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Curved momentum spaces from quantum groups with cosmological constant

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Quantum Spacetime and Physics Models (Corfu)

Overview

1. Introduction
2. The κ-dS Poisson-Hopf algebra
3. Momentum space for the κ-dS Poisson-Hopf algebra
4. Concluding remarks

Introduction

Nontrivial geometry on momentum space

The idea that the **momentum space** (and not only spacetime) could have a **nontrivial geometry** has a long history.

- Originally proposed by Max Born[1]
- A general feature of Doubly Special Relativity theories[2], where the Planck energy is a second relativistic invariant generating curvature in momentum space
- In (2+1)D the effective description of quantum gravity coupled to point particles is given by a theory with curved momentum space and noncommutative spacetime coordinates[3]

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[1] Born M, *Proc. R. Soc. Lond.*, (1938).
[2] Amelino-Camelia G, *Phys. Lett. B*, (2001).
Kowalski-Glikman J, Nowak S, *Class. Quant. Grav.*, (2003).
[3] Matschull H J, Welling M, *Class. Quant. Grav.*, (1998).

- Directly related with the results here presented are the models in which momentum space is generated by coordinates associated to the generators of the Lie algebras of symmetries of spacetimes[4]
- Here we will generalize previous results to the case in which **spacetime itself is curved** and construct explicitly the momentum space using physically adapted coordinates which allow us to give a physical interpretation of the results
- Finally we give a **geometrical description of our momentum space** allowing us to obtain deformed relation dispersions

[4] Kowalski-Glikman J, *Int. J. Mod. Phys. A*, (2013).

The κ-dS Poisson-Hopf algebra

The (2+1) dS algebra

Poisson-Lie brackets of the undeformed (2+1)-dS algebra

$$\{J, P_i\} = \epsilon_{ij} P_j, \qquad \{J, K_i\} = \epsilon_{ij} K_j, \qquad \{J, P_0\} = 0,$$
$$\{P_i, K_j\} = -\delta_{ij} P_0, \qquad \{P_0, K_i\} = -P_i, \qquad \{K_1, K_2\} = -J,$$
$$\{P_0, P_i\} = -\Lambda\, K_i, \qquad \{P_1, P_2\} = \Lambda\, J,$$

where $i, j = 1, 2$, and ϵ_{ij} is a skew-symmetric tensor with $\epsilon_{12} = 1$.

Quadratic Casimir functions

$$\mathcal{C} = P_0^2 - \mathbf{P}^2 - \Lambda(J^2 - \mathbf{K}^2), \qquad \mathcal{W} = -JP_0 + K_1 P_2 - K_2 P_1,$$

where $\mathbf{P}^2 = P_1^2 + P_2^2$ and $\mathbf{K}^2 = K_1^2 + K_2^2$.

The undeformed Hopf algebra structure is given by the coproduct:
$\Delta_0(X_i) = X_i \otimes 1 + 1 \otimes X_i$.

Deforming: The (2+1) $\kappa-$dS algebra

Poisson-Lie brackets of the (2+1) $\kappa-$dS algebra

$$\{J,P_0\}=0, \qquad \{J,P_1\}=P_2, \qquad \{J,P_2\}=-P_1,$$

$$\{J,K_1\}=K_2, \qquad \{J,K_2\}=-K_1, \qquad \{K_1,K_2\}=-\frac{\sin(2z\sqrt{\Lambda}J)}{2z\sqrt{\Lambda}},$$

$$\{P_0,P_1\}=-\Lambda\,K_1, \qquad \{P_0,P_2\}=-\Lambda\,K_2, \qquad \{P_1,P_2\}=\Lambda\,\frac{\sin(2z\sqrt{\Lambda}J)}{2z\sqrt{\Lambda}},$$

$$\{K_1,P_0\}=P_1, \qquad \{K_2,P_0\}=P_2,$$

$$\{P_2,K_1\}=z\,(P_1P_2-\Lambda K_1K_2) \qquad \{P_1,K_2\}=z\,(P_1P_2-\Lambda K_1K_2)\,,$$

$$\{K_1,P_1\}=\frac{1}{2z}\left(\cos(2z\sqrt{\Lambda}J)-e^{-2zP_0}\right)+\frac{z}{2}\left(P_2^2-P_1^2\right)-\frac{z\Lambda}{2}\left(K_2^2-K_1^2\right)$$

$$\{K_2,P_2\}=\frac{1}{2z}\left(\cos(2z\sqrt{\Lambda}J)-e^{-2zP_0}\right)+\frac{z}{2}\left(P_1^2-P_2^2\right)-\frac{z\Lambda}{2}\left(K_1^2-K_2^2\right)$$

Quadratic deformed Casimir function $\mathcal{C}_z$

$$\mathcal{C}_z=\frac{2}{z^2}\left[\cosh(zP_0)\cos(z\sqrt{\Lambda}J)-1\right]-e^{zP_0}\left(\mathbf{P}^2-\Lambda\,\mathbf{K}^2\right)\cos(z\sqrt{\Lambda}\,J)-2\,\Lambda\,e^{zP_0}\,\frac{\sin(z\sqrt{\Lambda}J)}{\sqrt{\Lambda}}\,R_3,$$

with $R_3=\epsilon_{3bc}K_bP_c$.

