Gravity as a gauge theory in non-commutative spaces

George Manolakos

National Technical University of Athens

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Joint work (in progress) with: G. Zoupanos, A. Chatzistavrakidis, L. Jonke, P. Manousselis

1. Gravity as gauge theory in 3+1 and 2+1 dimensions
2. Gauge theory on nc spaces
3. Application on the nc dS3 ~~and fuzzy sphere~~

Gravity in four dimensions as a gauge theory

Use of alternative approach of GR - vielbein formulation

$\downarrow$

4-d gravity described as a gauge theory of the Poincare group $\mathrm{ISO}(3,1)$

$\downarrow$

consists of 10 generators:

see for details Kibble - Stelle '85, Utiyama '56, McDowell-Mansuri '79

- 4 of local translations P_a
- 6 Lorentz transformations M_{ab}

The generators satisfy the commutation relations:

$$[M_{ab}, M_{cd}] = \eta_{ac}M_{db} - \eta_{bc}M_{da} - \eta_{ad}M_{cb} + \eta_{bd}M_{ca}$$
$$[P_a, M_{bc}] = \eta_{ab}P_c - \eta_{ac}P_b$$
$$[P_a, P_b] = 0$$

where $\eta_{ab} = \mathrm{diag}(-1, +1, +1, +1)\,.$

- Gauging: For each generator $\rightsquigarrow$ introduction of a gauge field:
    - Vielbein ${e_\mu}^a$ corresponding to translations
    - Spin connection ${\omega_\mu}^{ab}$ corresponding to Lorentz transformations

- Therefore, the gauge connection is expanded as:

$$A_\mu = {e_\mu}^a(x)P_a + \frac{1}{2}{\omega_\mu}^{ab}(x)M_{ab}$$

- A_μ transforms in the adjoint rep:

$$\delta A_\mu = \partial_\mu \epsilon + [A_\mu, \epsilon]\,,$$

where ϵ is a parameter valued in $\mathbf{iso}(3,1)$:

$$\epsilon = \xi^a(x)P_a + \frac{1}{2}\lambda^{ab}(x)M_{ab}$$

- The transformations of the gauge fields, e, ω are:

$$\delta e_\mu{}^a = \partial_\mu \xi^a - e_\mu{}^b \lambda^a{}_b + \omega_\mu{}^{ab} \xi_b$$
$$\delta \omega_\mu{}^{ab} = \partial_\mu \lambda^{ab} - \lambda^a{}_c \omega_\mu{}^{cb} + \lambda^b{}_c \omega_\mu{}^{ca}$$

- Curvature tensors are obtained using the standard formula:

$$R_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

- Writing $R_{\mu\nu} = R_{\mu\nu}{}^a P_a + \frac{1}{2} R_{\mu\nu}{}^{ab} M_{ab}$, we obtain:

$$R_{\mu\nu}{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + e_\mu{}^b \omega_{\nu b}{}^a - e_\nu{}^b \omega_{\mu b}{}^a$$
$$R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} - \omega_\mu{}^{cb} \omega_\nu{}^a{}_c + \omega_\mu{}^{ac} \omega_{\nu c}{}^b$$

Action is needed to complete the picture:

- Built out of Poincare invariants
- Analogy with Y-M theory suggests an action of the form:

$$\mathcal{S} = \int d^4\xi R_{ab}{}^{cd} R^{ab}{}_{cd}$$

- The right choice is:

$$\mathcal{S}_E = \frac{1}{16\pi G}\int d^4x\sqrt{-g} R_{ab}{}^{ab} \,,$$

which can be written as:

$$\mathcal{S}_E = \frac{1}{16\pi G}\int d^4x\sqrt{-g} e^{\mu}{}_a e^{\nu}{}_b (\partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\mu{}^{ac}\omega_{\nu c}{}^{b} - \omega_\nu{}^{ac}\omega_{\mu c}{}^{b})$$

⇝ Functional of both the vielbeins and the spin connections

⇝ First order formulation of GR equations

Varying with respect to the fields $\rightsquigarrow$ e.o.m.:

- with respect to ω $\rightsquigarrow$ torsion-free condition
  - ✓ Torsion-free condition holds when scalars coupled to gravity
  - ✗ Torsion non-zero when spinors coupled to gravity
- with respect to e $\rightsquigarrow$ Einstein field equations (no matter)

Therefore, we conclude:

- Form of Einstein action: $A^2(dA + A^2)$
- Such action does not exist in gauge theories
- Gravity *cannot be* considered as a gauge theory.

