New fuzzy spheres through confining potentials and energy cutoffs

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Introduction

Noncommutative space(time) algebras are introduced and studied:

- To avoid UV divergences in QFT [Snyder 1947].
- As an arena to formulate QG, inducing $\Delta x \gtrsim L_p$ predicted by QG arguments [Mead 1966, Doplicher et al 1994-95].
- As an arena for unification of interactions [Connes-Lott,....]
- ...

Fuzzy spaces are particularly appealing: a FS is a family $\mathcal{A}_{n\in\mathbb{N}}$ of *finite-dimensional* algebras such that $\mathcal{A}_n \stackrel{n\to\infty}{\longrightarrow} \mathcal{A}$ $\equiv$algebra of regular functions on an ordinary manifold.
First, seminal example: the Fuzzy Sphere (FS) of Madore [1991]: $\mathcal{A}_n \simeq M_n(\mathbb{C})$, generated by coordinates x^i $(i=1,2,3)$ fulfilling

$$[x^i,x^j] = \frac{2i}{\sqrt{n^2-1}}\varepsilon^{ijk}x^k, \quad r^2 := x^ix^i = 1, \qquad n\in\mathbb{N}\setminus\{1\}; \quad (1)$$

(1) are covariant under $SO(3)$, but not under the whole $O(3)$; in particular not under parity $x^i \mapsto -x^i$.

In fact $L^i = x^i\sqrt{n^2-1}/2$ make up the standard basis of $so(3)$ in the irrep (π_l, V_l) characterized by $L^iL^i = l(l+1)$, $l = 2n+1$. Does the FS approximate the configuration space algebra of a particle on S^2? Problems: a) parity; b) V_l is irreducible, whereas

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Here fuzzy approximations of QM on S^d $(d = 1, 2)$ solving a),b):

• Ordinary quantum particle in $\mathbb{R}^D$ $(D = d+1)$, subject to a potential $V(r)$ with a very sharp minimum on the sphere $r = 1$.

• By low enough energy-cutoff $E \leq \overline{E}$ we *'freeze' radial excitations*, make only a finite-dimensional Hilbert subspace $\mathcal{H}_{\overline{E}}$ accessible, and on it the x^i noncommutative *à la Snyder*; the x^i generate the whole algebra of observables. $O(D)$-covariant by construction.

• Making $\overline{E}$, $V''(1) \gg 0$ diverge with $\Lambda \in \mathbb{N}$ (while $E_0 = 0$), we get a sequence $\mathcal{A}_\Lambda$ of fuzzy approximations of ordinary QM on S^d.

• On $\mathcal{H}_{\overline{E}}$ the square distance $\mathcal{R}^2$ from the origin is not identically 1, but a function of L^2 which collapses to 1 in the $\Lambda \to \infty$ limit.

Remarks:

- Our construction is inspired by the Landau model: there noncommuting x, y obtained projecting QM with a strong uniform magnetic field B on the lowest energy subspace.
- *Physically sound method*, applicable to more general contexts. Imposing a cutoff $\overline{E}$ on an existing theory can be used to:
  • can yield an effective description of a system when our measurements, or the interactions with the environment, cannot bring its state to energies $E > \overline{E}$; or even
  • may be a necessity if we believe $\overline{E}$ represents the threshold for the onset of new physics not accountable by that theory.
- If H is invariant under some symmetry group, then the projection $P_{\overline{E}}$ on $\mathcal{H}_{\overline{E}}$ is invariant as well, and the projected theory will inherit that symmetry.

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General framework

Consider a quantum particle in $\mathbb{R}^D$ configuration space with Hamiltonian

$$H = -\frac{1}{2}\Delta + V(r); \qquad (3)$$

we fix the minimum $V_0 = V(1)$ of the the confining potential $V(r)$ so that the ground state has energy $E_0 = 0$. Choose an energy cutoff $\overline{E}$ fulfilling

$$V(r) \simeq V_0 + 2k\,(r-1)^2 \qquad (4)$$

if $V(r) \leq \overline{E}$; so that $V(r)$ has a harmonic behavior for $|r-1| \leq \sqrt{\frac{\overline{E}-V_0}{2k}}$.