Hopf algebra structure

Compatible coproduct

$$
\begin{aligned}
\Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \qquad \Delta(J) = J \otimes 1 + 1 \otimes J, \\
\Delta(P_1) &= P_1 \otimes \cos(z\sqrt{\Lambda}J) + e^{-zP_0} \otimes P_1 + \Lambda\, K_2 \otimes \frac{\sin(z\sqrt{\Lambda}J)}{\sqrt{\Lambda}}, \\
\Delta(P_2) &= P_2 \otimes \cos(z\sqrt{\Lambda}J) + e^{-zP_0} \otimes P_2 - \Lambda\, K_1 \otimes \frac{\sin(z\sqrt{\Lambda}J)}{\sqrt{\Lambda}}, \\
\Delta(K_1) &= K_1 \otimes \cos(z\sqrt{\Lambda}J) + e^{-zP_0} \otimes K_1 + P_2 \otimes \frac{\sin(z\sqrt{\Lambda}J)}{\sqrt{\Lambda}}, \\
\Delta(K_2) &= K_2 \otimes \cos(z\sqrt{\Lambda}J) + e^{-zP_0} \otimes K_2 - P_1 \otimes \frac{\sin(z\sqrt{\Lambda}J)}{\sqrt{\Lambda}}
\end{aligned}
$$

Limit $\Lambda \to 0$

- $\kappa-$dS reduces to $\kappa-$Poincaré in the limit $\Lambda \to 0$
- $\{P_0, P_1, P_2\}$ forms an **Abelian Poisson-Hopf algebra**
- Also, $\mathcal{C}_z = \mathcal{C}_z(P_0, P_1, P_2)$ and it can be interpreted as a **modified dispersion relation** (hopefully observable!) in a curved momentum space

$\Lambda \neq 0$

When $\Lambda \neq 0$ Lorentz generators $\{K_1, K_2, J\}$ get intertwined with translations generators $\{P_0, P_1, P_2\}$, so it was not clear how to extend the construction to curved spacetimes.

Our proposal

Here we propose to **enlarge the momentum space** to include the coordinates associated to boosts in addition to the ones associated to the canonical momenta.

Momentum space for the κ-dS Poisson-Hopf algebra

Sketch of the construction of momentum space

- Applying the **quantum duality principle**
- Calculate the dual Lie algebra to the κ-dS algebra
- Calculate the full dual quantum group to the κ-dS algebra
- Define a linear action of this dual quantum group in a Minkowski space
- Consider the orbit of a certain point (origin of the momentum space)

The (2+1) κ-dS Poisson-Hopf algebra

Denoting by $\{X^0, X^1, X^2, L^1, L^2, R\}$ the generators dual to, respectively, $\{P_0, P_1, P_2, K_1, K_2, J\}$, the Lie brackets defining the Lie algebra g^* of the dual Poisson-Lie group G^*_Λ are

$$
\begin{aligned}
&[X^0, X^1] = -z\, X^1, && [X^0, X^2] = -z\, X^2, && [X^1, X^2] = 0,\\
&[X^0, L^1] = -z\, L^1, && [X^0, L^2] = -z\, L^2, && [L^1, L^2] = 0,\\
&[R, X^2] = -z\, L^1, && [R, L^1] = z\,\Lambda\, X^2, && [L^1, X^2] = 0,\\
&[R, X^1] = z\, L^2, && [R, L^2] = -z\,\Lambda\, X^1, && [L^2, X^1] = 0,\\
&[R, X^0] = 0, && [L^1, X^1] = 0, && [L^2, X^2] = 0.
\end{aligned}
$$

Dual Lie group

Exponentiating the dual Lie algebra we obtain

$$
\begin{aligned}
G^*_\Lambda = {} & \exp\left(\theta\rho(R)\right) \exp\left(p_1\rho(X^1)\right) \exp\left(p_2\rho(X^2)\right)\\
& \exp\left(\chi_1\rho(L^1)\right) \exp\left(\chi_2\rho(L^2)\right) \exp\left(p_0\rho(X^0)\right)
\end{aligned}
$$

where $\rho : g^* \to \mathrm{M}(6, \mathbb{R})$ is a faithful real representation of g^*.

