Pre-NQ-manifolds and derived brackets in generalized geometry and double field theory

Andreas Deser[1], Christian Sämann[2]

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[1]Istituto Nazionale di Fisica Nucleare, Torino
[2] Department of Mathematics, Heriot-Watt University, Edinburgh

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Motivation: What is double field theory?

Canonical momenta and winding

- Sigma model $X: \Sigma \to M = T^d$

$$S = \int_\Sigma h^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j G_{ij}\, d\mu_\Sigma + \int_\Sigma X^* B ,$$

  where $h \in \Gamma(\otimes^2 T^*\Sigma)$, $G \in \Gamma(\otimes^2 TM)$, $B \in \Gamma(\wedge^2 T^* M)$.

- Classical solutions to e.o.m. (take *closed* string $\Sigma = \mathbb{R} \times S^1$)

$$X^i_R = x^i_{0R} + \alpha^i_0(\tau - \sigma) + i \sum_{n \neq 0} \frac{1}{n}\alpha^i_n e^{-in(\tau-\sigma)} , \quad X^i_L = \dots ,$$

$$\alpha^i_0 = \frac{1}{\sqrt{2}} G^{ij}\Big(p_j - (G_{jk} + B_{jk})w^k\Big) ,$$

  - p_k: Canonical momentum zero modes
  - w^k: *Winding* zero modes, $w^k := \frac{1}{2\pi}\int_0^{2\pi} \partial_\sigma X^k d\sigma$.

Motivation: What is double field theory?

Two sets of coordinates

- Two sets of momenta in $\alpha_0^i \rightarrow$ differential operators:

$$p_k \simeq \tfrac{1}{i}\partial_k \;, \quad w^k \simeq \tfrac{1}{i}\tilde{\partial}^k \;.$$

- Level matching $L_0 - \bar{L}_0$, with $L_0 = \frac{1}{2}\alpha_0^i G_{ij}\alpha_0^j + N - 1$ gives

$$N - \bar{N} = \partial_i \tilde{\partial}^i$$

- Want: If two fields obey the constraint, then also their product. Thus choose a subset, which also has:

$$\partial_k \phi \, \tilde{\partial}^k \psi + \tilde{\partial}^k \phi \, \partial_k \psi = 0 \;,$$

for all elements ϕ, ψ of the subset.

Motivation: What is double field theory?

$O(d,d)$-transformations, generalized tangent bundle, Gualtieri, Hitchin

Observation 1: The strong constraint is given by

$$\eta^{MN}\,\partial_M\phi\,\partial_N\psi = 0\,, \quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

And stays the same if we apply a constant transformation that leaves η invariant:

$$A^t\,\eta\,A = \eta \quad \text{i.e.} \quad A \in O(d,d;\mathbb{R})$$

Observation 2: This is the structure group of the generalized tangent bundle, locally isomorphic to $TM \oplus T^*M$:

- Sections: $TM \oplus T^*M \ni s = s^i\,\partial_i + s_i\,dx^i$.
- Fundamental rep of $O(d,d)$: $s^M := (s^i, s_i)$.
- Bilinear pairing η: $\langle s, t\rangle = s^i t_i + s_i t^i = \eta_{MN} s^M t^N$.

Motivation: What is double field theory?

Action of DFT, C-bracket, Hohm, Hull, Zwiebach

Observation 3:

$$S_{DFT} = \int d^{2D}x\, e^{-2d}\Big(\tfrac{1}{8}\mathcal{H}_{MN}\partial^M\mathcal{H}_{KL}\partial^N\mathcal{H}^{KL} - \tfrac{1}{2}\mathcal{H}_{MN}\partial^M\mathcal{H}_{KL}\partial^L\mathcal{H}^{KN} - 2\partial^M d\,\partial^N\mathcal{H}_{MN} + 4\mathcal{H}_{MN}\partial^M d\,\partial^N d\Big)\,.$$

Properties:

- The action has a global $O(d,d;\mathbb{R})$-symmetry, and a gauge symmetry given by applying the **generalized Lie derivative**:

$$(\delta_X\mathcal{H})_{MN} := X^P\partial_P\mathcal{H}_{MN} + (\partial_M X^P - \partial^P X_M)\mathcal{H}_{PN} + (\partial_N X^P - \partial^P X_N)\mathcal{H}_{MP}\,.$$

- The commutator of two such transformations gives the **C-bracket**:

$$[V,W]_C^M := V^K\partial_K W^M - W^K\partial_K V^M - \tfrac{1}{2}\Big(V^K\partial^M W_K - W^K\partial^M V_K\Big)\,.$$

Questions

- What is the geometric meaning of double fields, such as $d(x, \tilde{x})$, $V^M(x, \tilde{x})$, $\mathcal{H}_{MN}(x, \tilde{x})$? At least locally?
- Is there an algebraic way to understand the C-bracket?
- How does this apply to the Lie- and Courant bracket?

