# Coset Space Dimensional Reduction (CSDR) over $SU(3)/U(1) \times U(1)$ (Review)

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# Table of contents

# Motivation

- Use of higher dimensional unified theories
  - 4 dimensional effective theory
- Unify gauge and Higgs sectors
- Unify fermion interactions with gauge and Higgs fields

# Introduction

$\mathcal{N}=1$, SYM of gauge group $G$ theory in $10D$ manifold $\mathcal{M}$

$$\mathcal{S}=\int dx^n\sqrt{-g}\left(\frac{i}{2}\bar{\psi}\Gamma_M D^M\psi-\frac{1}{2}Tr\{F_{MN}F^{MN}\}\right)$$

- $\mathcal{M}=\mathcal{M}_4\times B$
  - $B=S/R$ Coset Space
    - $S,R$ Lie Groups and $R$ is a subgroup of $S$
- $M,N=0,\ldots,9\ /\ \mu,\nu=0,\ldots,3\ /\ a,b=1,\ldots,6$
  - $4D\ coords\ x^\mu$
  - $6D\ coords\ y^a=x^{3+a}$
- $g^{MN}=\begin{pmatrix}\eta^{\mu\nu} & 0\\ 0 & -g^{ab}\end{pmatrix}$
  - $\eta^{\mu\nu}=diag\{1,-1,-1,-1\}$
  - $g^{ab}$ is the metric of the coset space

- $\Gamma^M: \ \{\Gamma^M, \Gamma^N\} = 2g^{MN} I_{32}$
- $\mathcal{N} = 1$ in $10D$
    - $\psi$ Weyl - Majorana spinor $\Rightarrow$ 8 d.o.f.
    - $A_M$ has 8 d.o.f.
- $D^M = \partial^M - igA^M - \theta^M$ , $F^{MN} = \partial^M A^N - \partial^N A^M + g[A^M, A^N]$
    - $\theta^M$ is the spin connection
- Fields: non trivial dependence from $y^a$, but we impose the condition that a symmetry transformation by an element of the isometry group $S$ of $B$ is compensated by a gauge transformation.
    - $\mathcal{L}$ is independent of $y^a$ just because is gauge invariant

**CSDR** $\Longrightarrow \mathcal{L}_{4D}$

# Coset Space Geometry

- Coset Space: $B = S / R$
  - $S$ isometry group
  - $R$ isotropy group
- Generators of R : $Q_i$
- Generators of S : $Q_i$, $Q_a$
- The commutation relations are:

$$
\begin{aligned}
[Q_i, Q_j] &= f_{ij}^{k} Q_k \\
[Q_i, Q_a] &= f_{ia}^{b} Q_b + f_{ia}^{j} Q_j \\
[Q_a, Q_b] &= f_{ab}^{i} Q_i + f_{ab}^{c} Q_c
\end{aligned}
$$

- 0 because $S$, $R$ reductive groups, since $R$ is compact
- 0 in the case that the coset space is symmetric

The general element of the group $S$ can be written as

$$
\begin{aligned}
s &= exp\{\omega^{\alpha} \delta_{\alpha}^{a} Q_a + \omega^{i} Q_i\} \\
&= exp\{y^{\alpha} \delta_{\alpha}^{a} Q_a\} exp\{\phi^{i} Q_i\} \\
&= L(y) r
\end{aligned}
$$

Where $r$ is the element of the group $R$.

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$$

Where $r$ is the element of the group $R$.

