Initial Value Formulation
of General Relativity

S. Ch. Papadoudis

Department of Physics
School of Applied Mathematics and Physical Sciences
National Technical University of Athens

Thesis Presentation, 2010
Outline

1. General Theory of Relativity
   - Principles of Einstein’s Theory
   - Einstein’s Theory of Gravity

2. The Initial Value Formulation
   - Preliminaries
   - Deploying the Problem
   - Initial Values and Cauchy Development
   - Development Equations (in vacuum, $R_{ab} = 0$)
The Problem, in classical field theory.

How to reformulate Einstein’s physical equation for gravity in an causal frame with well posed Cauchy problem for to be designated type of initial value information.

- initial value information for second order hyperbolic systems:
  - initial value of fields
  - initial value of first time order of fields
  - initial value constraints

Extension of the hyperbolic problem includes timelike initial value hypersurfaces serving as boundaries to the solution, promting to a:

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Special Covariance, spawing Special Relativity.

- Physics does not change under *isometries* of spacetime.
  - sense of inertial observer:

*Electromagnetism* is special covariant!

- inertial observer: shielded from electromagnetic fields.
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General Covariance, spawning General Relativity.

- Physics does not change under \textit{diffeomorphisms} of spacetime.
  - \textit{General Relativity} is general covariant... (and \textit{locally} special covariant!)

- No sense of Inertial observer!
  - Inertial observer:
    - no known method for shielding from gravitational fields.
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The theme here is that inertial observers cannot be designated with respect to gravity. Einstein proposed designating all observers inertial:

- Gravitational field vanishes in this perspective.
- Phenomenons linked to gravity are now put to the framework of curved spacetime.

Emergent general covariance, going by the name:

“Equivalence Principle”
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Spacetime intrinsic properties

Internal Structure

- **metric** $\langle \_ | \_ \rangle$: $g_{ab}$
- **Levi-Civita connection** $\nabla$:
  $$\Gamma^c_{ab} = (1/2)g^{cd}(\partial_a g_{bd} + \partial_b g_{da} - \partial_d g_{ab})$$
- **Riemann curvature tensor**:
  $$R^a_{bcd} = \partial_d \Gamma^a_{cb} - \partial_c \Gamma^a_{db} + \Gamma^a_{de} \Gamma^e_{cb} - \Gamma^a_{ce} \Gamma^e_{db}$$
- **Ricci tensor**: $R_{ab} = g^c_e R^e_{acb}$
- **curvature scalar**: $R = g^{ab} R_{ab}$
- **Einstein tensor**: $G_{ab} = R_{ab} - (1/2)R g_{ab}$
Spacetime

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Spacetime
physical properties

stress-energy-momentum tensor $T_{ab}$

Decomposed in energy $E$, momentum vector $\vec{p}$ and stress tensor:

\[
\begin{bmatrix}
T_{ab} \upsilon^a \upsilon^b &= E \\
T_{ab} \upsilon^a x^b &= p_x \\
T_{ab} x^a x^b &= \sigma_{xx} \\
T_{ab} \upsilon^a y^b &= p_y \\
T_{ab} x^a y^b &= \sigma_{xy} \\
T_{ab} \upsilon^a z^b &= p_x \\
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T_{ab} z^a z^b &= \sigma_{zz}
\end{bmatrix}
\]

for an orthonormal local coordinate system with

timelike basis vector $\upsilon^a$ and spacelike basis vectors $x^a$, $y^a$ and $z^a$

- is symmetric
- satisfies the energy condition: $T_{ab} \upsilon^a \upsilon^b \geq 0$
Spacetime physical properties

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Einstein’s Field Equation

in mass units \((c = G = 1)\)

Einstein’s Equation

\[ G_{ab} = 8\pi T_{ab} \]

The metric is implicit in \(T_{ab}\) as well as \(G_{ab}\)!

leading Einstein’s equation to comprise a coupled, non-linear, second order PDE system for the metric components.

Bianchi Identity

\[ \nabla^a G_{ab} = 0 \]

Equation of Motion

\[ \nabla^a T_{ab} = 0 \]
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Motives for an Initial Value Formulation of General Relativity

Einstein’s equation is a spacetime equation:

- Predictability is implicit.
- No experiment can be set prior to having a spacetime solution. Observations are spacelike instances!
- If a spacelike configuration is set, how is its evolution extracted from Einstein’s equation?

