Multiple M-Branes

Neil Lambert

XVIII European Workshop on String Theory

Corfu 22^{nd} September 2012

Introduction

In these lectures we want to discuss the (Conformal) Field Theories that appear on the worldvolumes of multiple M-branes.

We will mainly consider M2-branes since M5-branes remain very mysterious. The worldvolume theories are certain highly supersymmetric Chern-Simons-Matter Theories (BLG and ABJM). Such theories play a similar role to Yang-Mills theories on D-branes but there are curious differences such as:

- Matter fields are not in the adjoint representation, but yet a related to the gauge fields by susy.
- Susy is determined by the gauge group and matter representation.

References

Much of the lectures follow the reviews:

- Bagger, Lambert, Mukhi and Papageorgakis [arxiv:1203.3546];
- Lambert [arxiv:1203.4244]

which (hopefully) contain detail references. I will therefore not give references in these slides and hope that no one takes offence.

Furthermore the lectures are aimed at being pedagogical rather than historical.

PLAN

Roughly 3 lectures of 60 mins.

1. M2-branes and Chern-Simons-Matter Theories.
   - BLG and ABJM
   - 3-algebras
2. Physical Analysis
   - Vacuum Moduli Space
   - $AdS_4 \times \mathbb{C}P^3$
   - Novel Higgs reduction to D2-branes
   - Monopole ('t Hooft) operators and hidden symmetries
3. M5-branes and the (2,0) CFT
   - A (2,0) system
   - 5D SYM
   - DLCQ

Conventions

Indices: $m, n ... = 0, 1, 2, ...10 \qquad \mu, \nu, ... = 0, 1, 2$

$$I, J ... = 3, 4, ..., 10 \qquad A, B = 1, 2, 3, 4.$$

Metric: $\eta = \mathrm{diag}(-1, 1, ..., 1)$

Spinors:

Γ_m 32x32 real, $C = \Gamma_0$; Ψ real, $\bar{\Psi} = \Psi^T C$, $\Gamma_{012}\Psi = -\Psi$

γ_μ 2x2 real, $C = \gamma_0$; ψ_A complex, $\bar{\psi}^A = \psi_A^\dagger C$

M2-branes and Chern-Simons-Matter Theories

Consider an M2-brane along x^0, x^1, x^2:

$SO(1,10) \to SO(1,2) \times SO(8)$ and $32 \to 16$ susys

Susys that are preserved by the M2-branes: $\Gamma_{012}\epsilon = \epsilon$.

Worldvolume fermions are Goldstino modes $\Gamma_{012}\Psi = -\Psi$.

World volume scalars are Goldstone modes X^I.

Free theory Supersymmetry:

$$
\begin{aligned}
\delta X^I &= i\bar{\epsilon}\Gamma^I\Psi \\
\delta\Psi &= \partial_\mu X^I \Gamma^\mu \Gamma^I \epsilon \, .
\end{aligned}
$$

To describe multiple M2-branes we let the fields take values in a vector space with basis T^a:

$$X^I \to X^I_a T^a \qquad \Psi \to \Psi_a T^a$$

A natural guess for susy is:

$$\begin{aligned}\delta X^I_d &= i\bar{\epsilon}\Gamma^I \Psi_d \\ \delta \Psi_d &= \partial_\mu X^I_d \Gamma^\mu \Gamma^I \epsilon - \frac{1}{3!} X^I_a X^J_b X^K_c f^{abc}{}_d \Gamma^{IJK} \epsilon \,.\end{aligned}$$

$f^{abc}{}_d$ are just some sort of 'structure' constants.

Next we must check that this supersymmetry algebra closes:

$$[\delta_1, \delta_2]\, X^I_d = -2i\bar{\epsilon}_2\Gamma^\mu\epsilon_1\partial_\mu X^I_d - (2i\bar{\epsilon}_2\Gamma^{JK}\epsilon_1 X^J_a X^K_b f^{abc}{}_d)X^I_c \ .$$

The first term is simply a translation by $v^\mu = -2i\bar{\epsilon}_2\Gamma^\mu\epsilon_1$.

The second term must be interpreted as a new gauge symmetry

$$\delta X^I_d = \tilde{\Lambda}^c{}_d X^I_c \ , \qquad \tilde{\Lambda}^c{}_d = -2i\bar{\epsilon}_2\Gamma^{JK}\epsilon_1 X^J_a X^K_b f^{abc}{}_d \ .$$

So we have a gauge theory!

Next we must introduce a gauge field for this gauge symmetry. Following the standard procedure we define

$$D_\mu X^I_d = \partial_\mu X^I_d - \tilde{A}_\mu{}^c{}_d X^I_c \ ,$$

and similarly for Ψ_d. This is gauge covariant provided that

$$\delta \tilde{A}_\mu{}^c{}_d = \partial_\mu \tilde{\Lambda}^c{}_d + \tilde{A}_\mu{}^c{}_e \tilde{\Lambda}^e{}_d - \tilde{\Lambda}^c{}_e \tilde{A}_\mu{}^e{}_d$$

under a gauge transformation. We can also compute the field strength from $[D_\mu, D_\nu] X^I_b = \tilde{F}_{\mu\nu}{}^a{}_b X^I_a$ and find

$$\tilde{F}_{\mu\nu}{}^a{}_b = \partial_\nu \tilde{A}_\mu{}^a{}_b - \partial_\mu \tilde{A}_\nu{}^a{}_b - \tilde{A}_\mu{}^a{}_c \tilde{A}_\nu{}^c{}_b + \tilde{A}_\nu{}^a{}_c \tilde{A}_\mu{}^c{}_b \ .$$

In summary we find

$$
\begin{aligned}
\delta X_a^I &= i\bar{\epsilon}\Gamma^I \Psi_a \\
\delta \Psi_a &= D_\mu X_a^I \Gamma^\mu \Gamma^I \epsilon - \frac{1}{6} X_b^I X_c^J X_d^K f^{bcd}{}_a \Gamma^{IJK}\epsilon \\
\delta \tilde{A}_\mu{}^b{}_a &= i\bar{\epsilon}\Gamma_\mu \Gamma_I X_c^I \Psi_d f^{cdb}{}_a \ .
\end{aligned}
$$

Lagrangian

$$
\mathcal{L} = -\frac{1}{2} D_\mu X^{aI} D^\mu X_a^I + \frac{i}{2}\bar{\Psi}^a \Gamma^\mu D_\mu \Psi_a + \frac{i}{4}\bar{\Psi}_b \Gamma_{IJ} X_c^I X_d^J \Psi_a f^{abcd} - V + \mathcal{L}_{CS} \ ,
$$

Potential

$$
V = \frac{1}{12} X_a^I X_b^J X_c^K X_e^I X_f^J X_g^K f^{abcd} f^{efg}{}_d
$$

“twisted” Chern-Simons term

$$
\mathcal{L}_{CS} = \frac{1}{2}\varepsilon^{\mu\nu\lambda}\left(f^{abcd} A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3} f^{cda}{}_g f^{efgb} A_{\mu ab} A_{\nu cd} A_{\lambda ef} \right)
$$

The 'structure constants' must satisfy the Fundamental identity:

$$f^{abc}{}_g f^{efg}{}_d = f^{efa}{}_g f^{gbc}{}_d + f^{agc}{}_d f^{efb}{}_g + f^{abg}{}_d f^{efc}{}_g$$

and we required an inner-product on the vector space with

$$h^{de} f^{abc}{}_e = f^{abcd} = f^{[abcd]}$$

Thus we have constructed the BLG theory with 16 supersymmetries, an $SO(8)$ R-symmetry and conformal invariance.