The Mauren-Cartan (MC) 1-form

$$
\begin{aligned}
e(y) &= L^{-1}(y)dL(y) \\
&= e_\alpha^A Q_A dy^\alpha \\
&= e^a Q_a + e^i Q_i
\end{aligned}
$$

- $e^a$ : coframe
- $e^i$ : R-connection

$e^i$ can be expanded in the coset vielbeins as:

$$e^i = e^i_a(y)e^a \ .$$

# Reduction to the $4D$ theory

The Action in $10D$ theory of gauge group $G = E_8$ is given

$$\mathcal{S} = \int dx^4 dy^6 \sqrt{-g}\left(\frac{i}{2}\bar{\psi}\Gamma_M D^M\psi - \frac{1}{2}Tr\{F_{MN}F^{MN}\}\right)$$

Imply the Symmetric condition

$$\begin{aligned} A_\mu(x,y) =& g(s)A_\mu(x,s^{-1}y)g^{-1}(s) \\ A_\alpha(x,y) =& g(s)J_\alpha^\beta A_\beta(x,s^{-1}y)g^{-1}(s) + g(s)\partial_\alpha g^{-1}(s) \\ \psi(x,y) =& f(s)\Omega(y,s)\psi(x,s^{-1}y)f^{-1}(s) \end{aligned}$$

- ↪ $g(s)$ : gauge transformation for the gauge fields (*adj G*)
- ↪ $f(s)$ : gauge transformation for the fermion fields (*rep F of G*)
- ↪ $\Omega(s,y)$ : rotation of spinor field for coordinate transformation
- ↪ $J_\alpha^\beta$ : Jacobian for coordinate transformation

It connects transformation of $S/R$ coordinates and gauge transformation

Coordinate transformation is compensated by a gauge transformation

The $4D$ Lagrangian will have the form

$$\mathcal{L}_{eff} = \frac{i}{2}\bar{\psi}\Gamma_{\mu}D^{\mu}\psi - \frac{i}{2}\bar{\psi}\Gamma_{a}D^{a}\psi - \frac{1}{2}Tr\{F_{\mu\nu}F^{\mu\nu}\} + \frac{1}{2}(D_{\mu}\phi_a)(D^{\mu}\phi^a) - V(\phi)$$

$$V(\phi) = \frac{1}{2}g^{ac}g^{bd}Tr\{F_{ab}F_{cd}\}$$

$$F_{ab} = f_{ab}^{C}\phi_C - [\phi_a,\phi_b]$$

f structure constants of $S$

- $D_{\mu} = \partial_{\mu} - A_{\mu}$
- $D_a = \partial_a - \theta_a - \phi_a$
- $\theta_a = \frac{1}{2}\theta_{abc}\Sigma^{bc}$

Still $V(\phi)$ only formal since $\phi_a$ must satisfy

$$f_{ai}^{D}\phi_D - [\phi_a,\phi_i] = 0$$

# Gauge group and it's reps in $4D$

**Gauge bosons in $4D$**

The gauge group $H$ in $4D$ is the centralizer of $R$ in $G$

$$G \supset R_G \times H \hookrightarrow H = C_G(R)$$

Then the gauge group is $H$.

$$E_8 \supset U(1) \times U(1) \times E_6$$

In the case when $R$ $(= U(1) \times U(1))$ is Abelian group then the centralizer of $G$ $(= E_8)$ is

$$H = C_{E_8}(U(1) \times U(1)) = E_6 \times U(1) \times U(1)$$

## Scalar fields in $4D$

$$G \supset R_G \times H$$

The $adj\mathbf{G}$ decomposes under $R_G \times H$ as:

$$adj\mathbf{G} = (adj\mathbf{R}, 1) + (1, adj\mathbf{H}) + \sum (r_i, h_i)$$

$$S \supset R$$

The $adj\mathbf{S}$ decomposes under $R$ as:

$$adj\mathbf{S} = adj\mathbf{R} + \sum s_i$$

$\forall s_i \ : s_i = r_j \implies h_j$ is a representation of the scalar fields

