Generalized Complex Geometry

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Outline

- Formulations of Generalized Kähler Geometry
- The corresponding Sigma Models

M. Göteman, C. Hull, M. Roček, I. Ryb, R. von Unge, M. Zabzine.

Outline II: Sigma Model geometry

- Sigma models
- SUSY sigma models and geometry
- Complex geometry
- Kähler
- Bihermitean geometry
- Generalized complex geometry
- Generalized Kähler geometry
- Superspace descriptions

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Outline III: Relation to supergravity

- Pure Spinors
- Generalized Calabi Yau
- Supergravity
- Relation to the Sigma Model formulation

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Sigma Models

$$\phi^i : \Sigma \to \mathcal{T}$$

$$\xi \mapsto \phi^i(\xi)$$

$$S = \int_\Sigma d\phi^i g_{ij}(\phi) \star d\phi^j$$

$$\nabla^2\phi^i := \partial^2\phi^i + \partial\phi^j \Gamma_{jk}{}^i \partial\phi^k = 0$$

$$S = \int_{\Sigma_B} d\xi \left\{ \eta^{\mu\nu}\partial_\mu X^i g_{ij}(X)\partial_\nu X^j + \dots \right\}$$

Susy Sigma Models

$$\boxed{\text{Susy } \sigma \text{ models} \iff \text{Geometry of } \mathcal{T}}$$

d=	6	4	2	Geometry
N=	1	2	4	Hyperkähler
N=		1	2	Kähler
N=			1	Riemannian

(Odd dimensions have the same structure as the even dimension lower.)

Sigma models in d=2

The (1,1) analysis by Gates Hull and Roček gives:

Susy	(0,0) (1,1)	(2,2)	(2,2)	(4,4)	(4,4)
Bgd	G, B	G	G, B	G	G, B
Geom	Riem.	Kähler	biherm.	hyperk.	bihyperc.

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Complex Geometry

Manifold (M^{2d}, J)

Complex structure: $J \in End(TM) \qquad J^2 = -1$

Projectors: $\quad \pi_\pm := \frac{1}{2}(\mathbf{1} \pm iJ)$

These define an involutive distribution if

$$\pi_\mp[\pi_\pm u, \pi_\pm v] = 0 \quad \Longleftrightarrow \quad \mathcal{N}(J) = 0 \text{ (Nijenhuis)}$$

This is integrability of J.
Local holomorphic coordinates, $M^{2d} \supset \mathcal{O} \approx \mathbb{C}^{2k}$, and holomorphic transition functions.

Hermitean

Hermitean Metric: $J^t g J = g$

$(g \to g = g + J^t g J)$

Symplectic 2-form: $\omega := gJ$

Kähler: $d\omega = 0, \quad \nabla J = 0, \qquad g_{z\bar{z}} = \partial_z \partial_{\bar{z}} K(z, \bar{z})$

Hyperkähler: $J^A, A = 1, 2, 3 \quad J^A J^B = -\delta^{AB} + \epsilon^{ABC} J^C$

Kähler Geometry

Erich Kähler 1906-2000

$K3$

$$(M, g, J_{(\pm)}, H)$$

$$\boxed{\begin{array}{c} J^2_{(\pm)} = -\mathbf{1}\ , \quad J^t_{(\pm)} g J_{(\pm)} = g\ , \quad \nabla^{(\pm)} J_{(\pm)} = 0 \\ \Gamma^{(\pm)} = \Gamma^0 \pm \tfrac{1}{2} g^{-1} H\ , \quad H = dB\ . \end{array}}$$

$$E := g + B$$

$$(M, g, J_{(\pm)})$$

$$\boxed{\begin{gathered} J^2_{(\pm)} = -\mathbf{1}\,, \quad J^t_{(\pm)} g J_{(\pm)} = g\,, \quad \omega_{(\pm)} := gJ_{(\pm)} \\ d^c_{(+)}\omega_{(+)} + d^c_{(-)}\omega_{(-)} = 0\,, \quad dd^c_{(\pm)}\omega_{(\pm)} = 0\,, \\ H := d^c_{(+)}\omega_{(+)} = -d^c_{(-)}\omega_{(-)} \end{gathered}}$$

Generalized Complex Geometry

Complex structure:

$$\mathcal{J} \in End(TM \oplus T^*M), \qquad \mathcal{J}^2 = -1$$

$$\Pi_\pm := \tfrac{1}{2}(\mathbf{1} \pm \mathcal{J})$$

"Nijenhuis":

$$\mathcal{N}_C(\mathcal{J}) = 0 \quad \Longleftrightarrow \quad \Pi_\mp[\Pi_\pm U, \Pi_\pm V]_C = 0$$