But for a positive definite choice of h^{ab} there is just one finite-dimensional solution:

$$f^{abcd} = \frac{2\pi}{k}\varepsilon^{abcd} \qquad a, b, c, d = 1, 2, 3, 4$$

Gauge algebra generated by $\tilde{\Lambda}^c{}_d = \Lambda_{ab} f^{abc}{}_d$ is $so(4) = su(2) \oplus su(2)$

Fields X^I_a, Ψ_a are in the $\mathbf{4} = \mathbf{2} + \overline{\mathbf{2}} =$ bifundamental.

$$\mathcal{L}_{CS} = \frac{k}{8\pi}\varepsilon^{\mu\nu\lambda} tr\left(\tilde{A}^+_\mu \partial_\nu \tilde{A}^+_\lambda + \frac{2}{3}\tilde{A}^+_\mu \tilde{A}^+_\nu \tilde{A}^+_\lambda - \tilde{A}^-_\mu \partial_\nu \tilde{A}^-_\lambda - \frac{2}{3}\tilde{A}^-_\mu \tilde{A}^-_\nu \tilde{A}^-_\lambda\right)$$

where $\tilde{A}^{\pm}_\mu{}^a{}_b$ are the (anti)-self-dual parts of $\tilde{A}_\mu{}^a{}_b$.

A standard result tells us that $k \in \mathbb{Z}$ - no continuous parameter.

Thats great! Its the only example of a maximally supersymmetric non-gravitational lagrangian that is not Yang-Mills.

But rather limited as it turns out to only describe 2 M2's (see later). To do better we need to generalize:

- consider less supersymmetry: 12 supercharges
- $SO(8)$ is reduced to $SU(4) \times U(1)$

Play a similar game

X_a^I written as 4 complex scalar fields Z_a^A in $\mathbf{4}$ of $SU(4)$ with $U(1)$ charge 1

- $(Z_a^A)^\dagger = \bar{Z}_A^a$ in $\bar{\mathbf{4}}$ of $SU(4)$ with $U(1)$ charge -1

Ψ_a written as 4 complex fermions ψ_{Aa} in $\bar{\mathbf{4}}$ with $U(1)$ charge 1

- $(\psi_{Aa})^\dagger = \psi^{Aa}$ in $\mathbf{4}$ of $SU(4)$ with $U(1)$ charge -1

The 16 components of ϵ are reduced to $\epsilon^{AB} = -\epsilon^{BA}$ in $\mathbf{6}$ of $SU(4)$ with $U(1)$ charge 0.

- $(\epsilon^{AB})^* = \epsilon_{AB} = \frac{1}{2}\varepsilon_{ABCD}\epsilon^{CD}$

Supersymmetry:

$$
\begin{aligned}
\delta Z_d^A &= i\bar{\epsilon}^{AB}\psi_{Bd} \\
\delta\psi_{Bd} &= \gamma^\mu D_\mu Z_d^A \epsilon_{AB} + f^{ab}{}_{cd} Z_a^C Z_b^A \bar{Z}_C^c \epsilon_{AB} + f^{ab}{}_{cd} Z_a^C Z_b^D \bar{Z}_B^c \epsilon_{CD} \\
\delta \tilde{A}_\mu{}^c{}_d &= -i\bar{\epsilon}_{AB}\gamma_\mu Z_a^A \psi^{Bb} f^{ca}{}_{bd} + i\bar{\epsilon}^{AB}\gamma_\mu \bar{Z}_A^b \psi_{Ba} f^{ca}{}_{bd}
\end{aligned}
$$

Lagrangian

$$
\begin{aligned}
\mathcal{L} &= -D^\mu \bar{Z}_A^a D_\mu Z_a^A - i\bar{\psi}^{Aa}\gamma^\mu D_\mu \psi_{Aa} - V + \mathcal{L}_{CS} \\
&\quad -if^{ab}{}_{cd}\,\bar{\psi}^{Ad}\psi_{Aa}\,Z_b^B \bar{Z}_B^c + 2if^{ab}{}_{cd}\,\bar{\psi}^{Ad}\psi_{Ba}\,Z_b^B \bar{Z}_A^c \\
&\quad +\frac{i}{2}\varepsilon_{ABCD} f^{ab}{}_{cd}\,\bar{\psi}^{Ad}\psi^{Bc}\,Z_a^C Z_b^D - \frac{i}{2}\varepsilon^{ABCD} f^{cd}{}_{ab}\,\bar{\psi}_{Ac}\psi_{Bd}\,\bar{Z}_C^a \bar{Z}_D^b\ .
\end{aligned}
$$

The potential is

$$
\begin{aligned}
V &= \frac{2}{3}\,\Upsilon^{CD}_{Bd}\,\bar{\Upsilon}^{Bd}_{CD} \\
\Upsilon^{CD}_{Bd} &= f^{ab}{}_{cd}\,Z^C_a Z^D_b \bar{Z}^c_B - \frac{1}{2}\delta^C_B f^{ab}{}_{cd}\,Z^E_a Z^D_b \bar{Z}^c_E + \frac{1}{2}\delta^D_B f^{ab}{}_{cd}\,Z^E_a Z^C_b \bar{Z}^c_E
\end{aligned}
$$

The 'twisted' Chern-Simons term $\mathcal{L}_{CS}$ is given by

$$
\mathcal{L}_{CS} = \frac{1}{2}\varepsilon^{\mu\nu\lambda}\left(f^{ab}{}_{cd}\,A_\mu{}^c{}_b\,\partial_\nu A_\lambda{}^d{}_a + \frac{2}{3}f^{ac}{}_{dg}f^{ge}{}_{fb}\,A_\mu{}^b{}_a\,A_\nu{}^d{}_c\,A_\lambda{}^f{}_e\right)
$$

And the 'structure constants' satisfy

$$
\begin{aligned}
&f^{ef}{}_{gb}f^{cb}{}_{ad} + f^{fe}{}_{ab}f^{cb}{}_{gd} + f^{*}_{ga}{}^{fb}f^{ce}{}_{bd} + f^{*}_{ag}{}^{eb}f^{cf}{}_{bd} = 0 \\
&(f^{ab}{}_{cd})^* = f^{cd}{}_{ab} \\
&f^{ab}{}_{cd} = -f^{ba}{}_{cd}
\end{aligned}
$$

The structure constants define a triple product:

$$[T^a, T^b; T_c] = f^{ab}{}_{cd} T^d$$

An finite class of solutions are given by $m \times n$ complex matrices:

$$[A, B; C] = \frac{2\pi}{k}(AC^\dagger B - BC^\dagger A)$$

Gauge group generated by $\delta Z^A_d = \Lambda^c{}_b f^{ab}{}_{cd} Z^A_a$ is

$$\delta Z^A = MZ^A - Z^A N$$

where M, N are $m \times m$ and $n \times n$ matrices respectively.