For $G = E_8$ and $R = U(1)\times U(1)$

$$E_8 \supset U(1)\times U(1)\times E_6$$

The $\mathbf{248} = adj\mathbf{E_8}$ decomposes under $U(1)\times U(1)\times E_6$

$$\begin{aligned}\mathbf{248} = &\boxed{\mathbf{1}_{(0,0)} + \mathbf{1}_{(0,0)} + \mathbf{78}_{(0,0)}}\\ &+\mathbf{1}_{(3,\frac{1}{2})} + \mathbf{1}_{(-3,\frac{1}{2})} + \mathbf{1}_{(0,-1)}\\ &+\mathbf{27}_{(3,\frac{1}{2})} + \mathbf{27}_{(-3,\frac{1}{2})} + \mathbf{27}_{(0,-1)}\\ &+\mathbf{1}_{(-3,-\frac{1}{2})} + \mathbf{1}_{(3,-\frac{1}{2})} + \mathbf{1}_{(0,1)}\\ &+\overline{\mathbf{27}}_{(-3,-\frac{1}{2})} + \overline{\mathbf{27}}_{(3,-\frac{1}{2})} + \overline{\mathbf{27}}_{(0,1)}\end{aligned}$$

$adj\mathbf{U(1)} + adj\mathbf{U(1)} + adj\mathbf{E_6}$

$S = SU(3)$ and $SU(3) \supset U(1)\times U(1)$. The $adj\mathbf{SU(3)} = \mathbf{8}$ decomposes under $U(1)\times U(1)$

$$\begin{aligned}\mathbf{8} = &\boxed{(0,0) + (0,0)}\\ &+(3,\frac{1}{2}) + (-3,\frac{1}{2}) + (0,-1)\\ &+(-3,-\frac{1}{2}) + (3,-\frac{1}{2}) + (0,1)\end{aligned}$$

$adj\mathbf{U(1)} + adj\mathbf{U(1)}$

The $U(1)\times U(1)$ charges will survive in $4D$ theory

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$$+\overline{\mathbf{27}}_{(-3,-\frac{1}{2})} + \overline{\mathbf{27}}_{(3,-\frac{1}{2})} + \overline{\mathbf{27}}_{(0,1)}$$

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$S = SU(3)$ and $SU(3)\supset U(1)\times U(1)$. The $adj\mathbf{SU(3)} = \mathbf{8}$ decomposes under $U(1)\times U(1)$

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$adj\mathbf{U(1)} + adj\mathbf{U(1)}$

The $U(1)\times U(1)$ charges will survive in $4D$ theory

# Scalar Potential in $4D$

$$
\begin{aligned}
V(\alpha^i,\alpha,\beta^i,\beta,\gamma^i,\gamma) = const. &+ \left(\frac{4R_1^2}{R_2^2R_3^2}-\frac{8}{R_1^2}\right)\alpha^i\alpha_i + \left(\frac{4R_1^2}{R_2^2R_3^2}-\frac{8}{R_1^2}\right)\overline{\alpha}\alpha \\
&+ \left(\frac{4R_2^2}{R_1^2R_3^2}-\frac{8}{R_2^2}\right)\beta^i\beta_i + \left(\frac{4R_2^2}{R_1^2R_3^2}-\frac{8}{R_2^2}\right)\overline{\beta}\beta \\
&+ \left(\frac{4R_3^2}{R_1^2R_2^2}-\frac{8}{R_3^2}\right)\gamma^i\gamma_i + \left(\frac{4R_3^2}{R_1^2R_2^2}-\frac{8}{R_3^2}\right)\overline{\gamma}\gamma \\
&+ \left[\sqrt{280}\left(\frac{R_1}{R_2R_3}+\frac{R_2}{R_1R_3}+\frac{R_3}{R_2R_1}\right)d_{ijk}\alpha^i\beta^j\gamma^k\right. \\
&\left.+\sqrt{280}\left(\frac{R_1}{R_2R_3}+\frac{R_2}{R_1R_3}+\frac{R_3}{R_2R_1}\right)\alpha\beta\gamma + h.c\right] \\
&+\frac{1}{6}\left(\alpha^i(G^\alpha)_i^j\alpha_j + \beta^i(G^\alpha)_i^j\beta_j + \gamma^i(G^\alpha)_i^j\gamma_j\right)^2 \\
&+\frac{10}{6}\left(\alpha^i(3\delta_i^j)\alpha_j + \overline{\alpha}(3)\alpha + \beta^i(-3\delta_i^j)\beta_j + \overline{\beta}(-3)\beta\right)^2 \\
&+\frac{40}{6}\left(\alpha^i(\frac{1}{2}\delta_i^j)\alpha_j + \overline{\alpha}(\frac{1}{2})\alpha + \beta^i(\frac{1}{2}\delta_i^j)\beta_j + \overline{\beta}(\frac{1}{2})\beta + \gamma^i(-1\delta_i^j)\gamma_j + \overline{\gamma}(-1)\gamma\right)^2 \\
&+ 40\alpha^i\beta^j d_{ijk}d^{klm}\alpha_l\beta_m + 40\beta^i\gamma^j d_{ijk}d^{klm}\beta_l\gamma_m + 40\alpha^i\gamma^j d_{ijk}d^{klm}\alpha_l\gamma_m \\
&+40(\overline{\alpha}\overline{\beta})(\alpha\beta) + 40(\overline{\beta}\overline{\gamma})(\beta\gamma) + 40(\overline{\gamma}\overline{\alpha})(\gamma\alpha)
\end{aligned}
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$\alpha \sim 1_{(3,\frac{1}{2})}$

