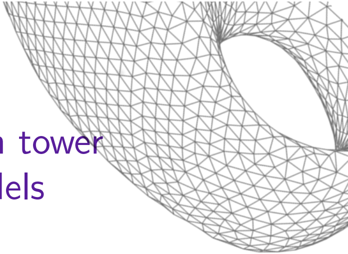
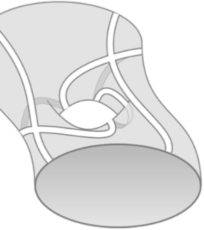


# The full Schwinger-Dyson tower for coloured tensor models



Carlos. I. Pérez-Sánchez

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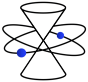
Mathematics Institute,
University of Münster

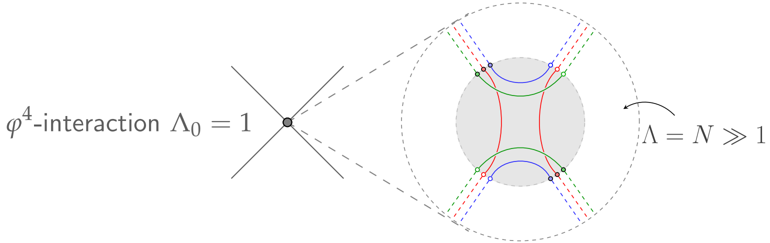
Groups, Geometry & Actions SFB 878

Corfu 17. 19 September
COST Training School, Qspace (MP 1405)



# Outline

- Non-perturbative approach to quantum (coloured) tensor fields




  - correlation functions $G_{\mathcal{B}}^{(2k)}$

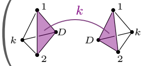
$$G_\bullet : \{\text{coloured graphs}\} \longrightarrow \text{ function space}$$

  - full Ward-Takahashi Identities
  - Schwinger-Dyson equations (joint work with Raimar Wulkenhaar)

# Motivation

- Tensor models generalize the random 2D geometry of random matrices ("Quantum Gravity")

$$\mathcal{Z} = \sum\!\!\!\!\!\!\int_{\substack{\text{topologies}\\ \text{geometries}}} \mathcal{D}[g]\mathrm{e}^{-S_{\mathrm{EH}}[g]} \sim \sum\!\!\!\!\!\!\int_{(\cdot)\in\substack{\text{topologies}\\ \text{geometries}}} \mu\left(\cdot\right)$$

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- tensor models also useful in $\mathrm{AdS}_2/\mathrm{CFT}_1$ (Gurau-Witten Sachdev-Ye–Kitaev-like model; course by V. Rivaseeau)

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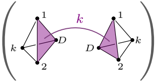
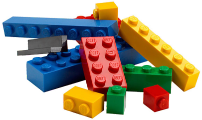



$$\left\{ 1\square 1\,,\ 2\square 2\,,\ 3\square 3 \right\} = \left\{ \quad \right\}$$



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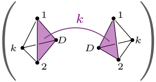


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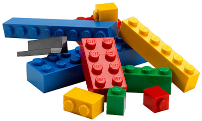

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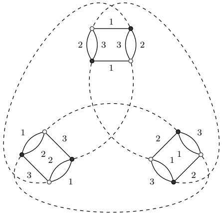


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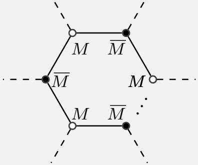
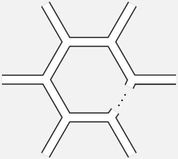
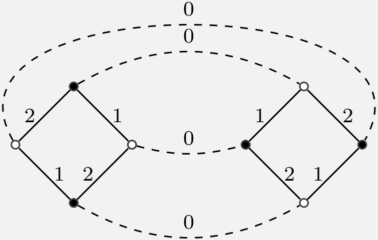
# Matrix models$^{\mathbb{C}}$ as "Rank-2 tensor models"

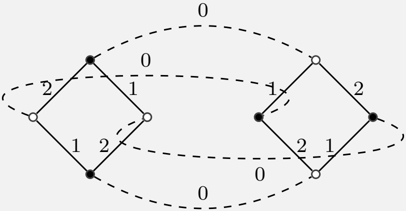
- For complex matrix models $\int \mathcal{D}[M,\overline{M}]\mathrm{e}^{-\mathrm{Tr}(MM^\dagger)-\lambda V(M,M^\dagger)}$




$=$ dual triangulation to  .

