Lattice Gauge Theory - Gravity duality (and Coulomb's constant in five dimensions)

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Gravity

D. Giataganas & N.I., PRD85 (2012) 046001

Consider the background due to N D4-branes (Witten, 1998)

$$ds^2 = \left(\frac{u}{R}\right)^{3/2} (\eta_{\mu\nu}dx^\mu dx^\nu + f(u)dx_4^2) + \left(\frac{R}{u}\right)^{3/2} (\frac{du^2}{f(u)} + u^2 d\Omega_4^2)$$

$$e^\phi = g_s \left(\frac{u}{R}\right)^{3/4} , \; F_4 = dC_3 = \frac{2\pi N}{V_4}\,\epsilon_4 \;, \; f(u) = 1 - \frac{u_k^3}{u^3} \;, \; R^3 = \pi g_s N l_s^3$$

$$x^4 \sim x^4 + 2\pi\rho \;, \qquad \rho = \frac{2}{3}\frac{R^{3/2}}{u_k^{1/2}}$$

Dual to a 5d SU(N) pure gauge theory in the large N limit

$$g_5^2 = (2\pi)^2 g_s l_s \;, \quad g_4^2 = \frac{g_5^2}{2\pi\rho} = 3\sqrt{\pi}\left(\frac{g_s u_k}{N l_s}\right)^{1/2}, \quad \lambda_5 = g_5^2 N, \quad \lambda = g_4^2 N$$

The Wilson Loop

Maldacena (98), Rey & Yee (98), Brandhuber, Itzaki, Sonnenschein & Yankelowicz (98), Kinar, Schreiber & Sonnenschein (00), Kol & Sonnenschein (10)

$$ds^2 = g_{00}d\tau^2 + g_{ii}dx_i^2 + g_{uu}du^2$$

static gauge: $x_0 = \tau, \qquad x_1 = \sigma$

ansatz: $u = u(\sigma)$

induced metric $G_{\alpha\beta} = g_{MN}\partial_\alpha X^M \partial_\beta X^N$

$$G_{00} = g_{00}, \qquad G_{11} = g_{11} + g_{uu}u'^2$$

the Nambu-Goto action

$$S = \frac{1}{2\pi\alpha'}\int d\sigma d\tau \sqrt{-g_{00}(g_{11} + g_{uu}u'^2)} = \frac{1}{2\pi\alpha'}\int d\sigma d\tau \sqrt{D}$$

the Hamiltonian $H = \frac{g_{00} g_{11}}{\sqrt{D}}$ is a constant of the motion

setting it equal to c we solve the turning point equation

$$u' = \pm\sqrt{-\frac{(g_{00} g_{11} + c^2) g_{11}}{c^2 g_{uu}}}$$

if we solve u'=0: $g_{00} g_{11} = -c^2$ This yields the turning point u_0

The length of the string with its endpoints separated by r on the boundary is

$$r = 2\int_{\infty}^{u_0} \frac{du}{u'} = 2\int_{u_0}^{\infty} du \sqrt{\frac{-g_{uu}c^2}{(g_{00}g_{11}+c^2)g_{11}}}$$

And its energy, with the masses of the two heavy quarks subtracted out is

$$2\pi\alpha' V = 2\left(\int_{u_0}^{\infty} d\sigma \mathcal{L} - \int_{u_0}^{\infty} du \sqrt{g_{00}g_{uu}}\right.$$

$$= cr + 2\left|\int_{u_0}^{\infty} du\sqrt{-g_{uu}g_{00}}\left(\sqrt{1+\frac{c^2}{g_{11}g_{00}}}-1\right) - \int_{u_k}^{u_0} du\sqrt{-g_{00}g_{uu}}\right.$$

Defining $A = u_k/u_0 \quad y = u/u_0$ we can rewrite

$$\hat{r} = \frac{r}{3\rho} = \sqrt{A}\int_1^\infty \frac{dy}{\sqrt{(y^3 - A^3)(y^3 - 1)}}$$

$$\hat{V} = \frac{2\pi\alpha'}{u_k}V = \frac{2}{A}\left\{\int_1^\infty dy \left[\frac{y^3}{\sqrt{(y^3 - A^3)(y^3 - 1)}} - \frac{1}{\sqrt{1 - \frac{A^3}{y^3}}}\right] - \int_A^1 dy \frac{1}{\sqrt{1 - \frac{A^3}{y^3}}}\right.$$