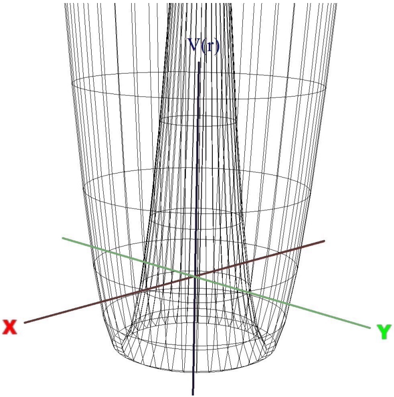
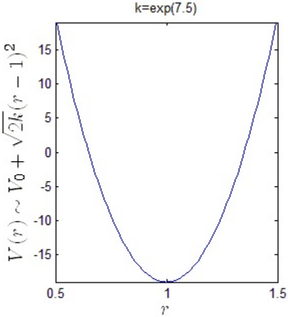


Figure 1 : Three-dimensional plot of $V(r)$

Then we restrict to $\mathcal{H}_{\overline{E}} \subset \mathcal{H} \equiv \mathcal{L}^2(\mathbb{R}^D)$ spanned by ψ with $E \leq \overline{E}$. This entails replacing every observable A by $\overline{A}$:

$$A \mapsto \overline{A} := P_{\overline{E}} A P_{\overline{E}},$$



where $P_{\overline{E}}$ is the projection on $\mathcal{H}_{\overline{E}}$. Because of the behavior of $V(r)$ as $k \to +\infty$, we expect that when both k, $\overline{E}$ diverge $\dim(\mathcal{H}_{\overline{E}})$ diverges and we recover standard QM on the sphere $\mathbb{S}^{D-1}$. The Laplacian in D dimensions decomposes as follows

$$\Delta = \partial_r^2 + (D-1)\frac{1}{r}\partial_r - \frac{1}{r^2}L^2. \qquad (5)$$

where $L_{ij} := ix^j\partial_i - ix^i\partial_j$ are the angular momentum components (in normalized units), and $L^2 = L_{ij}L_{ij}$ is the square angular momentum, i.e. the Laplacian on the sphere $\mathbb{S}^{D-1}$.

$H, L_{ij}, P_{\overline{E}}$ commute. As known, the eigenvalues of L^2 are $j(j+D-2)$; the Ansatz $\psi = f(r)Y(\varphi,...)$ (Y are eigenfunctions of L^2 and of the elements of a Cartan subalgebra of $so(D)$; $r, \varphi, ...$ are polar coordinates) transforms the eigenvalue equation $H\psi = E\psi$ into this auxiliary ODE in the unknown $f(r)$:

$$\left[-\partial_r^2 - \frac{D-1}{r}\partial_r + \frac{j(j+D-2)}{r^2} + V(r)\right] f(r) = Ef(r); \qquad (6)$$

we must stick to solutions f leading to square-integrable ψ. To obtain the lowest eigenvalues we don't need to solve it exactly: condition (4) allows us to approximate (6) with the eigenvalue equation of a $1-$dimensional harmonic oscillator, by Taylor expanding $V(r)$, $1/r$, $1/r^2$ around $r=1$.

D=2: $O(2)$-covariant fuzzy circle

For convenience we look for ψ in the form $\psi = e^{im\varphi} f(\rho)$, $\rho = \ln r$; $m \in \mathbb{Z} \equiv$ spectrum of $L \equiv L_{12}$. Expand around $\rho = 0$; the harm. osc. approx. of (6) has eigenvalues and (Hérmite) eigenfunctions

$$E = E_{n,m} = 2n\sqrt{2k} - 8n(n+1) + m^2 + O\left(1/\sqrt{k}\right) \tag{7}$$

$$\begin{aligned} f_{n,m}(\rho) &= N_{n,m} \exp\left[-\frac{(\rho-\widetilde{\rho}_{n,m})^2 \sqrt{k_{n,m}}}{2}\right] H_n\left[(\rho-\widetilde{\rho}_{n,m}) \sqrt[4]{k_{n,m}}\right], \\ k_{n,m} &= 2(k-E_{n,m}+V_0), \qquad \widetilde{\rho}_{n,m} = \frac{E_{n,m}-V_0}{k_{n,m}}, \end{aligned} \tag{8}$$

with $n \in \mathbb{N}_0$, $V_0 = -\sqrt{2k}+2+O\left(\frac{1}{\sqrt{k}}\right)$. Up to $O\left(\frac{1}{\sqrt{k}}\right)$, (7) gives

$$E_m \equiv E_{0,m} = m^2 \tag{9}$$

i.e. the eigenvalues of the Laplacian L^2 on S^1, while $E_{n,m} \to \infty$ as $k \to \infty$ if $n > 0$; can eliminate them by a cutoff $\mathbf{E} \leq \overline{\mathbf{E}} < 2\sqrt{2\mathbf{k}} - 2$.

