Beyond the spectral Standard Model: Pati-Salam unification

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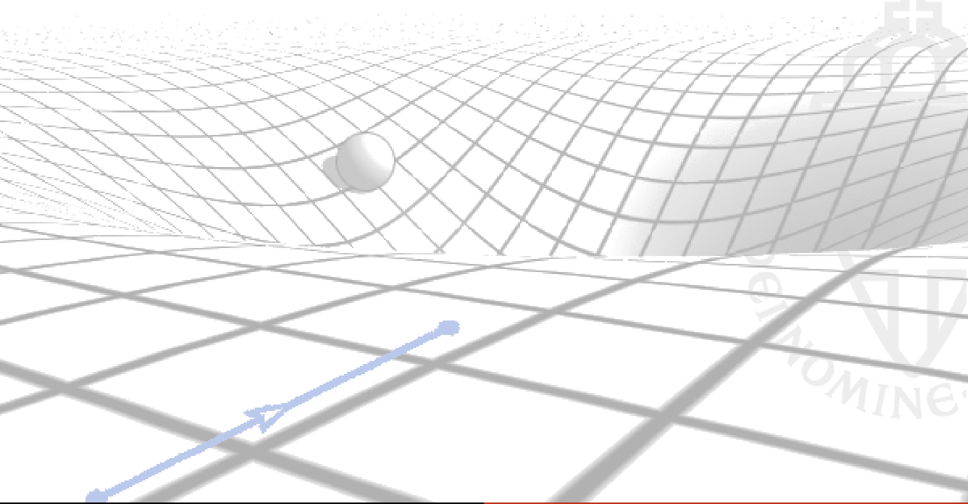
Overview

- Motivation: NCG and HEP
- Noncommutative Riemannian spin manifolds (aka spectral triples)
- Gauge theory from spectral triples: gauge group, gauge fields
- The spectral Standard Model and Beyond

A fermion in a spacetime background



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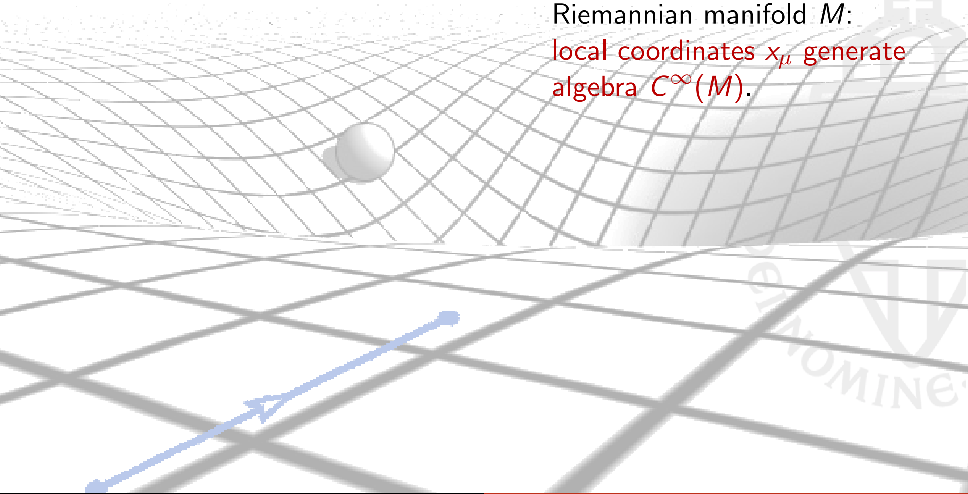

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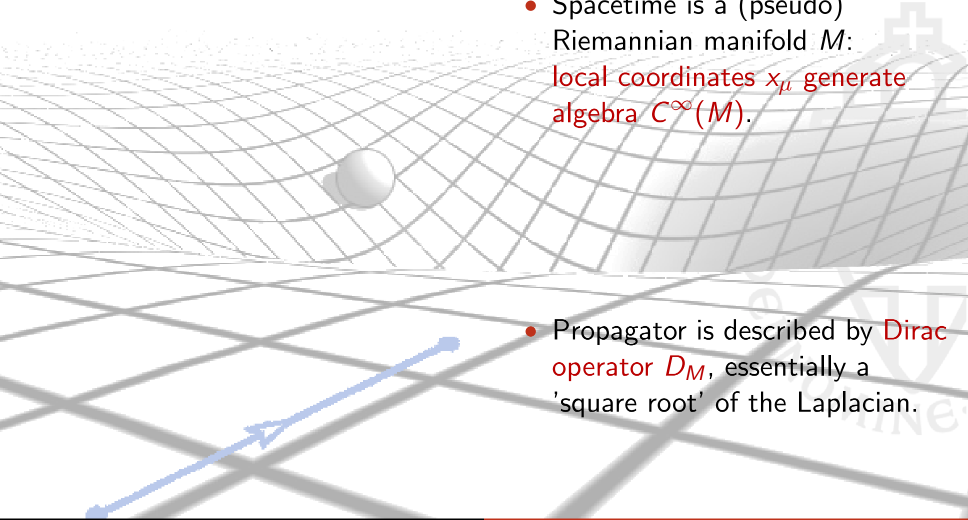

