Initial Value Formulation

of General Relativity

S. Ch. Papadoudis

Department of Physics
School of Applied Mathematics and Physical Sciences
National Technical University of Athens

Thesis Persentation, 2010

Outline

The Problem,
in classical field theory.

How to reformulate Einstein's physical equation for gravity in an causal frame with well posed Cauchy problem for to be designated type of initial value information.

- initial value information for second order hyperbolic systems:
  - initial value of fields
  - initial value of first time order of fields
  - initial value constraints

Extension of the hyperbolic problem includes timelike initial value hypersurfaces serving as boundaries to the solution, promting to a:

"Initial Value - Boundary Condition Problem"

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Special Covariance,

spawing Special Relativity.

- Physics does not change under *isometries* of spacetime.
  - sense of inertial observer:
- *Electromagnetism* is special covariant!
  - inertial observer: shielded from electromagnetic fields.

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The theme here is that inertial observers cannot be designated with respect to gravity.

Einstein proposed designating all observers inertial:

- Gravitational field vanishes in this perspective.
  - Phenomenons linked to gravity are now put to the framework of curved spacetime.

Emergent general covariance, going by the name:

"Equivalence Principle"

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Spacetime
intrinsic properties

Internal Structure

- metric $\langle \_ | \_ \rangle$: g_{ab}
- Levi-Civita connection ∇:
  $\Gamma^c{}_{ab} = (1/2) g^{cd} (\partial_a g_{bd} + \partial_b g_{da} - \partial_d g_{ab})$
- Riemann curvature tensor:
  $R^a{}_{bcd} = \partial_d \Gamma^a{}_{cb} - \partial_c \Gamma^a{}_{db} + \Gamma^a{}_{de} \Gamma^e{}_{cb} - \Gamma^a{}_{ce} \Gamma^e{}_{db}$
- Ricci tensor: $R_{ab} = g^c{}_e R^e{}_{acb}$
- curvature scalar: $R = g^{ab} R_{ab}$
- Einstein tensor: $G_{ab} = R_{ab} - (1/2) R g_{ab}$

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Spacetime
physical properties

stress-energy-momentum tensor T_{ab}

Decomposed in energy E, momentum vector $\overline{p}$ and stress tensor:

$$\begin{bmatrix} T_{ab}\upsilon^a\upsilon^b = E & T_{ab}\upsilon^a x^b = p_x & T_{ab}\upsilon^a y^b = p_y & T_{ab}\upsilon^a z^b = p_x \\ & T_{ab}x^a x^b = \sigma_{xx} & T_{ab}x^a y^b = \sigma_{xy} & T_{ab}x^a z^b = \sigma_{xz} \\ & & T_{ab}y^a y^b = \sigma_{yy} & T_{ab}y^a z^b = \sigma_{yz} \\ & & & T_{ab}z^a z^b = \sigma_{zz} \end{bmatrix}$$

for an orthonormal local coordinate system with *timelike* basis vector υ^a and *spacelike* basis vectors x^a, y^a and z^a

- is symmetric
- satisfies the energy condition: $T_{ab}\upsilon^a\upsilon^b \geq 0$

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Einstein's Field Equation
in mass units ($c = G = 1$)

Einstein's Equation

$$G_{ab} = 8\pi T_{ab}$$

The metric is implicit in T_{ab} as well as G_{ab}!

leading Einstein's equation to comprise a coupled, non-linear, second order PDE system for the metric components.

Bianchi Identity

$$\nabla^a G_{ab} = 0$$

Equation of Motion

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Motives

for an Initial Value Formulation of General Relativity

Einstein's equation is a spacetime equation:

- Predictability is implicit.
- No experiment can be set prior to having a spacetime solution. Observations are spacelike instances!
  - If a spacelike configuration is set, how is its evolution extracted from Einstein's equation?

The last question demonstrates the already known and accepted property that all Physical Theories have:

an "Initial Value Formulation"

which stands for the time evolution nature of all theories.

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Deployment

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$$

$$= \frac{1}{2} \sum_{\sigma} \sum_{\rho} g^{\sigma\rho} (\partial_{\sigma} \partial_{\rho} g_{\alpha\beta} + \partial_{\alpha} \partial_{\beta} g_{\sigma\rho} - 2 \partial_{\rho} \partial_{(\alpha} g_{\beta)\sigma})$$

$$- \frac{1}{2} \sum_{\sigma} \sum_{\rho} g^{\sigma\rho} g_{\alpha\beta} \sum_{\mu} \sum_{\nu} g^{\mu\nu} (\partial_{\sigma} \partial_{\rho} g_{\mu\nu} - \partial_{\rho} \partial_{\mu} g_{\nu\sigma}) + \dots$$

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Theorems

Theorem (Cauchy-Kowalewski)

All second time order PDE systems

$$\frac{\partial^2 \phi_i}{\partial t^2} = F_i\left(t, x^\mu; \phi_i; \frac{\partial \phi_i}{\partial t}, \frac{\partial \phi_i}{\partial x^\mu}; \frac{\partial^2 \phi_i}{\partial t \partial x^\mu}, \frac{\partial^2 \phi_i}{\partial x^\mu \partial x^\nu}\right)$$

endowed with arbitrary *analytic* initial values

$$\left(\phi_i(0, x^\mu) = f_i(x^\mu) \text{ and } \frac{\partial \phi_i}{\partial t}(0, x^\mu) = g_i(x^\mu)\right) \in \mathscr{C}^\omega[\mathbb{R}^{\dim M - 1}|\mathbb{R}]$$

constitute a well posed Cauchy problem with *analytic* solution.