The eigenfunctions of H corresponding to $E = E_m$ are

$$\psi_m(\rho,\varphi) = N_m e^{im\varphi} e^{-\frac{(\rho-\widetilde{\rho}_m)^2\sqrt{k_m}}{2}}.$$

Setting $\Lambda := \left[\sqrt{\overline{E}}\right]$, $E_m \le \overline{E}$ implies

$$\mathbf{m^2 \le \Lambda^2 < 2\sqrt{2k} - 2} \qquad (10)$$

so that all E_m are smaller than the energy levels corresponding to $n > 0$ (see figure). We can recover the whole spectrum of L^2 on S^1 by allowing $\sqrt{\overline{E}}$, or equivalently Λ, to diverge with k respecting (10). We abbreviate $\mathcal{H}_\Lambda \equiv \mathcal{H}_{\overline{E}}$; clearly $\dim(\mathcal{H}_\Lambda) = 2\Lambda+1$.

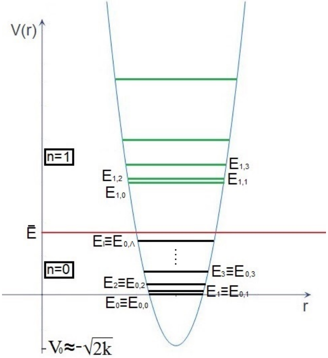


Figure 2 : Two-dimensional plot of $V(r)$ including the energy-cutoff

Let $x^\pm := \frac{x\pm iy}{\sqrt{2}} = re^{\pm i\varphi}$. By explicit computations

$$\langle\psi_n, x^\pm\psi_m\rangle = \frac{a}{\sqrt{2}}\left[1 + \frac{m(m\pm 1)}{2k}\right]\delta^n_{m\pm 1} \qquad (11)$$

with $a = 1+\frac{9}{4}\frac{1}{\sqrt{2k}}+\frac{137}{64k}+\ldots$. To get rid of a we rescale $\xi^\pm := \frac{\overline{x}^\pm}{a}$. $\overline{x}^-, \xi^-$ are resp. the adjoints of $\overline{x}^+, \xi^+$. Then, up to terms $O(1/k^{3/2})$

$$\begin{aligned}
\xi^\pm\psi_m &= \begin{cases} \frac{1}{\sqrt{2}}\left[1 + \frac{m(m\pm1)}{2k}\right]\psi_{m\pm1} & \text{if } -\Lambda \le \pm m \le \Lambda - 1 \\ 0 & \text{otherwise,} \end{cases} \qquad (12) \\
\overline{L}\,\psi_m &= m\,\psi_m.
\end{aligned}$$

Let $\mathcal{R}^2 := \xi^+\xi^- + \xi^-\xi^+$, and $\widetilde{P}_m$ be the projection over the 1-dim subspace spanned by ψ_m. Eq. (12) implies at leading order

$$[\xi^+,\xi^-] = -\frac{\overline{L}}{k} + \left[1+\frac{\Lambda(\Lambda+1)}{k}\right]\frac{\widetilde{P}_\Lambda - \widetilde{P}_{-\Lambda}}{2}. \tag{13}$$

$$\prod_{m=-\Lambda}^{\Lambda}\left(\overline{L}-mI\right) = 0, \qquad \left(\overline{L}\right)^\dagger = \overline{L}, \tag{14}$$

$$\left[\overline{L},\xi^\pm\right] = \pm\xi^\pm, \quad \xi^{+\dagger} = \xi^-, \quad \left(\xi^\pm\right)^{2\Lambda+1} = 0. \tag{15}$$

$$\mathcal{R}^2 = 1 + \frac{\overline{L}^2}{k} - \left[1+\frac{\Lambda(\Lambda+1)}{k}\right]\frac{\widetilde{P}_\Lambda + \widetilde{P}_{-\Lambda}}{2}. \tag{16}$$

Eq. (13-16) are *exact* if we adopt (12) as *definitions* of $\xi^+,\xi^-,\overline{L}$. To obtain a fuzzy space we can choose k as a function of Λ fulfilling (10), for example $k=\Lambda^2(\Lambda+1)^2$, and the commutative limit will be $\Lambda\to\infty$. Then e.g. (13) becomes

$$[\xi^+,\xi^-] = \frac{-\overline{L}}{\Lambda^2(\Lambda+1)^2} + \left[1+\frac{1}{\Lambda(\Lambda+1)}\right]\frac{\widetilde{P}_\Lambda - \widetilde{P}_{-\Lambda}}{2}. \tag{17}$$

