"Topologically Massive" theories in and beyond 3D

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Motivation and overview

This work belongs to a series of attempts to generalize 3D higher-derivative gravity (at the linearized level), motivated by higher spin gauge theories.

Part 1: The 3D theories

- Fierz-Pauli equations and the linearized New Massive Gravity (NMG)
- $\sqrt{\text{Fierz-Pauli}}$ equations and the linearized Topologically Massive Gravity (TMG)

Part 2: Attempts to go beyond 3D

- General attempts
- An example: Linearized TMG $\rightarrow$ a 7D spin-2 theory

Part 1: The 3D theories

Fierz-Pauli equations for bosonic spin-s

- Massive spin-s field: $\varphi_{(\mu_1\mu_2\cdots\mu_s)}$

$$\left.\begin{aligned} \left(\Box - m^2\right)\varphi_{\mu_1\mu_2\cdots\mu_s} &= 0 \\ \eta^{\mu_1\mu_2}\varphi_{\mu_1\mu_2\cdots\mu_s} &= 0 \\ \partial^{\mu_1}\varphi_{\mu_1\mu_2\cdots\mu_s} &= 0 \end{aligned}\right\} \quad \text{Fierz-Pauli equations}$$

4D: (2s+1) propagating d.o.f. (little group SO(3))

Example: massive spin-2 in 4D

d.o.f. = 2×2+1 = 5

· Use a rank-2 symmetric tensor: $\varphi_{(\mu\nu)}$, 10 d.o.f.

· Traceless condition: $\eta^{\mu\nu}\varphi_{\mu\nu} = 0$, -1 d.o.f.

· Divergenceless condition: $\partial^{\mu}\varphi_{\mu\nu} = 0$, -4 d.o.f.

3D: 2 propagating d.o.f. (little group SO(2))

- Gauge symmetry?

3D linearized New Massive Gravity as a spin-2 FP theory

- 3D New Massive Gravity e.o.m. :

$$G_{\mu\nu} + \frac{1}{m^2}\,(\text{ 4th order derivatives }) = 0$$

Einstein's tensor

$$\Downarrow \; g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

Perturbative approximation on Minkowski background

$$\left(\Box - m^2\right) G_{\mu\nu}(h) = 0 \;,\; \eta^{\mu\nu} G_{\mu\nu}(h) = 0$$

Linearized Einstein's tensor : $G_{\mu_1\mu_2}(h) = \varepsilon_{\mu_1}{}^{\nu_1\rho_1}\varepsilon_{\mu_2}{}^{\nu_2\rho_2}\partial_{\nu_1}\partial_{\nu_2}h_{\rho_1\rho_2}$

Gauge symmetry : $\delta h_{\rho_1\rho_2} = \partial_{(\rho_1}\xi_{\rho_2)}$

- Compare with spin-2 Fierz-Pauli equations:

$$\left(\Box - m^2\right)\varphi_{\mu\nu} = 0 \;,\; \eta^{\mu\nu}\varphi_{\mu\nu} = 0 \;,\; \partial^\mu\varphi_{\mu\nu} = 0$$

$$\Updownarrow$$

$$\varphi_{\mu\nu} = G_{\mu\nu}(h)$$

- Equivalent by solving the divergenceless condition.
- Not only for spin-2 …

3D NMG-like higher spin theories

- For arbitrary spin-s Fierz-Pauli equations in 3D

$$\left(\Box - m^2\right)\varphi_{\mu_1\mu_2\cdots\mu_s} = 0\ ,\ \eta^{\mu_1\mu_2}\varphi_{\mu_1\mu_2\cdots\mu_s} = 0\ ,\ \underline{\partial^{\mu_1}\varphi_{\mu_1\mu_2\cdots\mu_s} = 0}$$

we can always solve the divergenceless condition by

$$\begin{aligned}\varphi_{\mu_1\cdots\mu_s} &= G_{\mu_1\mu_2\cdots\mu_s}(h) \\ &= \varepsilon_{\mu_1}{}^{\nu_1\rho_1}\cdots\varepsilon_{\mu_s}{}^{\nu_s\rho_s}\partial_{\nu_1}\cdots\partial_{\nu_s}h_{\rho_1\cdots\rho_s}\end{aligned}$$

and hence convert the FP equations into

$$\left(\Box - m^2\right)G_{\mu_1\mu_2\cdots\mu_s}(h) = 0\ ,\ \eta^{\mu_1\mu_2}G_{\mu_1\mu_2\cdots\mu_s}(h) = 0$$

with higher order derivatives and with a gauge symmetry:

$$\delta h_{\rho_1\rho_2\cdots\rho_s} = \partial_{(\rho_1}\xi_{\rho_2\cdots\rho_s)}$$