- For $V(M,\bar{M}) = \lambda \mathrm{Tr}((MM^\dagger)^2)$, different connected $\mathcal{O}(\lambda^2)$-graphs are




- rectangular matrices, $M \in \mathbb{M}_{N_1\times N_2}(\mathbb{C})$ and $M \mapsto W^{(1)}M(W^{(2)})^t$ ($W^{(a)} \in \mathrm{U}(N_a)$). $\mathrm{U}(N_1)\times\mathrm{U}(N_2)$-invariants are $\mathrm{Tr}((MM^\dagger)^q)$, $q \in \mathbb{Z}_{\geq 1}$

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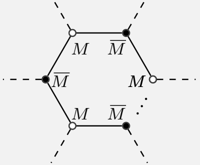


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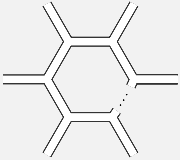

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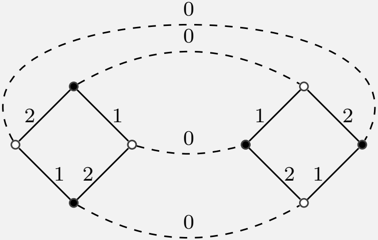


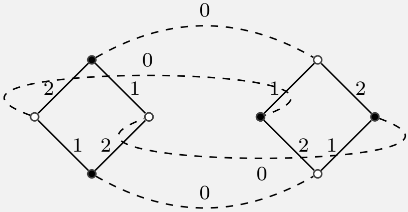


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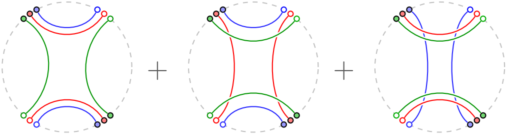
# Coloured Tensor Models

- a QFT for tensors $\varphi_{a_1\dots a_D}$ and $\overline{\varphi}_{a_1\dots a_D}$, whose indices transform *independently* under each factor of $G = \mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \dots \times \mathrm{U}(N_D)$
- for each $g = (W^{(1)}, \dots, W^{(D)}) \in G$, $W^{(a)} \in \mathrm{U}(N_a)$,

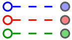
$$\varphi_{a_1 a_2 \dots a_D} \overset{g}{\mapsto} (\varphi')_{a_1 a_2 \dots a_D} = W^{(1)}_{a_1 b_1} W^{(2)}_{a_2 b_2} \dots W^{(D)}_{a_D b_D} \varphi_{b_1 \dots b_D}$$

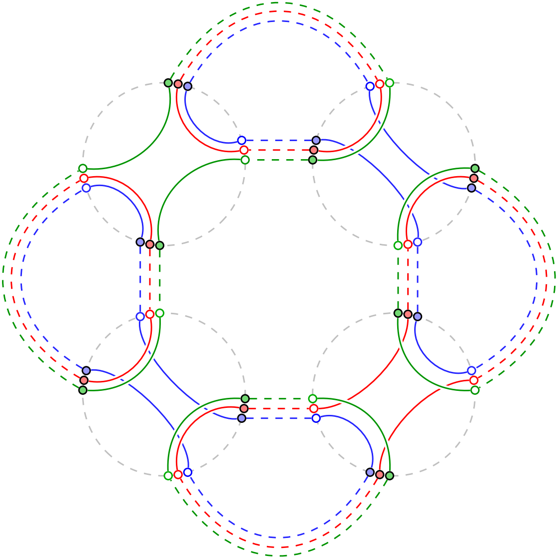
$$\overline{\varphi}_{a_1 a_2 \dots a_D} \overset{g}{\mapsto} (\overline{\varphi}')_{a_1 a_2 \dots a_D} = \overline{W}^{(1)}_{a_1 b_1} \overline{W}^{(2)}_{a_2 b_2} \dots \overline{W}^{(D)}_{a_D b_D} \overline{\varphi}_{b_1 b_2 \dots b_D}$$

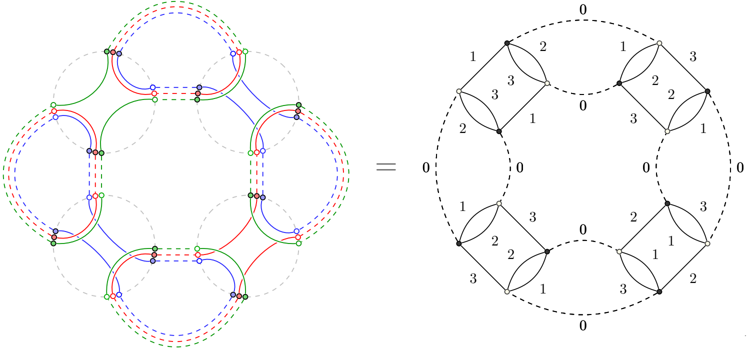
- $S_0[\varphi, \bar\varphi] = \mathrm{Tr}_2(\varphi, \bar\varphi) =$  is the kinetic term and higher $G$-invariants serve as *interaction vertices*. For instance, the $\varphi^4_3$-theory has:

$$S_{\mathrm{int}}[\varphi, \bar\varphi] = \lambda \Big($$  $$\Big)$$

- $Z = \int \mathcal{D}[\varphi, \bar\varphi]\, \mathrm{e}^{-N^2 (S_0 + S_{\mathrm{int}})[\varphi, \overline{\varphi}]}$ (with $\mathcal{D}[\varphi, \bar\varphi] = \prod_a \frac{\mathrm{d}\varphi_a \mathrm{d}\overline{\varphi}_a}{2\pi \mathrm{i}}$)

An example of $\mathcal{O}(\lambda^4)$ Feynman graph of the $\varphi_3^4$-model, where  is the propagator:





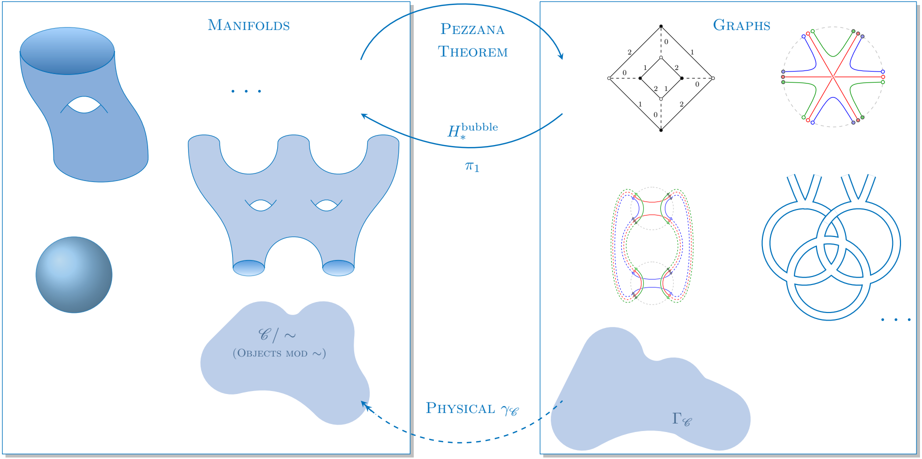
## Vertex bipartite regularly edge–$D$-coloured graphs

- (Vacuum) Feynman graphs of a model $V$, $\mathrm{Feyn}_D^{\mathrm{vac.}}(V)$, are $(D+1)$-coloured graphs. PL-manifolds can be crystallized [Pezzana, '74] by such graphs
- $1/N$-expansion

$$\mathcal{A}(\mathcal{G}) = \lambda^{V(\mathcal{G})/2} N^{\overbrace{F(\mathcal{G}) - \frac{D(D-1)}{4} V(\mathcal{G})}} = \exp(-S_{\mathrm{Regge}}[N, D, \lambda])$$

$$=: D - \frac{2}{(D-1)!}\omega(\mathcal{G}) \quad \leftsquigarrow \quad \text{generalizes } g\text{; not topol. invariant}$$

[Gurău, '09], [Bonzom, Gurău, Riello, Rivasseau, '11]




On the low-dim topology of colored tensor models without Pezzana's theorem [CP 2017, J.Geom.Phys. 120 (2017) (arXiv:1608.00246)]

# Correlation functions

- replace the expansion of $\log Z_{\mathrm{QFT}}[J]$ by the respective CTM-one

$$\log Z_{\mathrm{QFT}}[J] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}x_1 \cdots \mathrm{d}x_n \, G^{(n)}_{\mathrm{conn.}}(x_1, x_2, \ldots, x_n) J(x_1) J(x_2) \cdots J(x_n)$$

- $\mathrm{Feyn}_D(V) = \{$open Feynman diagrams of the (rank-$D$) tensor model $V\}$.
- The boundary $\partial\mathcal{G}$ of $\mathcal{G} \in \mathrm{Feyn}_D(V)$ has as vertex-set the external legs of $\mathcal{G}$ and as $a$-coloured edges $0a$-paths in $\mathcal{G}$ between them.
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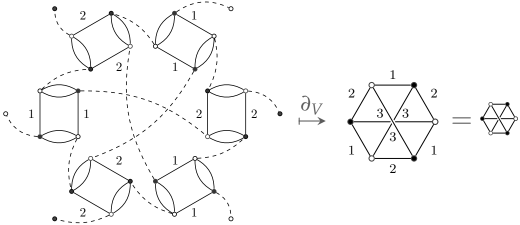


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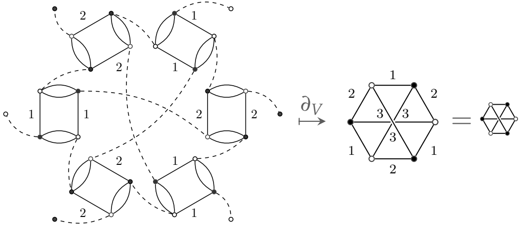
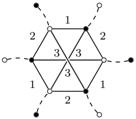



- Also $\mathcal{F} =$  has as boundary , but $\mathcal{F} \notin \mathrm{Feyn}(\varphi^4_3)$.