The last question demonstrates the already known and accepted property that all Physical Theories have:

an “Initial Value Formulation”

which stands for the time evolution nature of all theories.
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Deployment

\[
G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}
\]

\[
= \frac{1}{2} \sum_{\sigma} \sum_{\rho} g^{\sigma\rho} (\partial_\sigma \partial_\rho g_{\alpha\beta} + \partial_\alpha \partial_\beta g_{\sigma\rho} - 2 \partial_\rho \partial_{(\alpha} g_{\beta)\sigma})
\]

\[
- \frac{1}{2} \sum_{\sigma} \sum_{\rho} g^{\sigma\rho} g_{\alpha\beta} \sum_{\mu} \sum_{\nu} g^{\mu\nu} (\partial_\sigma \partial_\rho g_{\mu\nu} - \partial_\rho \partial_\mu g_{\nu\sigma}) + \ldots
\]
Deployment

\[ G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} \]

\[ = \frac{1}{2} \sum_{\sigma} \sum_{\rho} g^{\sigma\rho} \left( \partial_{\sigma} \partial_{\rho} g_{\alpha\beta} + \partial_{\alpha} \partial_{\beta} g_{\sigma\rho} - 2 \partial_{\rho} \partial_{(\alpha} g_{\beta)\sigma} \right) \]

\[ - \frac{1}{2} \sum_{\sigma} \sum_{\rho} g^{\sigma\rho} g_{\alpha\beta} \sum_{\mu} \sum_{\nu} g^{\mu\nu} \left( \partial_{\sigma} \partial_{\rho} g_{\mu\nu} - \partial_{\rho} \partial_{\mu} g_{\nu\sigma} \right) + \ldots \]
Theorem (Cauchy-Kowalewski)

All second time order PDE systems

\[
\frac{\partial^2 \phi_i}{\partial t^2} = F_i \left( t, x^\mu; \phi_i, \frac{\partial \phi_i}{\partial t}, \frac{\partial \phi_i}{\partial x^\mu}, \frac{\partial^2 \phi_i}{\partial t \partial x^\mu}, \frac{\partial^2 \phi_i}{\partial x^\mu \partial x^v} \right)
\]

endowed with arbitrary analytic initial values

\[
\left( \phi_i(0, x^\mu) = f_i(x^\mu) \text{ and } \frac{\partial \phi_i}{\partial t}(0, x^\mu) = g_i(x^\mu) \right) \in \mathcal{C}_a^\omega[\mathbb{R}^{\text{dim} M-1}\mid \mathbb{R}]
\]

constitute a well posed Cauchy problem with analytic solution.
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constitute a well posed Cauchy problem with \textit{analytic} solution.
Assumptions,

Spacetime is globally hyperbolic:
- it admits a monparametric foliation of diffeomorphic Cauchy hypersurfaces
  - all of spacetime is either future or past time-depended on events on a Cauchy hypersurface
  - a Cauchy hypersurface cuts through spacetime separating in a past and a future connected component
- Analytic solutions do not work on causal spacetimes.
  - globally hyperbolic spacetimes are stably casual
  - assuming at most differential initial conditions and solutions
  - no generic theorems for it!
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Theorems

Theorem

All linear, diagonal, second order hyperbolic PDE systems on $M$

$$g^{ab} \nabla_a \nabla_b \phi_i + \sum_j (A_{ij})^a \nabla_a \phi_j + \sum_j B_{ij} \phi_j + C_i = 0$$

endowed with arbitrary smooth initial values on $\Sigma$, $\phi_i$ and $n^a \nabla_a \phi_i$ constitute a well posed Cauchy problem with smooth solution.
Theorem

All linear, diagonal, second order hyperbolic PDE systems on $M$

$$g^{ab}\nabla_a \nabla_b \phi_i + \sum_j (A_{ij})^a \nabla_a \phi_j + \sum_j B_{ij} \phi_j + C_i = 0$$

endowed with arbitrary smooth initial values on $\Sigma$, $\phi_i$ and $n^a \nabla_a \phi_i$ constitute a well posed Cauchy problem with smooth solution.
All quasi-linear, diagonal, second order hyperbolic systems on $M$

$$g^{ab}(\phi_j|\nabla_c \phi_j)\nabla_a(\phi_j|\nabla_c \phi_j)\nabla_b(\phi_j|\nabla_c \phi_j)\phi_i = F_i(\phi_j|\nabla_c \phi_j)$$

endowed with smooth initial values on $\Sigma$

$$(\phi_i \text{ and } n^a \nabla_a \phi_i) \in C^\infty[\Sigma|\mathbb{R}^n]$$

locally sufficiently close to those of a background solution, constitute a well posed Cauchy problem with smooth solution.
All quasi-linear, diagonal, second order hyperbolic systems on $M$

$$g^{ab}(\phi_j | \nabla_c \phi_j) \nabla_a (\phi_j | \nabla_c \phi_j) \nabla_b (\phi_j | \nabla_c \phi_j) \phi_i = F_i (\phi_j | \nabla_c \phi_j)$$