The infinitely long string $(A \to 1)$ has tension $\hat{\sigma}^{(0)} = 2$

since $\hat{V} = 2\hat{r}$

We solve these equations for A<1

$$\left(1-\frac{A^3}{y^3}\right)^{-1/2}=\sum_{k=0}^{\infty}c_k\left(\frac{A^3}{y^3}\right)^k \qquad c_k=\binom{-1/2}{k}$$

and using

$$\int_1^{\infty}dy\left[\frac{y^{3+\lambda}}{\sqrt{y^3(y^3-1)}}-my^{\lambda}\right]\sum_{k=0}^{\infty}c_k\left(\frac{A^3}{y^3}\right)^k$$

$$=\sum_{k=0}^{\infty}c_kA^{3k}\left[\frac{m}{\lambda+1-3k}+\frac{\sqrt{\pi}}{3}\frac{\Gamma(k-1/3-\lambda/3)}{\Gamma(k+1/6-\lambda/3)}\right]$$

$$\hat{r}:l=-3,m=0$$

$$\hat{V}:l=0,m=1$$

we obtain

$$\hat{r} = \frac{\sqrt{\pi}}{3} \sum_{k=0}^{\infty} c_k \frac{\Gamma(k+2/3)}{\Gamma(k+7/6)} A^{3k+1/2}$$

$$\hat{V} = \text{const.} + \frac{2\sqrt{\pi}}{3} \sum_{k=0}^{\infty} c_k \ \frac{\Gamma(k-1/3)}{\Gamma(k+1/6)} A^{3k-1}$$

now it is straightforward to invert

$$\hat{V} = -\frac{\pi^{3/2}}{27} \left(\frac{\Gamma(2/3)}{\Gamma(7/6)} \right)^3 \frac{1}{\hat{r}^2} + \cdots$$

$$V = -\frac{2}{54\pi} \left(\frac{\sqrt{\pi}\Gamma(2/3)}{\Gamma(7/6)} \right)^3 \frac{\lambda\rho}{r^2}$$

$$V = const. - \frac{c_2}{r^2} \quad c_2 = \frac{2}{54\pi}\left(\frac{\sqrt{\pi}\Gamma(2/3)}{\Gamma(7/6)}\right)^3 \lambda\rho \quad \lambda\rho = 2\sqrt{\frac{\pi N g_s R^3}{l_s}}$$

divide by the lattice spacing (use $\beta \equiv \frac{2Na}{g_5^2}, \lambda N_5 = \frac{2N^2}{\beta}$):

$$\frac{c_2}{a} \equiv \overline{c_2} = \frac{2}{54\pi^2}\left(\frac{\sqrt{\pi}\Gamma(2/3)}{\Gamma(7/6)}\right)^3 \frac{N^2}{\beta} = 0.0649\frac{N^2}{\beta}$$

and define $w_N(\beta) \equiv \bar{c}_2 \frac{\beta}{N^2} = 0.0649$,valid for any N, large N

Summarize

$$w_N(\beta) = \bar{c}_2 \frac{\beta}{N^2} = \frac{2c_2}{\lambda_5} = 0.06498... = \frac{2}{54\pi^2}\left(\frac{\sqrt{\pi}\Gamma(2/3)}{\Gamma(7/6)}\right)^3$$

Lattice Gauge Theory

Continuum Gauge Theory

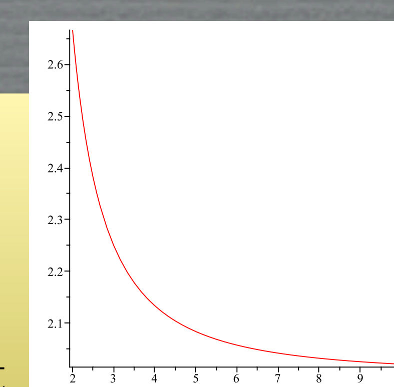
Gravity

Confined — Coulomb

β_c — lattice mean-field — perturbation theory

in $4d$: $V_4 = -\frac{c_1}{r},\ c_1 = \frac{g^2 C_F}{4\pi},\ C_F = \frac{N^2-1}{2N} \longrightarrow \frac{Nc_1}{C_F\lambda} = \frac{1}{4\pi} = 0.0795$