# Expansion of the free energy

- $\mathrm{im}\,\partial_V = \partial\,\mathsf{Feyn}_D(V)$ is the *boundary sector* of the model $V$

$$W[J,\bar{J}] = \sum_{k=1}^{\infty} \sum_{\substack{\mathcal{B}\in\partial\mathsf{Feyn}_D(V(\varphi,\bar\varphi)) \\ 2k=\#(\mathcal{B}^{(0)})}} \frac{1}{|\mathrm{Aut}_c(\mathcal{B})|} \; G^{(2k)}_{\mathcal{B}} \star \mathbb{J}(\mathcal{B}) \,.$$

- Coloured automorphisms of $\mathcal{B}$

$$(\mathbb{J}(\mathcal{B}))(\underbrace{\mathbf{x}^1,\ldots,\mathbf{x}^k}_{(\mathbb{Z}^D)^k}) = J_{\mathbf{x}^1}\cdots J_{\mathbf{x}^k}\bar{J}_{\mathbf{y}^1}\cdots\bar{J}_{\mathbf{y}^k}$$

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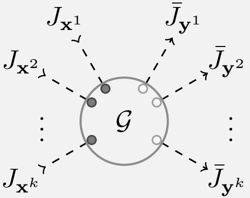
- 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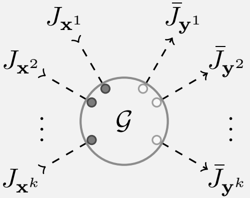


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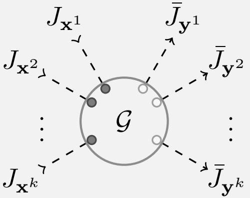
- 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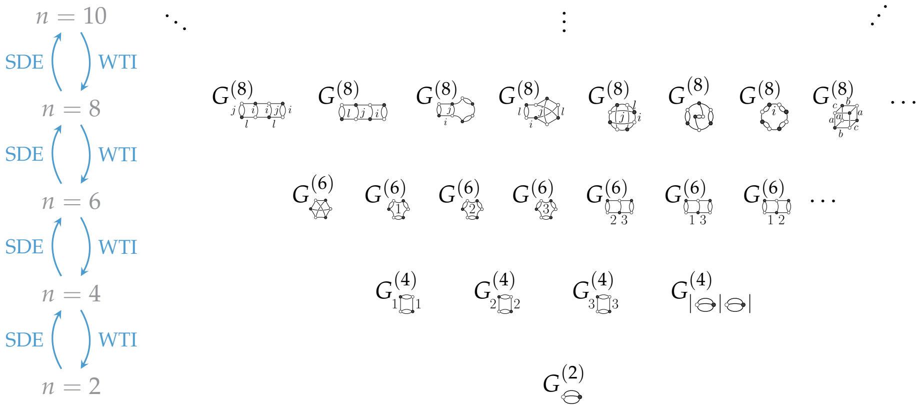



[CP](arXiv:1608.08134) [CP, R. Wulkenhaar](arXiv:1706.07358)