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Spacetime is globally hyperbolic:

- it admits a monparametric foliation of diffeomorphic Cauchy hypersurfaces
  - all of spacetime is either future or past time-depended on events on a Cauchy hypersurface
  - a Cauchy hypersurface cuts through spacetime seperating in in a past and a future connected component

Analytic solutions do not work on causal spacetimes.

- globally hyperbolic spacetimes are stably casual
- assuming at most differential initial conditions and solutions
- no generic theorems for it!

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All linear, diagonal, second order hyperbolic PDE systems on M

$$g^{ab}\nabla_a\nabla_b\phi_i + \sum_j (A_{ij})^a\nabla_a\phi_j + \sum_j B_{ij}\phi_j + C_i = 0$$

endowed with arbitrary *smooth* initial values on Σ, ϕ_i and $n^a\nabla_a\phi_i$ constitute a well posed Cauchy problem with *smooth* solution.

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All *quasi*-linear, diagonal, second order hyperbolic systems on M

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ADM decomposition,

of spacetime metric g_{ab} into a spatial metric h_{ab} and more...

$\forall \upsilon^a$ such, that $\upsilon^a \nabla_a t = 1$:

$$h_{ab} = g_{ab} + n_a n_b$$
$$N = -\upsilon^a n_a = (n^a \nabla_a t)^{-1}$$
$$N_a = h_{ab} \upsilon^b$$

covariant decomposition?

$$g_{00} = h_{ij} N^i N^j - NN$$
$$g_{i0} = N_i / g_{0j} = N_j$$
$$g_{ij} = h_{ij}$$

in adapted coordinates

$$\left[\begin{array}{c|ccc} g_{tt} = h_{ij}N^iN^j - NN & g_{tx} = N_x & g_{ty} = N_y & g_{tz} = N_z \\ \hline g_{xt} = N_x & g_{xx} = h_{xx} & g_{xy} = h_{xy} & g_{xz} = h_{xz} \\ g_{yt} = N_y & g_{yx} = h_{yx} & g_{yy} = h_{yy} & g_{yz} = h_{yz} \\ g_{zt} = N_z & g_{zx} = h_{zx} & g_{zy} = h_{zy} & g_{zz} = h_{zz} \end{array}\right]$$

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$$N_a = h_{ab} \upsilon^b$$

covariant decomposition?

$$g_{00} = h_{ij} N^i N^j - NN$$
$$g_{i0} = N_i / g_{0j} = N_j$$
$$g_{ij} = h_{ij}$$

in adapted coordinates

$$\left[\begin{array}{c|ccc} g_{tt} = h_{ij}N^iN^j - NN & g_{tx} = N_x & g_{ty} = N_y & g_{tz} = N_z \\ \hline g_{xt} = N_x & g_{xx} = h_{xx} & g_{xy} = h_{xy} & g_{xz} = h_{xz} \\ g_{yt} = N_y & g_{yx} = h_{yx} & g_{yy} = h_{yy} & g_{yz} = h_{yz} \\ g_{zt} = N_z & g_{zx} = h_{zx} & g_{zy} = h_{zy} & g_{zz} = h_{zz} \end{array}\right]$$

ADM decomposition,

of spacetime metric g_{ab} into a spatial metric h_{ab} and more...

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Einstein's Equation
ADM decomposition

equations $a0$

$$G_{ab}n^b = 8\pi T_{ab}n^b$$

equations ab

$$G_{ab} = 8\pi T_{ab}$$

equation 00

$$G_{ab}n^a n^b = 8\pi\rho$$

$$\rho = T_{ab}\, n^a n^b$$

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Outline

1. General Theory of Relativity
   - Principles of Einstein's Theory
   - Einstein's Theory of Gravity
2. The Initial Value Formulation
   - Preliminaries
   - Deploying the Problem
   - Initial Values and Cauchy Development
   - Development Equations (in vacuum, $R_{ab} = 0$)

Initial Values,

on Σ_0 of the foliation as initial value space.

spatial metric

Initialization on the lines of ADM decomposition.

exterior curvature

$$K_{ab} := D_a n_b = \frac{1}{2} \pounds_n h_{ab}$$

covariant time derivative

$$\frac{1}{2} \pounds_t h_{ab} = N K_{ab} + \frac{1}{2} \pounds_N h_{ab}$$

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contraint *i*

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4 metric non-development equations allowing 4 degrees of freedom:

- relevant to general covariance of solution,
- employ coordinated to fix guage.