Gravity in three dimensions as a gauge theory

The Einstein action for 3-d gravity is: *Witten '88*

$$\mathcal{S} = \int_M \epsilon^{\mu\nu\rho}(e_{\mu a}(\partial_\nu \omega_\rho{}^a - \partial_\rho \omega_\nu{}^a + \epsilon_{abc}\omega_\nu{}^b \omega_\rho{}^c))$$

- Consideration of e and ω as gauge fields
- The above action, $\mathcal{S}$, is $AdA + A^3$ in general form
- Interpretation of $\mathcal{S}$ as a Chern-Simons 3-form

Commutation relations of $\mathrm{ISO}(2,1)$:

$$[J_a, J_b] = \epsilon_{abc} J^c \quad [J_a, P_b] = \epsilon_{abc} P^c \quad [P_a, P_b] = 0$$

Construction of a gauge theory for $\mathrm{ISO}(2,1)$:

- Gauge field - Lie valued one form: $A_\mu = e_\mu{}^a P_a + \omega_\mu{}^a J_a$
- Infinitesimal gauge parameter: $u = \rho^a P_a + \tau^a J_a$
- Transformation of A_μ under a gauge trans/tion: $\delta A_\mu = -D_\mu u$
- Covariant derivative: $D_\mu = \partial_\mu u + [A_\mu, u]$

- Standard procedure $\rightsquigarrow$ transformations of the fields:

$$\delta e_\mu{}^a = -\partial_\mu \rho^a - e_\mu{}^b \tau^c \epsilon_{abc} - \omega_\mu{}^b \rho^c \epsilon_{abc}$$
$$\delta \omega_\mu{}^a = -\partial_\mu \tau^a - \omega_\mu{}^b \tau^c \epsilon_{abc}$$

- Curvature tensor $\rightsquigarrow$ commutator of covariant derivatives:

$$F_{\mu\nu} = [D_\mu, D_\nu] = P_a(\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \epsilon^{abc}(\omega_{\mu b} e_{\nu c} + e_{\mu b}\omega_{\nu c})) + J_a(\partial_\mu \omega_\nu{}^a - \partial_\nu \omega_\mu{}^a + \epsilon^{abc}\omega_{\mu b}\omega_{\nu c})$$

- IF we had considered ISO$(2,1)$ gauge theory on a 4-d manifold, Y, we would form a topological invariant:

$$Tr(T^a T^b) \int F^a F^b$$

- Calculations lead to the expression:

$$\frac{1}{2}\int_Y \epsilon^{\mu\nu\rho\sigma}(\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \epsilon^{abc}(\omega_{\mu b} e_{\nu c} + e_{\mu b}\omega_{\nu c})\times (\partial_\rho \omega_{\sigma a} - \partial_\sigma \omega_{\rho a} + \epsilon_{ade}\omega_\rho{}^d \omega_\sigma{}^e)$$

- Integrand can be written as a total derivative
- Integral reduces to integral on the 3-d boundary of Y, M.
- This integral is by definition the Chern-Simons action:

$$\mathcal{S}_{CS} = \int_M \epsilon^{\mu\nu\rho}(e_{\mu a}(\partial_\nu \omega_\rho{}^a - \partial_\rho \omega_\nu{}^a + \epsilon_{abc}\omega_\nu{}^b \omega_\rho{}^c))$$

- Therefore, $\mathrm{ISO}(2,1)$ Chern - Simons action is identical to 3-d Einstein action
- 3-d gravity ***is*** a Chern - Simons gauge theory of $\mathrm{ISO}(2,1)$

