Projective Superspace

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Outline

- Properties of Twistor Space
- Sigma Model derivation

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Hyperkähler space $\mathcal{M}$, $I, J, K | IJ = -JI = K$,
$aI + bJ + cK$ is again a Kähler structure on $\mathcal{M}$ if
$a^2 + b^2 + c^2 = 1$, *i.e.*, if $\{a, b, c\} \in S^2 \simeq \mathbb{P}^1$.
The Twistor space $\mathcal{Z}$ of a hyperkähler space $\mathcal{M}$ is the product of $\mathcal{M}$ with this two-sphere $\mathcal{Z} = \mathcal{M} \times \mathbb{P}^1$.
ζ coordinate on $\mathbb{P}^1$
A choice of ζ corresponds to a choice of a preferred complex structure, *e.g.*, J, with Kähler form $\omega^{(1,1)}$
I and K can be used to construct the holomorphic and antiholomorphic symplectic two-forms $\omega^{(2,0)}$ and $\omega^{(0,2)}$.

$$\Omega(\zeta) \equiv \omega^{(2,0)} + \zeta\omega^{(1,1)} - \zeta^2\omega^{(0,2)} ,$$

4*d* Hyperkähler space obeys the Monge-Ampère equation,

$$2\,\omega^{(2,0)}\,\omega^{(0,2)} = (\omega^{(1,1)})^2 ,$$

$\Leftrightarrow$

$$\Omega^2 = 0 .$$

Higher dimensions

$$\Omega^{n+1} = 0$$

$d\Omega = 0$, Ω nondegenerate,
$\implies \exists$ Darboux coordinates Υ^p and $\tilde{\Upsilon}_p$:

$$\Omega(\zeta) = i\, d\Upsilon^p(\zeta)\, d\tilde{\Upsilon}_p(\zeta)$$

Real-structure $\mathfrak{R}$ on $\mathbb{P}^1$ defined by complex conjugation composed with the antipodal map.
Reality condition

$$\Omega(\zeta) = -\zeta^2 \mathfrak{R}(\Omega(\zeta))\ ;$$

$$\mathfrak{R}(\Upsilon^p(\zeta)) = \bar{\Upsilon}^p(-\frac{1}{\zeta})$$

$$
i\, d\Upsilon^p(\zeta)\, d\tilde{\Upsilon}_p(\zeta) = i\,\zeta^2 d\bar{\Upsilon}^p(-\frac{1}{\zeta})\, d\bar{\tilde{\Upsilon}}_p(-\frac{1}{\zeta})
$$

$\Upsilon, \tilde{\Upsilon}$ related to $\bar{\Upsilon}, \bar{\tilde{\Upsilon}}$ by a twisted symplectomorphism.
Generating function $f(\Upsilon, \bar{\Upsilon}; \zeta)$

$$
\tilde{\Upsilon}_p = \zeta \frac{\partial f}{\partial \Upsilon^p}\,, \quad \bar{\tilde{\Upsilon}}_p = -\frac{1}{\zeta}\frac{\partial f}{\partial \bar{\Upsilon}^p}\;;
$$

then

$$
i\, d\Upsilon^p\, d\tilde{\Upsilon}_p = i\,\zeta \frac{\partial^2 f}{\partial \Upsilon^p \partial \bar{\Upsilon}^q} d\Upsilon^p\, d\bar{\Upsilon}^q \equiv i\,\zeta \partial \bar{\partial} f\;,
$$

The reality condition on Ω and the relation btw the two Darboux sets imply:

$$
\oint \frac{d\zeta}{2\pi i \zeta}\zeta^i \frac{\partial f}{\partial \Upsilon^p} = 0\,, \quad i \geqslant 2\,,
$$

Sigma Model

$$\{\mathbb{D}_{a\pm}, \bar{\mathbb{D}}^b_\pm\} = \pm i\delta^b_a \partial_{\pm\!\!\!\!=} \ , \quad \{\mathbb{D}_{a\pm}, \mathbb{D}_{b\pm}\} = 0$$
$$\{\mathbb{D}_{a\pm}, \mathbb{D}_{b\mp}\} = 0 \ , \qquad \{\mathbb{D}_{a\pm}, \bar{\mathbb{D}}^b_\mp\} = 0$$

$$\nabla(\zeta) = \mathbb{D}_2 + \zeta\mathbb{D}_1 \ , \quad \bar{\nabla}(\zeta) = \bar{\mathbb{D}}^1 - \zeta\bar{\mathbb{D}}^2$$

The bar on ∇ denotes conjugation with respect to a real structure $\mathfrak{R}$ defined as complex conjugation composed with the antipodal map on $\mathbb{P}^1 \backsimeq S^2$.

$$\{\nabla, \bar{\nabla}\} = 0$$

They may be used to introduce constraints on superfields similarly to how the $\mathcal{N} = (2, 2)$ derivatives are used to impose chirality constraints. Superfields now live in an extended superspace with coordinates x, ζ, θ.