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ADM decomposition, of spacetime metric $g_{ab}$ into a spatial metric $h_{ab}$ and more...

∀ $\nu^a$ such, that $\nu^a \nabla_a t = 1$:

\[
\begin{align*}
  h_{ab} &= g_{ab} + n_a n_b \\
  N &= -\nu^a n_a = (n^a \nabla_a t)^{-1} \\
  N_a &= h_{ab} \nu^b
\end{align*}
\]

Covariant decomposition?

\[
\begin{align*}
  g_{00} &= h_{ij} N^i N^j - NN \\
  g_{i0} &= N_i / g_{0j} = N_j \\
  g_{ij} &= h_{ij}
\end{align*}
\]

In adapted coordinates:

\[
\begin{bmatrix}
  g_{tt} = h_{ij} N^i N^j - NN \\
  g_{xt} = N_x \\
  g_{yt} = N_y \\
  g_{zt} = N_z
\end{bmatrix}
\begin{bmatrix}
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ADM decomposition, of spacetime metric $g_{ab}$ into a spatial metric $h_{ab}$ and more...

$\forall \nu^a$ such, that $\nu^a \nabla_a t = 1$:

$h_{ab} = g_{ab} + n_a n_b$

$N = -\nu^a n_a = (n^a \nabla_a t)^{-1}$

$N_a = h_{ab} \nu^b$

Covariant decomposition?

$g_{00} = h_{ij} N^i N^j - N N$

$g_{i0} = N_i / g_{0j} = N_j$

$g_{ij} = h_{ij}$

in adapted coordinates:

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\begin{bmatrix}
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ADM decomposition

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2. The Initial Value Formulation
   - Preliminaries
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   - Development Equations (in vacuum, $R_{ab} = 0$)
Initial Values, on $\Sigma_0$ of the foliation as initial value space.

spatial metric
Initialization on the lines of ADM decomposition.

exterior curvature

$$K_{ab} := D_a n_b = \frac{1}{2} \nabla_n h_{ab}$$

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4 metric non-development equations allowing 4 degrees of freedom:
- relevant to general covariance of solution,
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$$\Box x^{\mu} = g_{ab} \nabla^a \nabla^b x^{\mu} = \sum_v \partial_v g^{v\mu} + \frac{1}{2} \sum_v \sum_{\alpha} \sum_{\beta} g^{\alpha\beta} \partial_v g_{\alpha\beta} = 0$$

**Einstein Reduced Equation**

$$R_{\mu\nu} = F_{\mu\nu} + \frac{1}{2} \sum_{\alpha} \sum_{\beta} g^{\alpha\beta} \partial_{\alpha} \partial_{\beta} g_{\mu\nu} = 0$$

Compatibility with Einstein's equation, yields a well posed local, linear, diagonal, second order hyperbolic PDE system for $\Box x^{\mu}$. 
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- Solve vacuum equations locally on all events on $\Sigma$.
  - thus generating a solved film proxima to entire $\Sigma$.
- Take all such (locally) diffeomorphic solutions on entire $\Sigma$.
- Compare any pair of classes of diffeomorphic solutions, with respect to $\subseteq$.
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  - scalar fields
  - electromagnetism
  - perfect fluid
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    (only for some state equations \( P = P(\rho) \))
  - some other specific \( T_{ab} \)
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\[ T_{ab} = \rho \nu_a \nu_b + P(g_{ab} + \nu_a \nu_b) \]

(only for some state equations \( P = P(\rho) \))

some other specific \( T_{ab} \)...

also has a well posed initial value formulation.

linear fields of spin > 1 fail to have well posed initial value formulation.
Q: It has an well posed initial value formulation, why can't I solve my configuration?

A: Your configuration has matter so it doesn’t necessarily have a well posed initial value formulation.

If a theory is prooved not to have a well posed initial value formulation (different from simply not knowing), then that particular theory is hard to be interpreted as physical!
Outlook

- More $T_{ab}s$...