3–d Gravity with cosmological constant as a gauge theory

Generalization of the previous case, with action:

$$\mathcal{S} = \int_M \epsilon^{\mu\nu\rho}(e_{\mu a}(\partial_\nu \omega_\rho{}^a - \partial_\rho \omega_\nu{}^b) + \epsilon_{abc} e_\mu{}^a \omega_\nu{}^b \omega_\rho{}^c + \frac{1}{3}\lambda \epsilon_{abc} e_\mu{}^a e_\nu{}^b e_\rho{}^c)$$

We note:

- Not Minkowski but dS or AdS depending on the sign of λ
- Their symmetry is *not* ISO$(2,1)$ but SO$(3,1)$ and SO$(2,2)$
- Since 3–d gravity $\leftrightarrow$ gauging ISO$(2,1)$ $\Rightarrow$ 3–d gravity with $\lambda \rightarrow$ gauging SO$(3,1)$ and SO$(2,2)$, respectively (?)

First thing to do, generalization of the algebra:

$$[J_a, J_b] = \epsilon_{abc} J^c \quad [J_a, P_b] = \epsilon_{abc} P^c \quad [P_a, P_b] = \lambda \epsilon_{abc} J^c$$

- Repeating the procedure of the gauging $\rightsquigarrow$ generalization of the transformations of the fields:

$$\delta e_\mu{}^a = -\partial_\mu \rho^a - e_\mu{}^b \tau^c \epsilon_{abc} - \omega_\mu{}^b \rho^c \epsilon_{abc}\,.$$
$$\delta \omega_\mu{}^a = -\partial_\mu \tau^a - \omega_\mu{}^b \tau^c \epsilon_{abc} - \lambda \epsilon^{abc} e_{\mu b} \rho_c$$

- The expression for the curvature is:

$$F_{\mu\nu} = P_a(\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \epsilon_{abc}(\omega_\mu{}^b e_\nu{}^c + \omega_\nu{}^c e_\mu{}^b)) + J_a(\partial_\mu \omega_\nu{}^a - \partial_\nu \omega_\mu{}^a + \epsilon^{abc}(\omega_{\mu b}\omega_{\nu c} + \lambda e_{\mu b} e_{\nu c}))$$

- Chern - Simons 3−form is precisely the Einstein action.
- E.o.m. $\rightsquigarrow$ vanishing of the field strength tensor:
  - vanishing of the coefficient of P_a $\Rightarrow$ ω is Levi Civita connection (torsionless condition)
  - vanishing of the coefficient of J_a $\Rightarrow$ Einstein equation with cosmological constant

Gauge theories on nc spaces

- Algebra $\mathcal{A}$ of operators $X^\mu \rightarrow$ nc space with nc coords. *Madore - Schraml - Schupp - Wess '00*
- Operators X^μ satisfy the comm relation $[X_\mu, X_\nu] = i\theta_{\mu\nu}$, $\theta_{\mu\nu}$ not specified.
- Introduction of nc gauge theories through *covariant nc coordinates*, defined as: $\mathcal{X}_\mu = X_\mu + A_\mu$, obeying a covariant gauge transformation rule: $\delta\mathcal{X}_\mu = [\epsilon, \mathcal{X}_\mu]$ ($\sim$ cov der)
- $A_\mu$transforms as: $\delta A_\mu = -[X_\mu, \epsilon] + [\epsilon, A_\mu]$ ($\sim$ gauge connection)
- A_μ is used to define an nc *covariant field strength* which defines the nc gauge theory:

$$F_{\mu\nu} = [\mathcal{X}_\mu, \mathcal{X}_\nu] - i\bar{\theta}_{\mu\nu}\,, \quad F_{\mu\nu} = [\mathcal{X}_\mu, \mathcal{X}_\nu] - C_{\mu\nu\rho}\mathcal{X}_\rho\,,$$

cases of constant and linear noncommutativity.

- Gauge theory could be abelian or nonabelian:
  - Abelian if ϵ is a function in $\mathcal{A}$.
  - Nonabelian if ϵ is matrix valued.