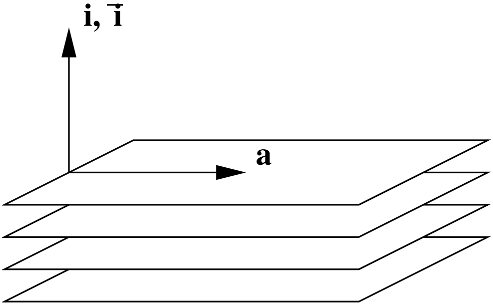
where

$$U = (u, \xi)\ , \quad V = (v, \rho)$$

$$[U, V]_C = [u, v] + \mathcal{L}_u\rho - \mathcal{L}_v\xi - \tfrac{1}{2}d(\imath_u\rho - \imath_v\xi)$$

"N-N"

When $\mathcal{J}$ is integrable there are local holomorphic and Darboux coordinates such that M^{2d} looks like $\mathbb{C}^k \times \mathbb{R}^{2d-k}$.



The automorphisms of this courant bracket are diffeomorphisms and *b*-transforms:

$$e^{b}(u,\xi)=(u,\xi+\imath_{u}b)\ ,\quad db=0\ .$$

In a coordinate basis (∂_x, dx) a b-transform acts on $\mathcal{J}$ as follows:

$$\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\mathcal{J}\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}\ ,$$

In such a basis, the natural pairing

$$< (u, \xi), (v, \rho) >= \imath_u \rho + \imath_v \xi$$

is represented by the matrix

$$\mathcal{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

A final requirement of GCG is that

$$\mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I}$$

Generalized Kähler Geometry

Two commuting generalized complex structures

$$\mathcal{J}^2_{(1,2)} = -\mathbf{1} \; , \qquad [\mathcal{J}_{(1)}, \mathcal{J}_{(2)}] = 0 \; ,$$

$$\mathcal{J}^t_{(1,2)} \mathcal{I} \mathcal{J}_{(1,2)} = \mathcal{I} \; , \qquad \mathcal{G} := -\mathcal{J}_{(1)} \mathcal{J}_{(2)}$$

Ex. Kähler $(\omega = gJ)$:

$$\mathcal{J}_1 = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix} \quad \mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \quad \mathcal{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$$

The G-map

GKG $\leftrightarrow$ Bi-Hermitean :

$\mathcal{J}_{(1,2)} =$

$$\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} J_{(+)} \pm J_{(-)} & -(\omega^{-1}_{(+)} \mp \omega^{-1}_{(-)}) \\ \omega_{(+)} \mp \omega_{(-)} & -(J^t_{(+)} \pm J^t_{(-)}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}$$

$$[U, V]_H = [U, V]_C + \imath_U \imath_V H$$

$$(M, J_{(\pm)})$$

Locally, $\exists$ "symplectic" two-forms $\mathcal{F}_{(\pm)}$ such that

$$\boxed{\mathcal{F}_{(\pm)}(v, J_{(\pm)}v) > 0\ , \quad d(\mathcal{F}_{(+)}J_{(+)} - J^t_{(-)}\mathcal{F}_{(-)}) = 0\ .}$$

$$\mathcal{F}_{(\pm)} = \tfrac{1}{2}i(B^{(2,0)}_{(\pm)} - B^{(0,2)}_{(\pm)}) \mp \omega_{(\pm)}$$

$$\mathcal{F}_{(+)} = -\frac{1}{2}E^t_{(+)}J_{(+)}\ , \quad \mathcal{F}_{(-)} = -\frac{1}{2}J^t_{(-)}E^t_{(-)}$$

Geometric data: $(M, g, H, J_{(\pm)})$ or $(M, g, J_{(\pm)})$ or $(M, \mathcal{F}_{(\pm)}, J_{(\pm)})$. In each case, there is a complete description in terms of a Generalized Kähler potential K. Unlike the Kähler case, the expressions are non-linear in second derivatives of K. E.g.,

$$J_{(+)} = \begin{pmatrix} J & 0 \\ (K_{LR})^{-1}[J, K_{LL}] & (K_{LR})^{-1} J K_{LR} \end{pmatrix}$$

$$g = \Omega[J_{(+)}, J_{(-)}]$$

$$\mathcal{F}_{(+)} = d\lambda_{(+)} , \quad \lambda_{(+)\ell} = iK_R J (K_{LR})^{-1} K_{L\ell} , \ldots$$

Generating function

There are two special sets of Darboux coordinates for the symplectic form Ω. One set, $(\mathbb{X}^L, \mathbb{Y}_L)$, is also canonical coordinates for $J_{(+)}$ and the other set, $(\mathbb{X}^R, \mathbb{Y}_R)$ is canonical coordinates for $J_{(-)}$. The symplectomorphism that relates the two sets of coordinates has thus a generating function. This generating function is in fact the generalized Kähler-potential $K(\mathbb{X}^L, \mathbb{X}^R)$.