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Gauge fixing,

for *vacuum* equations $R_{ab} = 0$.

gauge freedom (fixed by employing harmonic coordinates)

$$\Box x^{\mu} = g_{ab}\nabla^{a}\nabla^{b}x^{\mu} = \sum_{\nu}\partial_{\nu}g^{\nu\mu} + \frac{1}{2}\sum_{\nu}g^{\nu\mu}\sum_{\alpha}\sum_{\beta}g^{\alpha\beta}\partial_{\nu}g_{\alpha\beta} = 0$$

Einstein Reduced Equation

$$R_{\mu\nu} = F_{\mu\nu} + \frac{1}{2}\sum_{\alpha}\sum_{\beta}g^{\alpha\beta}\partial_{\alpha}\partial_{\beta}g_{\mu\nu} = 0$$

Compatibility with Einstein's equation, yields a *well posed local, linear, diagonal, second order hyperbolic* PDE *system* for $\Box x^{\mu}$.

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- The rest 6 equations are *quasi*-linear, diagonal, second order hyperbolic for the purely spatial metric components.
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Maximal Cauchy Development,

solving *vacuum* equations $R_{ab} = 0$.

- Solve vacuum equations locally on all events on Σ.
  - thus generating a solved film proxima to entire Σ.
- Take all such (locally) diffeomorphic solutions on entire Σ.
- Compare any pair of *classes* of diffeomorphic solutions, with respect to $\subseteq$,
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not enough?

- Cauchy development breaks down in the presence of singularities,
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- Einstein's theory of gravity for:
  - scalar fields
  - electromagnetism
  - perfect fluid
    $$T_{ab} = \rho v_a v_b + P(g_{ab} + v_a v_b)$$
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- More T_{ab}s...
- Automation
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  - A: Your configuration has matter so it doesn't necessarily have a well posed initial value formulation.

If a theory is prooved *not* to have a well posed initial value formulation (different from simply not knowing), then that particular theory is hard to be interpreted as physical!

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Derivatives.

covariant derivative

$$\nabla_a T^{c_1 \dots c_k}{}_{b_1 \dots b_l} = \partial_a T^{c_1 \dots c_k}{}_{b_1 \dots b_l}$$

$$+ \sum_{i=1}^{k} \Gamma^{c_i}{}_{ad} T^{c_1 \dots c_{i-1} d c_{i+1} \dots c_k}{}_{b_1 \dots b_l} - \sum_{j=1}^{l} \Gamma^{d}{}_{a b_j} T^{c_1 \dots c_k}{}_{b_1 \dots b_{j-1} d b_{j+1} \dots b_l}$$

ξ-Lie derivative

$$\pounds_\xi T^{a_1 \dots a_k}{}_{b_1 \dots b_l} = \xi^c \nabla_c T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$$

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Derivatives.

covariant derivative

$$\nabla_a T^{c_1\ldots c_k}{}_{b_1\ldots b_l} = \partial_a T^{c_1\ldots c_k}{}_{b_1\ldots b_l}$$

$$+\sum_{i=1}^{k}\Gamma^{c_i}{}_{ad}T^{c_1\ldots c_{i-1}dc_{i+1}\ldots c_k}{}_{b_1\ldots b_l} - \sum_{j=1}^{l}\Gamma^{d}{}_{ab_j}T^{c_1\ldots c_k}{}_{b_1\ldots b_{j-1}db_{j+1}\ldots b_l}$$

ξ-Lie derivative

$$\pounds_\xi T^{a_1\ldots a_k}{}_{b_1\ldots b_l} = \xi^c\nabla_c T^{a_1\ldots a_k}{}_{b_1\ldots b_l}$$

$$-\sum_{i=1}^{k}T^{a_1\ldots a_{i-1}ca_{i+1}\ldots a_k}{}_{b_1\ldots b_l}\nabla_c\xi^{a_i} + \sum_{j=1}^{l}T^{a_1\ldots a_k}{}_{b_1\ldots b_{j-1}cb_{j+1}\ldots b_l}\nabla_{b_j}\xi^c$$

Appendix

covariant derivative commutation

$$\nabla_c \nabla_d T^{a_1 \dots a_k}{}_{b_1 \dots b_l} = \nabla_d \nabla_c T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$$

$$- \sum_{i=1}^{k} R^{a_i}{}_{ecd} T^{a_1 \dots a_{i-1} e a_{i+1} \dots a_k}{}_{b_1 \dots b_l} + \sum_{j=1}^{l} R^{e}{}_{b_j cd} T^{a_1 \dots a_k}{}_{b_1 \dots b_{j-1} e b_{j+1} \dots b_l}$$

projected tensors

$$T^{a_1 \dots a_k}{}_{b_1 \dots b_l} \longrightarrow h^{a_i}{}_{c_i} \dots h^{a_i}{}_{c_i} h_{b_i}{}^{d_i} \dots h_{b_i}{}^{d_i} T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$$

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$$D_{a\_} \equiv h_a{}^b \nabla_{b\_}$$

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For Further Reading I

- Robert M. Wald.
  General Relativity.
  The University of Chigago Press, first edition, 1984.
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  The large scale structure of space-time.
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