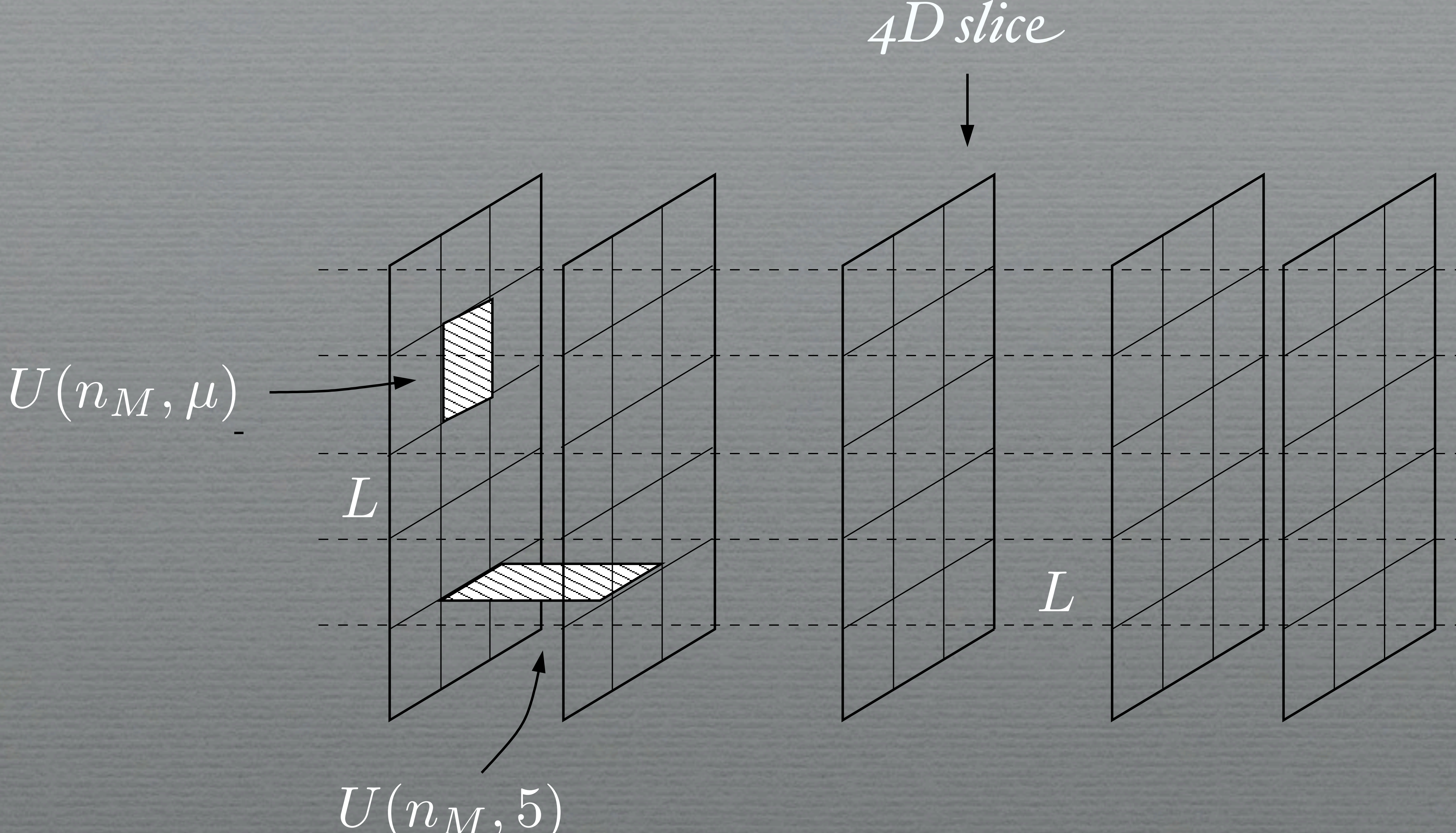
Coulomb's constant



Lattice Gauge Theory

1. N.I. & F. Knechtli, NPB 822 (2009) 1
2. N.I., PRD 85 (2012) 066007
3. N.I. & G. Koutsoumbas, JHEP 1208 (2012) 103

The 5d Lattice



The mean-field expansion

parameters: L, β

$$Z = \int DU \int DV \int DH e^{(1/N)\mathrm{Re}[\mathrm{tr}H(U-V)]} e^{-S_G[V]}$$

$$Z = \int DV \int DH e^{-Seff[V,H]}, \quad Seff = S_G[V] + u(H) + (1/N)\mathrm{Retr}HV$$

$$e^{-u(H)} = \int DU e^{(1/N)\mathrm{Retr}UH}$$

To 0'th order

The background is determined by

$$\overline{V} = -\frac{\partial u}{\partial H}\Big|_{\overline{H}} \qquad \overline{H} = -\frac{\partial S_G[V]}{\partial V}|_{\overline{V}}$$