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# Scalar Potential in $4D$

## Soft Breaking Terms

$$
\begin{aligned}
V(\alpha^i,\alpha,\beta^i,\beta,\gamma^i,\gamma) = const. &+ \left(\frac{4R_1^2}{R_2^2R_3^2}-\frac{8}{R_1^2}\right)\alpha^i\alpha_i + \left(\frac{4R_1^2}{R_2^2R_3^2}-\frac{8}{R_1^2}\right)\overline{\alpha}\alpha \\
&+ \left(\frac{4R_2^2}{R_1^2R_3^2}-\frac{8}{R_2^2}\right)\beta^i\beta_i + \left(\frac{4R_2^2}{R_1^2R_3^2}-\frac{8}{R_2^2}\right)\overline{\beta}\beta \\
&+ \left(\frac{4R_3^2}{R_1^2R_2^2}-\frac{8}{R_3^2}\right)\gamma^i\gamma_i + \left(\frac{4R_3^2}{R_1^2R_2^2}-\frac{8}{R_3^2}\right)\overline{\gamma}\gamma \\
&+ \Bigg[\sqrt{280}\left(\frac{R_1}{R_2R_3}+\frac{R_2}{R_1R_3}+\frac{R_3}{R_2R_1}\right)d_{ijk}\alpha^i\beta^j\gamma^k \\
&+ \sqrt{280}\left(\frac{R_1}{R_2R_3}+\frac{R_2}{R_1R_3}+\frac{R_3}{R_2R_1}\right)\alpha\beta\gamma + h.c\Bigg] \\
&+ \frac{1}{6}\left(\alpha^i(G^\alpha)_i^j\alpha_j + \beta^i(G^\alpha)_i^j\beta_j + \gamma^i(G^\alpha)_i^j\gamma_j\right)^2 \\
&+ \frac{10}{6}\left(\alpha^i(3\delta_i^j)\alpha_j + \overline{\alpha}(3)\alpha + \beta^i(-3\delta_i^j)\beta_j + \overline{\beta}(-3)\beta\right)^2 \\
&+ \frac{40}{6}\left(\alpha^i(\frac{1}{2}\delta_i^j)\alpha_j + \overline{\alpha}(\frac{1}{2})\alpha + \beta^i(\frac{1}{2}\delta_i^j)\beta_j + \overline{\beta}(\frac{1}{2})\beta + \gamma^i(-1\delta_i^j)\gamma_j + \overline{\gamma}(-1)\gamma\right)^2 \\
&+ 40\alpha^i\beta^j d_{ijk}d^{klm}\alpha_l\beta_m + 40\beta^i\gamma^j d_{ijk}d^{klm}\beta_l\gamma_m + 40\alpha^i\gamma^j d_{ijk}d^{klm}\alpha_l\gamma_m \\
&+ 40(\overline{\alpha}\overline{\beta})(\alpha\beta) + 40(\overline{\beta}\overline{\gamma})(\beta\gamma) + 40(\overline{\gamma}\overline{\alpha})(\gamma\alpha)
\end{aligned}
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$\alpha \sim 1_{(3,\frac{1}{2})}$

