Beyond the Standard Model with noncommutative geometry

Pierre Martinetti

Università di Genova

Corfou

Testing Fundamental Physics Principal

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Introduction

Noncommutative geometry [NCG] provides a common geometrical framework for the standard model of elementary particles [SM] and (Euclidean) general relativity.

Assuming space-(time) is the product of a Riemannian manifold by some "matrix geometry", then the SM Lagrangian together with Einstein-Hilbert action follow from a single action formula: the spectral action. Chamseddine, Connes 1996

Bonus: the Higgs field comes out as the noncommutative part of the connection. Its mass is a function of the other parameters of the theory, and can be calculated.

- Under the big desert hypothesis: $m_H = 170$ Gev.
- Physical motivations to question the big desert (instability in the Higgs potential).

How to go beyond the Standard Model with noncommutative geometry ?

1. **Noncommutative geometry of the Standard Model in a nutshell**

2. **Grand symmetry and twisted spectral triple**

3. **Gauge transformation**

4. **Lorentz signature**

1. The noncommutative geometry of the Standard Model in a nutshell

Gelfand duality

commutative C^*-algebras $\Longleftrightarrow$ locally compact topological spaces

- Noncommutative C^*-algebra should play the role of "functions on a noncommutative space".
- To go beyond topology (e.g. differential structure, metric, homology, integration etc) one needs more than just an algebra.

Spectral triple

A $*$-algebra $\mathcal{A}$, faithful representation on $\mathcal{H}$, operator D on $\mathcal{H}$ with compact resolvent such that $[D, a]$ is bounded for all $a \in \mathcal{A}$. With extra-conditions:

Theorem Connes 1996-2008-2013

i. $\mathcal{M}$ compact Riemann spin manifold, then

$$(C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), \partial\!\!\!/)$$

is a spectral triple, where $L^2(\mathcal{M}, S)$ is the Hilbert space of square integrable spinors on $\mathcal{M}$ while

$$\partial\!\!\!/ := -i \sum_{\mu=1}^{\dim\mathcal{M}} \gamma^\mu \nabla_\mu \quad \text{with} \quad \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{I}$$

is the Dirac operator, where $\nabla_\mu := \partial_\mu + \omega_\mu$ with ω_μ is the spin connection.

ii. $(\mathcal{A}, \mathcal{H}, D)$ a spectral triple with $\mathcal{A}$ unital commutative, then there exists a compact Riemannian spin manifold $\mathcal{M}$ such that $\mathcal{A} = C^\infty(\mathcal{M})$.

A noncommutative geometry is a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is non necessarily commutative.

$$\begin{array}{ccc} \text{commutative spectral triple} & \rightarrow & \text{noncommutative spectral triple} \\ \updownarrow & & \downarrow \\ \text{Riemannian geometry} & & \text{non-commutative geometry} \end{array}$$

The spectral triple of the Standard Model

$$\mathcal{A}_{sm} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_F, \quad \mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F, \quad D = \partial\!\!\!/ \otimes \mathbb{I}_F + \gamma \otimes D_F.$$

with $\mathcal{M}$ a closed *Riemannian* spin manifold, while

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \quad \mathcal{H}_F = \mathbb{C}^{96}, \quad D_F = \text{matrix of fermion masses.}$$

The gauge fields are obtained by fluctuation of the metric:

$$D \to D_A := D + A + \epsilon' JAJ^{-1}$$

where $\epsilon' = \pm 1$, while $A = A^*$ is a generalized 1-form, element of

$$\Omega^1_D(\mathcal{A}) := \left\{ \sum_i a_i [D, b_i],\ a_i, b_i \in \mathcal{A}_{sm} \right\}.$$

Explicitly,

$$A = \gamma \otimes H - i \sum_\mu \gamma^\mu \otimes A_\mu,$$

- H: scalar field on $\mathcal{M}$ with value in $\mathcal{A}_F$ $\to$ Higgs.
- A_μ: 1-form field with value in $Lie(U(\mathcal{A}_F))$ $\to$ gauge field.