There are other examples which have gauge group $Sp(n) \times O(2)$. And also examples with less susy (10,8,...) with other choices of gauge groups and matter representations.

Gauge group is $U(m) \times U(n)$ with matter in the bi-fundamental.

- $m = n$ gives $SU(n) \times SU(n)$ theories
- Add by hand $U(1)$ gauge fields to get $U(n) \times U(n)$ ABJM
- $m \neq n$ gives the ABJ theories

In the special case of $SU(2) \times SU(2)$ we recover the BLG theory in complex notation.

In these lectures we will restrict attention to these BLG, ABJM and ABJ theories.

Let us look more closely at this triple product: vector space $\mathcal{V}$ generated by T^a admits a triple product - a 3-algebra:

$$[\cdot,\cdot;\cdot] : \mathcal{V}\otimes\mathcal{V}\otimes\bar{\mathcal{V}}\to\mathcal{V}$$

Key idea is that the analog of adjoint map

$$\varphi_{U,\bar{V}}(X) = [X, U; \bar{V}] \qquad \varphi_{U,\bar{V}}(\bar{X}) = -[\bar{X}, \bar{V}; U]$$

is a derivation

$$\varphi_{U,\bar{V}}([X, Y; \bar{Z}]) = [\varphi_{U,\bar{V}}(X), Y; \bar{Z}] + [X, \varphi_{U,\bar{V}}(Y); \bar{Z}] + [X, Y; \varphi_{U,\bar{V}}(\bar{Z})]$$

And this is the fundamental identity that we encountered.

The fundamental identity tells us that the action of φ on $\mathcal{V}$ is that of a lie-algebra $\mathcal{G}$ generated by $\varphi_{U,\bar{V}}$ for all $U, V \in \mathcal{V}$

- i.e. $\mathcal{V}$ is representation of $\mathcal{G}$.
- thus a 3-algebra defines a lie-algebra (e.g. $u(m) \oplus u(n)$) along with a preferred representation (e.g. the bi-fundamental)

In fact the reverse is also true: Given a Lie-algebra and a representation (along with invariant inner-products) one can always construct a triple product satisfying the fundamental identity (via the so-called Faulkner map).

Thus one need not think of a 3-algebra and just think of the gauge group and matter representation. However susy fixes the symmetry properties of the triple product

- and so which gauge algebras and representations arise
- leads to these rather esoteric choices (and indefinite inner-products on the lie-algebra)

Thus the amount of susy is determined by the gauge algebra and matter representations

- Whereas in super-Yang-Mills the gauge algebra is arbitrary and all fields are in the adjoint (for more than 8 susys)
- possible because in Chern-Simons theories there are no propagating gauge fields

Physical Analysis

The first thing to look at is the vacuum moduli space. This tells us the space of all the zero-energy configurations of the M2-branes.

Consider ABJM:

$$[Z^A, Z^B; \bar{Z}_C] = 0 \longleftrightarrow Z^A \bar{Z}_C Z^B = Z^B \bar{Z}_C Z^A$$

Generically this implies that all the Z^A commute (c.f. D-branes):

$$Z^A = \mathrm{diag}(z_1^A, ..., z_n^A)$$

To see that this is all requires one to evaluate the mass formula for small fluctuations which one finds is non-zero (generically: there are special points where extra massless modes arise but are expected to be lifted by non-perturbative effects).

We could set $g_L = g_R$ so that this is an adjoint action, as with D-branes. Thus allows us to put Z^A in diagonal form (as we have already done) and in addition acts as

$$z_i^A \leftrightarrow z_j^A \qquad \text{for any } i \neq j$$

e.g. for $i, j = 1, 2$ these are generated by

$$g_L = g_R = \begin{pmatrix} 0 & i & & & \\ i & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

These generate the action of the symmetric group S_n on z_i^A.

Unlike D-branes we also have continuous gauge transformations:

$$z_i^A \to e^{i\theta_i} z_i^A$$

These arise from taking

$$g_L = g_R^{-1} = \mathrm{diag}(e^{i\theta_1/2}, ..., e^{i\theta_n/2})$$

To see the effect of this on the vacuum moduli space we must examine the Lagrangian for the moduli z_i^A, including the gauge fields:

$$\mathcal{L} = -\frac{1}{2}\sum_i D_\mu z_i^A D^\mu \bar{z}_{Ai} + \frac{k}{4\pi}\varepsilon^{\mu\nu\lambda}\sum_i A^L_{\mu i}\partial_\nu A^L_{\lambda i} - \frac{k}{4\pi}\varepsilon^{\mu\nu\lambda}\sum_i A^R_{\mu i}\partial_\nu A^R_{\lambda i}$$

where $A^L_\mu = \mathrm{diag}(A^L_{\mu 1}, ..., A^L_{\mu n})$, $A^R = \mathrm{diag}(A^R_{\mu 1}, ..., A^R_{\mu n})$ and $D_\mu z_i^A = \partial_\mu z_i^A - i(A^L_{\mu i} - A^R_{\mu i})z_i^A$.

Note that z_i^A only couples to $B_{\mu i} = A^L_{\mu i} - A^R_{\mu i}$ and not to $Q_{\mu i} = A^L_{\mu i} + A^R_{\mu i}$:

$$\mathcal{L} = -\frac{1}{2}\sum_i D_\mu z_i^A D^\mu \bar{z}_{Ai} + \frac{k}{4\pi}\varepsilon^{\mu\nu\lambda}\sum_i B_{\mu i}\partial_\nu Q_{\lambda i}$$

with $D_\mu z_i^A = \partial_\mu z_i^A - iB_{\mu i} z_i^A$.

It's helpful to dualize $Q_{\mu i}$:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\sum_i D_\mu z_i^A D^\mu \bar{z}_{Ai} + \frac{k}{8\pi}\varepsilon^{\mu\nu\lambda}\sum_i B_{\mu i}H_{\nu\lambda i} - \frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\sigma_i\partial_\mu H_{\nu\lambda i} \\ &\cong -\frac{1}{2}\sum_i D_\mu z_i^A D^\mu \bar{z}_{Ai} + \frac{k}{8\pi}\varepsilon^{\mu\nu\lambda}\sum_i B_{\mu i}H_{\nu\lambda i} + \frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\partial_\mu\sigma_i H_{\nu\lambda i}\end{aligned}$$

where $H_{\nu\lambda i} = \partial_\nu Q_{\lambda i} - \partial_\lambda Q_{\nu i}$.