- Spacetime is a (pseudo) Riemannian manifold M: local coordinates x_μ generate algebra $C^\infty(M)$.
- Propagator is described by Dirac operator D_M, essentially a 'square root' of the Laplacian.

The circle

- The Laplacian on the circle $\mathbb{S}^1$ is given by

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$$D_{\mathbb{S}^1} = -i\frac{d}{dt}$$

with square $\Delta_{\mathbb{S}^1}$.

The 2-dimensional torus

- Consider the two-dimensional torus $\mathbb{T}^2$ parametrized by two angles $t_1, t_2 \in [0, 2\pi)$.
- The Laplacian reads

$$\Delta_{\mathbb{T}^2} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}.$$

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- At first sight it seems difficult to construct a differential operator that squares to $\Delta_{\mathbb{T}^2}$:

$$\left(a\frac{\partial}{\partial t_1} + b\frac{\partial}{\partial t_2} \right)^2 = a^2\frac{\partial^2}{\partial t_1^2} + 2ab\frac{\partial^2}{\partial t_1 \partial t_2} + b^2\frac{\partial^2}{\partial t_2^2}$$

- This puzzle was solved by Dirac who considered the possibility that a and b be complex *matrices*:

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

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- The Dirac operator on the torus is

$$D_{\mathbb{T}^2} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} \\ -\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} & 0 \end{pmatrix},$$

which satisfies $(D_{\mathbb{T}^2})^2 = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}$.

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- The relations $ij = -ji$, $ik = -ki$, *et cetera* imply that its square coincides with $\Delta_{\mathbb{T}^4}$.

Hearing the shape of a drum: motivation from math

Kac (1966), Connes (1989)

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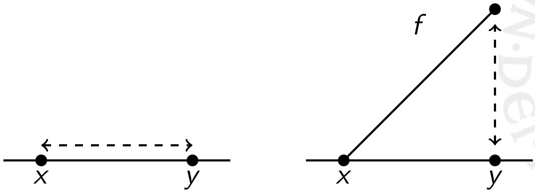

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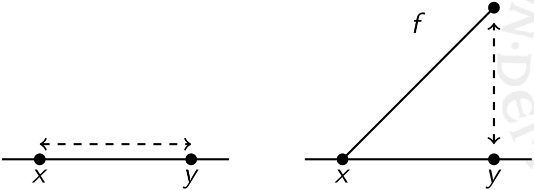

- The gradient of f is given by the commutator $[D_M, f] = D_M f - f D_M$ (*e.g.* $[D_{\mathbb{S}^1}, f] = -i\frac{df}{dt}$)

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Replace spacetime by **spacetime** $\times$ **finite (nc) space***:*
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- F is considered as internal space (Kaluza–Klein like)
- F is described by a noncommutative algebra, such as $M_3(\mathbb{C})$, just as spacetime is described by coordinate functions $x_\mu(p)$.
- 'Propagation' of particles in F is described by a Dirac-type operator D_F which is actually simply a hermitian matrix.

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- Finite space F, discrete topology

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- The finite Dirac operator is an arbitrary hermitian matrix D_F, giving rise to a distance function on F as

$$d(p,q) = \sup_{f \in C^\infty(F)} \{ |f(p) - f(q)| : \|[D_F, f]\| \leq 1 \}$$

Example: two-point space

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- Then the algebra of smooth functions

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- The distance formula then becomes

$$d(1,2) = \frac{1}{|c|}$$

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- A finite Dirac operator is still given by a hermitian matrix.