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# Scalar Potential in $4D$

## D-Terms

$$
\begin{aligned}
V(\alpha^i,\alpha,\beta^i,\beta,\gamma^i,\gamma) = const. &+ \left(\frac{4R_1^2}{R_2^2R_3^2}-\frac{8}{R_1^2}\right)\alpha^i\alpha_i + \left(\frac{4R_1^2}{R_2^2R_3^2}-\frac{8}{R_1^2}\right)\overline{\alpha}\alpha \\
&+ \left(\frac{4R_2^2}{R_1^2R_3^2}-\frac{8}{R_2^2}\right)\beta^i\beta_i + \left(\frac{4R_2^2}{R_1^2R_3^2}-\frac{8}{R_2^2}\right)\overline{\beta}\beta \\
&+ \left(\frac{4R_3^2}{R_1^2R_2^2}-\frac{8}{R_3^2}\right)\gamma^i\gamma_i + \left(\frac{4R_3^2}{R_1^2R_2^2}-\frac{8}{R_3^2}\right)\overline{\gamma}\gamma \\
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&+ \frac{1}{6}\left(\alpha^i(G^\alpha)_i^j\alpha_j + \beta^i(G^\alpha)_i^j\beta_j + \gamma^i(G^\alpha)_i^j\gamma_j\right)^2 \\
&+ \frac{10}{6}\left(\alpha^i(3\delta_i^j)\alpha_j + \overline{\alpha}(3)\alpha + \beta^i(-3\delta_i^j)\beta_j + \overline{\beta}(-3)\beta\right)^2 \\
&+ \frac{40}{6}\left(\alpha^i(\frac{1}{2}\delta_i^j)\alpha_j + \overline{\alpha}(\frac{1}{2})\alpha + \beta^i(\frac{1}{2}\delta_i^j)\beta_j + \overline{\beta}(\frac{1}{2})\beta + \gamma^i(-1\delta_i^j)\gamma_j + \overline{\gamma}(-1)\gamma\right)^2 \\
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## Fermion fields in $4D$

The spinor representation of $G$ decomposes under $R \times H$

$$F = \sum (t_i, h_i)$$

and the spinor representation of $SO(d)$ decomposes under $R$

$$\sigma_d = \sum \sigma_i$$

$\forall\, \sigma_j : \; \sigma_j = t_i \implies h_i$ is a representation of $4D$ theory

The spinor in **D** $-$ *Dimenssions* can be decomposed

$$\begin{aligned} SO(1, D-1) &\supset SO(1,3) \times SO(d) \\ &= SU(2)_L \times SU(2)_R \times SO(d) \end{aligned}$$

$d = D - 4$ if $d$ odd

$$\sigma_D = (2, 1, \sigma_d) + (1, 2, \sigma_d)$$

and if $d$ even

$$\begin{aligned} \sigma_D =&(2, 1, \sigma_d) + (1, 2, \bar{\sigma}_d) \\ &+(2, 1, \bar{\sigma}_d) + (1, 2, \sigma_d) \end{aligned}$$

In even dimensions we can impose the **Weyl condition** ($\Gamma^* \Psi = \pm \Psi$) and that leads