▷ *In nonabelian case, where are the gauge fields valued?*

- Let us consider the relation:

$$[\epsilon, A] = [\epsilon^A T^A, A^B T^B] = \frac{1}{2}\{\epsilon^A, A^B\}[T^A, T^B] + \frac{1}{2}[\epsilon^A, A^B]\{T^A, T^B\},$$

- Cannot restrict to a matrix algebra - last term neither 0 nor algebra element in nc.

see Prof. Castellani's lectures

- There are two options to overpass the difficulty:
  - Consider the universal enveloping algebra
  - Extending the generators and/or fixing the rep so that the anticommutators close.

▷ *We employ the second option.*

3d gravity with cosmological constant in nc

- The cov coord should accommodate info of nc vielbein and spin connection (analogy with the gauging of Poincare/(A)dS group)

Nair '03, Abe - Nair '03, Nair '06

- We consider the $3-$d case with positive λ.
- The relevant isometry group is $\mathrm{SO}(3,1)$ ($\mathrm{SL}(2,\mathbf{C})$ the corresponding spin group)
- Nonabelian group $\rightarrow$ focus on the spinor rep with generators: $\Sigma_{AB} = \frac{1}{2}\gamma_{AB} = \frac{1}{4}[\gamma_A, \gamma_B]$, $A, B = 1 \ldots 4$.

Due to the product relation:

$$\gamma_{AB}\gamma^{CD} = 2\delta^{[C}_{[B}\delta^{D]}_{A]} + 4\delta^{[C}_{[B}\gamma_{A]}{}^{D]} + i\varepsilon_{AB}{}^{CD}\gamma_5 \ ,$$

one finds the commutation and anticommutation relations:

$$[\gamma_{AB}, \gamma_{CD}] = 8\eta_{[A[C}\gamma_{D]B]}$$
$$\{\gamma_{AB}, \gamma_{CD}\} = 4\eta_{C[B}\eta_{A]D}\mathbf{1} + 2i\epsilon_{ABCD}\gamma_5$$

- γ_5 and $\mathbf{1}$ have to be included in the algebra
- Extension by these two elements $\rightarrow$ 8−dimensional algebra $\rightarrow$ $\mathrm{SL}(2, \mathbf{C})$ to $\mathrm{GL}(2, \mathbf{C})$ with generators $\{\gamma_{AB}, \gamma_5, i\mathbf{1}\}$

P. Aschieri - L. Castellani '09

- In SO(3) notation we have the generators γ_{ab} and $\gamma_a = \gamma_{a4}$ with $a = 1, \ldots, 3$. We can also define: $\widetilde{\gamma}^a = \epsilon^{abc}\gamma_{bc}$.
- Commutation and anticommutation relations for γ and $\widetilde{\gamma}$:

$$[\tilde{\gamma}^a, \tilde{\gamma}^b] = -4\epsilon^{abc}\tilde{\gamma}_c\,,\ [\gamma_a, \tilde{\gamma}_b] = -4\epsilon_{abc}\gamma^c\,,\ [\gamma_a, \gamma_b] = \epsilon_{abc}\tilde{\gamma}^c\,,\ [\gamma^5, \gamma^{AB}] = 0$$
$$\{\tilde{\gamma}^a, \tilde{\gamma}^b\} = -8\eta^{ab}\mathbf{1}\,,\ \{\gamma_a, \tilde{\gamma}^b\} = 4i\delta_a^b\gamma_5\,,\ \{\gamma_a, \gamma_b\} = 2\eta_{ab}\mathbf{1}\,,\ \{\gamma^5, \gamma^{AB}\} = 0$$

- We consider $\mathrm{GL}(2, \mathbf{C})$ as the gauge group. The covariant coordinate is:

$$\mathcal{X}_\mu = e_\mu{}^a(X) \otimes \gamma_a + \omega_\mu{}^a(X) \otimes \widetilde{\gamma}_a + \mathcal{A}_\mu \otimes i\mathbf{1} + \widetilde{A}_\mu(X) \otimes \gamma_5\ ,$$

with $\mathcal{A}_\mu = e_\mu^a X_a + A_\mu(X)$.