What is a derived bracket?

Motivation: An easy calculation...

Given a manifold M, consider $T[1]M$ with local coordinates (x^μ, ξ^μ). Its shifted cotangent bundle $T^*[1]T[1]M$ locally has $(x^\mu, \xi^\mu, \zeta_\mu, p_\mu)$ of degree $(0, 1, 0, 1)$ and is Poisson:

$$\{p_\mu, x^\nu\} = \delta_\mu^\nu \qquad \{\zeta_\mu, \xi^\nu\} = \delta_\mu^\nu \ .$$

In general, consider $T^*[n]T[1]M$ (Vinogradov algebroid of degree n). Let us call a degree $(n-1)$ object a *extended vector field*. E.g. above $X = X^\mu \zeta_\mu$.

What is a derived bracket?

Motivation: An easy calculation...

Let's take the operator $Q = \xi^\mu p_\mu$, and vector fields $X = X^\mu \zeta_\mu$, $Y = Y^\nu \zeta_\nu$, then we can do the following exercise:

$$
\begin{aligned}
\Big\{\{Q, X\}, Y\Big\} &= \Big\{\{\xi^\mu p_\mu, X^\nu \zeta_\nu\}, Y^\rho \zeta_\rho\Big\} \\
&= \Big\{\xi^\mu \partial_\mu X^\nu \zeta_\nu + X^\nu p_\nu, Y^\rho \zeta_\rho\Big\} \\
&= - Y^\rho \partial_\rho X^\nu \zeta_\nu + X^\rho \partial_\rho Y^\nu \zeta_\nu \\
&= [X, Y]^\nu_{\mathrm{Lie}} \zeta_\nu \ .
\end{aligned}
$$

We say, that the Lie bracket is a **derived bracket** (due to Kosmann-Schwarzbach, Roytenberg, Voronov).

The Courant bracket as derived bracket

Roytenberg, Weinstein

For a manifold M, take now $T^*[2]T[1]M$. Locally, coordinates are $(x^\mu, \xi^\mu, \zeta_\mu, p_\mu)$ of degrees $0, 1, 1, 2$. We get

- $\mathcal{Q} = \xi^\mu p_\mu$ squares to zero.
- Generalized vectors are degree 1 objects, i.e. $V = X^\mu \zeta_\mu + \alpha_\mu \xi^\mu$, $W = Y^\mu \zeta_\mu + \beta_\mu \xi^\mu$.
- $\mathcal{Q}$ on functions gives the de Rham differential.
- The derived bracket, i.e. $\{\{\mathcal{Q}, V\}, W\} - V \leftrightarrow W$ results in

$$[X, Y]^\mu \zeta_\mu + (L_X\beta - L_Y\alpha - \tfrac{1}{2} d(\iota_X\beta - \iota_Y\alpha))_\mu \xi^\mu$$

  i.e. we get the Courant bracket.

So we recover generalized geometry on a Courant algebroid.

More interesting: C-bracket as derived bracket

New result: Interpretation of the C-bracket

We take the same setting as before, but instead of M as base, we take T^*M, i.e. we take $T^*[2]T[1](T^*M)$. Local coordinates are now $(x^M, \xi^M, \zeta_M, p_M)$ of degree $(0, 1, 1, 2)$.

Problem: We now have too many "vectors". We solve this by defining

$$\theta^M := \tfrac{1}{\sqrt{2}}(\xi^M + \eta^{MN}\zeta_N) \quad \text{and} \quad \beta^M := \tfrac{1}{\sqrt{2}}(\xi^M - \eta^{MN}\zeta_N) \,,$$

and taking only θ^M as degree-1 coordinates. Taking

$$\left\{\theta^M, \theta^N\right\} = \eta^{MN} \,,$$

we get a pre-NQ-manifold (no time to explain...$\mathcal{Q}$ gives a sh Lie algebra).

More interesting: C-bracket as derived bracket

New result: Interpretation of the C-bracket

With this we get the following results:

- $\{\mathcal{Q}, f\} = \theta^M \partial_M f$, i.e. the de Rham on the doubled space.
- For vectors $X = X_M \theta^M$, $Y = Y_M \theta^M$ we get, using $\eta^{MN} X_M \partial_N = X^N \partial_N$, the derived bracket $\{\{\mathcal{Q}, X\}, Y\} - X \leftrightarrow Y$ gives

$$(X^M \partial_M Y_K - Y^M \partial_M X_K - \tfrac{1}{2}(Y^M \partial_K X_M - X^M \partial_K Y_M))\theta^K ,$$

i.e. the C-bracket of double field theory.