Integrating out $H_{\nu\lambda i}$ tells us $B_{\mu i} = -k^{-1}\partial_\mu\sigma_i$ and everything is pure gauge:

$$\mathcal{L} = -\frac{1}{2}\sum_i \partial_\mu w_i^A \partial^\mu \bar{w}_{Ai}$$

where $w_i^A = e^{i\sigma_i/k} z_i^A$ is gauge invariant.

But σ_i is periodic:

$$
\begin{aligned}
\int \mathcal{L}(\sigma_i + 2\pi) - \int \mathcal{L}(\sigma_i) &= -\frac{1}{4}\sum_i \int \varepsilon^{\mu\nu\lambda}\partial_\mu H_{\nu\lambda i} \\
&= -\frac{1}{2}\sum_i \int dH \\
&= -\frac{1}{2}\sum_i \int dF^L + dF^R \\
&\in 2\pi\mathbb{Z}
\end{aligned}
$$

because of the Dirac quantization rule

$$\int dF \in 2\pi\mathbb{Z}$$

and the fact that $B_i = -k^{-1} d\sigma_i$ implies $dB_i = F_i^L - F_i^R = 0$

This means that (recall $w_i^A = e^{i\sigma_i/k} z_i^A$)

$$w_i^A \cong e^{2\pi i/k} w_i^A$$

Thus there is an extra orbifold action in spacetime

$$\mathbb{R}^8 \to \mathbb{C}^4/\mathbb{Z}_k$$

and the vacuum moduli space is

$$\mathcal{M} = \mathrm{Sym}^n \left(\mathbb{C}^4/\mathbb{Z}_k\right)$$

Corresponding to n M2-branes in an $\mathbb{C}^4/\mathbb{Z}_k$ transverse space.

And indeed this orbifold preserves 12 supersymmeties.

What about ABJ with gauge group $U(n) \times U(m)$, $n \neq m$?

Write $m = n + l$

- Vacuum moduli space is still $\mathrm{Sym}^n \left(\mathbb{C}^4/\mathbb{Z}_k\right)$
- describes n M2's in $\mathbb{C}^4/\mathbb{Z}_k$
- Party is broken.

Interpreted as l 'fractional' M2-branes stuck at the fixed point.

- Corresponds to including l-units of discrete torsion in the background four-form ($H^4(\mathbb{C}^4/\mathbb{Z}_k, \mathbb{Z}) = \mathbb{Z}_k$).
- requires that $l \leq k$
- Conjecture: $U(n) \times U(n+l)$ ABJ has no vacuum for $l > k$
- Conjecture: $U(n) \times U(n+k)$ ABJ is dual to $U(n) \times U(n)$

Let us look more closely. The orbifold acts as

$$(x^0, x^1, x^2) \to (x^0, x^1, x^2)\ , \qquad z^A \to e^{2\pi i/k} z^A$$

here z^A are complex coords for the $\mathbb{R}^8$ spanned by $x^3, ..., x^{10}$.

Write $\mathbb{R}^8$ in 'spherical' coordinates

$$ds^2_{\mathbb{R}^8} = dr^2 + r^2 ds^2_{S^7}$$

and then S^7 as a Hopf fibration over $\mathbb{C}P^3$:

$$ds^2_{S^7} = (d\psi + \omega)^2 + ds^2_{\mathbb{C}P^3}$$

In these coordinates the orbifold simply acts on the fibre as

$$\psi \to \psi + 2\pi i/k$$

At large k, which is weak coupling, the orbifold shrinks the fibre and we have the type IIA background.

Let use consider the large n limit. The supergravity solution solution is ($H = 1 + nr_0^6/r^6$, $r_0 \propto l_p$)

$$\begin{aligned} ds_{11}^2 &= H^{-2/3}(-(dx^0)^2 + (dx^1)^2 + (dx^2)^2) + H^{1/3} ds^2_{\mathbb{C}^4/\mathbb{Z}_k} \\ &= \frac{r^4}{n^{\frac{2}{3}} r_0^4}(-(dx^0)^2 + (dx^1)^2 + (dx^2)^2) + \frac{n^{\frac{1}{3}} r_0^2}{r^2} dr^2 + n^{\frac{1}{3}} r_0^2 ds^2_{S^7/\mathbb{Z}_k} \\ &= \frac{n^{\frac{1}{3}} r_0^2}{4} \left(\frac{-(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + dz^2}{z^2} \right) + n^{\frac{1}{3}} r_0^2 ds^2_{S^7/\mathbb{Z}_k} \end{aligned}$$

Therefore we obtain, in the large k limit, an $AdS_4 \times \mathbb{C}P^3$ dual

$$ds_{10}^2 = \frac{\sqrt{n} r_0^3}{4k} ds^2_{AdS_4} + \frac{\sqrt{n} r_0^3}{k^3} ds^2_{\mathbb{C}P^3}$$

with $e^{2\phi} = \sqrt{n} r_0^3 k^{-3}$, $C^{(1)} = k\omega$

Next we consider a bunch of M2-branes located at $z^4 = iv$, far from the origin.

$$Z^A = iv\delta^{A4} + \frac{1}{\sqrt{2}}X^A + i\frac{1}{\sqrt{2}}X^{A+4} .$$

This leads to a 'novel Higgs' effect since $v \neq 0$ breaks$(U(n) \times U(n) \to U(n)$.

$B_\mu = A^L_\mu - A^R_\mu$ has no kinetic term and can integrated out yeilding a kinetic term for A_μ.

Furthermore $A_\mu = A^L_\mu + A^R_\mu$ eats X^8 and becomes dynamical.

Resulting action has a dynamical vector and 7 scalars (plus fermions) and must preserve 16susys

- Must be 3D MSYM at leading order (as a calculation shows)

$$\mathcal{L} = \frac{1}{g_{YM}^2}\mathcal{L}_{3DSYM} + \mathcal{O}(1/v^3)$$

where $g_{YM}^2 = 8\pi^2 v^2/k$.

Corrections correspond to the fact that finite fluctuations sense that spacetime is not $\mathbb{R}^7 \times S^1$ but $\mathbb{R}^8/\mathbb{Z}_k$.

Let us return to the moduli space. It follows that we can think of

$$Z^A = \begin{pmatrix} z_i^A & & \\ & \ddots & \\ & & z_n^A \end{pmatrix}$$

as describing the positions of n M2-branes in $\mathbb{C}^4/\mathbb{Z}_k$. Furthermore the natural circle for the M-theory direction is the over-all phase.