- The matched *confining potential* and *energy cutoff* lead to a non-vanishing commutator (between the coordinates) of the Snyder's Lie algebra type (apart from the sign and the term containing the projections), i.e. proportional to L.
- $\mathcal{R}^2 \neq 1$, but its spectrum (except the highest eigenvalue) is close to 1 and collapses to 1 as $\Lambda \to \infty$.
- Relations (13-16) are $O(2)$-invariant, because in the original model both the commutation relations and H (hence also $P_{\overline{E}}$) are $O(2)$-invariant.
- The ordered monomials $(\xi^+)^h(\overline{L})^l(\xi^-)^n$ [degrees h, l, n bounded by (14-15)] make up a basis of the $(2\Lambda+1)^2$-dim vector space $\mathcal{A}_\Lambda := End(\mathcal{H}_\Lambda)$ ($\widetilde{P}_m$ can be expressed as polynomials in $\overline{L}$).

- ξ^+, ξ^- (or equivalently $\overline{x}^+, \overline{x}^-$) generate the $*$-algebra $\mathcal{A}_\Lambda$ (also $\overline{L}$ can be expressed as a non-ordered polynomial in ξ^+, ξ^-). Below we determine as an alternative set of generators E^+, E^- in the $(2\Lambda+1)$-dimensional representation of $su(2)$.
- As $\Lambda \to \infty$ $[\xi^+, \xi^-] \to 0$, $\dim(\mathcal{H}_\Lambda) \to 0$, $\psi_m \to \delta(\rho) e^{im\varphi}$.

What about $\overline{\partial}_\pm$?

As seen, they are not needed as generators of $\mathcal{A}_\Lambda$.
In fact, as expected, $\overline{\partial}_\pm$ do not go to $\partial_\pm$ as $\Lambda \to \infty$.

On the contrary, $\overline{L} \to L$; this is welcome, because in the limit $\Lambda \to \infty$ all vector fields tangential to S^1 are $\propto L$.

Realization of the algebra of observables through $Uso(3)$

$$\mathcal{A}_\Lambda := End(\mathcal{H}_\Lambda) \simeq M_N(\mathbb{C}) \simeq \pi_\Lambda[Uso(3)], \qquad N = 2\Lambda+1, \quad (18)$$

where π_Λ is the N-dimensional unitary representation of $Uso(3)$. This is characterized by the condition $\pi_\Lambda(C) = \Lambda(\Lambda+1)$, where $C = E^a E^{-a}$ is the Casimir, and E^a $(a \in \{+,0,-\})$ make up the Cartan-Weyl basis E^a of $so(3)$,

$$[E^+, E^-] = E^0, \qquad [E^0, E^\pm] = \pm E^\pm, \qquad E^{a\dagger} = E^{-a}. \quad (19)$$

To simplify notation drop π_Λ. We can realize $\xi^+, \overline{L}, \xi^-$ by setting

$$\overline{L} = E^0, \qquad \overline{\xi}^\pm = f_\pm(E^0) E^\pm,$$
$$f_+(s) = \frac{1}{\sqrt{2}} \sqrt{\frac{1+s(s-1)/k}{\Lambda(\Lambda+1)-s(s-1)}} = f_-(s-1). \quad (20)$$

Within the group $SU(N)$ of $*$-automorphisms of $M_N(\mathbb{C}) \simeq \mathcal{A}_\Lambda$

$$a \mapsto g\,a\,g^{-1}, \qquad a \in \mathcal{A}_\Lambda \simeq M_N, \quad g \in SU(N), \tag{21}$$

a special role is played by the subgroup $SO(3)$ acting through the representation π_Λ, namely $g = \pi_\Lambda\left[e^{i\alpha}\right]$, where $\alpha \in so(3)$ is a combination with real coefficients of $E^0, E^+ + E^-, i(E^- - E^+)$.