3D NMG-like spin-1,2,3 actions

$$S_{\text{spin-1}} = \int d^3x \left\{ \frac{1}{2m^3} h^{\mu} \left(\Box - m^2 \right) G_{\mu}(h) \right\}$$

$$S_{\text{spin-2}} = \int d^3x \left\{ \frac{1}{2m^4} h^{\mu\nu} \left(\Box - m^2 \right) G_{\mu\nu}(h) + \frac{1}{4m^2} \left(G^{\text{tr}}(h) \right)^2 \right\}$$

(denote $G^{\text{tr}}(h) \equiv \eta^{\mu\nu} G_{\mu\nu}(h)$)

$$S_{\text{spin-3}} = \int d^3x \left\{ \frac{1}{2m^5} h^{\mu\nu\rho} \left(\Box - m^2 \right) G_{\mu\nu\rho}(h) + \frac{3}{4m^5} h^{\mu\nu\rho} \left(\partial_\mu \partial_\nu - \eta_{\mu\nu} \Box \right) G^{\text{tr}}_{\rho}(h) \right\}$$

(denote $G^{\text{tr}}_{\rho}(h) \equiv \eta^{\mu\nu} G_{\mu\nu\rho}(h)$)

(arXiv: 0911.3061)

- For spin≥4 , auxiliary fields are needed…

3D NMG-like spin-4 action

$$
\begin{aligned}
S_{\text{spin-4}}[h,\pi,\phi] \;=\; \int d^3x \Big\{ & \frac{1}{2m^6} h^{\mu\nu\rho\sigma}\left(\Box - m^2\right) G_{\mu\nu\rho\sigma}(h) \\
& + \frac{1}{m^4}\pi^{\mu\nu} G^{\text{tr}}_{\mu\nu}(h) \\
& - \frac{1}{2m^2}\pi^{\mu\nu} G_{\mu\nu}(\pi) - \frac{1}{2}\left(\pi^{\mu\nu}\pi_{\mu\nu} - \pi^2\right) \\
& + \phi\pi + \frac{13}{12}\phi^2 + \frac{1}{12m^2}\phi\Box\phi \Big\}
\end{aligned}
$$

(denote $\pi \equiv \eta^{\mu\nu}\pi_{\mu\nu}$ and $G^{\text{tr}}_{\mu\nu}(h) \equiv \eta^{\rho\sigma} G_{\mu\nu\rho\sigma}(h)$)

$$
\begin{aligned}
G_{\mu_1\mu_2\mu_3\mu_4}(h) &\equiv \varepsilon_{\mu_1}{}^{\nu_1\rho_1}\varepsilon_{\mu_2}{}^{\nu_2\rho_2}\varepsilon_{\mu_3}{}^{\nu_3\rho_3}\varepsilon_{\mu_4}{}^{\nu_4\rho_4}\partial_{\nu_1}\partial_{\nu_2}\partial_{\nu_3}\partial_{\nu_4} h_{\rho_1\rho_2\rho_3\rho_4} \\
G_{\mu_1\mu_2}(\pi) &\equiv \varepsilon_{\mu_1}{}^{\nu_1\rho_1}\varepsilon_{\mu_2}{}^{\nu_2\rho_2}\partial_{\nu_1}\partial_{\nu_2}\pi_{\rho_1\rho_2}
\end{aligned}
$$

Gauge symmetry: $\delta h_{\rho_1\rho_2\rho_3\rho_4} = \partial_{(\rho_1}\xi_{\rho_2\rho_3\rho_4)}$

(arXiv: 1109.0382)

NMG → TMG

- The cost of going to higher derivatives: ghosts

  In the NMG-like action, when the spin number s is odd, the 2 d.o.f. have different signs in front of their kinetic terms, i.e. one of them is a ghost! (arXiv:1109.0382)

- A cure: construct a model on only 1 d.o.f.