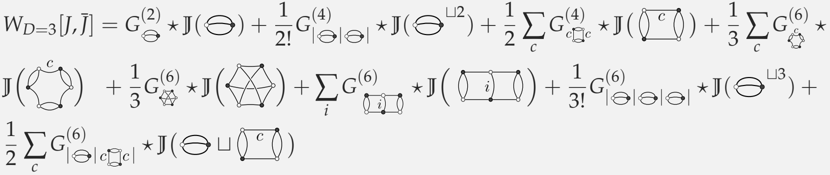
## $W_{D=3}[J,\bar{J}]$

$W_{D=3}[J,\bar{J}] = G^{(2)}$  $\star\, \mathbb{J}($  $) +$

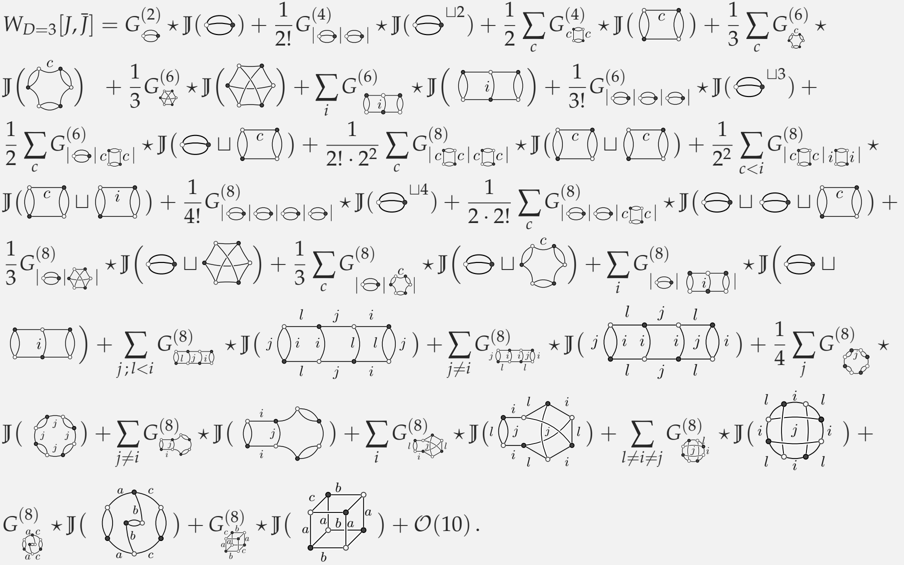
# $W_{D=3}[J,\bar{J}]$

$$W_{D=3}[J,\bar{J}] = G^{(2)}_{\ominus} \star \mathbb{J}(\ominus) + \frac{1}{2!} G^{(4)}_{|\ominus|\ominus|} \star \mathbb{J}(\ominus^{\sqcup 2}) + \frac{1}{2} \sum_{c} G^{(4)}_{c \square c} \star \mathbb{J}(\square_c)$$

# $W_{D=3}[J,\bar{J}]$



# $W_{D=3}[J,\bar{J}]$



W_{D=3}[J, J̄] = G^{(2)} ⋆ 𝕁( ) + 1/2! G^{(4)} ⋆ 𝕁( ⊔2) + 1/2 Σ_c G^{(4)} ⋆ 𝕁( c ) + 1/3 Σ_c G^{(6)} ⋆
𝕁( c ) + 1/3 G^{(6)} ⋆ 𝕁( ) + Σ_i G^{(6)} ⋆ 𝕁( i ) + 1/3! G^{(6)} ⋆ 𝕁( ⊔3) +
1/2 Σ_c G^{(6)} ⋆ 𝕁( ⊔ c ) + 1/(2!·2^2) Σ_c G^{(8)} ⋆ 𝕁( c ⊔ c ) + 1/2^2 Σ_{c<i} G^{(8)} ⋆
𝕁( c ⊔ i ) + 1/4! G^{(8)} ⋆ 𝕁( ⊔4) + 1/(2·2!) Σ_c G^{(8)} ⋆ 𝕁( ⊔ ⊔ c ) +
1/3 G^{(8)} ⋆ 𝕁( ⊔ ) + 1/3 Σ_c G^{(8)} ⋆ 𝕁( ⊔ c ) + Σ_i G^{(8)} ⋆ 𝕁( ⊔
i ) + Σ_{j;l<i} G^{(8)} ⋆ 𝕁( l j i ) + Σ_{j≠i} G^{(8)} ⋆ 𝕁( l j l ) + 1/4 Σ_j G^{(8)} ⋆
𝕁( j ) + Σ_{j≠i} G^{(8)} ⋆ 𝕁( i j ) + Σ_i G^{(8)} ⋆ 𝕁( l i j ) + Σ_{l≠i≠j} G^{(8)} ⋆ 𝕁( l i j ) +
G^{(8)} ⋆ 𝕁( a b c ) + G^{(8)} ⋆ 𝕁( a b c ) + 𝒪(10).

## Boundary graphs and bordisms interpretation

- for quartic melonic theories, the boundary sector is the set of all (possible disconnected) coloured graphs
- since $\mathcal{B} = \partial\mathcal{G}$ represents the 'boundary of a simplicial complex that $\mathcal{G}$ triangulates', one can give a bordism-interpretation to the Green's functions

for instance, if $|\Delta(\mathcal{B})| = \mathbb{S}^3 \sqcup (\mathbb{S}^2 \times \mathbb{S}^1) \sqcup L_{3,1} = \mathcal{M}$, then $G_{\mathcal{B}} = \partial W[J,\bar{J}]/\partial\mathcal{B}$ describes the bulk compatible with the triangulation of $\mathcal{M}$

Each correlation function supports also a $1/N$-expansion, $G_{\mathcal{B}}^{(2k)} = \sum_{\omega} G_{\mathcal{B}}^{(2k,\omega)}$.