The free energy

$$F^{(0)} = -\frac{1}{\mathcal{N}} \ln(Z[\overline{V}, \overline{H}]) = \frac{S_{\text{eff}}[\overline{V}, \overline{H}]}{\mathcal{N}}$$

To 1st order

$$H = \bar{H} + h \qquad V = \bar{V} + v$$

$$Seff = Seff[\bar{V}, \bar{H}] + \frac{1}{2}\left(\left.\frac{\delta^2 Seff}{\delta H^2}\right|_{\overline{V},\overline{H}} h^2 + 2 \left.\frac{\delta^2 Seff}{\delta H \delta V}\right|_{\overline{V},\overline{H}} hv + \left.\frac{\delta^2 Seff}{\delta V^2}\right|_{\overline{V},\overline{H}} v^2 \right)$$

$$\left.\frac{\delta^2 Seff}{\delta H^2}\right|_{\overline{V},\overline{H}} h^2 = h_i K_{ij}^{(hh)} h_j = h^T K^{(hh)} h$$

$$\left.\frac{\delta^2 Seff}{\delta V^2}\right|_{\overline{V},\overline{H}} v^2 = v_i K_{ij}^{(vv)} v_j = v^T K^{(vv)} v$$

$$\left.\frac{\delta^2 Seff}{\delta V \delta H}\right|_{\overline{V},\overline{H}} v^2 = v_i K_{ij}^{(vh)} h_j = v^T K^{(vh)} h$$

The Gaussian fluctuations

$$z = \int Dv \int Dh e^{-S^{(2)}[v,h]} \qquad S^{(2)}[v,h] = \frac{1}{2}\left(h^T K^{(hh)} h + 2v^T K^{(vh)} h + v^T K^{(vv)} v\right)$$

$$z = \frac{(2\pi)^{|h|/2}(2\pi)^{|v|/2}}{\sqrt{\det[(-\mathbf{1} + K^{(hh)}K^{(vv)})]}}$$

$$Z^{(1)} = Z[\overline{V}, \overline{H}] \cdot z = e^{-S_{\mathrm{eff}}[\overline{V}, \overline{H}]} \cdot z$$

The free energy to first order

$$F^{(1)} = F^{(0)} - \frac{1}{\mathcal{N}} \ln(z) = F^{(0)} + \frac{1}{2\mathcal{N}} \ln\Big[\det\left(-\mathbf{1} + K^{(hh)}K^{(vv)}\right)\Delta_{\mathrm{FP}}^{-2}\Big]$$

Observables

$$\mathcal{O}[V] = \mathcal{O}[\overline{V}] + \left.\frac{\delta \mathcal{O}}{\delta V}\right|_{\overline{V}} v + \frac{1}{2} \left.\frac{\delta^2 \mathcal{O}}{\delta V^2}\right|_{\overline{V}} v^2 + \ldots$$

$$= \mathcal{O}[\overline{V}] + \frac{1}{2}\left.\frac{\delta^2 \mathcal{O}}{\delta V^2}\right|_{\overline{V}} \frac{1}{z} \int Dv \int Dh v^2 e^{-S^{(2)}[v,h]}$$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int Dv \int Dh \left(\mathcal{O}[\overline{V}] + \frac{1}{2} \left.\frac{\delta^2 \mathcal{O}}{\delta V^2}\right|_{\overline{V}} v^2 \right) e^{-\left(Seff[\overline{V},\overline{H}] + S^{(2)}[v,h]\right)}$$

$$< v_i v_j >= \frac{1}{z} \int Dv \int Dh\, v_i v_j e^{-S^{(2)}[v,h]} = (K^{-1})_{ij}$$

$$K = -K^{(vh)} K^{(hh)^{-1}} K^{(vh)} + K^{(vv)}$$

$$< \mathcal{O} >= \mathcal{O}[\overline{V}] + \frac{1}{2} \mathrm{tr} \left\{ \left.\frac{\delta^2 \mathcal{O}}{\delta V^2}\right|_{\overline{V}} K^{-1} \right\}$$