  → linearized TMG

3D $\sqrt{\mathrm{FP}}$ (e.g. spin-1)

$$\left(\Box - m^2\right)\varphi_\mu = 0\ , \quad \partial^\mu \varphi_\mu = 0$$

3D $\longrightarrow$

$$\left(\varepsilon_\mu{}^{\nu\rho}\partial_\nu \pm m\delta_\mu^\rho\right)\left(\varepsilon_\rho{}^{\sigma\tau}\partial_\sigma \mp m\delta_\rho^\tau\right)\varphi_\tau = 0$$

$\sqrt{\mathrm{FP}}$ equations:

$$\varepsilon_\rho{}^{\sigma\tau}\partial_\sigma\varphi_\tau = +\, m\varphi_\rho$$ ← Helicity "+"

$$\varepsilon_\rho{}^{\sigma\tau}\partial_\sigma\varphi_\tau = -\, m\varphi_\rho$$ ← Helicity "-"

- Any solution of either equation is a solution to the Klein-Gordon equation.
- Each contains one helicity state.
- Two equations are interchanged by parity transformation.

3D $\sqrt{\mathrm{FP}}$ for spin-s

$$\left(\Box - m^2\right) \varphi_{\mu_1 \cdots \mu_s} = 0 \ , \quad \partial^{\mu_1} \varphi_{\mu_1 \cdots \mu_s} = 0 \ , \quad \eta^{\mu_1 \mu_2} \varphi_{\mu_1 \cdots \mu_s} = 0$$

3D $\longrightarrow$

$$\left(\varepsilon_{\mu_1}{}^{\nu\rho} \partial_\nu \pm m \delta^\rho_{\mu_1}\right) \left(\varepsilon_\rho{}^{\sigma\tau} \partial_\sigma \mp m \delta^\tau_\rho\right) \varphi_{\tau \mu_2 \cdots \mu_s} = 0$$

Pick out one helicity:

$$\left(\varepsilon_{\mu_1}{}^{\sigma\tau} \partial_\sigma - m \delta^\tau_{\mu_1}\right) \varphi_{\tau \mu_2 \cdots \mu_s} = 0$$

$$\Updownarrow$$

$$\begin{cases} \left(\varepsilon_{(\mu_1|}{}^{\sigma\tau} \partial_\sigma - m \delta^\tau_{(\mu_1|}\right) \varphi_{\tau|\mu_2 \cdots \mu_s)} = 0 \\ \partial^{\mu_1} \varphi_{\mu_1 \cdots \mu_s} = 0 \\ \eta^{\mu_1 \mu_2} \varphi_{\mu_1 \cdots \mu_s} = 0 \end{cases}$$

3D linearized TMG and spin-2 $\sqrt{\mathrm{FP}}$

- 3D Topologically Massive Gravity e.o.m. :

$$G_{\mu\nu} - \frac{1}{m}\ (\text{ 3rd order derivatives })=0$$

Einstein's tensor

$$\Downarrow \ g_{\mu\nu}=\eta_{\mu\nu}+\kappa h_{\mu\nu}$$

Perturbative approximation on Minkowski background

$$\left(\varepsilon_{(\mu|}{}^{\rho\sigma}\partial_\rho - m\delta^\sigma_{(\mu|}\right) G_{\sigma|\nu)}(h)=0\ ,\ \eta^{\mu\nu}G_{\mu\nu}(h)=0$$

Linearized Einstein's tensor : $G_{\mu_1\mu_2}(h)=\varepsilon_{\mu_1}{}^{\nu_1\rho_1}\varepsilon_{\mu_2}{}^{\nu_2\rho_2}\partial_{\nu_1}\partial_{\nu_2}h_{\rho_1\rho_2}$

Gauge symmetry : $\delta h_{\rho_1\rho_2}=\partial_{(\rho_1}\xi_{\rho_2)}$

- Equivalent to 3D spin-2 "$\sqrt{\mathrm{FP}}$ "

$$\left(\varepsilon_{(\mu|}{}^{\rho\sigma}\partial_\rho - m\delta^\sigma_{(\mu|}\right)\varphi_{\sigma|\nu)}=0\ ,\ \eta^{\mu\nu}\varphi_{\mu\nu}=0\ ,\ \partial^\mu\varphi_{\mu\nu}=0$$

$$\Updownarrow$$

$$\varphi_{\mu\nu} \doteq G_{\mu\nu}(h)$$

3D TMG-like theories for generic spin-s

$$\left(\varepsilon_{(\mu_1|}{}^{\rho\sigma}\partial_\rho - m\delta^\sigma_{(\mu_1|}\right) G_{\sigma|\mu_2\cdots\mu_s)}(h) = 0 \ , \ \eta^{\mu_1\mu_2} G_{\mu_1\mu_2\cdots\mu_s}(h) = 0$$