Example: **noncommutative** two-point space

The two-point space can be given a noncommutative structure by considering the algebra $\mathcal{A}_F$ of 3×3 block diagonal matrices of the following form

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix}$$

A finite Dirac operator for this example is given by a hermitian 3×3 matrix, for example

$$D_F = \begin{pmatrix} 0 & \overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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  - $J : \mathcal{H} \to \mathcal{H}$ real structure (charge conjugation)

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- D is said to satisfy first-order condition if

$$[[D, a], b^{\mathrm{op}}] = 0$$

$$\text{Trace}\, f(D/\Lambda) + \frac{1}{2}\langle J\widetilde{\psi}, D\widetilde{\psi}\rangle$$

Spectral invariants
Chamseddine–Connes (1996, 1997)

$$\text{Trace}\, f(D/\Lambda) + \frac{1}{2}\langle J\widetilde{\psi}, D\widetilde{\psi}\rangle$$

- Invariant under unitaries $u \in \mathcal{U}(\mathcal{A})$ acting as

$$D \mapsto UDU^*; \qquad U = u(u^*)^{\mathrm{op}}$$

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- Gauge group: $\mathcal{G}(\mathcal{A}) := \{u(u^*)^{\mathrm{op}} : u \in \mathcal{U}(\mathcal{A})\}$.
- Compute *rhs*:

$$D \mapsto D + u[D, u^*] \pm Ju[D, u^*]J^{-1}$$

Semigroup of inner perturbations
Chamseddine–Connes-vS (2013)

Extend this to more general perturbations:

$$\mathrm{Pert}(\mathcal{A}) := \left\{ \sum_j a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \;\middle|\; \begin{array}{l} \sum_j a_j b_j = 1 \\ \sum_j a_j \otimes b_j^{\mathrm{op}} = \sum_j b_j^* \otimes a_j^{*\mathrm{op}} \end{array} \right\}$$

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- $\mathcal{U}(\mathcal{A})$ maps to $\mathrm{Pert}(\mathcal{A})$ by sending $u \mapsto u \otimes u^{*\mathrm{op}}$.
- $\mathrm{Pert}(\mathcal{A})$ acts on D:

$$D \mapsto \sum_j a_j D b_j = D + \sum_j a_j [D, b_j]$$

and this also extends to real spectral triples via the map

$$\mathrm{Pert}(\mathcal{A}) \to \mathrm{Pert}(\mathcal{A} \otimes J\mathcal{A}J^{-1})$$

Proposition

Let $\mathcal{A}_F$ be the algebra of block diagonal matrices (fixed size). Then the perturbation semigroup of $\mathcal{A}_F$ is

$$\mathrm{Pert}(\mathcal{A}_F) \simeq \left\{ \sum_j A_j \otimes B_j \in \mathcal{A}_F \otimes \mathcal{A}_F \;\middle|\; \begin{array}{l} \sum_j A_j (B_j)^t = \mathbb{I} \\ \sum_j A_j \otimes B_j = \sum_j \overline{B_j} \otimes \overline{A_j} \end{array} \right\}$$

Perturbation semigroup for matrix algebras

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The semigroup law in $\mathrm{Pert}(\mathcal{A}_F)$ *is given by the matrix product in* $\mathcal{A}_F \otimes \mathcal{A}_F$*:*

$$(A \otimes B)(A' \otimes B') = (AA') \otimes (BB').$$

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$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

we can write an arbitrary element of $\mathrm{Pert}(\mathbb{C}^2)$ as

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- More generally, $\mathrm{Pert}(\mathbb{C}^N) \simeq \mathbb{C}^{N(N-1)/2}$ with componentwise product.

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- We can identify $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ with $M_4(\mathbb{C})$ so that elements in $\mathrm{Pert}(M_2(\mathbb{C}))$ are 4×4-matrices satisfying the normalization and self-adjointness condition. In a suitable basis:

$$\mathrm{Pert}(M_2(\mathbb{C})) = \left\{ \begin{pmatrix} 1 & v_1 & v_2 & iv_3 \\ 0 & x_1 & x_2 & ix_3 \\ 0 & x_4 & x_5 & ix_6 \\ 0 & ix_7 & ix_8 & x_9 \end{pmatrix} \middle| \begin{array}{l} v_1, v_2, v_3 \in \mathbb{R} \\ x_1, \ldots x_9 \in \mathbb{R} \end{array} \right\}$$

