Complex Action Problem

Silver-Blaze Phenomenon in the relativistic Bose gas

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Outline

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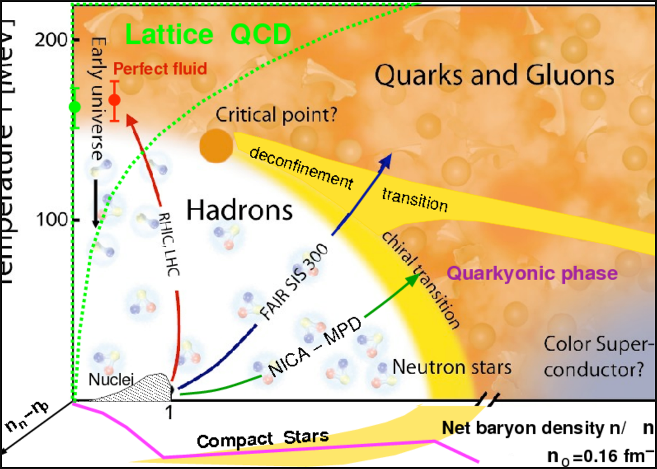
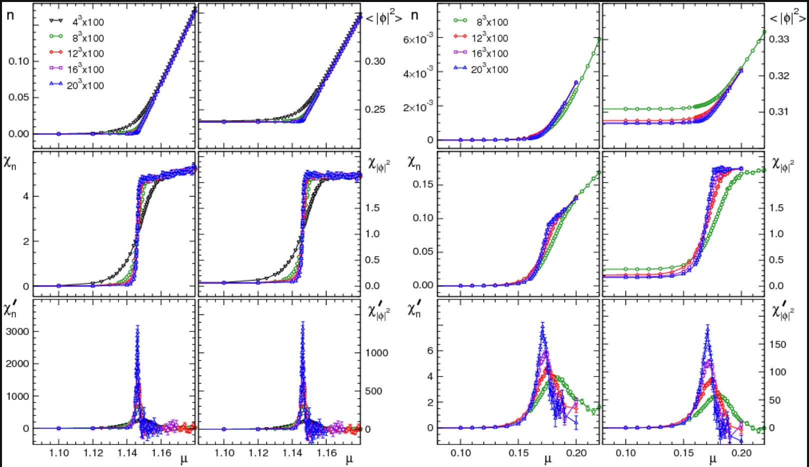
QCD phase diagram

Larry McLerran, arXiv:0906.2651v1 [hep-ph]



Silver-Blaze phenomenon

Christof Cattringer, Thomas Kloiber, arXiv:1206.2954v2 [hep-lat]



General Applications

Stochastic numerical calculations of integrals

- calculate expectation value integrals

$$\langle f \rangle_0 = \frac{\int_X f(x)\varrho(x)dx}{\int_X \varrho(x)dx}$$

  by sampling configuration space via Monte Carlo
- improve calculation time by following markovian chains
- maximize calculation efficiency by sampling integration space with appropriate probability
- ϱ while natural is *not* always the best! (overlap problem)

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- expectation values now include a sign or phase!

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- implies signed or complex probability which makes no sense in either case

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Re-weighting

Partial solution to the sign problem

- partial solution comes with re-weighting

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- using phase-quenched weights probability makes sense again

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Overlap problem

Important integration domain

The subset of X that contributes the most to the integral $\langle f \rangle$. Grows with sample size.

The "important" integration domains of

$$\int_X f(x)\varrho(x)dx \text{ and } \int_X \varrho(x)dx$$

do not generally coincide, creating a "conflict" in the important integration domain of $\langle f \rangle$.