Generalized Einstein tensor: $G_{\mu_1\mu_2\cdots\mu_s}(h) = \varepsilon_{\mu_1}{}^{\nu_1\rho_1}\cdots\varepsilon_{\mu_s}{}^{\nu_s\rho_s}\partial_{\nu_1}\cdots\partial_{\nu_s}h_{\rho_1\cdots\rho_s}$

gauge symmetry: $\delta h_{\rho_1\rho_2\cdots\rho_s} = \partial_{(\rho_1}\xi_{\rho_2\cdots\rho_s)}$

Actions:

$$S_{\text{spin-1}} = \int d^3x \left\{ \frac{1}{2} h^{\mu_1} \left(\varepsilon_{\mu_1}{}^{\nu\rho}\partial_\nu - m\delta^\rho_{\mu_1}\right) G_\rho(h) \right\}$$

$$S_{\text{spin-2}} = \int d^3x \left\{ \frac{1}{2} h^{\mu_1\mu_2} \left(\varepsilon_{\mu_1}{}^{\nu\rho}\partial_\nu - m\delta^\rho_{\mu_1}\right) G_{\rho\mu_2}(h) \right\}$$

$$S_{\text{spin-3}} = \int d^3x \left\{ \frac{1}{2} h^{\mu_1\mu_2\mu_3} \left(\varepsilon_{\mu_1}{}^{\nu\rho}\partial_\nu - m\delta^\rho_{\mu_1}\right) G_{\rho\mu_2\mu_3}(h) + \pi^{\mu_3} G^{\text{tr}}_{\mu_3}(h) + \cdots \right\}$$

…

Ghost-free!

Part 2: Attempts to go beyond 3D

Two crucial tasks

In order to go beyond 3D, there are two important things to do:

- For both NMG and TMG-like theories, we must further generalize

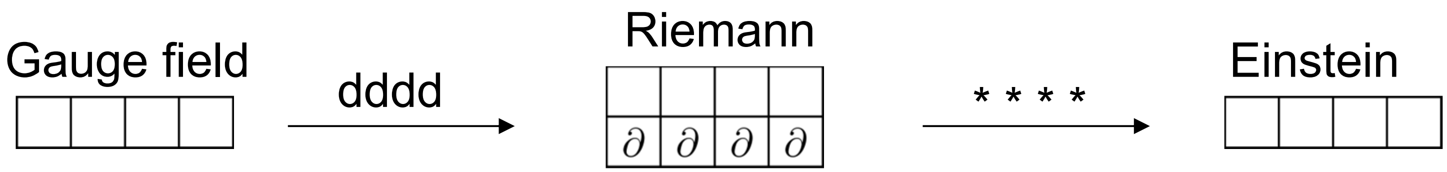
$$G_{\mu_1\mu_2\cdots\mu_s}(h) = \varepsilon_{\mu_1}{}^{\nu_1\rho_1}\cdots\varepsilon_{\mu_s}{}^{\nu_s\rho_s}\partial_{\nu_1}\cdots\partial_{\nu_s}h_{\rho_1\cdots\rho_s}$$

- For TMG-like theories, we must generalize the factorization of the K-G operator

$$\left(\varepsilon_{\mu_1}{}^{\nu\rho}\partial_\nu \pm m\delta^\rho_{\mu_1}\right)\left(\varepsilon_\rho{}^{\sigma\tau}\partial_\sigma \mp m\delta^\tau_\rho\right)G_{\tau\mu_2\cdots\mu_s}(h) = 0$$

Generalizing the Einstein tensors

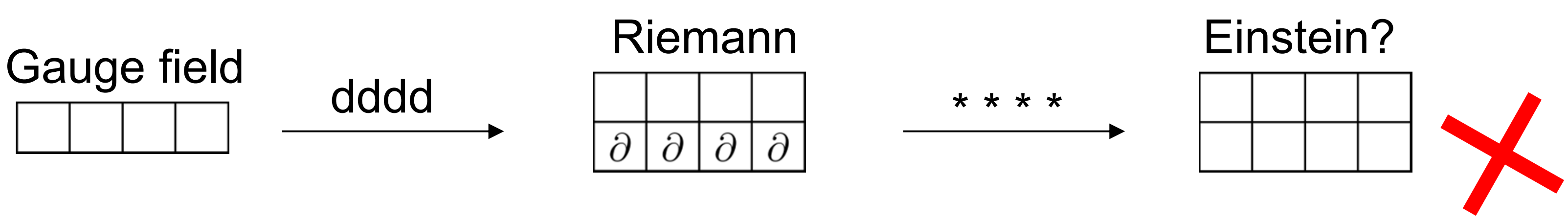
- D = 3 $$G_{\mu_1\mu_2\cdots\mu_s}(h) = \varepsilon_{\mu_1}{}^{\nu_1\rho_1}\cdots\varepsilon_{\mu_s}{}^{\nu_s\rho_s}\partial_{\nu_1}\cdots\partial_{\nu_s}h_{\rho_1\cdots\rho_s}$$