Suppose we wanted to describe n M2-branes moving along the M-theory circle with different speeds. One might expect that this corresponds to

$$Z^A = \begin{pmatrix} z_i^A e^{i\omega_1 t} & & \\ & \ddots & \\ & & z_n^A e^{i\omega_n t} \end{pmatrix}$$

But this is pure gauge! We can un-do it by taking

$$g_L = g_R^{-1} = \begin{pmatrix} e^{-i\omega_1 t/2} & & \\ & \ddots & \\ & & e^{-i\omega_n t/2} \end{pmatrix}$$

(Note that this gauge transformation is not allowed for D-branes where the scalars are in the adjoint.) So how do the M2-branes 'explore' the full transverse space? Let us set the fermions to zero and construct the hamiltonian

$$\begin{aligned} H &= \mathrm{tr}\int d^2x\ \Pi_{Z^A}\Pi_{\bar{Z}_A} + D_iZ^AD^i\bar{Z}_A + V \\ &\quad + \left(iZ^A\Pi_{Z^A} - i\Pi_{\bar{Z}_A}\bar{Z}_A - \frac{k}{2\pi}F^L_{12}\right)A^L_0 \\ &\quad + \left(i\bar{Z}_A\Pi_{\bar{Z}_A} - i\Pi_{Z^A}Z^A + \frac{k}{2\pi}F^R_{12}\right)A^R_0 \end{aligned}$$

As usual the time-components of the gauge field give constraints:

$$\begin{aligned}\frac{k}{2\pi}F^L_{12} &= iZ^A\Pi_{Z^A} - i\Pi_{\bar{Z}_A}\bar{Z}_A \\ \frac{k}{2\pi}F^R_{12} &= i\Pi_{Z^A}Z^A - i\bar{Z}_A\Pi_{\bar{Z}_A}\end{aligned}$$

Consider the vacuum moduli again:

$$Z^A = \begin{pmatrix} \frac{1}{\sqrt{2}}R^A_1 e^{i\theta^A_1} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{2}}R^A_n e^{i\theta^A_n} \end{pmatrix}$$

The constraint is

$$\frac{k}{2\pi}F^L_{12} = \frac{k}{2\pi}F^R_{12} = \begin{pmatrix} \sum_A (R^A_1)^2\partial_0\theta^A_0 & & \\ & \ddots & \\ & & \sum_A (R^A_n)^2\partial_0\theta^A_n \end{pmatrix}$$

In other words the momentum around the M-theory circle is given by the magnetic flux.

This is, in spirit, the same as dualization:

$$\partial_\mu X^{10} = \frac{1}{2}\varepsilon_{\mu\nu\lambda}F^{\nu\lambda} \qquad \longleftrightarrow \qquad \partial_0 X^{10} = F_{12}$$

This raises the next question: how do we compute quantities with 11D momentum. In particular the gauge invariant observables (appear to) only carry vanishing $U(1)$ charges:

$$\begin{aligned} \mathcal{O} &= \mathrm{tr}(Z^A \bar{Z}_B Z^C \ldots) \qquad \text{OK} \\ \mathcal{O} &= \mathrm{tr}(Z^A Z^B Z^C \ldots) \qquad \text{not OK} \end{aligned}$$

and hence don't really explore all 11 dimensions.

This brings us to monopole or 't Hooft operators: We want to create states that carry magnetic charge.

These operators are defined as a prescription for computing correlators in the path-integral. They are not constructed as a local expression of the fields.

$$< \mathcal{M}(y)\mathcal{O}(z)... >= \int_{\oint_y F=2\pi Q_M} DZD\psi DA\mathcal{O}(z)e^{-S}$$

in other words we require the fields in the path integral to have a specific singularity

$$F = \star\frac{Q_M}{2}d\left(\frac{1}{|x-y|}\right) + \text{nonsingular}$$

$Q_M \in u(n) \times u(n)$ is the magnetic flux and is subject to the standard Dirac quantization condition

$$e^{2\pi i Q_M} = 1 \ .$$

There is a famous result of GNO that the set of Q_M which satisfy this, modulo gauge transformations, are in one-to-one correspondence with highest weights of representations of the dual gauge group (Langlands dual).

- the dual of $U(n)$ is $U(n)$.

We can therefore group together various choices of fluxes into states associated with those of a representation of $U(n)$.

Furthermore we will be interested in supersymmetric monopole operators where the fields near the insertion point are those of a supersymmetric Dirac monopole (so the scalars also have a singlularity).

Next we note that due to the Chern-Simons term monopole operators transform locally under a gauge transformation $\delta A_\mu^{L/R} = D_\mu \omega_{L/R}$ (with $\omega \to 0$ at infinity) as

$$
\begin{aligned}
\mathcal{M}_{Q_M}(x) &\to e^{(ik/2\pi)\, tr \int (D\omega_L \wedge F^L - D\omega_R \wedge F^R)} \mathcal{M}_{Q_M}(x) \\
&= e^{ik\, tr((\omega_L(x) - \omega_R(x))\, Q_M)} \mathcal{M}_{Q_M}(x)
\end{aligned}
$$

Note that by construction we have broken the gauge group to $U(1)^n \times U(1)^n$. This is enough to tell us that under full gauge transformations the monopole operators transform in the representation of $U(n) \times U(n)$ whose highest weight is

$$\vec{\Lambda} = k(\vec{Q}_M, -\vec{Q}_M)$$

(actually because of the sign the second factor is the lowest weight)

This is all very abstract (and tricky to calculate with). Consider the abelian case (from the moduli space calculation and Wick rotated)

$$\mathcal{L} = -\frac{1}{2}\sum_i D_\mu z_i^A D^\mu \bar{z}_{Ai} + \frac{k}{8\pi}\varepsilon^{\mu\nu\lambda}\sum_i B_{\mu i}H_{\nu\lambda i} - \frac{i}{8\pi}\varepsilon^{\mu\nu\lambda}\sigma_i\partial_\mu H_{\nu\lambda i}$$

The monopole operators are just

$$\mathcal{M}_i(y) = e^{i\sigma_i(y)}$$

Since

$$\begin{aligned} < \mathcal{M}_i(y)\mathcal{O}(z)... > &= \int DzDBDQ e^{i\sigma_i(y)}\mathcal{O}(z)e^{-\int d^3x\mathcal{L}(x)} \\ &= \int DzDBDQ\mathcal{O}(z)e^{-\int d^3x\mathcal{L}(x)-i\sigma_i(x)\delta(x-y)} \end{aligned}$$

which is the same as taking

$$\frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\partial_\mu H_{\nu\lambda i} \to \frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\partial_\mu H_{\nu\lambda i} + 8\pi\delta(x-y)$$

i.e. inserting a magnetic charge at $x = y$.

Thus our gauge invariant operator on the moduli space is just

$$w_i^A = e^{i\sigma_i/k} z_i^A = (\mathcal{M}_i)^{\frac{1}{k}} z_i^A$$

and indeed $\mathcal{M}_i$ has charge $(k, -k)$ under $U(1) \times U(1)$.

Classically $e^{i\sigma_i}$ has scaling dimension zero (because $B_i \propto d\sigma_i$ has scaling dimension one). What about quantum mechanically? You might think this was shifted because of normal ordering (c.f. e^{ikX} on the string worldsheet).

However the momentum conjugate to $B_{\mu i}$ is $A_{\mu i}$ and thus $B_{\mu i}$ and hence σ_i has vanishing OPE with itself. So no normal ordering effects and $e^{i\sigma_i}$ is dimension zero in the quantum theory.