$O(2) \subset SO(3)$ as isometry group. In particular, choosing $\alpha = \theta E^0$ amounts to a rotation by an angle θ in the $\overline{x}^1\overline{x}^2$ plane: $\overline{L} \mapsto \overline{L}$ and

$$\overline{x}^{\pm} \mapsto \overline{x}'^{\pm} = e^{\pm i\theta}\overline{x}^{\pm} \qquad \Leftrightarrow \qquad \begin{cases} \overline{x}'^1 = \overline{x}^1 \cos\theta + \overline{x}^2 \sin\theta \\ \overline{x}'^2 = -\overline{x}^1 \sin\theta + \overline{x}^2 \cos\theta \end{cases}.$$

Choosing $\alpha = \pi(E^+ + E^-)/\sqrt{2}$ we obtain a $O(2)$-transformation with determinant $= -1$ in such a plane: $E^0 \mapsto -E^0$, $E^{\pm} \mapsto E^{\mp}$.
As $f_{\pm}(-s) = f_{\pm}(1+s) = f_{\mp}(s)$, this is equivalent to $\overline{x}^1 \mapsto \overline{x}^1$, $\overline{x}^2 \mapsto -\overline{x}^2$, $\overline{L} \mapsto -\overline{L}$.

D=3: $O(3)$-covariant fuzzy sphere

Ansatz $\psi = \frac{f(r)}{r} Y_l^m(\theta,\varphi)$. Y_l^m are the spherical harmonics:

$$L^2\, Y_l^m(\theta,\varphi) = l(l+1)\, Y_l^m(\theta,\varphi)\,, \qquad L_3\, Y_l^m(\theta,\varphi) = m Y_l^m(\theta,\varphi)\,,$$

$l \in \mathbb{N}_0$, $m \in \mathbb{Z}, |m| \le l$. Under assumption (4) the harmonic oscillator approximation of (6) admits the (Hérmite) eigenfunctions

$$f_{n,l}(r) = N_{n,l}\, e^{-\frac{(r-\widetilde{r}_l)^2\sqrt{k_l}}{2}} H_n\left((r-\widetilde{r}_l)\sqrt[4]{k_l}\right), \qquad n = 0, 1, \ldots.$$

where $k_l := 2k+3l(l+1)$, $\widetilde{r}_l = \frac{2k+4l(l+1)}{2k+3l(l+1)}$. $E_{0,0} = 0 \Rightarrow V_0 = -\sqrt{2k}$; then the energies associated to $\psi_{n,l,m} = \frac{f_{n,l}(r)}{r} Y_l^m(\theta,\varphi)$ are

$$E_{n,l} = 2n\sqrt{2k} + l(l+1) + O\left(1/\sqrt{2k}\right)$$

Again $E_{0,l} = l(l+1) =: E_l$ are the eigenvalues of the Laplacian L^2 on S^2, while $E_{n,l} \to \infty$ as $k \to \infty$ if $n > 0$.

We can eliminate them (constrain $n=0$) imposing a cutoff

$$\mathbf{E} \leq \boldsymbol{\Lambda}(\boldsymbol{\Lambda}+\mathbf{1}) \equiv \overline{\mathbf{E}} < \mathbf{2}\sqrt{\mathbf{2k}}, \tag{22}$$

i.e. projecting the theory on the subspace $\mathcal{H}_\Lambda \subset \mathcal{L}^2(\mathbb{R}^3)$ spanned by

$$\psi_l^m := \psi_{0,l,m} \simeq \frac{N_l}{r} e^{-\frac{(r-\widetilde{r}_l)^2\sqrt{k_l}}{2}} Y_l^m(\theta,\varphi), \quad |m| \leq l, \quad l \leq \Lambda. \tag{23}$$

Clearly $\dim(\mathcal{H}_\Lambda) = (\Lambda+1)^2$. Let $x^0 := z$, $x^\pm := \frac{x\pm iy}{\sqrt{2}}$. The action of $x^a = r\frac{x^a}{r}$ $(a=-,0,+)$ on ψ_l^m factorizes into the one of r on $\frac{f_{0,l}(r)}{r}$ and the one of $\frac{x^a}{r}$ on Y_l^m. After projection we find

$$\begin{aligned} \overline{x}^a \psi_l^m &= c_l A_l^{a,m} \psi_{l-1}^{m+a} + c_{l+1} A_{l+1}^{-a,m+a} \psi_{l+1}^{m+a}, \\ c_0 &= c_{\Lambda+1} = 0, \qquad c_l = \sqrt{1+\tfrac{l^2}{k}} \quad 1 \leq l \leq \Lambda \end{aligned} \tag{24}$$

up to $O\left(1/k^{\frac{3}{2}}\right)$, and $A_l^{a,m}, B_l^{a,m}$ are the coefficients determined by

$$\frac{x^a}{r} Y_l^m = A_l^{a,m} Y_{l-1}^{m+a} + A_{l+1}^{-a,m+a} Y_{l+1}^{m+a}.$$