- This is a general problem found in any weighting ϱ
- Re-weighting suffers from it too, though not as seriously as:

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Sign Problem

The expectation value and relative error estimated by N independent measurements of $e^{\imath\vartheta}$ scale with the number of degrees of freedom $\dim X$ as

$$\langle e^{\imath\vartheta}\rangle_0 \propto e^{-\dim X} \text{ and } \frac{\Delta\langle e^{\imath\vartheta}\rangle_0}{\langle e^{\imath\vartheta}\rangle_0} \propto \frac{1}{\sqrt{N}} e^{\dim X}$$

meaning $N \propto e^{2\dim X}$ *at least* which is prohibitive.

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- Assume a lattice $\mathbb{L}$ of
  - volume (number of sites) Ω (thermodynamic limit)
  - spacing (link size) $\ell = 1$ (continuum limit)
- thermodynamic/continuum limit $X \leftarrow \lim_{\Omega\to\infty} \lim_{\ell\to 0} \mathbb{L}$

continuous	discrete
fields $\phi(x)$	vectors ϕ_x
$\int dx$	$\sum_x$
$\int \mathcal{D}\phi$	$\prod_x \int d\phi_x$
$\partial_\mu \phi$	$\phi_{x+\mu} - \phi_x$ and $\phi_{x-\mu} - \phi_x$
field operators	vector matrices

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(Scalar) Quantum Field Theory

Action

$$S = \int_X dx \mathcal{L}(x) \longleftarrow \sum_x \mathcal{L}_x$$

Partition function (path integral)

$$Z = \int \mathcal{D}\phi \exp(-S[\phi]) \longleftarrow \prod_x \int d\phi_x \exp(-S[\phi])$$

Observables ((ground) expectation values)

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}\phi \mathcal{O}[\phi] \exp(-S[\phi]) \longleftarrow \prod_x \int d\phi_x \mathcal{O}[\phi] \exp(-S[\phi])$$

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- re-weighting (modified)
- Taylor expansion
- imaginary chemical potential

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at high density

- (complex) Langevin equation (stochastic quantization)
- Lefschetz thimbles (sister to complex Langevin equation)
- worm algorithms (and various supplementary ideas)
- effective 3D theories
- histogram method
- factorization (or density of state) method (among us!)
- imaginary chemical potential (generalized)
- fugacity expansion
- dimensional reduction
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G. Parisi, Y.-S. Wu, Sci. Sinica 24 (1981) 4S3

So what is stochastic quantization anyway?

Instead of scanning the pre-existent configurations space...
...we let a (fictional) time τ-process procude it instead.

Bonus! We get a configuration markovian chain in one package.

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Langevin equation

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(real) Langevin equation ($\phi \in \mathbb{R}$, $S \in \mathbb{R}$)

$$\frac{\partial}{\partial \tau}\phi(x,\tau) = K(\phi(x,\tau)) + \eta(x,\tau) \qquad \phi(x,\tau_0) = \phi_0(x)$$

(real) drift ($K \in \mathbb{R}$)

$$K(\phi(x)) = -\frac{\delta}{\delta\phi(x)}S[\phi]$$

(real gaussian) noise

$$\langle \eta(x,\tau)\eta(x',\tau')\rangle = 2\delta(x-x')\delta(\tau-\tau')$$

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(real) drift ($K \in \mathbb{R}$) with (hermitian positive-definite) kernel $\mathcal{K}$

$$K(\phi(x)) = -\int dx' \mathcal{K}(x,x') \frac{\delta}{\delta\phi(x')} S[\phi]$$

(real gaussian) noise

$$\langle \eta(x,\tau)\eta(x',\tau')\rangle = 2\mathcal{K}(x,x')\delta(\tau-\tau')$$

$$\mathcal{K}(x,x') = \delta(x-x')$$

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$$K(\phi(x)) = -\int dx' \mathcal{K}(x,x') \frac{\delta}{\delta\phi(x')} S[\phi]$$

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Poul H. Damgaard, Helmuth Hüffel 152, Nos. 5 & 6 (1987) 227-398

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Stochastic quantization postulate

Poul H. Damgaard, Helmuth Hüffel 152, Nos. 5 & 6 (1987) 227-398

- At the limit of large Langevin time $\tau \to \infty$:
  - equilibrium is reached at which...
  - ...target field theory (defined by action S) emerges

$$\lim_{\tau\to\infty}\left\langle \prod_{i=1}^{N}\phi(x_i,\tau)\right\rangle = \left\langle \prod_{i=1}^{N}\phi_\infty(x_i)\right\rangle$$

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G. Parisi and Y.-S. Wu, Sci. Sinica 24 (1981) 4S3

Langevin equation is stochastic.