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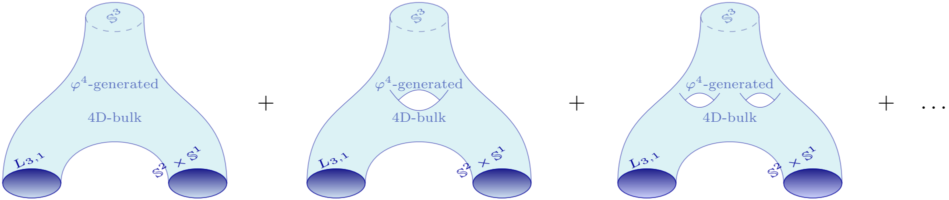


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## Theorem (CP) (Full Ward-Takahashi Identity for arbitrary tensor models)

*The partition function* $Z[J,\bar{J}]$ *of a tensor model with* $S_0 = \mathrm{Tr}_2(\bar{\varphi}, E\varphi)$ *such that*

$$E_{p_1 \dots p_{a-1} m_a p_{a+1} \dots p_D} - E_{p_1 \dots p_{a-1} n_a p_{a+1} \dots p_D} = E(m_a, n_a) \qquad \text{for each } a = 1, \dots, D,$$

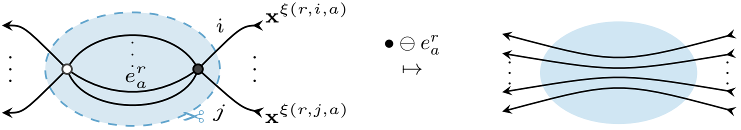
*satisfies, as consequence of the* $\mathrm{U}(N)$ *invariance of the path-integral measure,*

$$\sum_{p_i \in \mathbb{Z}} \frac{\delta^2 Z[J,\bar{J}]}{\delta J_{p_1 \dots p_{a-1} m_a p_{a+1} \dots p_D} \delta \bar{J}_{p_1 \dots p_{a-1} n_a p_{a+1} \dots p_D}} - \left( \delta_{m_a n_a} Y^{(a)}_{m_a}[J,\bar{J}] \right) \cdot Z[J,\bar{J}]$$

$$= \sum_{p_i \in \mathbb{Z}} \frac{1}{E(m_a, n_a)} \left( \bar{J}_{p_1 \dots m_a \dots p_D} \frac{\delta}{\delta \bar{J}_{p_1 \dots n_a \dots p_D}} - J_{p_1 \dots n_a \dots p_D} \frac{\delta}{\delta J_{p_1 \dots m_a \dots p_D}} \right) Z[J,\bar{J}]$$

*where*

$$Y^{(a)}_{m_a}[J,\bar{J}] = \sum_{\mathcal{C}} \mathfrak{f}^{(a)}_{\mathcal{C}, m_a} \star \mathbb{J}(\mathcal{C}) \,.$$



# Schwinger-Dyson equations

## SDEs for the $\varphi^4_{\mathrm{mel},D}$-model ($k \geq 2$) [CP, R. Wulkenhaar]

*Let* $D \geq 3$ *and let* $\mathcal{B}$ *be a connected boundary graph* $\partial\mathcal{G} = \mathcal{B} \in \mathsf{Grph}^{\mathrm{cl}}_D \subset \partial\mathsf{Feyn}_D(\varphi^4_{\mathrm{m},D})$.

*Let* $\mathbf{X} = (\mathbf{x}^1, \ldots, \mathbf{x}^k)$ *and* $\mathbf{s} = \mathbf{y}^1$. *Then* $G^{(2k)}_{\mathcal{B}}$ *obeys*

$$\Big(1 + \frac{2\lambda}{E_{\mathbf{s}}} \sum_{a=1}^{D} \sum_{\mathbf{q}_{\hat{a}}} \quad (s_a, \mathbf{q}_{\hat{a}})\Big) G^{(2k)}_{\mathcal{B}}(\mathbf{X})$$

$$= \frac{(-2\lambda)}{E_{\mathbf{s}}} \sum_{a=1}^{D} \Big\{ \sum_{\hat{\sigma} \in \mathrm{Aut_c}(\mathcal{B})} \sigma^* \mathfrak{f}^{(a)}_{\mathcal{B}, s_a}(\mathbf{X}) + \sum_{\rho > 1} \frac{1}{E(y^\rho_a, s_a)} Z_0^{-1} \frac{\partial Z[J, \bar{J}]}{\partial \varsigma_a(\mathcal{B}; 1, \rho)(\mathbf{X})}$$

$$- \sum_{b_a} \frac{1}{E(s_a, b_a)} \big[ G^{(2k)}_{\mathcal{B}}(\mathbf{X}) - G^{(2k)}_{\mathcal{B}}(\mathbf{X}|_{s_a \to b_a}) \big] \Big\}$$

*where* $(s_a, \mathbf{q}_a) = (q_1, q_2, \ldots, q_{a-1}, s_a, q_{a+1}, \ldots, q_D)$.