The $\overline{L}_i, \overline{x}^i$, $i \in \{1,2,3\}$, fulfill

$$\prod_{l=0}^{\Lambda}\left[\overline{L}^2 - l(l+1)I\right] = 0, \qquad \prod_{m=-l}^{l}\left(\overline{L}_3 - mI\right)\widetilde{P}_l = 0, \tag{25}$$

$$\overline{x}^{i\dagger} = \overline{x}^i, \quad \overline{L}_i^\dagger = \overline{L}_i, \quad [\overline{L}_i, \overline{x}^j] = i\varepsilon^{ijh}\overline{x}^h, \quad \left[\overline{L}_i, \overline{L}_j\right] = i\varepsilon^{ijh}\overline{L}_h, \tag{26}$$

$$\overline{x}^i\overline{L}_i = 0, \qquad [\overline{x}^i, \overline{x}^j] = i\varepsilon^{ijh}\left(-\frac{1}{k} + K\widetilde{P}_\Lambda\right)\overline{L}_h \tag{27}$$

where $K = \frac{1}{k} + \frac{1+\frac{\Lambda^2}{k}}{2\Lambda+1}$, $\overline{L}^2 := \overline{L}_i\overline{L}_i = \overline{L}_a\overline{L}_{-a}$ is L^2 projected on $\mathcal{H}_\Lambda$, and $\widetilde{P}_l$ is the projection on its eigenspace with eigenvalue $l(l+1)$. Moreover, the square distance from the origin is

$$\mathcal{R}^2 := \overline{x}^i\overline{x}^i = 1 + \frac{\overline{L}^2 + 1}{k} - \left[1 + \frac{(\Lambda+1)^2}{k}\right]\frac{\Lambda+1}{2\Lambda+1}\widetilde{P}_\Lambda. \tag{28}$$

These relations are *exact* if we adopt (24) as *exact* of $\overline{x}^a$.

Again:

- $[\overline{x},\overline{x}] = ...$ and $[\overline{L},\overline{x}] = ...$ are Snyder-like: $[\overline{x},\overline{x}] = -L/k$ (plus the term containing $\widetilde{P}_\Lambda$) and vanish as $\Lambda \to \infty$.
- Hence (25-27) are covariant under the whole $O(3)$, including parity $\overline{x}_i \mapsto -\overline{x}_i$, $\overline{L}_i \mapsto \overline{L}_i$, contrary to Madore FS.
- $\mathcal{R}^2 \neq 1$, its spectrum grows with l, but collapses to 1 as $\Lambda \to \infty$.
- The ordered monomials in $\overline{x_i}, \overline{L_i}$ make up a basis of the $(\Lambda+1)^4$-dim vector space $\mathcal{A} := End(\mathcal{H}_\Lambda) \simeq M_{(\Lambda+1)^2}(\mathbb{C})$ ($\widetilde{P}_l$ can be expressed as polynomials in $\overline{L}^2$).
- Actually, $\overline{x}_i$ *generate* the $*$-algebra $\mathcal{A}$ (also the $\overline{L}_i$ can be expressed as a non-ordered polynomial in the $\overline{x}_i$).

To obtain a fuzzy space we can choose k as a function of Λ fulfilling (22); one possible choice is $k = \Lambda^2(\Lambda+1)^2$, and the commutative limit will be $\Lambda \to +\infty$.

Realization of the algebra $\mathcal{A}$ of observables through $Uso(4)$

$so(4) \simeq su(2) \oplus su(2)$ is spanned by $\left\{E_i^1, E_i^2\right\}_{i=1}^3$ fulfilling

$$[E_i^1, E_j^2] = 0, \qquad [E_i^1, E_j^1] = i\varepsilon^{ijk} E_k^1, \qquad [E_i^2, E_j^2] = i\varepsilon^{ijk} E_k^2. \quad (29)$$

$L_i := E_i^1 + E_i^2$, $X_i := E_i^1 - E_i^2$ make up alternative basis of $so(4)$:

$$[L_i, L_j] = i\varepsilon^{ijk} L_k, \qquad [L_i, X_j] = i\varepsilon^{ijk} X_k, \qquad [X_i, X_j] = i\varepsilon^{ijk} L_k. \quad (30)$$