Therefore its solutions are as random as itself!

Every configuration ϕ in the full configuration space has a probability $\langle\phi|\wp(\tau,\tau_0)|\phi_0\rangle$ of being an instance at time τ of a solution $\phi(\tau)$ of said Langevin equation.

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$$\langle\phi|\wp(\tau,\tau'')|\phi''\rangle = \int \mathcal{D}\phi\langle\phi|\wp(\tau,\tau')|\phi'\rangle\langle\phi'|\wp(\tau',\tau'')|\phi''\rangle$$

or $\wp(\tau,\tau'') = \wp(\tau,\tau')\wp(\tau',\tau'')$ in operator notation.

Hint! Looks like a path integral makes sense in this context.

The whole formulation develops on Langevin time τ as well as spacetime X. (extra degrees of freedom)

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$\forall\tau$ in equilibrium (postulated)

$$\langle\mathcal{O}(\tau)\rangle = \int \mathcal{D}\eta\rho[\eta]\mathcal{O}[\phi(\tau)] = \int \mathcal{D}\phi\langle\phi|\wp(\tau,\tau_0)|\phi_0\rangle\mathcal{O}[\phi]$$

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Fokker-Planck equation with kernel $\mathcal{K}$

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Static (equilibrium) solution

Jean Zinn-Justin. International Series of Monographs on Physics 113.

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Outline

Extension to complex Langevin

Stochastic quantization is solid in theory for $\phi \in \mathbb{R}$ and $S \in \mathbb{R}$.

Does (Can) it break when $\phi \in \mathbb{C}$?

And what about $S \in \mathbb{C}$?

We already see a problem with $S \in \mathbb{C}$.

$\exp(-S)$ is complex and cannot be interpreted as probability!

But first things first...

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Complex index notation

$\forall \phi \in \mathbb{C}$, $\phi = \alpha^{-1}(\phi_0 + \imath\phi_1)$ where $\alpha > 0$ is a normalization.

abstract .... index notation

ϕ .... ϕ_a

$\phi\psi$ .... $\alpha^{-1}\bigcirc_{abc}\phi_b\psi_c$

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ϕ_a
ϕ_0
ϕ_1

$\bigcirc_{abc}$			
1	0	0	1
0	−1	1	0

$\lozenge_{ab}$	
[illegible]	[illegible]
[illegible]	[illegible]

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[illegible]	[illegible]	[illegible]	[illegible]
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$\lvert\phi\rvert^2 = \phi^\dagger\phi$	$\alpha^{-1}\bullet_{abc}\phi_b\phi_c = 1_a\alpha^{-1}\phi_d\phi_d \longrightarrow \alpha^{-2}\phi_d\phi_d \in \mathbb{R}$

ϕ_a: $\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix}$

$\bigcirc_{abc}$: $\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{array}\right)$

$\Diamond_{ab}$: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\bullet_{abc}$

Complex index notation

$\forall \phi \in \mathbb{C}$, $\phi = \alpha^{-1}(\phi_0 + \imath \phi_1)$ where $\alpha > 0$ is a normalization.

abstract	index notation
ϕ	ϕ_a
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ϕ_a
ϕ_0
ϕ_1

$\bigcirc_{abc}$			
1	0	0	1
0	−1	1	0

$\Diamond_{ab}$	
1	0
0	−1

$\bullet_{abc}$			
1	0	0	−1
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$\phi \in \mathbb{C}$ and $S \in \mathbb{R}$