and one can show that

$$\mathrm{Pert}(M_2(\mathbb{C})) \simeq \mathbb{R}^3 \rtimes S.$$

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- Let us consider a noncommutative example, $\mathcal{A}_F = M_2(\mathbb{C})$.
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$$\mathrm{Pert}(M_2(\mathbb{C})) = \left\{ \begin{pmatrix} 1 & v_1 & v_2 & iv_3 \\ 0 & x_1 & x_2 & ix_3 \\ 0 & x_4 & x_5 & ix_6 \\ 0 & ix_7 & ix_8 & x_9 \end{pmatrix} \;\middle|\; \begin{array}{l} v_1, v_2, v_3 \in \mathbb{R} \\ x_1, \ldots x_9 \in \mathbb{R} \end{array} \right\}$$

  and one can show that

$$\mathrm{Pert}(M_2(\mathbb{C})) \simeq \mathbb{R}^3 \rtimes S.$$

- More generally (B.Sc. thesis Niels Neumann),

$$\mathrm{Pert}(M_N(\mathbb{C})) \simeq W \rtimes S'.$$

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- Physicists call ϕ_1 and ϕ_2 the Higgs field.
- The group of unitary block diagonal matrices is now $U(1) \times U(2)$ and an element (λ, u) therein acts as

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto \overline{\lambda} u \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Example: perturbation semigroup of a manifold

Recall, for any involutive algebra $\mathcal{A}$

$$\mathrm{Pert}(\mathcal{A}) := \left\{ \sum_j a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \;\middle|\; \begin{array}{l} \sum_j a_j b_j = 1 \\ \sum_j a_j \otimes b_j^{\mathrm{op}} = \sum_j b_j^* \otimes a_j^{*\mathrm{op}} \end{array} \right\}$$

- We can consider functions in $C^\infty(M) \otimes C^\infty(M)$ as functions of two variables in $C^\infty(M \times M)$.

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- The action of $\mathrm{Pert}(C^\infty(M))$ on the partial derivatives appearing in a Dirac operator D_M is given by

$$\frac{\partial}{\partial x_\mu} \mapsto \frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial y_\mu} f(x,y)\Big|_{y=x} =: \partial_\mu + A_\mu$$

Applications to particle physics

- Combine (4d) Riemannian spin manifold M with finite noncommutative space F:

$$M \times F$$

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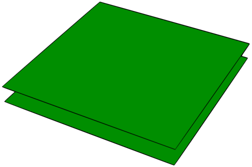
- F is internal space at each point of M



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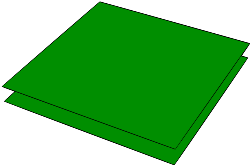

- Described by matrix-valued functions on M: algebra $C^\infty(M, \mathcal{A}_F)$

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- Using this, we can expand the heat trace:

$$\mathrm{Trace}\, e^{-D_{M\times F}^2/\Lambda^2} = \frac{\mathrm{Vol}(M)\Lambda^4}{(4\pi)^2} \mathrm{Trace}\left(1 - \frac{D_F^2}{\Lambda^2} + \frac{D_F^4}{2\Lambda^4}\right) + \mathcal{O}(\Lambda^{-1}).$$

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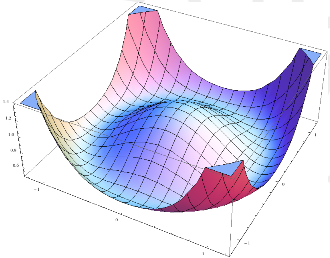
- Perturbation of finite Dirac operator D_F parametrized by ϕ_1, ϕ_2.
- Spectral action for the perturbed Dirac operator induces a potential:

$$V(\phi) = -2\Lambda^2(|\phi_1|^2 + |\phi_2|^2) + (|\phi_1|^2 + |\phi_2|^2)^2$$



The spectral Standard Model

Describe $M \times F_{SM}$ by [CCM 2007]

- Coordinates: $\hat{x}^{\mu}(p) \in \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ (with unimodular unitaries $U(1)_Y \times SU(2)_L \times SU(3)$).

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anti-particles

(ν_L, e_L) ν_R e_R (u_L, d_L) u_R d_R

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  where Y_ν, Y_e, Y_u and Y_d are 3×3 mass matrices acting on the three generations.
- The symmetric operator T only acts on the right-handed (anti)neutrinos, $T\nu_R = Y_R \overline{\nu_R}$ for a 3×3 symmetric Majorana mass matrix Y_R, and $Tf = 0$ for all other fermions $f \neq \nu_R$.