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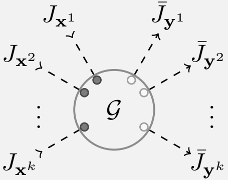
*Let $\mathbf{X} = (\mathbf{x}^1, \ldots, \mathbf{x}^k)$ and $\mathbf{s} = \mathbf{y}^1$. Then $G^{(2k)}_{\mathcal{B}}$ obeys*




$$\left(1 + \frac{2\lambda}{E_{\mathbf{s}}} \sum_{a=1}^{D} \sum_{\mathbf{q}_{\hat{a}}} G^{(2)}_{\ominus}(s_a, \mathbf{q}_{\hat{a}})\right) G^{(2k)}_{\mathcal{B}}(\mathbf{X})$$

$$= \frac{(-2\lambda)}{E_{\mathbf{s}}} \sum_{a=1}^{D} \Bigg\{ \sum_{\hat{\sigma} \in \mathrm{Aut}_c(\mathcal{B})} \sigma^* \mathfrak{f}^{(a)}_{\mathcal{B},s_a}(\mathbf{X}) + \sum_{\rho>1} \frac{1}{E(y^{\rho}_a, s_a)} Z_0^{-1} \frac{\partial Z[J,\bar{J}]}{\partial \varsigma_a(\mathcal{B};1,\rho)(\mathbf{X})}$$

$$- \sum_{b_a} \frac{1}{E(s_a, b_a)} \left[G^{(2k)}_{\mathcal{B}}(\mathbf{X}) - G^{(2k)}_{\mathcal{B}}(\mathbf{X}|_{s_a \to b_a})\right] \Bigg\}$$

*where* $(s_a, \mathbf{q}_a) = (q_1, q_2, \ldots, q_{a-1}, s_a, q_{a+1}, \ldots, q_D)$.

# A simple quartic model

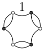
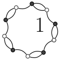
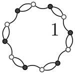
- Proposal of a model with $V[\varphi,\bar\varphi] = \lambda \cdot$ 

$$S_0[\varphi,\bar\varphi] = \mathrm{Tr}_2(\bar\varphi, E\varphi) = \sum_{\mathbf{x}\in\mathbb{Z}^3} \bar\varphi_{\mathbf{x}}(m^2 + |\mathbf{x}|^2)\varphi_{\mathbf{x}}\,, \quad |\mathbf{x}|^2 = x_1^2 + x_2^2 + x_3^2\,.$$

- The boundary sector is:

$\partial\,\mathsf{Feyn}_3($  $) = \{$3-coloured graphs with connected components in $\Theta\}$,

being

$\Theta = \{$  ,  ,  ,  ,  ,  $, \ldots\}\,.$

- Let $\mathcal{X}_{2k}$ be the graph in $\Theta$ with $2k$ vertices, and $G^{(2k)} = G^{(2k)}_{\mathcal{X}_{2k}}$:

$G^{(2)} = G^{(2)}$  $, \; G^{(4)} = G^{(4)}$  $, \; G^{(6)} = G^{(6)}$  $, \; G^{(8)} = G^{(8)}$  $, \; G^{(10)} = G^{(10)}$  $.$

- Full tower of exact equations obtained [CP, R. Wulkenhaar]

## The exact 2pt-function equation. Melonic (planar) limit, conjecturally

$$\Big(m^2 + |\mathbf{x}|^2 + 2\lambda \sum_{q,p\in\mathbb{Z}} G^{(2)}_{\mathrm{mel}}(x_1,q,p)\Big)\cdot G^{(2)}_{\mathrm{mel}}(\mathbf{x})$$
$$= 1 + 2\lambda \sum_{q\in\mathbb{Z}} \frac{1}{x_1^2-q^2}\big[G^{(2)}_{\mathrm{mel}}(x_1,x_2,x_3) - G^{(2)}_{\mathrm{mel}}(q,x_2,x_3)\big]$$