Langevin equation

$$\frac{\partial}{\partial \tau}\phi_a(x,\tau) = K_a(\phi(x,\tau)) + \eta_a(x,\tau)$$

$$K_a(\phi(x)) = -\frac{\delta}{\delta \phi_a(x)} S[\phi]$$

$$\langle \eta_a(x,\tau)\eta_{a'}(x',\tau')\rangle = 2\delta_{aa'}\delta(x-x')\delta(\tau-\tau')$$

$$\alpha_{aa} \leq \delta_{aa} = \dim_{\mathbb{R}} \mathbb{C} = 2$$

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$\phi \in \mathbb{C}$ and $S \in \mathbb{R}$

Fokker-Planck equation and equilibrium

Fokker-Planck hamiltonian

$$\int_X d^{\dim X} x \frac{\delta}{\delta \phi_a(x)} \left(\delta_{aa'} \frac{\delta}{\delta \phi_{a'}(x)} + \frac{\delta}{\delta \phi_a(x)} S[\phi] \right)$$

equilibrium distribution

$$\langle \phi | \wp_\infty | \phi_0 \rangle \propto \exp(-S[\phi])$$

$\phi \in \mathbb{C}$ and $S \in \mathbb{R}$

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Fokker-Planck hamiltonian

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$$\wp_\infty[\phi] \propto \exp(-\alpha\bot S[\phi])\exp(-(1-\alpha)\bot T[\phi])$$

The Feynman path integral is lost for the full theory!

Unless of course $\alpha_{aa'} = \delta_{aa'}$, i.e. *full noise* is taken.

$\phi \in \mathbb{C}$ and $S \in \mathbb{R}$

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We assume: $S = S_0 + \jmath S_1$

A distinct from field's complex unity $\jmath$ plus no normalization.

$\phi \in \mathbb{R}$ and $S \in \mathbb{C}$

Langevin equation

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Something's very wrong here!

$\phi \in \mathbb{C}$ and $S \in \mathbb{C}$

Langevin equation

$$\frac{\partial}{\partial \tau}\phi_a(x,\tau) = K_a(\phi(x,\tau)) + \eta_a(x,\tau)$$

$$K_a(\phi(x)) = -\beta^{-1}\bullet_{abc}\frac{\delta}{\delta\phi_b(x)}S_c[\phi]$$

$$\langle \eta_a(x,\tau)\eta_{a'}(x',\tau')\rangle = 2\beta_{aa'}\delta(x-x')\delta(\tau-\tau')$$

We assume: $S = S_0 + \jmath S_1$

We fix by extending real fields to complex: $\phi = \beta^{-1}(\phi_0 + \jmath\phi_1)$

$\phi \in \mathbb{C}$ and $S \in \mathbb{C}$

(modified) action

Technically, the action in its original from is non-writable in index form.

After field complexification however, S_a becomes a valid symbol.

Alas, S_0 is no longer the phase-quenched (more like phase-sqeezed) model but a whole new action involving full parameter information of the original action.

Even the original imaginary part! The parameters actually spread out even in both parts of the new action S_a.

$\phi \in \mathbb{C}$ and $S \in \mathbb{C}$

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Fokker-Planck hamiltonian

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$\beta_{aa} < \delta_{aa}$ doesn't necessarily mean loss of information.

The imaginary part ϕ_1 is auxiliary to start with.

However S_0 is yet "unrecognizable".