$(\mathbb{X}^L, \mathbb{Y}_L)$	$\leftarrow K(\mathbb{X}^L, \mathbb{X}^R) \rightarrow$	$(\mathbb{X}^R, \mathbb{Y}_R)$
$J_{(+)} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$		$J_{(-)} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

This fact is a key ingredient in the proof that we have a complete description or GKG.

$d = 2\ ,\ \ N = (2,2)$

Algebra:

$$\{\mathbb{D}_\pm, \bar{\mathbb{D}}_\pm\} = i\partial_{\substack{+\\=}}$$

Constrained superfields:

$$\begin{aligned} \bar{\mathbb{D}}_\pm \phi^a &= 0\ , \\ \bar{\mathbb{D}}_+ \chi^{a'} = \mathbb{D}_- \chi^{a'} &= 0\ , \\ \bar{\mathbb{D}}_+ \mathbb{X}^\ell &= 0\ , \\ \bar{\mathbb{D}}_- \mathbb{X}^r &= 0\ . \end{aligned}$$

Notation: $\quad c := a, \bar{a}\ , \quad t := a', \bar{a}'\ , \quad L := \ell, \bar{\ell}\ , \quad R := r, \bar{r}\ .$

The $(2,2)$ formulation uses the generalized Kähler Potential.

$$S = \int \mathbb{D}_+ \bar{\mathbb{D}}_+ \mathbb{D}_- \bar{\mathbb{D}}_- K(\phi^c, \chi^t, \mathbb{X}^L, \mathbb{X}^R)$$

$K \to K(\mathbb{X}^L, \mathbb{X}^R)$

Reduction to $(2,1)$ superspace

$\mathbb{D}_- =: D_- - iQ_-$, $\quad \mathbb{X}| =: X$, $\quad Q_- \mathbb{X}^L| =: \Psi^L_-$

$$S = \int \mathbb{D}_+ \bar{\mathbb{D}}_+ D_- \left(K_L \Psi^L_- + K_R J D_- X^R \right)$$

$$S = i \int \mathbb{D}_+ \bar{\mathbb{D}}_+ D_- (\lambda_{(+)\alpha} D_- \varphi^\alpha + c.c.)$$

which uses the "Liouville form" $(\mathcal{F}_{(+)} = d\lambda_{(+)})$

Superspace II

Reduction to $(1,1)$ finally yields

$$S = \int D_+ D_- \left(D_+ X E D_- X \right) \ .$$

The reduction goes via $\mathbb{D}_+ =: D_+ - iQ_+ \ , \quad Q_+ \mathbb{X}^R| =: \Psi_+^R$ and both the auxiliary spinors Ψ_-^L and Ψ_+^R have been eliminated.

The $(1,1)$ formulation uses $E = g + B$ directly.

Summary

Superspace encodes and dictates
all the geometric formulations
of Generalized Kähler Geometry.

Pure Spinors

Multi forms ρ on M are spinors of $T \oplus T^*$.

$U = (u, \xi) \in T \oplus T^*$ acts on a form ρ according to

$$U \cdot \rho = \imath_u \rho + \xi \wedge \rho$$

This satisfies the Clifford algebra identity for the indefinite metric $\mathcal{I}$:

$$\{U, V\} \cdot \rho = (U \cdot V + V \cdot U) \cdot \rho = 2\mathcal{I}(U, V)\rho$$

The null space of a spinor ρ

$$L_\rho = \{U \in TM \oplus T^*M | U \cdot \rho = 0\}$$

is isotropic. A spinor ρ is pure if its null space is maximally isotropic, rank d.

A GCS $\mathcal{J}$ may alternatively be defined via decomposition

$$(T \oplus T^*) \otimes \mathbb{C} = L + \bar{L}$$

where L is the $+i$ eigenbundle of $\mathcal{J}$.