Inner perturbations

- Inner perturbations of D_M give a matrix

$$A_\mu = \begin{pmatrix} B_\mu & 0 & 0 & 0 \\ 0 & W_\mu^3 & W_\mu^+ & 0 \\ 0 & W_\mu^- & -W_\mu^3 & 0 \\ 0 & 0 & 0 & (G_\mu^a) \end{pmatrix}$$

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$$\begin{pmatrix} Y_\nu & 0 \\ 0 & Y_e \end{pmatrix} \rightsquigarrow \begin{pmatrix} Y_\nu \phi_1 & -Y_e \overline{\phi}_2 \\ Y_\nu \phi_2 & Y_e \overline{\phi}_1 \end{pmatrix}$$

corresponding to SM-Higgs field. Similarly for Y_u, Y_d.

Dynamics and interactions

If we consider the spectral action:

$$\mathrm{Trace}\, f(D_M/\Lambda) \sim c_0 \int F_{\mu\nu}F^{\mu\nu} - c_2'|\phi|^2 + c_0'|\phi|^4 + \cdots$$

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- Moreover, the quartic Higgs coupling λ is related via

$$\lambda \approx 24 \frac{3+\rho^4}{(3+\rho^2)^2} g_2^2; \qquad \rho = \frac{m_\nu}{m_{\mathrm{top}}}$$

Phenomenology of the spectral Standard Model

This can be used to derive predictions as follows:

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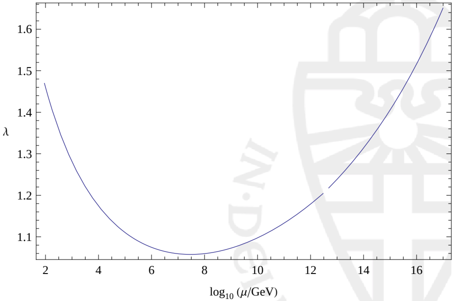

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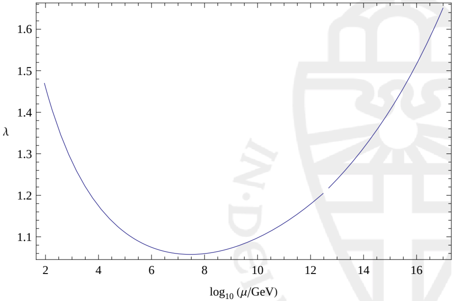

This gives [CCM 2007]

$$\boxed{167 \text{ GeV} \leq m_h \leq 176 \text{ GeV}}$$

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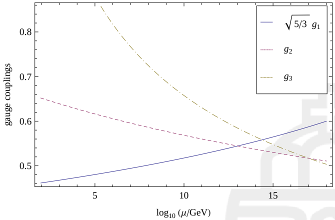

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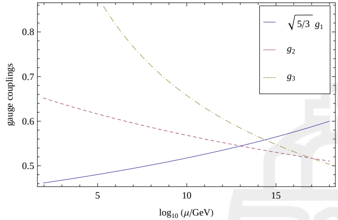
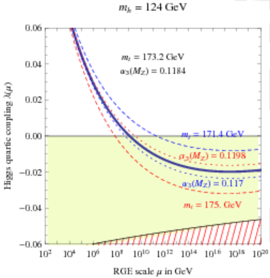
3. There is a problem with the low value of m_h, making the Higgs vacuum un/metastable [Elias-Miro et al. 2011].





- The matrix coordinates of the Standard Model arise naturally as a restriction of the following coordinates

$$\hat{x}^\mu(p) = \left(q_R^\mu(p), q_L^\mu(p), m^\mu(p)\right) \in \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$$

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- Again the finite Dirac operator is a 96×96-dimensional matrix (details in [CCS 2013]).

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$$W_R^\mu, \qquad W_L^\mu, \qquad V^\mu$$

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  - II A more general D_F with zero $\overline{f}_L - f_L$-interactions.