## The exact 2$k$-pt-function equation. Melonic limit, conjecturally

$$\Big(1 + \frac{2\lambda}{m^2+|\mathbf{s}|^2}\sum_{q,p\in\mathbb{Z}} G^{(2)}_{\mathrm{mel}}(x_1^1,q,p)\Big)\cdot G^{(2k)}_{\mathrm{mel}}(\mathbf{x}^1,\ldots,\mathbf{x}^k)$$
$$= \frac{(-2\lambda)}{m^2+|\mathbf{s}|^2}\Bigg[\sum_{\rho=2}^{k}\frac{1}{(x_1^\rho)^2-(x_1^1)^2}\Big(G^{(2\rho-2)}_{\mathrm{mel}}(\mathbf{x}^1,\ldots,\mathbf{x}^{\rho-1})\cdot G^{(2k-2\rho+2)}_{\mathrm{mel}}(\mathbf{x}^\rho,\ldots,\mathbf{x}^k)\Big)$$
$$-\sum_{q\in\mathbb{Z}}\frac{G^{(2k)}_{\mathrm{mel}}(x_1^1,x_2^1,x_3^1,\mathbf{x}^2,\ldots,\mathbf{x}^k) - G^{(2k)}_{\mathrm{mel}}(q,x_2^1,x_3^1,\mathbf{x}^2,\ldots,\mathbf{x}^k)}{(x_1^1)^2-q^2}\Bigg]$$

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$$= 1 + 2\lambda \sum_{q\in\mathbb{Z}} \frac{1}{x_1^2 - q^2}\left[G^{(2)}_{\mathrm{mel}}(x_1,x_2,x_3) - G^{(2)}_{\mathrm{mel}}(q,x_2,x_3)\right]$$

## The exact $2k$-pt-function equation. Melonic limit, conjecturally

$$\left(1 + \frac{2\lambda}{m^2+|\mathbf{s}|^2} \sum_{q,p\,\in\mathbb{Z}} G^{(2)}_{\mathrm{mel}}(x_1^1,q,p)\right) \cdot G^{(2k)}_{\mathrm{mel}}(\mathbf{x}^1,\ldots,\mathbf{x}^k)$$

$$= \frac{(-2\lambda)}{m^2+|\mathbf{s}|^2}\left[\sum_{\rho=2}^{k} \frac{1}{(x_1^\rho)^2-(x_1^1)^2}\left(G^{(2\rho-2)}_{\mathrm{mel}}(\mathbf{x}^1,\ldots,\mathbf{x}^{\rho-1}) \cdot G^{(2k-2\rho+2)}_{\mathrm{mel}}(\mathbf{x}^\rho,\ldots,\mathbf{x}^k)\right)\right.$$

$$\left. - \sum_{q\in\mathbb{Z}} \frac{G^{(2k)}_{\mathrm{mel}}(x_1^1,x_2^1,x_3^1,\mathbf{x}^2,\ldots,\mathbf{x}^k) - G^{(2k)}_{\mathrm{mel}}(q,x_2^1,x_3^1,\mathbf{x}^2,\ldots,\mathbf{x}^k)}{(x_1^1)^2-q^2}\right]$$

# Conclusions & outlook

- (Coloured) tensor field theories [Ben Geloun, Bonzom, Carrozza, Gurău, Krajewski, Oriti, Ousmane-Samary, Rivasseau, Ryan, Tanasa, Toriumi, Vignes-Tourneret,...] provide a framework for $3 \leq D$-dimensional random geometry
  - A bordism interpretation of the correlation functions was given
  - A (non-perturbative) Ward-Takahashi identity [CP] based that for matrix models has been found
  - It has been used to derive the full tower of SDE [CP-Wulkenhaar]
  - Closed equations: hope of a solvable theory for the simple 11-model (spherical 3-geometries)
- Outlook:
  - Apply these techniques SYK-like (Sachdev-Ye-Kitaev) models [Witten]
  - Applications to GFT
  - Gauge fields on colored graphs (random spaces) by representing graphs on finite spectral triples

# References


- M. Disertori, R. Gurău, J. Magnen, and V. Rivasseau.
  Phys. Lett. B, **649** 95–102, 2007.
  arXiv:hep-th/0612251.
- Bonzom, V. and Gurău, R. and Riello, A. and Rivasseau, V.
  Nucl. Phys. B **853**, 2011
  arXiv:1105.3122
- R. Gurău,
  Commun. Math. Phys. **304**, 69 (2011)
  arXiv:0907.2582 [hep-th] .
- C. I. Pérez-Sánchez, R. Wulkenhaar
  arXiv:1706.07358
- C. I. Pérez-Sánchez.
  J.Geom.Phys. **120** 262-289 (2017) (arXiv:1608.00246)
  and arXiv:1608.08134
- E. Witten
  arXiv:1610.09758v2 [hep-th].


*Thank you for your attention!*