- Automation
  - Q: It has an well posed initial value formulation, why can’t I solve my configuration?
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More $T_{ab}s$...

Automation

Q: It has an well posed initial value formulation, why can’t I solve my configuration?

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- More $T_{ab}s$...
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If a theory is proved not to have a well posed initial value formulation (different from simply not knowing), then that particular theory is hard to be interpreted as physical!
Outlook

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  Q: It has an well posed initial value formulation, why can’t I solve my configuration?
  
  A: Your configuration has matter so it doesn’t necessarily have a well posed initial value formulation.

If a theory is proved *not* to have a well posed initial value formulation (different from simply not knowing), then that particular theory is hard to be interpreted as physical!
Appendix

Derivatives.

covariant derivative

$$\nabla_a T^{c_1...c_k}_{b_1...b_l} = \partial_a T^{c_1...c_k}_{b_1...b_l}$$

$$+ \sum_{i=1}^{k} \Gamma^c_{ai} T^{c_1...c_{i-1}dc_{i+1}...c_k}_{b_1...b_l} - \sum_{j=1}^{l} \Gamma^d_{b_j} T^{c_1...c_k}_{b_1...b_{j-1}db_{j+1}...b_l}$$

ξ-Lie derivative

$$\mathcal{L}_{\xi} T^{a_1...a_k}_{b_1...b_l} = \xi^c \nabla_c T^{a_1...a_k}_{b_1...b_l}$$

$$- \sum_{i=1}^{k} T^{a_1...a_{i-1}ca_{i+1}...a_k}_{b_1...b_l} \nabla_c \xi^a_i + \sum_{j=1}^{l} T^{a_1...a_k}_{b_1...b_{j-1}cb_{j+1}...b_l} \nabla_b \xi^c_j$$

Stratos Ch. Papadoudis
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Appendix

Derivatives.

**covariant derivative**

\[ \nabla_a T^{c_1...c_k}_{b_1...b_l} = \partial_a T^{c_1...c_k}_{b_1...b_l} + \sum_{i=1}^{k} \Gamma^c_{ad} T^{c_1...c_{i-1}d c_i+1...c_k}_{b_1...b_l} - \sum_{j=1}^{l} \Gamma^d_{abj} T^{c_1...c_k}_{b_1...b_{j-1}db_{j+1}...b_l} \]

**\(\xi\)-Lie derivative**

\[ \mathcal{L}_\xi T^{a_1...a_k}_{b_1...b_l} = \xi^c \nabla_c T^{a_1...a_k}_{b_1...b_l} - \sum_{i=1}^{k} T^{a_1...a_{i-1}ca_i+1...a_k}_{b_1...b_l} \nabla_c \xi^a_i + \sum_{j=1}^{l} T^{a_1...a_k}_{b_1...b_{j-1}cb_{j+1}...b_l} \nabla_j \xi^c \]
The covariant derivative is given by:

\[ \nabla_a T^{c_1...c_k}_{b_1...b_l} = \partial_a T^{c_1...c_k}_{b_1...b_l} + \sum_{i=1}^{k} \Gamma^c_{ad} T^{c_1...c_{i-1}dc_{i+1}...c_k}_{b_1...b_l} - \sum_{j=1}^{l} \Gamma^d_{ab_j} T^{c_1...c_k}_{b_1...b_{j-1}db_{j+1}...b_l} \]

The \( \xi \)-Lie derivative is given by:

\[ \mathcal{L}_\xi T^{a_1...a_k}_{b_1...b_l} = \xi^c \nabla_c T^{a_1...a_k}_{b_1...b_l} - \sum_{i=1}^{k} T^{a_1...a_{i-1}ca_{i+1}...a_k}_{b_1...b_l} \nabla_c \xi^a_i + \sum_{j=1}^{l} T^{a_1...a_k}_{b_1...b_{j-1}cb_{j+1}...b_l} \nabla_b_j \xi^c \]
covariant derivative commutation

\[ \nabla_c \nabla_d T^{a_1...a_k}_{b_1...b_l} = \nabla_d \nabla_c T^{a_1...a_k}_{b_1...b_l} \]

\[ - \sum_{i=1}^{k} R^{a_i}_{ecd} T^{a_1...a_i-1e a_i+1...a_k}_{b_1...b_l} + \sum_{j=1}^{l} R^{e}_{b j e} T^{a_1...a_k}_{b_1...b_j-1e b_j+1...b_l} \]