The superfields Υ we shall be interested in satisfy the projective chirality constraint

$$\nabla\Upsilon = \bar{\nabla}\Upsilon = 0$$

and are taken to have the following ζ-expansion:

$$\Upsilon = \sum_i \Upsilon_i \zeta^i$$

When the index $i \in [0, \infty)$ the field Υ is analytic around the north pole of the $\mathbb{P}^1$ and consequently called an arctic multiplet. Real structure acting on superfields, $\Re(\Upsilon) \equiv \bar{\Upsilon}$, may be used to impose reality conditions on the superfields.

An $\mathcal{O}(2n)$

$$\Upsilon \equiv \eta_{(2n)} = (-)^n \zeta^{2n} \bar{\Upsilon}$$

The ζ-expansion is useful in displaying the $\mathcal{N} = 1$ content of the multiplets.

$$\eta_{(4)} = \phi + \zeta\Sigma + \zeta^2 X - \zeta^3 \overline{\Sigma} + \zeta^4 \bar{\phi}$$

$\mathcal{N} = 1$ fields: chiral ϕ, unconstrained X and complex linear Σ.

$$\bar{\mathbb{D}}^2 \Sigma = 0$$

and is dual to a chiral superfield.
A general arctic projective chiral Υ has the expansion

$$\Upsilon = \phi + \zeta\Sigma + \sum_{i=2}^{\infty} X_i \zeta^i$$

with all X_i's unconstrained.

The Generalized Legendre Transform

A $\mathcal{N} = 2$ invariant action is

$$S = \int \mathbb{D}^2 \bar{\mathbb{D}}^2 F$$

with

$$F \equiv \oint_C \frac{d\zeta}{2\pi i\zeta} f(\Upsilon, \bar{\Upsilon}; \zeta)$$

Eliminating the auxiliary fields X_i by their equations of motion will yield an $\mathcal{N} = 1$ model defined on the tangent bundle parametrized by (ϕ, Σ). Dualizing the complex linear fields Σ to chiral fields $\tilde{\phi}$ the final result is a $\mathcal{N} = 1$ sigma model in terms of $(\phi, \tilde{\phi})$ which is guaranteed by construction to have $\mathcal{N} = 2$ supersymmetry, and thus to define a hyperkähler metric.

These steps are:

Solve the equations of motion for the auxiliary fields:

$$\frac{\partial F}{\partial \Upsilon_i} = \oint_C \frac{d\zeta}{2\pi i \zeta} \zeta^i \left(\frac{\partial}{\partial \Upsilon} f(\Upsilon, \bar{\Upsilon}; \zeta) \right) = 0 \ , \quad i \geq 2$$

Solving these equations puts us on $\mathcal{N} = 2$-shell, which means that only the $\mathcal{N} = 1$ component symmetry remains off-shell. In $\mathcal{N} = 1$ superspace the resulting model, after eliminating X_i, is given by a Lagrangian $K(\phi, \bar{\phi}, \Sigma, \bar{\Sigma})$. This is finally dualized to $\tilde{K}(\phi, \bar{\phi}, \tilde{\phi}, \bar{\tilde{\phi}})$ via a Legendre transform

$$\tilde{K}(\phi, \bar{\phi}, \tilde{\phi}, \bar{\tilde{\phi}}) = K(\phi, \bar{\phi}, \Sigma, \bar{\Sigma}) - \tilde{\phi}\Sigma - \bar{\tilde{\phi}}\bar{\Sigma}$$

$$\tilde{\phi} = \frac{\partial K}{\partial \Sigma} \ , \quad \bar{\tilde{\phi}} = \frac{\partial K}{\partial \bar{\Sigma}}$$