There is a part D' of the mass matrix D_F that does not fluctuate:

$$[\gamma \otimes D', a] = 0 \qquad \forall a \in \mathcal{A}_{sm}.$$

- This became relevant after the discovery of the Higgs boson in 2012: instability in the Standard Model & wrong mass of the Higgs can be cured by turning the constant component of D' into a field. Chamseddine, Connes 2012.

How to justify this ?

- Drop out the first-order condition (Chamseddine, Connes, van Suijlekom),

$$a[D, JbJ^{-1}] = 0 \quad \forall a, b \in \mathcal{A}.$$

- Take advantage of the "over size" of the Hilbert space $\mathcal{H}$ in the spectral triple of the SM (the fermion doubling problem), so that to generate the field σ without violating the first-order condition.

2. Grand Symmetry and twisted spectral triple

Devastato, Lizzi, P.M. 2014.

The action of $C^\infty(\mathcal{M})$ on spinors is the direct sum of two representations

$$\pi(f)\psi = \begin{pmatrix} f\mathbb{I}_2 & 0_2 \\ 0_2 & f\mathbb{I}_2 \end{pmatrix} \begin{pmatrix} \psi_l \\ \psi_r \end{pmatrix}.$$

So on $L^2(\mathcal{M}, S)$ there is enough space to represent twice the algebra $C^\infty(\mathcal{M})$:

$$\pi(f,g)\psi = \begin{pmatrix} f\mathbb{I}_2 & 0_2 \\ 0_2 & g\mathbb{I}_2 \end{pmatrix} \begin{pmatrix} \psi_l \\ \psi_r \end{pmatrix}.$$

On $\mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F$, there is enough space to represent the grand algebra

$$\mathcal{A}_G := \mathcal{A}_{sm} \otimes \mathbb{C}^2 = (C^\infty(\mathcal{M}) \otimes \mathcal{A}_F) \otimes \mathbb{C}^2.$$

- $\gamma \otimes D'$ no longer commutes with $\mathcal{A}_G$: fluctuations generate the extra-field σ.
- Problem: $[\partial\!\!\!/ \otimes \mathbb{I}_F, a]$ is no longer bounded.

Actually, what is bounded is the twisted commutator

$$[\partial\!\!\!/ \otimes \mathbb{I}_F, a]_\rho := (\partial\!\!\!/ \otimes \mathbb{I}_F)\, a - \rho(a)\, (\partial\!\!\!/ \otimes \mathbb{I}_F)$$

where ρ is the flip

$$\rho((f,g)\otimes m) = (g,f)\otimes m \qquad \forall (f,g) \in C^\infty(\mathcal{M})\otimes \mathbb{C}^2,\; m \in \mathcal{A}_F.$$

One generates the required extra-scalar field (together with an additional vector field) by considering twisted fluctuations:

$$D \to D + A_\rho + \epsilon' J A_\rho J^{-1}$$

where A_ρ is in

$$\Omega^1_D(\mathcal{A}_G, \rho) := \left\{ \sum_i a_i [D, b_i]_\rho,\; a_i, b_i \in \mathcal{A}_G \right\}.$$

Twisted spectral triple: given a triple $(\mathcal{A}, \mathcal{H}, D)$, instead of asking $[D,a]$ to be bounded, one asks the boundedness of the twisted commutator Connes, Moscovici 2008

$$[D,a]_\rho := Da - \rho(a)D \qquad \text{for some} \quad \rho \in \mathrm{Aut}(\mathcal{A}).$$

- Relevant to deal with conformal transformation.
- Makes sense mathematically.

3. Gauge transformation

Non-twisted case

Fluctuations of the metric arise as a particular case of a general construction allowing to export a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ to a Morita equivalent algebra $\mathcal{B}$ (namely the case of self-Morita equivalence: $\mathcal{B} = \mathcal{A}$).

A connection is required on the $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{E}$ that implements the Morita equivalence. A gauge transformation is a change of this connection, implemented by a unitary endomorphism u of $\mathcal{E}$.

