Snyder-type spacetimes, twisted Poincaré algebra and addition of momenta

Rina Štrajn
In collaboration with D. Meljanac, S. Meljanac and S. Mignemi

Ruđer Bošković Institute,
Division of Theoretical Physics

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Contents

Introduction

The Snyder model

- The first proposed version of a noncommutative spacetime
- Preserves Lorentz invariance
- Given by the commutation relations

$$\begin{aligned}
[\hat{x}_\mu, \hat{p}_\nu] &= i(\eta_{\mu\nu} + \beta^2 \hat{p}_\mu \hat{p}_\nu), \\
[\hat{x}_\mu, \hat{x}_\nu] &= i\beta^2 \hat{J}_{\mu\nu}, \\
[\hat{p}_\mu, \hat{p}_\nu] &= 0,
\end{aligned} \tag{1}$$

where $\hat{x}_\mu, \hat{p}_\mu$ and $\hat{J}_{\mu\nu}$ correspond to the generators of position, momentum and angular momenta, respectively, $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ and β is a coupling constant assumed to be of order one in Planck units.

$\hat{J}_{\mu\nu}$ satisfy the usual commutation relations

$$[\hat{J}_{\mu\nu}, \hat{J}_{\rho\sigma}] = i\left(\eta_{\nu\rho}\hat{J}_{\mu\sigma} - \eta_{\mu\rho}\hat{J}_{\nu\sigma} - \eta_{\sigma\mu}\hat{J}_{\rho\nu} + \eta_{\sigma\nu}\hat{J}_{\rho\mu}\right), \tag{2}$$

$$[\hat{J}_{\mu\nu}, \hat{p}_\mu] = i\left(\eta_{\nu\lambda}\hat{p}_\mu - \eta_{\mu\lambda}\hat{p}_\nu\right), \quad [\hat{J}_{\mu\nu}, \hat{x}_\mu] = i\left(\eta_{\nu\lambda}\hat{x}_\mu - \eta_{\mu\lambda}\hat{x}_\nu\right)$$

Snyder-type spaces

- A deformation of phase space, generated by $\hat{x}_\mu$, $\hat{p}_\mu$ and $\hat{J}_{\mu\nu}$, which satisfy

$$
\begin{aligned}
&[\hat{x}_\mu, \hat{x}_\nu] = i\beta^2 \hat{J}_{\mu\nu}\psi(\beta^2\hat{p}^2), \quad [\hat{p}_\mu, \hat{p}_\nu] = 0, \quad [\hat{p}_\mu, \hat{x}_\nu] = -i\varphi_{\mu\nu}(\beta^2\hat{p}^2) \\
&[\hat{J}_{\mu\nu}, \hat{J}_{\rho,\sigma}] = i(\eta_{\mu\nu}\hat{J}_{\nu\sigma} - \eta_{\mu\sigma}\hat{J}_{\nu\rho} + \eta_{\nu\rho}\hat{J}_{\mu\sigma} - \eta_{\nu\sigma}\hat{J}_{\mu\rho}), \\
&[\hat{J}_{\mu\nu}, \hat{p}_\lambda] = i(\eta_{\mu\nu} - \eta_{\lambda\nu}\hat{x}_\mu), \quad [\hat{J}_{\mu\nu}, \hat{x}_\lambda] = i(\eta_{\mu\nu} - \eta_{\lambda\nu}\hat{x}_\mu)
\end{aligned}
\tag{3}
$$

- $\psi(\beta^2\hat{p}^2)$, $\varphi_{\mu\nu}(\beta^2\hat{p}^2)$ - constrained by the requirement that the Jacobi identities hold
- $\psi = const. \longrightarrow$ the original Snyder model
- A realisation of $\hat{x}_\mu$, $\hat{p}_\mu$ and $\hat{J}_{\mu\nu}$ in terms of commutative coordinates x_μ and p_μ

$$
\hat{x}_\mu = x_\mu\varphi_1(\beta^2p^2) + \beta^2 x\cdot p p_\mu\varphi_2(\beta^2p^2) + \beta^2 p_\mu\chi(\beta^2p^2), \tag{4}
$$

$$
\hat{p}_\mu = p_\mu, \quad \hat{J}_{\mu\nu} \equiv J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu. \tag{5}
$$

$$
\Longrightarrow \varphi_{\mu\nu} = \eta_{\mu\nu}\varphi_1 + \beta^2 p_\mu p_\nu\varphi_2, \quad \psi = -2\varphi_1\varphi_1' + \varphi_1\varphi_2 - 2\beta^2p^2\varphi_1'\varphi_2
$$

The generalised addition of momenta, coproduct and star product

The generalised addition of momenta and the coproduct

- It can be shown that

$$e^{ik\cdot\hat{x}} \triangleright 1 = e^{iK(k)\cdot x + ig(k)} \tag{6}$$

$$e^{ik\cdot\hat{x}} \triangleright e^{iq\cdot x} = e^{i\mathcal{P}(k,q)\cdot x + i\mathcal{Q}(k,q)} \tag{7}$$

with $\mathcal{P}_\mu(k,0) = K_\mu(k), \quad \mathcal{P}_\mu(0,q) = q_\mu$

- From

$$e^{-i\lambda k\cdot\hat{x}} p_\mu e^{i\lambda k\cdot\hat{x}} \triangleright e^{iq\cdot x} = \mathcal{P}_\mu(\lambda k, q) e^{iq\cdot x} \tag{8}$$