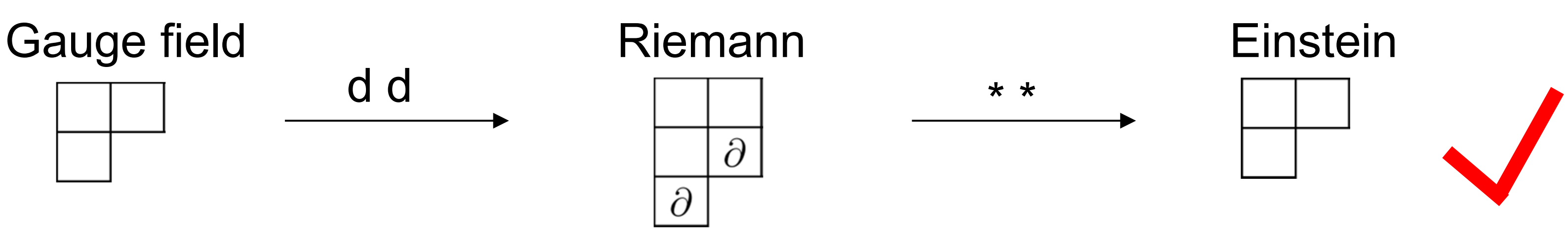


Two important features:

1. Vanishing Einstein tensor → pure gauge
2. Einstein tensor & gauge field in the same representation

- In D > 3, however, in general it's difficult to satisfy both, e.g. D = 4





$\sqrt{\mathrm{FP}}$ beyond 3D

- 3D:

$$\left(\varepsilon_{\mu}{}^{\alpha\nu}\partial_{\alpha} \pm m\delta^{\nu}_{\mu}\right) T_{\nu,\cdots} = 0$$

- D=4k-1 (k=1,2,3,… e.g. 3D, 7D, 11D)

$$\left(\frac{1}{(2k-1)!}\varepsilon_{\mu_1\cdots\mu_{2k-1}}{}^{\alpha\nu_1\cdots\nu_{2k-1}}\partial_{\alpha} \pm m\delta^{\nu_1\cdots\nu_{2k-1}}_{\mu_1\cdots\mu_{2k-1}}\right) T_{\nu_1\cdots\nu_{2k-1},\cdots} = 0$$

(D=4k+1 : tachyons)

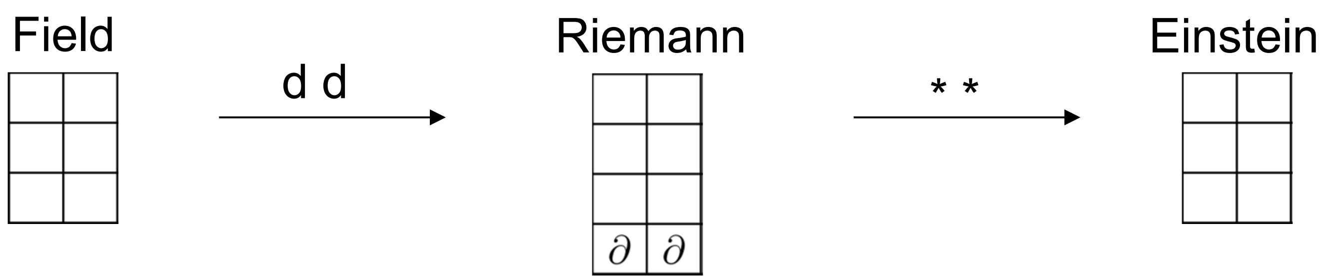
"Topologically Massive" spin-2 in 7D arXiv:1207.0192

- The model starts with the field $T_{\mu_1\mu_2\mu_3,\nu_1\nu_2\nu_3}$, which satisfies the symmetry

$$\begin{array}{|c|c|}\hline \mu_1 & \nu_1 \\ \hline \mu_2 & \nu_2 \\ \hline \mu_3 & \nu_3 \\ \hline \end{array}$$

$$\text{i.e. } T_{\mu_1\mu_2\mu_3,\nu_1\nu_2\nu_3} = \hat{Y}\left\{T_{\mu_1\mu_2\mu_3,\nu_1\nu_2\nu_3}\right\}$$