The L_i close another $su(2)$. Passing to generators labelled by $a \in \{-, 0, +\}$,

$$[L_+, L_-] = L_0, \quad [L_0, L_\pm] = \pm L_\pm = [X_0, X_\pm], \quad [X_+, X_-] = L_0, \quad (31)$$

$$[L_\pm, X_\mp] = \pm X_0, \quad [L_0, X_\pm] = \pm X_\pm = [X_0, L_\pm], \quad [L_a, X_a] = 0 \quad (32)$$

(no sum over a), where $L^2 = L_i L_i = L_a L_{-a}$, $X^2 = X_i X_i = X_a X_{-a}$.

In the tensor product representation $\boldsymbol{\pi}_\Lambda := \pi_{\frac{\Lambda}{2}} \otimes \pi_{\frac{\Lambda}{2}}$ of $Uso(4) \simeq Usu(2) \otimes Usu(2)$ on the Hilbert space $\mathbf{V}_\Lambda := V_{\frac{\Lambda}{2}} \otimes V_{\frac{\Lambda}{2}}$ it is $C^1 := E^1_i E^1_i = \frac{\Lambda}{2}(\frac{\Lambda}{2}+1) = E^2_i E^2_i =: C^2$, or equivalently

$$X \cdot L = L \cdot X = 0, \qquad X^2 + L^2 = \Lambda(\Lambda+2) \tag{33}$$

(we have dropped the symbols $\boldsymbol{\pi}_\Lambda$). $\mathbf{V}_\Lambda$ admits an orthonormal basis consisting of common eigenvectors of L^2 and L_0:

$$L_0 \,|l,m\rangle = m\,|l,m\rangle\,, \qquad L^2\,|l,m\rangle = l(l+1)\,|l,m\rangle \tag{34}$$

with $0 \le l \le \Lambda$ and $|m| \le l$. $\mathbf{V}_\Lambda, \mathcal{H}_\Lambda$ have the same dimension $(\Lambda+1)^2$ and decomposition in irreps of the L_i subalgebra; we identify them setting $\psi_l^m \equiv |l,m\rangle$. The action of X^a on $\mathbf{V}_\Lambda$ reads

$$X^a\,|l,m\rangle = d_l A_l^{a,m}\,|l-1,m+a\rangle + d_{l+1} B_l^{a,m}\,|l+1,m+a\rangle \tag{35}$$
$$d_l := \sqrt{(\Lambda+1)^2 - l^2}$$

We can naturally realize $\overline{L}_a, \overline{x}^a$ in $\pi_\Lambda[Usu(2)\otimes Usu(2)]$.
Define $\lambda := \frac{\sqrt{4L^2+1}-1}{2}$; then $\lambda\,|l,m\rangle = l\,|l,m\rangle$. The Ansatz

$$\overline{L}_a = L_a, \qquad \overline{x}^a = g(\lambda)\,X^a\,g(\lambda), \tag{36}$$

fulfills (24) and therefore (25-27) provided

$$\begin{aligned} g(l) &= \sqrt{\frac{\prod_{h=0}^{l-1}(\Lambda+l-2h)}{\prod_{h=0}^{l}(\Lambda+l+1-2h)}\prod_{j=0}^{\left[\frac{l-1}{2}\right]}\frac{1+\frac{(l-2j)^2}{k}}{1+\frac{(l-1-2j)^2}{k}}} \\ &= \sqrt{\frac{\Gamma\left(\frac{\Lambda+l}{2}+1\right)\Gamma\left(\frac{\Lambda-l+1}{2}\right)}{\Gamma\left(\frac{\Lambda+1+l}{2}+1\right)\Gamma\left(\frac{\Lambda-l}{2}+1\right)}\,\frac{\Gamma\left(\frac{l}{2}+1+\frac{i\sqrt{k}}{2}\right)\Gamma\left(\frac{l}{2}+1-\frac{i\sqrt{k}}{2}\right)}{\sqrt{k}\,\Gamma\left(\frac{l+1}{2}+\frac{i\sqrt{k}}{2}\right)\Gamma\left(\frac{l+1}{2}-\frac{i\sqrt{k}}{2}\right)}} \end{aligned} \tag{37}$$