To every GCS $\mathcal{J}$ with $+i$ eigenbundle $L_\mathcal{J}$ is associated a *complex* pure spinor $\rho_\mathcal{J}$:

$$L_\mathcal{J} = L_{\rho_\mathcal{J}}$$

Generalized CY

A *generalized Calabi-Yau structure*:

$$d\rho = 0\ , \quad (\rho, \bar{\rho}) \neq 0$$

A *generalized Calabi-Yau metric structure* is defined as a pair of closed pure spinors ρ_1 and ρ_2 such that the corresponding generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ give rise to a GKS and

$$(\rho_1, \bar{\rho}_1) = \alpha(\rho_2, \bar{\rho}_2) \neq 0$$

Mukai pairing:

$$(\rho_1, \rho_2) = \sum_j (-1)^j [\rho_1^{2j} \wedge \rho_2^{d-2j} + \rho_1^{2j+1} \wedge \rho_2^{d-2j-1}]\ ,$$

The conditions for Type II supergravity solutions in

$$ds^2_{(10)} = e^{2A(y)} ds^2_{(4)} + g_{mn} dy^m dy^n$$

is

$$\begin{aligned} d_H(e^{4A-\Phi} \Re \rho_1) &= e^{4A} \tilde{F} \\ d_H(e^{3A-\Phi} \rho_2) &= 0 \\ d_H(e^{2A-\Phi} \Im \rho_1) &= 0 \end{aligned}$$

where $\tilde{F}$ is (part of) the polyform of RR fields.

The generalized CY metric structure defines a Type II supersymmetric supergravity solution.
(No RR fluxes).

Use the Gualtieri map to find $(g_{\mu\nu}, H_{\mu\nu\rho}, \Phi)$.

$$(\rho_1, \bar{\rho}_1) = \alpha(\rho_2, \bar{\rho}_2) = e^{-2\Phi}\sqrt{g}\, dx^1 \wedge .... \wedge dx^D$$

$$R^{(+)}_{\mu\nu} + 2\nabla^{(-)}_{\mu}\partial_{\nu}\Phi = 0\ ,$$

automatically satisfied.

Construction from the sigma model

Ansatz:

$$\rho_{1,2} = N_{1,2} \wedge e^{R_{1,2}+iS_{1,2}} \ , \qquad (0.1)$$

where

$$\begin{aligned}
N_1 &= e^{f(\phi)} d\phi^1 \wedge \ldots \wedge d\phi^{d_c} \ , \\
N_2 &= e^{g(\chi)} d\chi^1 \wedge \ldots \wedge d\chi_{d_t} \ , \\
R_1 &= -d(K_L dX_L) \ , \\
R_2 &= -d(K_R dX_R) \ , \\
S_1 &= d(K_T J d\chi + K_L J dX_L - K_R J dX_R) \ , \\
S_2 &= -d(K_C J d\phi + K_L J dX_L + K_R J dX_R) \ ,
\end{aligned}$$

These are pure spinors with the correct properties.

The Generalized Monge-Ampère equation

$$(\rho_1, \rho_1) = \alpha(\rho_2, \rho_2) \implies$$

$$(-1)^{d_s d_c} e^{f(\phi)} e^{\bar{f}(\bar{\phi})} \det \begin{pmatrix} -K_{\bar{l}l} & -K_{lr} & -K_{l\bar{t}} \\ -K_{\bar{r}l} & -K_{\bar{r}r} & -K_{\bar{r}\bar{t}} \\ -K_{t\bar{l}} & -K_{tr} & -K_{t\bar{t}} \end{pmatrix} \tag{0.2}$$

$$= \alpha \; e^{g(\chi)} e^{\bar{g}(\bar{\chi})} \det \begin{pmatrix} K_{l\bar{r}} & K_{l\bar{l}} & K_{l\bar{c}} \\ K_{r\bar{r}} & K_{r\bar{l}} & K_{r\bar{c}} \\ K_{c\bar{r}} & K_{c\bar{l}} & K_{c\bar{c}} \end{pmatrix}$$

$$e^{2\Phi} = (-1)^{d_s d_c} \frac{e^{-f(\phi)} e^{-\bar{f}(\bar{\phi})}}{\det K_{LR}} \det \begin{pmatrix} -K_{\bar{l}l} & -K_{lr} & -K_{l\bar{t}} \\ -K_{\bar{r}l} & -K_{\bar{r}r} & -K_{\bar{r}\bar{t}} \\ -K_{t\bar{l}} & -K_{tr} & -K_{t\bar{t}} \end{pmatrix} \tag{0.3}$$

Outlook

- Include RR fields in the geometry.
  See recent work by Waldram and collaborators.
- Understand reduction of GKG.
  Cavalcanti, Gualtieri,....
- SKT (strong Kähler with Torsion).
  Cavalcanti,...,$(2,1)$ -models.
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THANK YOU FOR YOUR ATTENTION!