Let us now return to where we started: BLG. This is an $SU(2) \times SU(2)$ Chern-Simons-Matter theory with maximal supersymmetry and an $SO(8)$ R-symmetry.

How does it fit in? To cut a long(ish) story short:

- BLG $(SU(2) \times SU(2))/\mathbb{Z}_2$ at $k = 1$ is dual to ABJM $U(2) \times U(2)$ at $k = 1$, i.e. 2 M2's in $\mathbb{R}^8$
- BLG $SU(2) \times SU(2)$ at $k = 2$ is dual to ABJM $U(2) \times U(2)$ at $k = 2$, i.e. 2 M2's in $\mathbb{R}^8/\mathbb{Z}_2$
- BLG $(SU(2) \times SU(2))/\mathbb{Z}_2$ at $k = 4$ is dual to ABJ $U(2) \times U(3)$ at $k = 2$, i.e. 2 M2's in $\mathbb{R}^8/\mathbb{Z}_2$ with torsion

So it describes 2 M2-branes in $\mathbb{R}^8$ or $\mathbb{R}^8/\mathbb{Z}_2$ with all symmetries manifest.

Let us return to the maximally supersymmetric cases $k = 1, 2$ where the transverse space is

$$k = 1 : \ \mathbb{R}^8 \ , \qquad k = 2 : \ \mathbb{R}^8/\mathbb{Z}_2$$

M2-branes in these backgrounds preserve 16 supersymmetries.

For two M2's we could use BLG but for more we need ABJM.

What happened to the extra two supersymmetries and $SO(8)$ R-symmetry? The claim is that they are there but not manifest (this is okay since $k = 1, 2$ is strongly coupled).

The extra supersymmetry comes from the supercurrent

$$J_\mu = \mathcal{M}^{ab} D_\mu Z^A_a \psi_{Ab}$$

and extra R-symmetries from

$$J^{AB}_\mu = \mathcal{M}^{ab}(Z^A_a D_\mu Z^B_b - Z^B_a D_\mu Z^A_b + i\varepsilon^{ABCD}\bar{\psi}_{Ca}\gamma_\mu\psi_{Db})$$

Recall that Z^A and ψ_A are both in the $(\mathbf{n}, \overline{\mathbf{n}})$, so $D_\mu Z^A \psi_A$, $Z^A D_\mu Z^B$ and $\bar{\psi}_C \psi_D$ are not gauge invariant. In particular they are in the tensor product of two fundamental representations of the $U(n) \times U(n)$ with $U(1)$ charge 2.

So for these currents to exist we require that $\mathcal{M}^{ab}$ is in the tensor product of two anti-bi-fundamental representations of $U(n) \times U(n)$ with $U(1)$ charge -2.

When does such an $\mathcal{M}$ exist?

The (highest weight,lowest weight) for two-tensored anti-bi-funamental representation is

$$\vec{\Lambda} = (2\vec{\lambda}_{n-1}, -2\vec{\lambda}_{n-1})$$

According to our discussion about monopole operators we must have

$$\vec{\Lambda} = k(\vec{Q}_M, -\vec{Q}_M)$$

for a monopole charge vector $\vec{Q}_M$. Thus $\mathcal{M}$ only exists if

- $k = 1$, $\vec{Q}_M = 2\vec{\lambda}_{n-1}$
- $k = 2$, $\vec{Q}_M = \vec{\lambda}_{n-1}$

Precisely as required!

M5-branes and the (2,0) CFT

Can we try our luck with M5-branes?

Low-energy M5-brane dynamics governed by a 6D theory with:

- ◇ $(2,0)$ supersymmetry
- ◇ conformal invariance
- ◇ $\mathrm{SO}(5)$ R-symmetry

Multiplet contains 5 scalars and a selfdual antisymmetric 3-form field strength + fermions

Very rich and novel 6D CFT dual to $AdS_7 \times S^4$

At linearized level the free susy variations are

$$\begin{aligned} \delta X^I &= i\bar{\epsilon}\Gamma^I\Psi \\ \delta\Psi &= \Gamma^\mu\Gamma^I\partial_\mu X^I\epsilon + \frac{1}{3!}\frac{1}{2}\Gamma^{\mu\nu\lambda}H_{\mu\nu\lambda}\epsilon \\ \delta H_{\mu\nu\lambda} &= 3i\bar{\epsilon}\Gamma_{[\mu\nu}\partial_{\lambda]}\Psi \ , \end{aligned}$$

and the equations of motion are those of free fields with $dH = 0$ (and hence $dH = d\star H = 0$).

Reduction to the D4-brane theory sets $\partial_5 = 0$ and

$$F_{\mu\nu} = H_{\mu\nu 5}$$

We wish to generalise this algebra to nonabelian fields with

$$D_\mu X^I_A = \partial_\mu X^I_A - \tilde{A}_\mu{}^B{}_A X^I_B$$

Upon reduction we expect Yang-Mills susy:

$$\begin{aligned} \delta X^I &= i\bar{\epsilon}\Gamma^I\Psi \\ \delta\Psi &= \Gamma^\alpha\Gamma^I D_\alpha X^I\epsilon + \frac{1}{2}\Gamma^{\alpha\beta}\Gamma^5 F_{\alpha\beta}\epsilon - \frac{i}{2}[X^I, X^J]\Gamma^{IJ}\Gamma^5\epsilon \\ \delta A_\alpha &= i\bar{\epsilon}\Gamma_\alpha\Gamma_5\Psi \ , \end{aligned}$$

Thus we need a term in $\delta\Psi$ that is quadratic in X^I and which has a single Γ_μ:

- Invent a field C^μ_A

After starting with a suitably general anstaz we find closure of the susy algebra implies

$$
\begin{aligned}
\delta X^I_A &= i\bar{\epsilon}\Gamma^I\Psi_A \\
\delta\Psi_A &= \Gamma^\mu\Gamma^I D_\mu X^I_A\epsilon + \frac{1}{12}\Gamma_{\mu\nu\lambda}H^{\mu\nu\lambda}_A\epsilon - \frac{1}{2}\Gamma_\lambda\Gamma^{IJ}C^\lambda_B X^I_C X^J_D f^{CDB}{}_A\epsilon \\
\delta H_{\mu\nu\lambda\ A} &= 3i\bar{\epsilon}\Gamma_{[\mu\nu}D_{\lambda]}\Psi_A + i\bar{\epsilon}\Gamma^I\Gamma_{\mu\nu\lambda\kappa}C^\kappa_B X^I_C\Psi_D f^{CDB}{}_A \\
\delta\tilde{A}_\mu{}^B{}_A &= i\bar{\epsilon}\Gamma_{\mu\lambda}C^\lambda_C\Psi_D f^{CDB}{}_A \\
\delta C^\mu_A &= 0
\end{aligned}
$$

where $f^{ABC}{}_D$ are totally anti-symmetric structure constants of the $N=8$ 3-algebra (possibly Lorentzian). Can also have C^μ and $f^{ABC}{}_D \to f^{AB}{}_C$.

