariant inner product $\langle\cdot,\cdot\rangle : \mathcal{V}^j \times \mathcal{V}^j \to \mathbb{C}$ by

$$\langle f, g\rangle_{\mathcal{V}^j} := \int_{\mathbb{C}^2} \mathrm{d}^4\mathbf{z}\, \frac{\mathsf{e}^{-\langle\mathbf{z}|\mathbf{z}\rangle}}{\pi^2}\, \overline{f(\mathbf{z})}\, g(\mathbf{z}). \tag{B.46}$$

A nice property of this inner product is that it factorizes into separate integrations over z_0 and z_1. After a change to polar coordinates, one is left with simple integrals over Gaussian moments. It is therefore straight forward to check

$$\langle P^j_m, P^j_n\rangle = |C|^2\,\delta_{mn} \quad \Longrightarrow \quad |C|^2 = 1. \tag{B.47}$$

We choose $C = 1$ for simplicity. The inner product (B.46) allows us to write the resolution of identity on $\mathcal{H}^j$ in terms of the basis polynomials P^j_m and this in turn will allow us to express the Wigner matrices in terms of complex polynomials. As an intermediate step, we define the ket

$$|j\,\mathbf{z}\rangle := \sum_{|m|\leq j} \overline{P^j_m(\mathbf{z})}\,|j\,m\rangle \tag{B.48}$$

which by inspection has the property

$$\langle j\,\mathbf{z}|j\,m\rangle = P^j_m(\mathbf{z}). \tag{B.49}$$

We can then write the identity on $\mathcal{H}^j$ as

$$\int_{\mathbb{C}^2} \mathrm{d}^4\mathbf{z}\, \frac{\mathsf{e}^{-\langle\mathbf{z}|\mathbf{z}\rangle}}{\pi^2}\ |j\,\mathbf{z}\rangle\,\langle j\,\mathbf{z}| = \mathbb{1}_{\mathcal{H}^j}\,, \tag{B.50}$$

as can be checked by direct computation. Using the above resolution of identity we find for the Wigner matrices

$$\begin{aligned}
D^j_{mn}(h) &= \langle j\,m|\,h\,|j\,n\rangle = \langle j\,m|\int_{\mathbb{C}^2} \mathrm{d}^4\mathbf{z}\,\frac{\mathsf{e}^{-\langle\mathbf{z}|\mathbf{z}\rangle}}{\pi^2}\,h\,|j\,\mathbf{z}\rangle\,\langle j\,\mathbf{z}|j\,n\rangle \\
&= \int_{\mathbb{C}^2} \mathrm{d}^4\mathbf{z}\,\frac{\mathsf{e}^{-\langle\mathbf{z}|\mathbf{z}\rangle}}{\pi^2} h \rhd \langle j\,m|j\,\mathbf{z}\rangle\,\langle j\,\mathbf{z}|j\,n\rangle \\
&= \int_{\mathbb{C}^2} \mathrm{d}^4\mathbf{z}\,\frac{\mathsf{e}^{-\langle\mathbf{z}|\mathbf{z}\rangle}}{\pi^2}\overline{P^j_m(\mathbf{z})}\,P^j_n(h^\intercal\mathbf{z}).
\end{aligned} \tag{B.51}$$

To get from the first to the second line we used the fact that h is acting on $\overline{P^j_m(\mathbf{z})}$ inside $|j\,\mathbf{z}\rangle$. In the third line we used $h \rhd \overline{P^j_m(\mathbf{z})} = \overline{P^j_m(\overline{h}\mathbf{z})} = P^j_m(h\overline{\mathbf{z}})$ to perform the change of variables $\tilde{\mathbf{z}} = \overline{h}\mathbf{z}$ which produces the $h^\intercal\mathbf{z}$ argument of P^j_n (after dropping the tilde). As we will see in a moment, it is possible to generalize this method to the $SL(2,\mathbb{C})$ case.
We recall that the principal series of $SL(2,\mathbb{C})$ is labeled by two parameters $\chi \equiv (k,p) \in \mathbb{R}\times\frac{1}{2}\mathbb{Z}$. Let $\mathcal{V}^\chi$ be the (infinite-dimensional) vector space of homogeneous meromorphic functions in two complex variables $\mathbf{z} = (z_0, z_1)^\intercal \in \mathbb{C}^2$, where homogeneity now means

$$\Phi(\lambda\mathbf{z}) = \lambda^{ik+p-1}\,\overline{\lambda}^{ik-p-1}\,\Phi(\mathbf{z}) \quad \forall\lambda\in\mathbb{C}\backslash\{0\} \text{ and } \forall\Phi\in\mathcal{V}^\chi. \tag{B.52}$$

By defining the action of $g \in SL(2,\mathbb{C})$ on $\Phi \in \mathcal{V}^\chi$ as

$$g \rhd \Phi(\mathbf{z}) = \Phi(g^\intercal\mathbf{z}) \tag{B.53}$$

we obtain an infinite-dimensional irreducible representation of $SL(2,\mathbb{C})$ on $\mathcal{V}^\chi$. Moreover, the $\mathcal{V}^\chi$ representation splits into irreducible representations $\mathcal{V}^j$ of the $SU(2)$ subgroup

$$\mathcal{V}^\chi \simeq \bigoplus_{j=|p|}^{\infty} \mathcal{V}^j, \tag{B.54}$$

where j increases in integer steps. This fact allows us to define an injection at the fixed value $p = j$

$$\begin{aligned} \mathcal{J} : \mathcal{V}^j &\to \mathcal{V}^{(k,j)} \\ P(\mathbf{z}) &\mapsto \Phi(\mathbf{z}) = \langle \mathbf{z} | \mathbf{z} \rangle^{ik-j-1} P(\mathbf{z}), \end{aligned} \tag{B.55}$$

which has indeed the correct homogeneity properties.

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