- Gauge parameter expands in a similar way:

$$\epsilon = \xi^a(X) \otimes \gamma_a + \lambda^a(X) \otimes \widetilde{\gamma}_a + \epsilon_0(X) \otimes i\mathbf{1} + \widetilde{\epsilon}_0(X) \otimes \gamma_5$$

- Trans of component fields derive from $\delta \mathcal{X}_\mu = [\epsilon, \mathcal{X}_\mu]$ and are:

$$\begin{aligned}
\delta e_\mu{}^a &= -i[\mathcal{A}_\mu, \xi^a] - 2\{\xi_b, \omega_{\mu c}\}\epsilon^{abc} - 2\{\lambda_b, e_{\mu c}\}\epsilon^{abc} + i[\epsilon_0, e_\mu{}^a] \\
\delta \omega_\mu{}^a &= -i[\mathcal{A}_\mu, \lambda^a] + \frac{1}{2}\{\xi_b, e_{\mu c}\}\epsilon^{abc} - 2\{\lambda_b, \omega_{\mu c}\}\epsilon^{abc} + i[\epsilon_0, \omega_\mu{}^a] \\
\delta \mathcal{A}_\mu &= -i[\mathcal{A}_\mu, \epsilon_0] - i[\xi_a, e_\mu{}^a] + 4i[\lambda_a, \omega_\mu{}^a] - i[\tilde{\epsilon}_0, \widetilde{A}_\mu] \\
\delta \widetilde{A}_\mu &= -i[\mathcal{A}_\mu, \tilde{\epsilon}_0] + 2i[\xi_a, \omega_\mu{}^a] + 2i[\lambda_a, e_\mu{}^a] + i[\epsilon_0, \widetilde{A}_\mu]
\end{aligned}$$

- If we consider: $e_\mu{}^a = \delta_\mu{}^a$, $\omega_\mu{}^a = 0$, $\tilde{A}_\mu = 0$ (Y-M limit), we obtain:

$$\delta A_\mu = -i[X_\mu, \epsilon_0] + i[\epsilon_0, A_\mu]$$

  recovering the trans rule for a nc Y-M gauge field.

- If we consider: $A_\mu = 0$, $[\mathcal{A}_\mu, f] \to \partial_\mu f$ (comm limit), we obtain field trans of $3-$d comm case.

  After a redefinition:

  $\gamma_a \to \frac{2i}{\sqrt{\lambda}} P_a, \tilde{\gamma}_a \to -4J_a, 4\lambda^a \to \lambda^a, \xi^a \frac{2i}{\sqrt{\lambda}} \to -\xi^a, e^a_\mu \to \frac{\sqrt{\lambda}}{2i} e^a_\mu, \omega^a_\mu \to -\frac{1}{4}\omega^a_\mu$

Curvature tensors are:

$$\mathcal{R}_{\mu\nu} = [\mathcal{X}_\mu, \mathcal{X}_\nu] - \epsilon_{\mu\nu\rho}\mathcal{X}_\rho$$

The curvature tensor can be expanded as:

$$\mathcal{R}_{\mu\nu} = T^a_{\mu\nu} \otimes \gamma_a + R^a_{\mu\nu} \otimes \tilde{\gamma}_a + F_{\mu\nu} \otimes i\mathbf{1} + \tilde{F}_{\mu\nu} \otimes \gamma_5$$