(with "+")

$$\sigma_{D_W} = (2, 1, \sigma_d) + (1, 2, \bar{\sigma}_d)$$

(or with " $-$ " then $\sigma_{D_W} = (2, 1, \bar{\sigma}_d) + (1, 2, \sigma_d)$)

If **D** $=$ **4n** $+$ **2** dimensions, the spinor is further reducible, we can impose **also** **Majorana condition** ($\Psi^c = \Psi$)

- In case we have Dirac fermions in higher dimensions then it is imposible to get chiral fermions in $4D$. For odd dimensional theories there is no hope to obtain **chiral** fermions by that method.
- When $F$ is a vector like representation and we are in even dimensions then we can impose the **Weyl condition** and get a **chiral** theory but with 2 identical copies.
- In the case $F$ is a vector like representation, $\mathbf{D = 4n + 2}$, then we can aply **also** **Majorana condition** and we get **chiral** theory.

$\Longrightarrow$ In $10D$ with **Weyl-Majorana condition** we can obtain chiral theory with CSDR.

For $G = E_8$ and $R = U(1)\times U(1)$

$$E_8 \supset U(1)\times U(1)\times E_6$$

F vector like $\mathbf{248} = adj\mathbf{E_8}$ decomposes under $U(1)\times U(1)\times E_6$

$$\begin{aligned}\mathbf{248} = {} & \boxed{\mathbf{1}_{(0,0)} + \mathbf{1}_{(0,0)} + \mathbf{78}_{(0,0)}} \\ & + \mathbf{1}_{(3,\frac{1}{2})} + \mathbf{1}_{(-3,\frac{1}{2})} + \mathbf{1}_{(0,-1)} \\ & + \mathbf{27}_{(3,\frac{1}{2})} + \mathbf{27}_{(-3,\frac{1}{2})} + \mathbf{27}_{(0,-1)} \\ & + \mathbf{1}_{(-3,-\frac{1}{2})} + \mathbf{1}_{(3,-\frac{1}{2})} + \mathbf{1}_{(0,1)} \\ & + \overline{\mathbf{27}}_{(-3,-\frac{1}{2})} + \overline{\mathbf{27}}_{(3,-\frac{1}{2})} + \overline{\mathbf{27}}_{(0,1)}\end{aligned}$$

The **4** representation of $SO(6)$ decomposes under $U(1)\times U(1)$

$$\mathbf{4} = \boxed{(0,0)} \text{ Gauginos}$$

$$+ \boxed{(3,\tfrac{1}{2}) + (-3,\tfrac{1}{2}) + (0,-1)}$$

**CHIRAL**

Thus applying the CSDR rules we find that the surviving fields in four dimensions are three $\mathcal{N}=1$ vector multiplets $V^{\alpha}$, $V_{(1)}$, $V_{(2)}$, (where $\alpha$ is an $E_6$, 78 index and the other two refer to the two $U(1)'s$) containing the gauge fields of $E_6\times U(1)\times U(1)$. The matter content consists of three $\mathcal{N}=1$ chiral multiplets $(A^i, B^i, C^i)$ with $i$ an $E_6$, 27 index and three $\mathcal{N}=1$ chiral multiplets $(A, B, C)$ which are $E_6$ singlets and carry only $U(1)\times U(1)$ charges.

## Yukawa Terms and Gauginos masses in $4D$

$$\mathcal{L}_Y = \frac{i}{2}\overline{\psi}\Gamma^a D_a\psi = \frac{i}{2}\overline{\psi}\Gamma^a\nabla_a\psi + \overline{\psi}V\psi$$

- $\Gamma^a$ are the $\Gamma$-matrices $\rightarrowtail \{\Gamma^a, \Gamma^b\} = -2g^{ab}I_{32}$
  - $g^{ab} = diag\{\frac{1}{R_1^2}, \frac{1}{R_1^2}, \frac{1}{R_2^2}, \frac{1}{R_2^2}, \frac{1}{R_3^2}, \frac{1}{R_3^2}\}$
- $\nabla_a = -\partial_a + \frac{1}{2}f_{ibc}e^i_\Gamma e^\Gamma_a\sigma^{bc} + \phi_a$
  - $\partial_a\psi = 0, \; e^i_\Gamma = 0, \;$ at $y = 0$

  $\rightarrowtail$ Yukawa terms
- $V = \frac{i}{4}\Gamma^a G_{abc}\sigma^{bc}$
  - Depends on torsion $\tau$, which is free parameter.