$\phi \in \mathbb{C}$ and $S \in \mathbb{C}$

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equilibrium distribution

$$\wp_\infty[\phi] \propto \exp(-g[\phi])$$

$$g[\phi] = g_a S_a[\phi]$$

$\phi \in \mathbb{C}$ and $S \in \mathbb{C}$

Fokker-Planck equation and equilibrium

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$$\wp_\infty[\phi] \propto \exp(-g[\phi])$$

$$g[\phi] = g_a S_a[\phi]$$

$\phi \in \mathbb{C} \otimes \mathbb{C}$ and $S \in \mathbb{C}$

(bi)complex fields

$$\alpha_{aa} = \alpha^2 = 2$$

$$\beta_{aa} = \beta^2 = 1$$

$$\sqrt{2}\phi = (\phi_{00} + \jmath\phi_{01}) + \imath(\phi_{10} + \jmath\phi_{11})$$

$$\frac{\partial}{\partial \tau}\phi_{ab}(x,\tau) = K_{ab}(\phi(x,\tau)) + 1_b \eta_a(x,\tau)$$

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$\phi \in \mathbb{C} \otimes \mathbb{C}$ and $S \in \mathbb{C}$

(bi)complex fields

$$\alpha_{aa} = \alpha^2 = 2$$

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Fokker-Planck equation and equilibrium

Fokker-Planck hamiltonian

$$\int_X d^{\dim X}x \frac{\delta}{\delta \phi_a(x)} \frac{\delta}{\delta \phi_a(x)} + \bullet_{bcd} \int_X d^{\dim X}x \frac{\delta}{\delta \phi_{ab}(x)} \frac{\delta}{\delta \phi_{ac}(x)} S_d[\phi]$$

$$\phi_a = \phi_{a0}$$

equilibrium distribution

$$\wp_\infty[\phi] \propto \exp(-g_0[\phi]) \exp(-g_1[\phi])$$

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Langevin dynamics test

- In general the Fokker-Planck distribution is away from the entropic factor $\exp(-S)$.
  - It's expected, the entropic factor is complex and unsuitable for use as a probability.
  - So there is no way to extract the path integral as it is by a (complex) Langevin process.
- So is averaging over the Langevin ensemble still valid?
  - One way we can check (quickly): simulations!
  - There is also an interesting property regarding observables.

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observables

- Typical observables defined via specific parameters in action

$$\langle \mathcal{O} \rangle = \frac{\partial}{\partial \alpha} \log_e Z = \left\langle -\frac{\partial}{\partial \alpha} S \right\rangle$$

- For complex action S, after
  - complexification of the field $\phi_a \longrightarrow \phi_{ab}$
  - extension of the action $S \longrightarrow S_b$
- follows extension of observables $\mathcal{O} \longrightarrow \mathcal{O}_b$

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- Langevin equations respect the following symmetry

$$\phi_{ab} \longrightarrow -\Diamond_{ac}\Diamond_{bd}\phi_{cd} \text{ and } K_{ab} \longrightarrow -\Diamond_{ac}\Diamond_{bd}K_{cd}$$

- Applying this symmetry to correlation functions,

$$\langle \phi_{ab}(x)\phi_{a'b'}(x')\rangle \propto \delta_{aa'}\delta_{bb'} + \varepsilon_{aa'}\varepsilon_{bb'}$$

- and continuing with observables, $\langle \mathcal{O}_1 \rangle = 0$!
- An interesting property: the auxiliary information is gone.
- However, much like the action, $\langle \mathcal{O}_0 \rangle$ is not what we think: it contains data from the full original complex observable.

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Outline

Discretization of Langevin time τ

continuous	discrete
$\tau = n\epsilon$	
$\phi(\tau)$	$\phi_n = \epsilon^{-1}\int_\tau^{\tau+\epsilon} d\tau \phi(\tau)$
$\frac{\partial}{\partial\tau}\phi(\tau)$	$\epsilon^{-1}(\phi_{n+1} - \phi_n)$
$\int d\tau f(\tau)$	$\sum_n \epsilon f_n$
$\delta(\tau - \tau')$	$\epsilon^{-1}\delta_{nn'}$

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$$\phi_{ab,x,n+1} = \phi_{ab,x,n} + \epsilon K_{ab}(\phi_{x,n}) + \sqrt{\epsilon}\bar{\eta}_{a,x,n}$$

$$K_{ab}(\phi_{x,n}) = -\bigcirc_{bcd}\frac{\partial}{\partial \phi_{ac,x,n}} S_d[\phi]$$

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For big enough ϵ can have runaway solutions!