In case of self-Morita equivalence, this boils down to

$$D + A + \epsilon' JAJ^{-1} \quad \longrightarrow \quad D + A^u + \epsilon' JA_u J^{-1}$$

where u is a unitary of $\mathcal{A}$ and

$$A^u := u[D, u^*] + uAu^*.$$

Equivalently, a gauge transformation is the conjugate action of $U := uJuJ^{-1}$:

$$UD_A U^* = D + A^u + \epsilon' JA^u J^{-1}.$$

Twisted case

Landi, PM 2017

$$D + A_\rho + J\,A_\rho\,J^{-1} \quad \longrightarrow \quad D + A^u_\rho + \epsilon'\,J\,A^u_\rho J^{-1}$$

where

$$A^u_\rho := \rho(u)\,[D, u^*]\rho + \rho(u)\,A\,u^*.$$

Furthermore,

$$D_{A^u_\rho} = \rho(U) D_{A_\rho} U^{-1} \quad \text{for} \quad U = \mathrm{Ad(u)}.$$

The law of transformation of the twisted-gauge potential

$$A_\rho \to A^u_\rho$$

is simply the twisted version of the usual transformation $A \to A^u$. The same is true for the conjugate action of U.

- However, usual gauge transformations

$$D \to UDU^*$$

  preserve selfadjointness of the Dirac operator, whereas $D_{A^u_\rho}$ has no reason to be selfadjoint, even if D_{A_ρ} is.

4. Lorentz signature

in progress with Devastato, Farnsworth and Lizzi.

$\mathcal{H}$ an Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and ρ an automorphism of $\mathcal{B}(\mathcal{H})$.

Definition

A ρ-twisted inner product $\langle \cdot, \cdot \rangle_\rho$ is an inner product on $\mathcal{H}$ such that

$$\langle \Psi, \mathcal{O}\Phi \rangle_\rho = \langle \rho(\mathcal{O})^\dagger \Psi, \Phi \rangle_\rho \qquad \forall \mathcal{O} \in \mathcal{B}(\mathcal{H}),\ \Psi,\ \Phi \in \mathcal{H},$$

where † is the adjoint with respect to the initial inner product. We denote

$$\mathcal{O}^+ := \rho(\mathcal{O})^\dagger.$$

the ρ-adjoint of $\mathcal{O}$.

- The ρ-twisted inner product is non necessarily definite positive.

If ρ an inner automorphism of $\mathcal{B}(\mathcal{H})$,

$$\rho(\mathcal{O}) = R\mathcal{O}R^\dagger \qquad \forall \mathcal{O} \in \mathcal{B}(\mathcal{H})$$

for a unitary operator R on $\mathcal{H}$, then a natural ρ-product is

$$\langle \Psi, \Phi \rangle_\rho = \langle \Psi, R\Phi \rangle.$$

In the twisted spectral triple of the Standard Model, the flip ρ is an inner automorphism of $\mathcal{B}(L^2(\mathcal{M}, S))$, with $R = \gamma^0$ the first Dirac matrix.

- The ρ-twisted inner product is the Krein product for the space of spinors on a Lorentzian manifold.

- Furthermore, extending ρ to the whole of $\mathcal{B}(L^2(\mathcal{M}, S))$, one finds

$$\rho(\gamma^0) = \gamma^0, \quad \rho(\gamma^j) = -\gamma^j \quad \text{for} \quad j = 1, 2, 3.$$

  The flip is the square of the Wick rotation

$$W(\gamma^0) = \gamma^0, \quad W(\gamma^j) = i\gamma^j.$$

  that is $\rho = W^2$.

- Krein selfadjointness is preserved by twisted fluctuations.

Conclusion

- SM is obtained as the (untwisted) vacuum of higher symmetry theory, described by a twisted spectral triple.
- The field $\boldsymbol{\sigma}$ together with an additional vector field X_μ encode small excitations around this vacuum.
- Similar result as the one obtained by Chamseddine, Connes and van Suijlekom by considering "fluctuations without first-order condition".
- Fluctuations of the metric and gauge transformations straightforwardly generalized to twisted spectral triples.
- Twisted gauge transformations do not preserve selfadjointness, but do preserve Krein-adjointness of Lorentzian spinors.

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