- The reason to choose such type of fields:

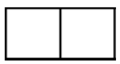
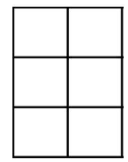


Dimension = 7 = 4k – 1

Height of the first column = 3 = 2k – 1

7D $\sqrt{\mathrm{FP}}$ spin-2

Notation $\bar{\mu}:\ [\mu_1\mu_2\mu_3]$ e.g. $T_{\bar{\mu},\bar{\nu}} \equiv T_{\mu_1\mu_2\mu_3,\nu_1\nu_2\nu_3}$

3D	7D
	
$T_{\mu,\nu} = T_{(\mu,\nu)}$	$T_{\bar{\mu},\bar{\nu}} = \hat{Y}\left\{T_{\bar{\mu},\bar{\nu}}\right\}$
$\eta^{\mu\nu}T_{\mu,\nu} = 0$	$\eta^{\mu_1\nu_1}T_{\bar{\mu},\bar{\nu}} = 0$
$\partial^{\mu}T_{\mu,\nu} = 0$	$\partial^{\mu_1}T_{\bar{\mu},\bar{\nu}} = 0$
$\left(\varepsilon_{\mu}{}^{\alpha\rho}\partial_\alpha \pm m\delta^{\rho}_{\mu}\right)T_{\rho,\nu} = 0$	$\left(\frac{1}{6}\varepsilon_{\bar{\mu}}{}^{\alpha\bar{\rho}}\partial_\alpha \pm m\delta^{\bar{\rho}}_{\bar{\mu}}\right)T_{\bar{\rho},\bar{\nu}} = 0$
2 physical d.o.f. $\to$ 1+1	70 physical d.o.f. $\to$ 35+35

7D spin-2 TMG-like theory

3D

$$T_{\mu,\nu} = G_{\mu,\nu}(h) = \varepsilon_\mu{}^{\alpha\rho}\varepsilon_\nu{}^{\beta\sigma}\partial_\alpha\partial_\beta h_{\rho,\sigma}$$

$$\left(\varepsilon_\mu{}^{\alpha\rho}\partial_\alpha - m\delta^\rho_\mu\right) G_{\rho,\nu}(h) = 0$$

$$S = \int d^3x \, \frac{1}{2} h^{\mu,\nu} \left(\varepsilon_\mu{}^{\alpha\rho}\partial_\alpha - m\delta^\rho_\mu\right) G_{\rho,\nu}(h)$$

$$\delta h_{\mu,\nu} = \partial_{(\mu}\xi_{\nu)}$$

1 physical d.o.f. , ghost-free

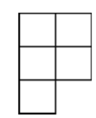
7D

$$T_{\bar\mu,\bar\nu} = G_{\bar\mu,\bar\nu}(h) = \varepsilon_{\bar\mu}{}^{\alpha\bar\rho}\varepsilon_{\bar\nu}{}^{\beta\bar\sigma}\partial_\alpha\partial_\beta h_{\bar\rho,\bar\sigma}$$

$$\left(\frac{1}{6}\varepsilon_{\bar\mu}{}^{\alpha\bar\rho}\partial_\alpha - m\delta^{\bar\rho}_{\bar\mu}\right) G_{\bar\rho,\bar\nu}(h) = 0$$

$$S = \int d^7x \, \frac{1}{2} h^{\bar\mu,\bar\nu} \left(\frac{1}{6}\varepsilon_{\bar\mu}{}^{\alpha\bar\rho}\partial_\alpha - m\delta^{\bar\rho}_{\bar\mu}\right) G_{\bar\rho,\bar\nu}(h)$$

$$\delta h_{\bar\mu,\bar\nu} = \hat{Y}\left\{\partial_{[\mu_1}\xi_{\mu_2\mu_3],\nu_1\nu_2\nu_3}\right\}$$

ξ : 

35 physical d.o.f. , ghost-free

Summary

- In 3D, the higher derivative massive gravity theories at the linearized level, can be generalized as free gauge theories of higher spins. The generalization of the linearized TMG is always ghost-free.
- For specific situations, they can be further generalized beyond 3D, e.g. 3D TMG-like theories can be generalized to 4k-1 dimensions (type of gauge fields: rectangular Young tableaux of height 2k-1).
- The original NMG and TMG (3D spin-2, full theories) do have interactions, but we are now only able to generalize them at the linearized level. We hope someday we will be able to go beyond that.

Thank you !