Has $(2,0)$ supersymmetry, SO(5) R-symmetry and scale symmetry (C^μ_A has dimensions of length)

The algebra closes with the on-shell conditions

$$
\begin{aligned}
0 &= D^2 X^I_A - C^\nu_B C_{\nu G} X^J_C X^J_E X^I_F f^{EFG}{}_D f^{CDB}{}_A + fermions \\
0 &= D_{[\mu} H_{\nu\lambda\rho]\ A} + \frac{1}{4}\epsilon_{\mu\nu\lambda\rho\sigma\tau} C^\sigma_B X^I_C D^\tau X^I_D f^{CDB}{}_A + fermions \\
0 &= \Gamma^\mu D_\mu \Psi_A + X^I_C C^\nu_B \Gamma_\nu \Gamma^I \Psi_D f^{CDB}{}_A \\
0 &= \tilde{F}_{\mu\nu}{}^B{}_A - C^\lambda_C H_{\mu\nu\lambda\ D} f^{CDB}{}_A \\
0 &= D_\mu C^\nu_A = C^\mu_C C^\nu_D f^{BCD}{}_A \\
0 &= C^\rho_C D_\rho X^I_D f^{CDB}{}_A = C^\rho_C D_\rho \Psi_D f^{CDB}{}_A = C^\rho_C D_\rho H_{\mu\nu\lambda\ A} f^{CDB}{}_A
\end{aligned}
$$

Thus C^μ_A picks out a fixed direction in space and in the 3-algebra and $C^\mu_A D_\mu = 0$. So apparently we are simply pushed back to 5D.

But no so quick, we can look at the Conserved currents, *e.g.*

$$
\begin{aligned}
T_{\mu\nu} &= D_\mu X^I_A D_\nu X^{IA} - \frac{1}{2}\eta_{\mu\nu} D_\lambda X^I_A D^\lambda X^{IA} \\
&\quad + \frac{1}{4}\eta_{\mu\nu} C^\lambda_B X^I_A X^J_C C_{\lambda G} X^I_F X^J_E f^{CDBA} f^{EFG}{}_D + \frac{1}{4} H_{\mu\lambda\rho\, A} H_\nu{}^{\lambda\rho\, A} \\
&\quad + fermions
\end{aligned}
$$

And we also obtain six-dimensional expressions for the supercurrent and central charges.

Here we see that the system is 6-dimensional

$$
C^\mu P_\mu = \int d^5 x C^\mu T_{0\mu} \sim \mathrm{Tr} \int F \wedge H \in \mathbb{Z}
$$

but with a compact direction (i.e. $\mathbb{R}^5 \times S^1$).

M-theory momentum is 'topological and quantized' - just as with ABJM.

For $C_A^\mu = g_{YM}^2 \delta_5^\mu \delta_A^0$ the previous system reduces to 5D SYM

$$P_5 = -\frac{1}{8g_{YM}^2} \int d^4x \, \mathrm{tr}(F_{ij} F_{kl} \varepsilon_{ijkl}) = \frac{k}{R_5}$$

First recall some facts about 5D SYM:

- ◇ Power-counting non-renormalisable, $g_{YM}^2 \sim$ length
  $\Rightarrow$ naively new d.o.f. should appear at some scale
- ◇ M-theory says a UV (strong coupling) fixed point exists and is 6-dimensional: the M5-brane CFT
- ◇ Agrees with Nahm's classification of SCFT's: indeed UV-fixed point theory cannot be 5D

From String Theory the relation between D4- and M5-brane theories given by compactification on S^1.

5D SYM also has particle states that carry instanton charge k with mass

$$M \propto \frac{k}{g_{YM}^2} \propto \frac{k}{R_5}$$

Simplest such states just D0's in D4 worldvolume.

Interpretation as momentum on S^1 of compactified 6D theory. [Rozali, Berkooz-Rozali-Seiberg]

$\Rightarrow$ Even in Yang-Mills limit this tower of states knows something about M-theory direction.

We argued that, at least for $< X^6 >= v$ 5D SYM contains a complete spectrum of KK modes in the soliton spectrum

- KK tower of W-Bosons given by Dyonic instantons [NL,Tong]
- KK tower of strings given by Monopole strings
- KK tower of photons given by quantum-sized instantons $< \rho >\sim g_{YM}^2/v$

All smooth finite energy states in the correct representation of (2,0) supersymmetry.

So:

- ◇ No more room for any additional UV states.
- ◇ Natural conclusion: 5D SYM is the (2,0) CFT compactified on S^1 (see also [Douglas])
- ◇ ... and hence well-defined non-perturbatively (some how). (see also [Douglas])
  - Details of how this works out not clear since 6D CFT contains momentum states which are non-perturbative from the point of view of 5D theory

You might hope that 5DSYM is finite (but apparently it's not at 6 loops [Bern,Douglas,..] to appear)

But this is also naive since one normally says that solitons are suppressed by factors of

$$e^{-1/g^2_{YM}}$$

But g^2_{YM} has dimensions of length so we in fact must have

$$e^{-d/g^2_{YM}}$$

where d is a length-scale (e.g. instanton size, instanton/anti-instanton separation)

So no decoupling from perturbative physics if $d \leq \mathcal{O}(g^2_{YM})$ (e.g. photon KK tower).

We could also consider a null reduction, $x^\mu = (x^+, x^-, x^i)$:

$$C^\mu_A = g^2_{YM}\delta^\mu_+\delta^0_A$$

$$\begin{array}{rcl}
0 & = & D^2 X^I_a - \frac{ig}{2}\bar{\Psi}_c\Gamma_+\Gamma^I\Psi_d f^{cd}{}_a \\
0 & = & \Gamma^\mu D_\mu \Psi_a + g^2_{YM} X^I_c \Gamma_+\Gamma^I\Psi_d f^{cd}{}_a \\
0 & = & D_{[\mu}H_{\nu\lambda\rho]\ a} - \frac{g^2_{YM}}{4}\epsilon_{\mu\nu\lambda\rho\tau+}X^I_c D^\tau X^I_d f^{cd}{}_a + fermions \\
0 & = & \tilde{F}_{\mu\nu}{}^b{}_a - g^2_{YM}H_{\mu\nu+\ d}f^{db}{}_a
\end{array}$$

where $f^{ab}{}_c = f^{0ab}{}_c$. Curious system with 16 supersymmetries and an $SO(5)$ R-symmetry but $D_+ = 0$

We wish to view x^- as time, solve the non-dynamical equations, and then quantizing using the hamiltonian

$$\mathcal{H} = \mathcal{P}_- = \int d^4 x T_{--}$$

Setting the fermions to zero the non-trivial equations are:

$$\begin{aligned} 0 &= D_i D^i X^I \\ 0 &= D^i F_{i-} \\ 0 &= D_- F_{i-} - D^j G_{ij} - i g^4 [X^I, D_i X^I] \end{aligned}$$

with $F_{i-} = -g^2 H_{i-+}$, $F_{ij} = -g^2 H_{ij+}$, $G_{ij} = -g^2 H_{ij-}$ and hence

$$F = -\star F \qquad G = \star G$$

It follows that A_i is determined by the AHDM construction

- introduces moduli $A_i = A_i(m^\alpha)$
- Natural moduli space metric

$$g_{\alpha\beta} = \mathrm{Tr} \int d^4x \delta_\alpha A_i \delta_\beta A_i$$

Furthermore X^I can also be solved for explicitly in terms of m^α and their vev's:

$$X^I = v^I + \mathcal{O}\left(\frac{1}{x^2}\right)$$

Next we note that

$$\partial_- A_i = \frac{\partial A_i}{\partial m^\alpha}\partial_- m^\alpha + D_i \omega$$

where ω is a gauge transformation that we choose to ensure that $D^i \partial_- A_i = 0$.