$\Longrightarrow$

$$\frac{d\mathcal{P}_\mu(\lambda k, q)}{d\lambda} = k_\alpha \varphi^\alpha_\mu \left(\mathcal{P}(\lambda k, q)\right), \tag{9}$$

- The generalised addition of momenta is defined as

$$k_\mu \oplus q_\mu = \mathcal{D}_\mu(k,q), \tag{10}$$

where $\mathcal{D}_\mu(k,0) = k_\mu$, $\mathcal{D}_\mu(0,q) = q_\mu$, and

$$\mathcal{D}_\mu(k,q) = \mathcal{P}_\mu(K^{-1}(k), q) \tag{11}$$

- The coproduct of the momenta is defined as

$$\Delta p_\mu = \mathcal{D}_\mu(p\otimes 1, 1\otimes p). \tag{12}$$

The star product

- It can be shown that

$$e^{ik\cdot x} = e^{iK^{-1}(k)\cdot\hat{x} - ig(K^{-1}(k))} \triangleright 1 \tag{13}$$

$\Longrightarrow$ The star product of two plane waves is given by

$$\begin{aligned} e^{ik\cdot x} * e^{iq\cdot x} &= e^{iK^{-1}(k)\cdot\hat{x} - ig(K^{-1}(k))} \triangleright e^{iq\cdot x} \\ &= e^{i\mathcal{P}(K^{-1}(k),q)\cdot x + i\mathcal{Q}(K^{-1}(k),q) - ig(K^{-1}(k))} \end{aligned} \tag{14}$$

where $g(k) = \mathcal{Q}(k, 0)$

- Defining

$$\mathcal{G}(k, q) = \mathcal{Q}(K^{-1}(k), q) - \mathcal{Q}(K^{-1}(k), 0) \tag{15}$$

$\Longrightarrow$

$$e^{ik\cdot x} * e^{iq\cdot x} = e^{i\mathcal{D}(k,q)\cdot x + i\mathcal{G}(k,q)} \tag{16}$$

- It can be shown that

$$\frac{d\mathcal{Q}(\lambda k, q)}{d\lambda} = k_\alpha \chi^\alpha \left(\mathcal{P}(\lambda k, q)\right) \tag{17}$$

with $\mathcal{Q}(0, q) = 0$ and $\chi^\alpha \equiv p^\alpha \chi(\beta^2 p^2)$

The twist operator for the Snyder realisation

The Twist

- A bidifferential operator that relates the deformed and undeformed coproducts

$$\Delta p_\mu = \mathcal{F}\Delta_0 p_\mu \mathcal{F}^{-1} \tag{18}$$

- it uniquely determines the realisation of the deformed space

$$\hat{x}_\mu = m\left(\mathcal{F}^{-1}(\triangleright \otimes 1)(x_\mu \otimes 1)\right) \tag{19}$$

- defines the noncommutative star-product between functions

$$(f * g)(x) = m\left(\mathcal{F}^{-1}(\triangleright \otimes \triangleright)(f \otimes g)\right) \tag{20}$$

- It can be show that it is given by

$$\mathcal{F}^{-1} =: \exp\left\{i(1 \otimes x^\alpha)(\Delta - \Delta_0)p_\alpha + \mathcal{G}(p \otimes 1, 1 \otimes p)\right\} : \tag{21}$$

The twist operator for the Snyder space

- The Snyder realisation

$$\hat{x}_\mu = x_\mu + \beta^2 x \cdot p p_\mu \tag{22}$$

- The corresponding coproduct of the momenta

$$\Delta p_\mu = \frac{1}{1-\beta^2 p_\alpha \otimes p^\alpha}\left(p_\mu \otimes 1 - \frac{\beta^2}{1+\sqrt{1+A}} p_\mu p_\alpha \otimes p^\alpha + \sqrt{1+A} \otimes p_\mu\right), \tag{23}$$

  with $A = \beta^2 p^2$

- The coproduct is expanded with respect to the deformation parameter β^2, $\Delta p_\mu = \sum_{k=0}^{\infty} \Delta_k p_\mu$, with $\Delta_k p_\mu \propto (\beta^2)^k$
- We look for the twist operator in the form

$$\mathcal{F} = e^{f_1 + f_2 + f_3 + \cdots}, \tag{24}$$

  where $f_k \propto (\beta^2)^k$

- For each order we obtain the equation that f_k needs to satisfy

$$\begin{aligned}
&[f_1, \Delta_0 p_\mu] = \Delta_1 p_\mu, \\
&[f_2, \Delta_0 p_\mu] = \Delta_2 p_\mu - \frac{1}{2}\left[f_1, [f_1, \Delta_0 p_\mu]\right], \\
&\quad \dots
\end{aligned} \tag{25}$$