Outlook + Questions

- We gave a unified algebraic view on Lie-, Courant- and C-brackets.
- Everything was local. Global analysis? Gerbes, groupoids... What is the global description of double field theory? T-duality?
- For higher Vinogradov algebroids $\mathcal{V}_n(M)$, degree $n-1$-objects are

$$X = X^\mu \zeta_\mu + X_{\mu_1 \dots \mu_{n-1}} \xi^{\mu_1} \cdots \xi^{\mu_{n-1}} \,,$$

  i.e. sections of $TM \oplus \wedge^{n-1} T^* M$. For $n = 3$, we get the easiest case of *exceptional generalized geometry*, where the U-duality group is $SL(5, \mathbb{R})$. How about the other exceptional tangent bundles?
- What are torsion and Riemann tensors in exceptional generalized geometries? Do they have a meaning in Poisson geometry on certain Vinogradov algebroids?
- Quantization: If we can write the brackets in terms of Poisson brackets, we can do deformation quantization!

Definition 1.

A **symplectic pre-NQ-manifold of $\mathbb{N}$-degree** *n is an $\mathbb{N}$-graded manifold $\mathcal{M}$, together with symplectic form ω of degree n and a vector field Q of degree 1, satisfying $L_Q\omega = 0$.*

Examples

An important class where in addition $Q^2 = 0$, are the **Vinogradov Lie n-algebroids**:

$$\mathcal{V}_n(M) := T^*[n]T[1]M \,.$$

They have the following properties:

- Local coordinates $(x^\mu, \xi^\mu, \zeta_\mu, p_\mu)$ of degrees $0, 1, n-1, n$.
- Symplectic form $\omega = dx^\mu \wedge dp_\mu + d\xi^\mu \wedge d\zeta_\mu$
- Nilpotent vector field Q with Hamiltonian $\mathcal{Q} = \xi^\mu p_\mu$, i.e. $\{\mathcal{Q}, \mathcal{Q}\} = 0$.

Constructing the brackets

Getzler, Fiorenza, Manetti

Let $\mathcal{M}$ be a symplectic pre-NQ-manifold. Functions on the body M are degree-0 objects, i.e. $f \in C_0^\infty(\mathcal{M})$. We choose as analogue of vector fields degree $(n-1)$-objects $X \in \mathcal{C}_{n-1}^\infty(\mathcal{M})$, call them **extended vector fields**. Then define n-ary brackets by

$$\mu_1(V) = \begin{cases} \{\mathcal{Q}, V\}\,, & \text{if } V \text{ has degree } 0 \\ 0\,, & \text{otherwise} \end{cases}$$

$$\mu_2(V, W) = \tfrac{1}{2}(\{\delta V, W\} - \{\delta W, V\})$$

$$\mu_3(V, W, U) = -\tfrac{1}{12}(\{\{\delta V, W\}, U\} \pm \dots)$$

$$\dots$$

$$\text{where } \delta V := \begin{cases} \{\mathcal{Q}, V\}\,, & \text{if } V \text{ has degree } n-1 \\ 0\,, & \text{otherwise} \end{cases}$$

Conditions for L_∞-structure

If $Q^2 = 0$, the above brackets form an L_∞ structure. In our case we want to investigate conditions that this is also true, especially for $n = 2$, where we found the following

Theorem 1.

Consider the subset of $\mathcal{C}^\infty(\mathcal{M})$ consisting of functions and extended vector fiels, i.e. $\mathcal{C}_0^\infty(\mathcal{M}) \oplus \mathcal{C}_1^\infty(\mathcal{M})$. If the Poisson brackets and the maps μ_i close on this subset, the latter is an L_∞-algebra if and only if

$$\begin{aligned} \{Q^2 f, g\} + \{Q^2 g, f\} &= 0\,, \\ \{Q^2 X, f\} + \{Q^2 f, X\} &= 0\,, \\ \{\{Q^2 X, Y\}, Z\}_{[X,Y,Z]} &= 0\,, \end{aligned}$$

for all functions f, g and extended vector fields X, Y, Z. The notation $Q^2 f$ means $\{\mathcal{Q}, \{\mathcal{Q}, f\}\}$ and the subscript $[X, Y, Z]$ means the alternating sum over X, Y, Z.