  $\rightarrowtail$ Gaugino masses

# Wilson Flux

$$F^{S/R} = \mathbb{Z}_3 \subseteq \mathrm{W}$$

$$A_\mu = \gamma_3 A_\mu \gamma_3^{-1}$$

where $\gamma_3 = \mathrm{diag}(\mathbf{1}_9, \omega\mathbf{1}_9,\ \omega^2\mathbf{1}_9)$, $\omega = e^{2i\pi/3}$

$$\vec{\alpha} = \omega\gamma_3\vec{\alpha}\,,\ \ \vec{\beta} = \omega^2\gamma_3\vec{\beta}\,,\ \ \vec{\gamma} = \omega^3\gamma_3\vec{\gamma}$$

$$\alpha = \omega\alpha\,, \quad \beta = \omega^2\beta\,, \quad \gamma = \omega^3\gamma$$

After the $\mathbb{Z}_3$ projection the gauge group reduces to

$$A_\mu^A, \qquad A \in SU(3)_c \times SU(3)_L \times SU(3)_R$$

$$\alpha_3 \sim (\bar{3},1,3)_{(3,1/2)},\ \ \beta_2 \sim (3,\bar{3},1)_{(-3,1/2)},\ \ \gamma_1 \sim (1,3,\bar{3})_{(0,-1)}$$

Among the singlets, only $\gamma_{(0,-1)}$ survives.

$$(\bar{3},1,3)\longrightarrow (q^c)^{\alpha}_{p} \quad (3,\bar{3},1)\longrightarrow Q^{a}_{\alpha} \quad (1,3,\bar{3})\longrightarrow L^{p}_{a}$$

$$q^c = \begin{pmatrix} d_R & u_R & D_R \end{pmatrix},\; Q = \begin{pmatrix} d_L \\ u_L \\ D_L \end{pmatrix},$$

$$L = \begin{pmatrix} H_d^0 & H_u^+ & \nu_L \\ H_d^- & H_u^0 & e_L \\ \nu_R & e_R & S \end{pmatrix}$$

# Breakings of $SU(3)^3$ to $SM$ gauge group

$$L_0^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & V_S \end{pmatrix}, \qquad L_0^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ V_R & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} & SU(3)_c \times SU(3)_L \times SU(3)_R \times U(1)_A \times U(1)_B \\ \longrightarrow & SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1) \\ \longrightarrow & SU(3)_c \times SU(2)_L \times U(1) \end{aligned}$$

$$L_0^{MSSM} = \begin{pmatrix} \upsilon_d & 0 & 0 \\ 0 & \upsilon_u & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow SU(3)_c \times U(1)_{em}$$

## General remark about neutrino mass in a $SU(3)^3$

If we just assume that the vevs in the theory have big scale difference $V_S \gg V_R \gg \upsilon_u$ then it leads to neutrino mass

$$m_\nu = \frac{y^2}{5g^2}\frac{\upsilon_u^2}{V_R^2}M$$

- $y$ is the unified Yukawa coupling
- $g$ is the unified gauge coupling
- $M$ is the gaugino Mass

# Conclusions

- Elegant way to reduce from extra dimensions to a $4D$ theory.
- Chiral theory is achieved.
- $\mathcal{N}=1$ softly broken theory.
- Get $SM$ particle content.

*Thank you!*