We end up with the various tensors:

$$T^a_{\mu\nu} = i[\mathcal{A}_\mu, e_\nu{}^a] - i[\mathcal{A}_\nu, e_\mu{}^a] - 2\{e_{\mu b}, \omega_{\nu c}\}\epsilon^{abc} - 2\{\omega_{\mu b}, e_{\nu c}\}\epsilon^{abc} - \epsilon_{\mu\nu}{}^\rho e_\rho{}^a$$

$$R^a_{\mu\nu} = i[\mathcal{A}_\mu, \omega_\nu{}^a] - i[\mathcal{A}_\nu, \omega_\mu{}^a] - 2\{\omega_{\mu b}, \omega_{\nu c}\}\epsilon^{abc} + \frac{1}{2}\{e_{\mu b}, e_{\nu c}\}\epsilon^{abc} - \epsilon_{\mu\nu}{}^\rho \omega_\rho{}^a$$

$$F_{\mu\nu} = i[\mathcal{A}_\mu, \mathcal{A}_\nu] - i[e_\mu{}^a, e_{\nu a}] + 4i[\omega_\mu{}^a, \omega_{\nu a}] - i[\widetilde{A}_\mu, \widetilde{A}_\nu] - \epsilon_{\mu\nu}{}^\rho \mathcal{A}_\rho$$

$$\widetilde{F}_{\mu\nu} = i[\mathcal{A}_\mu, \widetilde{A}_\nu] - i[\mathcal{A}_\nu, \widetilde{A}_\mu] + 2i[e_\mu{}^a, \omega_{\nu a}] + 2i[\omega_\mu{}^a, e_{\nu a}] - \epsilon_{\mu\nu}{}^\rho \widetilde{A}_\rho$$

- If we consider the comm limit: same tensors as comm case
- If we consider the Y-M limit,we obtain:

$$F_{\mu\nu} = i[\mathcal{A}_\mu, \mathcal{A}_\nu] - \epsilon_{\mu\nu}{}^\rho \mathcal{A}_\rho\,, \quad \mathcal{A}_\mu \to X_\mu + A_\mu\,,$$

field strength tensor with $\mathcal{A}$ interpreted as a cov coord.

Final step $\rightarrow$ write down the action:

$$\mathcal{S} = \frac{1}{2}\mathrm{Trtr}(\epsilon^{\mu\nu\rho}\mathcal{X}_\mu\mathcal{X}_\nu\mathcal{X}_\rho - \mathcal{X}_\mu\mathcal{X}^\mu) = \frac{1}{4}\mathrm{Trtr}(\epsilon^{\mu\nu\rho}\mathcal{X}_\mu\mathcal{R}_{\nu\rho}),$$

where

- Tr is trace over matrices
- tr is trace over the algebra

Using the following form of the algebra trace:

$$\mathrm{tr}(\gamma_a\gamma_b) = 4\eta_{ab}, \quad \mathrm{tr}(\tilde{\gamma}_a\tilde{\gamma}_b) = -16\eta_{ab},$$

it takes the form:

$$\mathcal{S} = \mathrm{Tr}\epsilon^{\mu\nu\rho}(e_{\mu a}T^a_{\nu\rho} - 4\omega_{\mu a}R^a_{\nu\rho} - \mathcal{A}_\mu F_{\nu\rho} + \tilde{A}_\mu\tilde{F}_{\nu\rho}).$$

Varying the action wrt $\mathcal{X}$, we obtain:

$$\mathcal{R}_{\nu\rho} + \frac{1}{3}\epsilon_{\nu\rho}{}^{\mu}\mathcal{X}_\mu = 0\,,$$

which decompose to the set of e.o.m.:

$$\begin{aligned} T^a_{\nu\rho} - \frac{1}{3}\epsilon_{\nu\rho}{}^{\mu}e_\mu{}^a &= 0\ , \quad R^a_{\nu\rho} - \frac{1}{3}\epsilon_{\nu\rho}{}^{\mu}\omega_\mu{}^a = 0\ , \\ F_{\nu\rho} - \frac{1}{3}\epsilon_{\nu\rho}{}^{\mu}\mathcal{A}_\mu &= 0\ , \quad \widetilde{F}_{\nu\rho} - \frac{1}{3}\epsilon_{\nu\rho}{}^{\mu}\widetilde{A}_\mu = 0\ . \end{aligned}$$

Comments

- Derive e.o.m. after variation wrt the gauge fields - expecting the same expressions
- Working on a model for gravity on a fuzzy sphere
- Next step - include matter fields
- Purpose is to learn the tools and move on to more realistic scenarios

Thank you!