Flat Jordan modules and particle physics[2]

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[2] Based on a joint work with L. Dabrowski and M. Dubois-Violette

Despite its many success, Connes's noncommutative formulation of standard model exhibits some flaws:

- Unimodularity of $SU(3)$.
- No quark-lepton symmetry (e.g. $d \leftrightarrow e^-, u \leftrightarrow \nu_e, ...$).
- No natural way to get three generations of particles (e^-, μ, τ).

In *"Exceptional quantum geometry and particle physics"*[5] M. Dubois-Violette shows how all of these problems might be overcome using modules of the exceptional Jordan algebra.

[5] Nuclear PhysicsB912(2016)426–449

Definition:

A **Jordan algebra** $(J, \circ)$, *is a vector space J together with a bilinear product* $\circ : J \times J \to J$, *such that* $\forall a, b \in J$:

- $a \circ b = b \circ a$
- $(a \circ b) \circ a^2 = a \circ (b \circ a^2)$

We will always consider unital Jordan algebras.

- Let A be an associative algebra, equip it with the product:

$$a \circ b = \frac{1}{2}(ab + ba).$$

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  $(A, \circ)$ is a Jordan algebra. Every Jordan algebra isomorphic to an algebra of this kind is called a **special Jordan algebra**.
- The **exceptional Jordan algebra** (sometimes called **Albert Algebra**) $(J_3^8, \circ)$:

$$J_3^8 = \{x \in M_3(\mathbb{O}) \mid x = x^*\}$$
$$x \circ y = \tfrac{1}{2}(xy + yx).$$

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Definition:

Let J be a Jordan algebra, let M be a vector space equipped with a right and left bilinear maps:

$$J \otimes M \to M \quad x \otimes \Phi \mapsto x\Phi$$
$$M \otimes J \to M \quad \Phi \otimes x \mapsto \Phi x$$

On $J \oplus M$, define the bilinear product $(x, \Phi)(x', \Phi') = (xx', x\Phi' + \Phi x')$, then M is a **Jordan bimodule** *if $J \oplus M$ endowed with this product is a Jordan algebra*

Definition:

Let J be a Jordan algebra, its **center** *$Z(J)$ is the (associative) subalgebra:*

$$Z(J) = \{z \in J \mid [x, y, z] = [x, z, y] = 0 \ \forall x, y \in J\}$$

where we have defined the **associator***:*

$$[x, y, z] = (xy)z - x(yz).$$

Definition:

A **connection** *on a Jordan module M is a linear map*

$$\nabla : Der(J) \to End(M)$$
$$X \mapsto \nabla_X$$

such that $\forall x \in J$, $m \in M$ *and* $z \in Z(J)$:

$$\begin{cases} \nabla_X(xm) = X(x)m + x\nabla_X(m) \\ \nabla_{zX}(m) = z\nabla_X(m) \end{cases}$$

Definition:

The **curvature** *of a connection* ∇ *is*

$$R_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Consider the Hilbert space $\mathbb{C}^3$ equipped with the usual vector product $\times$.
This product is nonassociative and $SU(3)-$invariant.
Equip $\mathbb{C}$ with the trivial representation of $SU(3)$

$$(g, z) \mapsto z \quad g \in SU(3),\ z \in \mathbb{C}.$$

Let $A = \mathbb{C} \oplus \mathbb{C}^3$, with the $SU(3)-$invariant product:

$$(z, Z)(z', Z') = (zz' - \langle Z, Z' \rangle, \bar{z}Z' - z'Z + iZ \times Z')\,.$$

Proposition

The algebra A is isomorphic to the algebra of octonions $\mathbb{O}$*. SU(3) is the subgroup of Aut(*$\mathbb{O}$*) which preserves the decomposition* $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$.

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If we interpret $\mathbb{C}^3$ as the **internal colour space of quarks**, and $\mathbb{C}$ as the **(trivial) internal colour space of leptons** $\Rightarrow$ quark-lepton symmetry is just a consequence of $SU(3)-$colour symmetry.