Thermalization time is unknown.

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Dynamic Langevin time step ϵ

standard drift average

$$\phi_{ab,x,n+1} = \phi_{ab,x,n} + \epsilon_n K_{ab}(\phi_{x,n}) + \sqrt{\epsilon_n}\bar{\eta}_{a,x,n}$$

$$\epsilon_n K_n = \epsilon K$$

$$\tau = N\epsilon = \sum_n \epsilon_n \qquad K_n = \sqrt{\Omega^{-1}\sum_x K_a(\phi_{x,n})K_a(\phi_{x,n})}$$

$$K = N^{-1}\sum_n K_n$$

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$$\log K = \sum_n \tau^{-1}\epsilon_n \log K_n$$

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Outline

Preliminaries

$$\bigcirc_{aa_1 \ldots a_n} = \delta_{ab_0} \prod_{i=1}^{n-1} \bigcirc_{b_{i-1} a_i b_i} \delta_{b_{n-1} a_n}$$

$$S = \sum_x \mathcal{L}_x$$

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$$\bigcirc_{aa_1\ldots a_n} = \delta_{ab_0}\prod_{i=1}^{n-1}\bigcirc_{b_{i-1}a_ib_i}\delta_{b_{n-1}a_n}$$

$$S = \sum_x \mathcal{L}_x$$

$$\mathcal{O}_n = \Omega^{-1}\sum_x \mathcal{O}_{x,n}$$

$$\langle\mathcal{O}\rangle = \sum_n \tau^{-1}\epsilon_n\mathcal{O}_n$$

Relativistic Bose gas

action

$$\varkappa_{\dim X} = 2\dim X + m^2$$

$$\mathcal{L}_{d,x} = \frac{1}{2}\varkappa_{\dim X}\bigcirc_{def}\phi_{ge,x}\phi_{gf,x} + \frac{1}{4}\lambda\bigcirc_{defgh}\phi_{ie,x}\phi_{jf,x}\phi_{ig,x}\phi_{jh,x} - \sum_{\alpha=1}^{\dim\mathbb{L}}\cosh(\ell\mu\delta_{\alpha\dim\mathbb{L}})\delta_{de}\bigcirc_{efg}\delta_{hi}\phi_{hf,x}\phi_{ig,x+\hat{\alpha}} - \sum_{\alpha=1}^{\dim\mathbb{L}}\sinh(\ell\mu\delta_{\alpha\dim X})\varepsilon_{de}\bigcirc_{efg}\varepsilon_{hi}\phi_{hf,x}\phi_{ig,x+\hat{\alpha}}$$

Relativistic Bose gas

drift

$$\varkappa_{\dim X} = 2\dim X + m^2$$

$$K_{ab}(\phi_x) = -\varkappa_{\dim X}\phi_{ab,x} - \lambda \bigcirc_{bcde}\phi_{ac,x}\phi_{fd,x}\phi_{fe,x} + \sum_{\alpha=1}^{\dim \mathbb{L}} \cosh(\mu\delta_{\alpha \dim X})\delta_{ac}\delta_{bd}(\phi_{cd,x+\hat{\alpha}} + \phi_{cd,x-\hat{\alpha}}) + \sum_{\alpha=1}^{\dim \mathbb{L}} \sinh(\mu\delta_{\alpha \dim X})\varepsilon_{ac}\varepsilon_{bd}(\phi_{cd,x+\hat{\alpha}} - \phi_{cd,x-\hat{\alpha}})$$