Generating Function

$$\Omega \equiv i\zeta\partial\bar{\partial}f = i\zeta \frac{\partial^2}{\partial\Upsilon^a\partial\bar{\Upsilon}^{\bar{b}}} f(\Upsilon,\bar{\Upsilon};\zeta)\, d\Upsilon^a d\bar{\Upsilon}^{\bar{b}}$$

$$\Omega = i d\Upsilon d\tilde{\Upsilon} = i\zeta^2 d\bar{\Upsilon} d\bar{\tilde{\Upsilon}}$$

where $\bar{\tilde{\Upsilon}} = -\frac{1}{\zeta}\frac{\partial}{\partial\bar{\Upsilon}} f$. Note that because $\Upsilon, \tilde{\Upsilon}$ are arctic and $\bar{\Upsilon}, \bar{\tilde{\Upsilon}}$ are antarctic, equation this *implies* that Ω is a section of an $\mathcal{O}(2)$ bundle.

This relation has the form of a twisted symplectomorphism, and therefore there should exist a generating function for this transformation. It is the $N = 2$ superspace Lagrangian $f(\Upsilon, \bar{\Upsilon}; \zeta)$.

The generalized Legendre transform has been used to find metrics on the Hermitian symmetric spaces listed in the following table:

Compact	Non-Compact
$U(n+m)/U(n)\times U(m)$	$U(n,m)/U(n)\times U(m)$
$SO(2n)/U(n);\ Sp(n)/U(n)$	$SO^*(2n)/U(n);\ Sp(n,\mathbb{R})/U(n)$
$SO(n+2)/SO(n)\times SO(2)$	$SO_0(n+2)/SO(n)\times SO(2)$

Example Kuzenko

$\mathbb{CP}^n \equiv G_{1,n+1}(\mathbb{C}) = \mathrm{U}(n+1)/\mathrm{U}(n) \times \mathrm{U}(1)$
Start from a solution at the origin

$$\Upsilon^{(0)} = \zeta\Sigma^{(0)}$$

Choose coset representative $L(\phi, \bar{\phi})$ to extend the solution from the origin to an arbitrary point.

$$\Upsilon^* = \frac{\Upsilon^{(0)} + \phi}{1 - \Upsilon^{(0)}\bar{\phi}} = \frac{\zeta\Sigma^{(0)} + \phi}{1 - \zeta\Sigma^{(0)}\bar{\phi}}$$

$$\Sigma \equiv \frac{d\Upsilon^*}{d\zeta}|_{\zeta=0} = (1 + \phi\bar{\phi})\Sigma^{(0)}$$

yields

$$\Upsilon^* = \frac{(1 + \phi\bar{\phi})\phi + \zeta\Sigma}{(1 + \phi\bar{\phi}) - \zeta\Sigma\bar{\phi}}$$

$$K(\Upsilon^*, \bar{\Upsilon}^*) = K(\phi, \bar{\phi}) + ln(1 - g_{\phi\bar{\phi}}\Sigma\bar{\Sigma})$$

The final Legendre transform replacing the linear multiplet by a new chiral field $\Sigma \to \tilde{\phi}$ produces the Kähler potential $K(\phi, \bar{\phi}, \tilde{\phi}, \bar{\tilde{\phi}})$ for the Eguchi–Hanson metric.

Generalization

Doubly projective superspace ($d = 2$): At each point in ordinary superspace we introduce one $\mathbb{P}^1$ for each chirality and denote the corresponding coordinates by ζ_L and ζ_R.

$$\nabla_+(\zeta_L) = \mathbb{D}_{2+} + \zeta_L \mathbb{D}_{1+}$$

$$\nabla_-(\zeta_R) = \mathbb{D}_{2+} + \zeta_R \mathbb{D}_{1-}$$

$\mathfrak{R}$ acting on both ζ_L and ζ_R.

$$\Upsilon = \sum_{i,j} \Upsilon_{i,j} \zeta_L^i \zeta_R^j$$

Both left and right projectively chiral.

We may also impose reality conditions using $\mathfrak{R}$, as well as particular conditions on the components, such as the "cylindrical" condition

$$\Upsilon_{i,j+k} = \Upsilon_{i,j}$$

for some k. Actions are formed in analogy to previous. The $\mathcal{N} = (2,2)$ components of such a model include twisted chiral fields χ, as well as semi-chiral ones $\mathbb{X}_{L,R}$. This is the context in which the semi-chiral $\mathcal{N} = (2,2)$ superfields were introduced (T.Busher, U.L and M. Roček 1987)