The inverse of (36) is clearly $X^a = [g(\lambda)]^{-1}\,\overline{x}^a\,[g(\lambda)]^{-1}$.
We have thus explicitly constructed a *-algebra map

$$\mathcal{A}_\Lambda := End(\mathcal{H}_\Lambda) \simeq M_N(\mathbb{C}) \simeq \pi_\Lambda[Uso(4)], \quad N := (\Lambda+1)^2. \tag{38}$$

As known, the group of $*$-automorphisms of $M_N(\mathbb{C}) \simeq \mathcal{A}_\Lambda$ is

$$b \to gbg^{-1}, \qquad b \in \mathcal{A}_\Lambda, \quad g \in SU(N).$$

Again a special role is played by the subgroup $SO(4)$ acting through the representation π_Λ, namely $g = \pi_\Lambda\left[e^{i\alpha}\right]$, $\alpha \in so(4)$.

$O(3) \subset SO(4)$ plays the role of isometry subgroup.

In particular, choosing $\alpha = \alpha_i L_i$ ($\alpha_i \in \mathbb{R}$) the automorphism amounts to a $SO(3)$ transf. (a rotation in 3-dimensional space).

An $O(3)$ transformation with determinant -1 in the $X^1X^2X^3$ space is parity $(L_i, X^i) \mapsto (L_i, -X^i)$, or equivalently $E_i^1 \leftrightarrow E_i^2$, the only automorphism of $so(4)$ (corresponding to the exchange of the two nodes in the Dynkin diagram).

Final remarks and conclusions

For $d=1,2$ we have built a sequence $(\mathcal{A}_\Lambda,\mathcal{H}_\Lambda)$ of finite-dim, $O(D)$-covariant ($D=d+1$) approximations of QM of a spinless particle on the sphere S^d; $\mathcal{R}^2 \gtrsim 1$ collapses to 1 as $\Lambda\to\infty$.

Achieved imposing $E\le\Lambda(\Lambda+d-1)$ on QM of a particle in $\mathbb{R}^D$ subject to a sharp confining potential $V(r)$ on the sphere $r=1$.

$\mathcal{A}_\Lambda$ are fuzzy approximations of the *whole algebra of observables* of the particle on S^d (phase space algebra).

$\mathcal{A}_\Lambda\simeq\pi_\Lambda[Uso(D+1)]$, with a suitable irrep π_Λ of $Uso(D+1)$ on $\mathcal{H}_\Lambda$.

$\mathcal{H}_\Lambda$ carries a *reducible* representation of the $Uso(D)$ subalgebra generated by the $\overline{L}_{ij}$: $\mathcal{H}_\Lambda=\bigoplus$irreps fulfilling $L^2\le\Lambda(\Lambda+d-1)$.

The same decomposition holds for the subspace $\mathcal{C}_\Lambda\subset\mathcal{A}_\Lambda$ of completely symmetrized polynomials in the $\overline{x}^i$.

As $\Lambda\to\infty$ these resp. become the decompositions (2) of $\mathcal{L}^2(S^d)$ and of $C(S^d)$ acting on $\mathcal{L}^2(S^d)$.

Approach seems applicable to $d \geq 3$ $\rightsquigarrow$ comparison with literature.

The fuzzy spheres of dimension $d = 4$ [Grosse, Klimcik, Presnajder 1996], $d \geq 3$ [Ramgoolam 2001], are based on $End(V)$ where V carries a particular *irrep* of $SO(d+1)$; $\mathcal{R}^2$ is central, can be set=1. Snyder-like commutation relations, hence $O(d+1)$-covariant.

In [Steinacker 2016-17] fuzzy 4-spheres S^4_N through reducible repr. of $Uso(5)$ obtained decomposing irreps π of $Uso(6)$ with suitable highest weights (N, n_1, n_2); so $End(V) \simeq \pi[Uso(6)]$, in analogy with our result. The elements X^i of a basis of $SO(6) \setminus SO(5)$ play the role of noncommutative cartesian coordinates.
Hence, the $SO(5)$-scalar $\mathcal{R}^2 = X^i X^i$ is no longer central, but its spectrum is still very close to 1 *only if* $N \gg n_1, n_2$;
if $n_1 = n_2 = 0$ then $\mathcal{R}^2 \equiv 1$, and one recovers the fuzzy 4-sphere [Grosse, Klimcik, Presnajder 1996].

In our approach $\mathcal{R}^2 \simeq 1$ is guaranteed by adopting $\overline{x}^i = g(L^2) X^i g(L^2)$ rather than X^i as noncommutative cartesian coordinates, $\mathcal{R}^2 = \overline{x}^i \overline{x}^i$.