projected tensors

\[ T^{a_1...a_k}_{b_1...b_l} \rightarrow h^{a_i}_{c_i} ... h^{a_i}_{c_i} h^{d_i}_{b_i} ... h^{d_i}_{b_i} T^{a_1...a_k}_{b_1...b_l} \]

projected derivative

\[ D_{a-} \equiv h^{b}_{a} \nabla_{b-} \]
covariant derivative commutation

\[
\nabla_c \nabla_d T^{a_1...a_k}_{b_1...b_l} = \nabla_d \nabla_c T^{a_1...a_k}_{b_1...b_l} - \sum_{i=1}^{k} R^{a_i}_{ecd} T^{a_1...a_{i-1}ea_{i+1}...a_k}_{b_1...b_l} + \sum_{j=1}^{l} R^{e}_{bjcd} T^{a_1...a_k}_{b_1...b_{j-1}eb_{j+1}...b_l}
\]

projected tensors

\[
T^{a_1...a_k}_{b_1...b_l} \rightarrow h^{a_i}_{c_i} \ldots h^{a_i}_{c_i} h^{d_i}_{b_i} \ldots h^{d_i}_{b_i} T^{a_1...a_k}_{b_1...b_l}
\]

projected derivative

\[
D_{a_-} \equiv h^b_a \nabla_bringe_-
\]
covariant derivative commutation

\[ \nabla_c \nabla_d T_{a_1...a_k}^{b_1...b_l} = \nabla_d \nabla_c T_{a_1...a_k}^{b_1...b_l} - \sum_{i=1}^{k} R_{ecd}^{a_i} T_{a_1...a_{i-1}e a_{i+1}...a_k}^{b_1...b_l} + \sum_{j=1}^{l} R_{b_{jcd}}^e T_{a_1...a_k}^{b_1...b_{j-1}e b_{j+1}...b_l} \]

projected tensors

\[ T_{a_1...a_k}^{b_1...b_l} \rightarrow h_{c_i}^{a_i} \cdots h_{c_i}^{a_i} h_{b_i}^{d_i} \cdots h_{b_i}^{d_i} T_{a_1...a_k}^{b_1...b_l} \]

projected derivative

\[ D_{a_--} \equiv h_{a}^{b} \nabla_{b--} \]
covariant derivative commutation

$$\nabla_c \nabla_d T^{a_1...a_k}_{b_1...b_l} = \nabla_d \nabla_c T^{a_1...a_k}_{b_1...b_l}$$

$$- \sum_{i=1}^{k} R^{a_i}_{ecd} T^{a_1...a_i-1ea_i+1...a_k}_{b_1...b_l} + \sum_{j=1}^{l} R^{e}_{bjcd} T^{a_1...a_k}_{b_1...b_j-1eb_j+1...b_l}$$

projected tensors

$$T^{a_1...a_k}_{b_1...b_l} \rightarrow h^{a_i}_{ci} \ldots h^{a_i}_{ci} h^{d_i}_{bi} \ldots h^{d_i}_{bi} T^{a_1...a_k}_{b_1...b_l}$$

projected derivative

$$D_{a-} \equiv h^{b}_{a} \nabla_{b-}$$
covariant derivative commutation

\[ \nabla_c \nabla_d T^{a_1 \ldots a_k}_{b_1 \ldots b_l} = \nabla_d \nabla_c T^{a_1 \ldots a_k}_{b_1 \ldots b_l} - \sum_{i=1}^{k} R^a_{i ec} T^{a_1 \ldots a_i-1 e a_i+1 \ldots a_k}_{b_1 \ldots b_l} + \sum_{j=1}^{l} R^e_{b jcd} T^{a_1 \ldots a_k}_{b_1 \ldots b_j-1 e b_j+1 \ldots b_l} \]

projected tensors

\[ T^{a_1 \ldots a_k}_{b_1 \ldots b_l} \rightarrow h^{a_i}_{c_i} \ldots h^{a_i}_{c_i} h^{d_i}_{b_i} \ldots h^{d_i}_{b_i} T^{a_1 \ldots a_k}_{b_1 \ldots b_l} \]

projected derivative

\[ D_a \equiv h^b_a \nabla_b \]
For Further Reading I

Robert M. Wald.  
*General Relativity.*  

Stephen W. Hawking and George F. R. Ellis.  
*The large scale structure of space-time.*  