Scalar sector of the spectral Pati–Salam model

Case I For a SM D_F, the resulting scalar fields are composite fields, expressed in scalar fields whose representations are:

	$SU(2)_R$	$SU(2)_L$	$SU(4)$
$\phi_{\dot{a}}^{b}$	2	2	1
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Case II For a more general finite Dirac operator, we have fundamental scalar fields:

particle	$SU(2)_R$	$SU(2)_L$	$SU(4)$
$\Sigma_{\dot{a}J}^{bJ}$	2	2	$1+15$
$H_{\dot{a}I\dot{b}J}$	3	1	10
	1	1	6

Action functional

As for the Standard Model, we can compute the spectral action which describes the usual Pati–Salam model with

- unification of the gauge couplings

$$g_R = g_L = g.$$

- A rather involved, fixed scalar potential, still subject to further study

Phenomenology of the spectral Pati–Salam model

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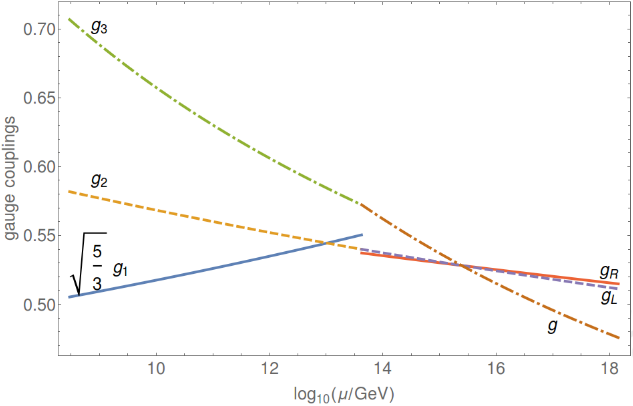
3. Vary m_R in a search for a unification scale Λ where

$$g_R = g_L = g$$

which is where the spectral action is valid as an effective theory.

For the Standard Model Dirac operator, we have found that with $m_R \approx 4.25 \times 10^{13}$ GeV there is unification at $\Lambda \approx 2.5 \times 10^{15}$ GeV:



In this case, we can also say something about the scalar particles that remain after SSB:

	$U(1)_Y$	$SU(2)_L$	$SU(3)$
$\begin{pmatrix} \phi_1^0 \\ \phi_1^+ \end{pmatrix} = \begin{pmatrix} \phi_{\dot{1}}^{\dot{1}} \\ \phi_{\dot{1}}^{\dot{2}} \end{pmatrix}$	1	2	1
$\begin{pmatrix} \phi_2^- \\ \phi_2^0 \end{pmatrix} = \begin{pmatrix} \phi_{\dot{2}}^{\dot{1}} \\ \phi_{\dot{2}}^{\dot{2}} \end{pmatrix}$	-1	2	1
σ	0	1	1
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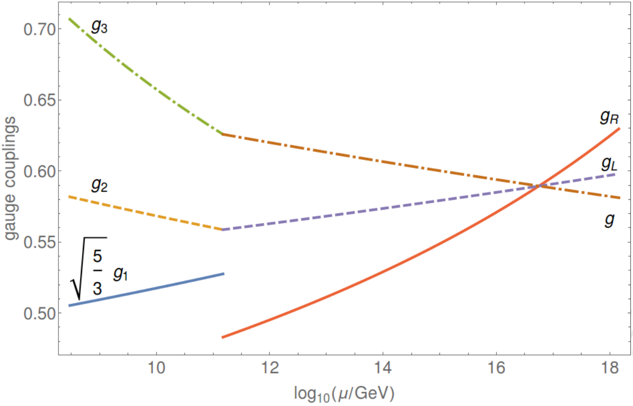
- It turns out that these scalar fields have a little influence on the running of the SM-gauge couplings (at one loop).
- However, this sector contains the real scalar singlet σ that allowed for a realistic Higgs mass and that stabilizes the Higgs vacuum [CC 2012].

For the more general case, we have found that with $m_R \approx 1.5 \times 10^{11}$ GeV there is unification at $\Lambda \approx 6.3 \times 10^{16}$ GeV:



Conclusion

We have arrived at a spectral Pati–Salam model that

- goes beyond the Standard Model
- has a fixed scalar sector once the finite Dirac operator has been fixed (only a few scenarios)
- exhibits grand unification for all of these scenarios (confirmed by [Aydemir–Minic–Sun–Takeuchi 2015])
- the scalar sector has the potential to stabilize the Higgs vacuum and allow for a realistic Higgs mass.

Further reading

A. Chamseddine, A. Connes, WvS.

Beyond the Spectral Standard Model: Emergence of Pati-Salam Unification. *JHEP* 11 (2013) 132. [arXiv:1304.8050]

Grand Unification in the Spectral Pati-Salam Model. *JHEP* 11 (2015) 011. [arXiv:1507.08161]

WvS.

Noncommutative Geometry and Particle Physics. Mathematical Physics Studies, Springer, 2015.

and also: `http://www.noncommutativegeometry.nl`