$\Longrightarrow$

$$\begin{aligned}
f_1 &= -i\beta^2 \left(p^2 \otimes x \cdot p + \frac{1}{2} p_\alpha p_\beta \otimes x^\alpha p^\beta + p_\alpha \otimes x \cdot p p^\alpha \right) \\
f_2 &= i\frac{\beta^4}{2} \left(\frac{1}{2} p^4 \otimes x \cdot p + \frac{1}{2} p_\alpha p_\beta p^2 \otimes x^\alpha p^\beta + p_\alpha p^2 \otimes x \cdot p p^\alpha \right) \\
& \quad \dots
\end{aligned} \tag{26}$$

For the closed form of the twist we get

$$\begin{aligned}
\mathcal{F} &= \exp\left\{ -i \left(\frac{1}{2} p^2 \otimes x \cdot p + \frac{1}{2} p_\alpha p_\beta \otimes x^\alpha p^\beta + p_\alpha \otimes x \cdot p p^\alpha \right) \times \right. \\
& \qquad \left. \left(\frac{ln(1+\beta^2 p^2)}{p^2} \otimes 1 \right) \right\}.
\end{aligned} \tag{27}$$

- This twist gives the right realisation of the Snyder space

$$m\left(\mathcal{F}^{-1} \triangleright x_\mu \otimes 1\right) = x_\mu + \beta^2 x \cdot p p_\mu \tag{28}$$

- An independent verification - starting from (21) $\longrightarrow$ the results agree
- For the Lorentz generators $\longrightarrow$ primitive coproduct (as it should be)

$$\Delta J_{\mu\nu} = \mathcal{F}(\Delta_0 J_{\mu\nu})\mathcal{F}^{-1} = \Delta_0 J_{\mu\nu} \tag{29}$$

- The coproduct for the Snyder space is non-co-associative $\Longrightarrow$ the twist for the Snyder space does not satisfy the cocycle condition

First order expansion of the general form

- The realisation

$$\hat{x}_\mu = x_\mu + \beta^2(s_1 x_\mu p^2 + s_2 x \cdot p p_\mu + c p_\mu) + O(\beta^4) \tag{30}$$

The commutation relations

$$\begin{aligned} [\hat{x}_\mu, \hat{x}_\nu] &= i\beta^2 s J_{\mu\nu} + O(\beta^4) \\ [p_\mu, \hat{x}_\nu] &= -i\left(\eta_{\mu\nu}(1 + \beta^2 s_1 p^2) + \beta^2 s_2 p_\mu p_\nu\right) + O(\beta^4) \end{aligned} \tag{31}$$

$s_1 = 0$, $s_2 = 1 \longrightarrow$ the exact Snyder realisation
$s_1 = -1/2$, $s_2 = 0 \longrightarrow$ the first order expansion of the Maggiore realisation
$s_2 = 2s_1$ commutative spacetime to first order in β^2

- The generalised addition of momenta

$$(k \oplus q)_\mu = \mathcal{D}_\mu(k,q) = k_\mu + q_\mu + \beta^2 \Bigg(s_2 k \cdot q q_\mu + s_1 q^2 k_\mu + \left(s_1 + \frac{s_2}{2}\right) k \cdot q k_\mu + \frac{s_2}{2} k^2 q_\mu \Bigg) + O(\beta^4) \quad (32)$$

for $s_2 = 2s_1 \neq 0$, $s = 0$, spacetime is commutative up to the first order in β^2, but the addition of momenta is deformed

$$(k \oplus q)_\mu \neq k_\mu + q_\mu \quad (33)$$

- The coproduct

$$\Delta p_\mu = \Delta_0 p_\mu + \beta^2 \Bigg(s_1 p_\mu \otimes p^2 + s_2 p_\alpha \otimes p^\alpha p_\mu + \left(s_1 + \frac{s_2}{2}\right) p_\mu p_\alpha \otimes p^\alpha + \frac{s}{2} p^2 \otimes p_\mu \Bigg) + O(\beta^4) \quad (34)$$

- The twist operator

$$\mathcal{F}^{-1} = 1 \otimes 1 + i(1 \otimes x_\alpha)(\Delta - \Delta_0)p^\alpha + ic\beta^2 p_\alpha \otimes p^\alpha + O(\beta^4) \quad (35)$$

Remarks and outlook

- In general:
  - the twist will not satisfy the cocycle condition
  - the corresponding star product will be non-associative
  - the coproducts Δp_μ, $\Delta J_{\mu\nu}$ will be non-coassociative

  exception: $s_2 = 2s_1$ (the commutative case) $\longrightarrow$ the star product is commutative and associative, but not local and the corresponding coproduct Δp_μ is cocommutative and coassociative
- Using the twist (35) to calculate the coproduct of $J_{\mu\nu} \longrightarrow$ $\Delta J_{\mu\nu} = \Delta_0 J_{\mu\nu} + O(\beta^4)$
- An important development of the work is the study of quantum field theory in Snyder spaces (free, interacting)
- A future work is the precise elaboration of the Hopf algebroid structure of the Snyder spacetime
- ...

Thank you for your attention!