Dual Poisson-Hopf structure

From the composition law of the dual group we have

$$
\begin{aligned}
\Delta(p_0) &= p_0 \otimes 1 + 1 \otimes p_0, \qquad \Delta(\theta) = \theta \otimes 1 + 1 \otimes \theta, \\
\Delta(p_1) &= p_1 \otimes \cos(z\sqrt{\Lambda}\,\theta) + e^{-zp_0} \otimes p_1 + \Lambda\,\chi_2 \otimes \frac{\sin(z\sqrt{\Lambda}\,\theta)}{\sqrt{\Lambda}}, \\
\Delta(p_2) &= p_2 \otimes \cos(z\sqrt{\Lambda}\,\theta) + e^{-zp_0} \otimes p_2 - \Lambda\,\chi_1 \otimes \frac{\sin(z\sqrt{\Lambda}\,\theta)}{\sqrt{\Lambda}}, \\
\Delta(\chi_1) &= \chi_1 \otimes \cos(z\sqrt{\Lambda}\,\theta) + e^{-zp_0} \otimes \chi_1 + p_2 \otimes \frac{\sin(z\sqrt{\Lambda}\,\theta)}{\sqrt{\Lambda}}, \\
\Delta(\chi_2) &= \chi_2 \otimes \cos(z\sqrt{\Lambda}\,\theta) + e^{-zp_0} \otimes \chi_2 - p_1 \otimes \frac{\sin(z\sqrt{\Lambda}\,\theta)}{\sqrt{\Lambda}}.
\end{aligned}
$$

Note that this coproduct coincides with the one for the original Hopf algebra under the identification:
$\{p_0 \to P_0, p_1 \to P_1, p_2 \to P_2, \chi_1 \to K_1, \chi_2 \to K_2, \theta \to J\}$

Geometric interpretation of the momentum space

Consider the left linear action of G^*_Λ on a 6 dimensional Minkowski space $G^*_\Lambda \triangleright \mathbb{R}^{1,5}$. Then the orbit of the point $(0,0,0,0,0,1) \in \mathbb{R}^{1,5}$ is given by $(S_0, S_1, S_2, S_3, S_4, S_5)$, where

$$
\begin{aligned}
S_0 &= \sinh(zp_0) + \frac{1}{2} e^{z\,p_0} z^2 \left(p_1^2 + p_2^2 + \Lambda\left(\chi_1^2 + \chi_2^2\right)\right),\\
S_1 &= e^{z\,p_0} z \left(\cos(z\sqrt{\Lambda}\,\theta)\, p_1 - \sqrt{\Lambda}\sin(z\sqrt{\Lambda}\,\theta)\,\chi_2\right),\\
S_2 &= e^{z\,p_0} z \left(\cos(z\sqrt{\Lambda}\,\theta)\, p_2 + \sqrt{\Lambda}\sin(z\sqrt{\Lambda}\,\theta)\,\chi_1\right),\\
S_3 &= e^{z\,p_0} z \left(-\sin(z\sqrt{\Lambda}\,\theta)\, p_2 + \sqrt{\Lambda}\cos(z\sqrt{\Lambda}\,\theta)\,\chi_1\right),\\
S_4 &= e^{z\,p_0} z \left(\sin(z\sqrt{\Lambda}\,\theta)\, p_1 + \sqrt{\Lambda}\cos(z\sqrt{\Lambda}\,\theta)\,\chi_2\right),\\
S_5 &= \cosh(zp_0) - \frac{1}{2} e^{z\,p_0} z^2 \left(p_1^2 + p_2^2 + \Lambda\left(\chi_1^2 + \chi_2^2\right)\right),
\end{aligned}
$$

and they satisfy the conditions

$$
-S_0^2 + S_1^2 + S_2^2 + S_3^2 + S_4^2 + S_5^2 = 1 \text{ and } S_0 + S_5 = e^{z\,p_0} > 0
$$

which is the defining relation for (half of) the (4+1)-dimensional dS space M_{dS} embedded in $\mathbb{R}^{1,5}$.

Geometric interpretation of the momentum space

- The **topology of the dual Lie group** is $\mathbb{R}^5 \times S^1$
- The **rotation subgroup** is the **stabilizer** of the origin of momentum space
- The **projection of the Casimir** operator $\mathcal{C}_z$ to M_{dS} can be interpreted as a distance from the origin of momentum space, thus providing a **geometrical interpretation of deformed dispersion relations** (in the spirit of relative locality)

Concluding remarks

- We have presented here the explicit construction of the curved momentum space related with the (2+1) κ-dS deformation
- In the same manner the momentum space related with the (2+1) κ-AdS and the (1+1) κ-(A)dS deformation can be constructed[5]
- The same procedure can be applied to the (3+1)D case[6], where some subtleties related with the $\kappa-$deformation have to be taken into account
- The approach presented is completely general and can be employed for any other deformation

[5]Ballesteros A, Gubitosi G, G-S, Herranz F J, *Physics Letters B*, (2017).
[6]Ballesteros A, Gubitosi G, G-S, Herranz F J, *In preparation*.