Relativistic Bose gas

observables

$$n_{a,x} = -\frac{\partial}{\partial(\ell\mu)}\mathcal{L}_{a,x} = \\ = \sum_{\alpha=1}^{\dim \mathbb{L}} \sinh(\ell\mu\delta_{\alpha \dim X})\delta_{ab}\bigcirc_{bcd}\delta_{ef}\phi_{ec,x}\phi_{fd,x+\hat{\alpha}} \\ + \sum_{\alpha=1}^{\dim \mathbb{L}} \cosh(\ell\mu\delta_{\alpha \dim X})\varepsilon_{ab}\bigcirc_{bcd}\varepsilon_{ef}\phi_{ec,x}\phi_{fd,x+\hat{\alpha}}$$

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Relativistic Bose gas

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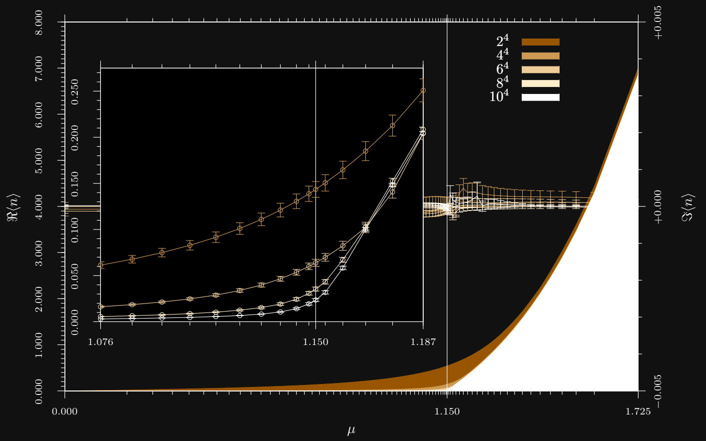
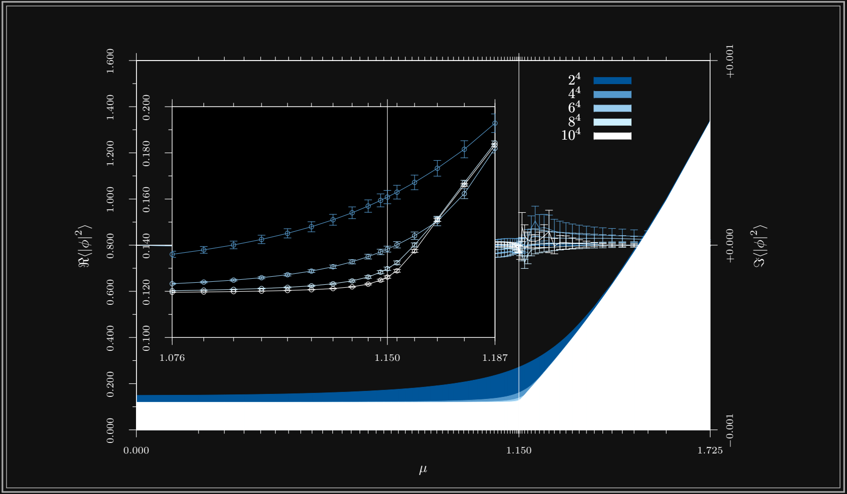
$\langle n \rangle$



$\langle|\phi|^2\rangle$



Outline

Conclusions

- Stochastic quantization agrees with Feynman path integral
  - as long as the action is real
  - and the process has full noise (all components)
- Stochastic quantization breaks with Feynman path integral
  - when noise is not full (not important)
  - when action is *complex*
- Looks like Stochastic quantization sees the Silver-Blaze phenomenon of the relativistic Bose gas
  - results agree with other methods like dual methods
    Christof Gattringer, Thomas Kloiber,
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Outlook

- As long as stochastic quantization gives promising results
  - a search for other models and in particular QCD are in order
  - but not only Standard Model physics can be studied
    - there are various Beyond the Standard Model physics with complex action to be tested (simulations)
    - the method is applicable to general systems with a complex action!
- Gert Aarts et. al. are onto working S. M. stochastic quantization
  - technicalities arise when starting to work with gauge symmetries
- What about (non-perturbative) renormalization?
- and other QFT and beyond aspects/problems to be adjusted to stochastic quantization?

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