[14] Based on a joint work with L. Dabrowski and M. Dubois-Violette

From the decomposition $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ one gets $J_3^8 = J_3^2 \oplus M_3(\mathbb{C})$:

$$\begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_3 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & z_3 & \overline{z}_2 \\ \overline{z}_3 & \xi_2 & z_1 \\ z_2 & \overline{z}_3 & \xi_3 \end{pmatrix} \oplus (Z_1, Z_2, Z_3)$$

$$x_i = z_i + Z_i \quad x_i \in \mathbb{O}, z_i \in \mathbb{C}, Z_i \in \mathbb{C}^3, \xi_i \in \mathbb{R}.$$

Proposition

The subgroup of $Aut(J_3^8)$ *which preserves the decomposition above is* $(SU(3) \times SU(3))/\mathbb{Z}_3$, *with action of* $(U, V) \in (SU(3) \times SU(3))/\mathbb{Z}_3$:

$$H \mapsto VHV^* \;\; M \mapsto UMV^*$$

$$(H, M) \in J_3^2 \oplus M_3(\mathbb{C}).$$

There are two familes for each generations $\Rightarrow$ take the module $M = J_3^8 \oplus J_3^8$, with the particle assignment:

$$J^u = \begin{pmatrix} \alpha_1 & \nu_\tau & \overline{\nu}_\mu \\ \overline{\nu}_\tau & \alpha_2 & \nu_e \\ \nu_\mu & \overline{\nu}_e & \alpha_3 \end{pmatrix} + (u, c, t) \quad J^d = \begin{pmatrix} \beta_1 & \tau & \overline{\mu} \\ \overline{\tau} & \beta_2 & e \\ \mu & \overline{e} & \beta_3 \end{pmatrix} + (d, s, b)$$

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$$H \mapsto VHV^* \ M \mapsto UMV^*$$
$$(H, M) \in J_3^2 \oplus M_3(\mathbb{C}).$$

- The action of $U \in SU(3)$ is responsible the usual color mixing.
- The action of $V \in SU(3)$ is responsible for the mixing of different generations of leptons.

[17] Based on a joint work with L. Dabrowski and M. Dubois-Violette

Any module of J_3^8 is **free** $\Rightarrow$ the lift of d defines a connection by:

$$\nabla^0_X(m) := dm(X)$$

For example on $J_3^8 \otimes \mathbb{R}^2 = J_3^8 \oplus J_3^8$:

$$d \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} (X) := \begin{pmatrix} dx_1(X) \\ dx_2(X) \end{pmatrix}$$

Where $X \in \mathfrak{f}_4 = Der(J_3^8)$.

For $M = J_3^8 \otimes E$, where E is a finite $n-$dimensional real vector space, we have:

Proposition (A.C., L.Dabrowski, M. Dubois-Violette)

A connection on M is

$$\nabla = \nabla^0 + A$$

where the map $A : \mathfrak{f}_4 \to M_n(\mathbb{R})$ is linear.

As for flat connection we get the following characterization:

Proposition (A.C., L.Dabrowski, M. Dubois-Violette)

Flat connections on M are in one to one correspondence with Lie algebra homomorphisms $A : \mathfrak{f}_4 \to M_n(\mathbb{R})$. That is, for a basis $\{X_\mu\} \subset \mathfrak{f}_4$ with structure constants $[X_\mu, X_\nu] = c^{\tau}_{\mu\nu} X_\tau$:

$$[A(X_\mu), A(X_\nu)] = c^{\tau}_{\mu\nu} A(X_\tau).$$

The study of connections on Jordan modules has revealed some interesting nonassociative geometry. These might be crucial in a future reformulation of standard model in this new context.
Some further studies:

- On the mathematical side:
  - Study connections for other kind of modules (such as modules of J_2^4).
  - Study Jordan module homomorphism.
  - Get deeper geometrical details.
- On the physical side:
  - Get $SU(2) \times U(1)$ gauge symmetry.
  - Study what happens when coupling with space-time degrees of freedom.
  - Write down a suitable action and study some dynamics.

- M. Dubois-Violette "*Exceptional quantum geometry and particle physics*" Nuclear Physics B, Volume 912, November 2016, Pages 426 – 449
- Shane Farnsworth and Latham Boyle "*Non-Associative Geometry and the Spectral Action Principle*" Journal of High Energy Physics, July 2015, 2015:23 "*Exceptional Lie Groups*" arXiv:0902.0431, 2009.
- M. Dubois-Violette, R. Kerner, J. Madore, "*Gauge bosons in a non-commutative geometry*", Phys. Lett. B 217 (1989) 485–488.