- $\Rightarrow D_i D^i A_- = 0 \qquad \Rightarrow \qquad A_- = w + \mathcal{O}\left(\frac{1}{x^2}\right)$

All fields are reduced to functions of the vev's v^I, w and instanton moduli m^α

'Time' dependence arises by letting these be functions of x^-.

Now

$$\mathcal{P}_+ = \int d^4x T_{-+} = -\frac{2}{g^2}\mathrm{Tr}\int F\wedge F \in \frac{4\pi^2}{g^2}\mathbb{Z}$$

gives the instanton number and

$$\mathcal{P}_i = \int d^4x T_{-i} = \mathrm{Tr}\int F_{ij}F_-{}^j \propto \text{total instanton momentum}$$

The hamiltonian $\mathcal{P}_-$ is:

$$\mathcal{P}_- = \frac{1}{2g^2}g_{\alpha\beta}(\partial_- m^\alpha - L^\alpha)(\partial_- m^\beta - L^\beta) + V$$

where

$$L_\alpha \partial_- m^\alpha = \mathrm{Tr}\oint \partial_- A_r A_- \qquad V = \frac{g^2}{2}\mathrm{Tr}\oint X^I D_r X^I \ .$$

The Superalgebra for this theory is

$$\begin{aligned}
\{\mathcal{Q}_-, \mathcal{Q}_-\} &= -\,2\mathcal{P}_-(\Gamma^- C^{-1}) + \mathcal{Z}_+^I(\Gamma^-\Gamma^I C^{-1}) + \mathcal{Z}_{ij+}^{IJ}(\Gamma^{ij}\Gamma^-\Gamma^{IJ} C^{-1}) \\
\{\mathcal{Q}_+, \mathcal{Q}_+\} &= -\,2\mathcal{P}_+(\Gamma^+ C^{-1}) \\
\{\mathcal{Q}_-, \mathcal{Q}_+\} &= -\,2\mathcal{P}_i(\Gamma^i C^{-1}) + \mathcal{Z}_i^I(\Gamma^i\Gamma^I C^{-1})\,,
\end{aligned}$$

where the central charges are

$$\begin{aligned}
\mathcal{Z}_+^I &= -\,2\mathrm{Tr}\int d^4x\ F_{-i}D^iX^I \\
\mathcal{Z}_i^I &= -\,\mathrm{Tr}\int d^4x\ G_{ij}D^jX^I \\
\mathcal{Z}_{ij+}^{IJ} &= -\,g^2\mathrm{Tr}\int d^4x\ D_{[i}X^I D_{j]}X^J\ .
\end{aligned}$$

Thus we obtained the [Aharony, Berkooz, Kachru, Seiberg, Silverstein] proposal of $(2,0)$ theory along with explicit expressions for $\mathcal{Z}$, $\mathcal{P}_\mu$ and generalized to include L^α, V.

Arises naturally from an infinite boost of 5D SYM or D4-branes using the (2,0) theory we constructed above.

So it follows from the conjecture that (2,0) on S^1 is 5D SYM in a [Seiberg] limit that C^μ becomes null:

$$F_{\mu\nu} \to F^-_{\mu\nu} \oplus F^+_{\mu\nu}$$

and $F^+_{\mu\nu}$ decouples as $C^\mu C_\mu \to 0$.

To summarize, the $(2,0)$ system that we constructed leads to the following:

- ◇ Proposal (2,0) theory on S^1 is exactly 5D SYM for any value of the coupling (see also [Douglas])
- ◇ Rederived and generalized the $(2,0)$ DLCQ theory as the quantum mechanics of instantons [Aharony, Berkooz, Kachru, Seiberg, Silverstein].

In some sense, since quantum 5D SYM isn't defined (unless one can make non-perturbative sense of the Lagrangian), the conjecture that it is $(2,0)$ on S^1 is just a definition.

Can we define 5D SYM another way? Deconstruction comes to mind [NL,CP,MSS] in progress:

Quiver

Start with $\mathcal{N}=2$, $N_f=2K$ $SU(K)$ SCFT with $<\Phi>=v$ breaking $SU(K)^N \to SU(K)$.

This theory deconstructs 5D SYM on S^1 with when $N \to \infty$:

$$2\pi R_5 = \frac{N}{g_4 v}$$

But actually, as noted in [Arkani-Hamed,Cohen,Karch,Motl], because of $SL(2,\mathbb{Z})$ this deconstructs a 6D theory on $S^1 \times S^1$

Extra circle (KK modes come from monopoles = wrapped versions of the states in 5D SYM above)

$$2\pi R_6 = \frac{g_4}{v} = \frac{g_{YM}^2}{2\pi}$$

So the conjecture that (2,0) on S^1 is 5DSYM comes out as limit of deconstruction conjecture.

Thus various proposals on the $(2, 0)$

- DLCQ of QM on instanton moduli space [Aharony, Berkooz, Kachru, Seiberg, Silverstein]
- Deconstruction from D=4 SCFT [Arkani-Hamed,Cohen,Karch,Motl]
- Strong coupling limit of 5D SYM [Douglas],[NL,CP,MSS]

all based on lower dimensional theories are interconnected. (See also recent six-dimensional proposals [Chu],[Ho, Huang, Matsuo].)

Altogether these paint a consistent, interconnected picture of the M5-brane in terms of lower dimensional theories

- Alternatively, what exactly is the difference between the $(2, 0)$

Subsequent work has greatly added to and enhanced this picture:

- S-duality for 5D SYM on S^1 [Tachikawa]
- Bound states in instanton QM [Kim,Kim,Koh,Lee,Lee],
- String Junctions in 5D SYM [Bolognesi, Lee]
- N^3 scaling of 5D SYM partition function [Kim,Kim],[Kallen,Minahan,Nedelin,Zabzine]

Some important technical points remain

- Is 5D SYM well-defined non-perturbatively
- In DLCQ picture instanton moduli space has singularities (but these are mild orbifold singularities)
- How much does deconstruction tell us about the full higher dimensional theory