1st order master formula

Correlators

$$C(t) =< \mathcal{O}(t_0+t)\mathcal{O}(t_0) > - < \mathcal{O}(t_0+t) >< \mathcal{O}(t_0) >= C^{(0)}(t) + C^{(1)}(t) + \cdots$$

$$C^{(0)}(t) = 0$$

$$< \mathcal{O}(t_0+t)\mathcal{O}(t_0) >= \mathcal{O}^{(0)}(t_0+t)\mathcal{O}^{(0)}(t_0) + \frac{1}{2}\mathrm{tr}\left\{ \frac{\delta^2(\mathcal{O}(t_0+t)\mathcal{O}(t_0))}{\delta^2 v} K^{-1} \right\} + \cdots$$

$$C^{(1)}(t) = \frac{1}{2}\mathrm{tr}\left\{ \frac{\delta^{(1,1)}(\mathcal{O}(t_0+t)\mathcal{O}(t_0))}{\delta^2 v} K^{-1} \right\} = \frac{1}{2}\mathrm{tr}\left\{ \frac{\tilde{\delta}^{(1,1)}(\mathcal{O}(t_0+t)\mathcal{O}(t_0))}{\delta^2 v} \tilde{K}^{-1} \right\}$$

$$C^{(1)}(t) = \sum_\lambda c_\lambda e^{-E_\lambda t} \qquad E_0 = m_H, \quad E_1 = m_H^*, \cdots$$

The SU(2) model

Mean-field parametrization

$$U(n,M) = u_0(n,M)\mathbf{1} + i\sum_k u_k(n,M)\sigma^k \qquad u_\alpha \in \mathbb{R}, u_\alpha u_\alpha = 1$$

$$V(n,M) = v_0(n,M)\mathbf{1} + i\sum_k v_k(n,M)\sigma^k\,, \quad v_\alpha \in \mathbb{C}$$

$$H(n,M) = h_0(n,M)\mathbf{1} - i\sum_k h_k(n,M)\sigma^k\,, \quad h_\alpha \in \mathbb{C}$$

The propagator (in momentum space)

$$\tilde{K}(p',M',\alpha';p'',M'',\alpha'') = \delta_{p'p''}\delta_{\alpha'\alpha''}C_{M'M''}(p',\alpha')$$

$$C_{M'M''}(p',\alpha') = [A\delta_{M'M''} + B_{M'M''}(1-\delta_{M'M''})]$$

$$A = -\left[\frac{1}{b_2}(1-\delta_{\alpha'0}) + \frac{1}{b_1}\delta_{\alpha'0}\right] - 2\beta\overline{v}_0^2\left[\sum_{N\neq M'}\cos(p'_N) + \frac{1}{\xi}\sin^2(p'_{M'}/2)\right]$$

$$B_{M'M''} = -4\beta\overline{v}_0{}^2\left[\delta_{\alpha'0}\cos\left(\frac{p'_{M'}}{2}\right)\cos\left(\frac{p'_{M''}}{2}\right) + (1-\delta_{\alpha'0})\sin\left(\frac{p'_{M'}}{2}\right)\sin\left(\frac{p'_{M''}}{2}\right)\right]$$

$$b_1 = -\frac{1}{\overline{h}_0 I_1(\overline{h}_0)}\left(I_2(\overline{h}_0) - \overline{h}_0\left(\frac{I_2(\overline{h}_0)^2}{I_1(\overline{h}_0)} - I_3(\overline{h}_0)\right)\right)$$

$$b_2 = -\frac{\overline{v}_0}{\overline{h}_0}$$

The static potential

$$V(r) = -2\log(\overline{v}_0) - \frac{1}{2\overline{v}_0^2}\frac{1}{L^3 N_5} \times \left\{ \sum_{{p'}_{M\neq 0}, p'_0=0} \left[\frac{1}{4}\sum_{N\neq 0}(2\cos(p'_N r)+2)\right] C_{00}^{-1}(p',0) \right]$$

$$+3 \sum_{{p'}_{M\neq 0}, p'_0=0} \left[\frac{1}{4}\sum_{N\neq 0}(2\cos(p'_N r)-2)\right] \frac{1}{C_{00}(p',1)} \right\}$$

The scalar mass

$$C_H^{(1)}(t) = \frac{1}{\mathcal{N}}(P_0^{(0)})^2 \sum_{p'_0} \cos(p'_0 t) \sum_{p'_5} |\tilde{\Delta}^{(\mathcal{N}_5)}(p'_5)|^2 \tilde{K}^{-1}\Big((p'_0,\vec{0},p'_5),5,0;(p'_0,\vec{0},p'_5),5,0\Big)$$

$$\Delta^{(m_5)}(n_5) = \sum_{r=0}^{m_5-1} \frac{\delta_{n_5 r}}{\overline{v}_0(\hat{r})}, \qquad \hat{r} = r + 1/2$$

The vector mass

$$C_V^{(2)}(t) = \frac{2304}{\mathcal{N}^2}(P_0^{(0)})^4(\overline{v}_0(0))^4 \sum_{\vec{p}'}\sum_k \sin^2(p'_k)\left(\overline{\overline{K}}^{-1}(t,\vec{p}',1)\right)^2$$

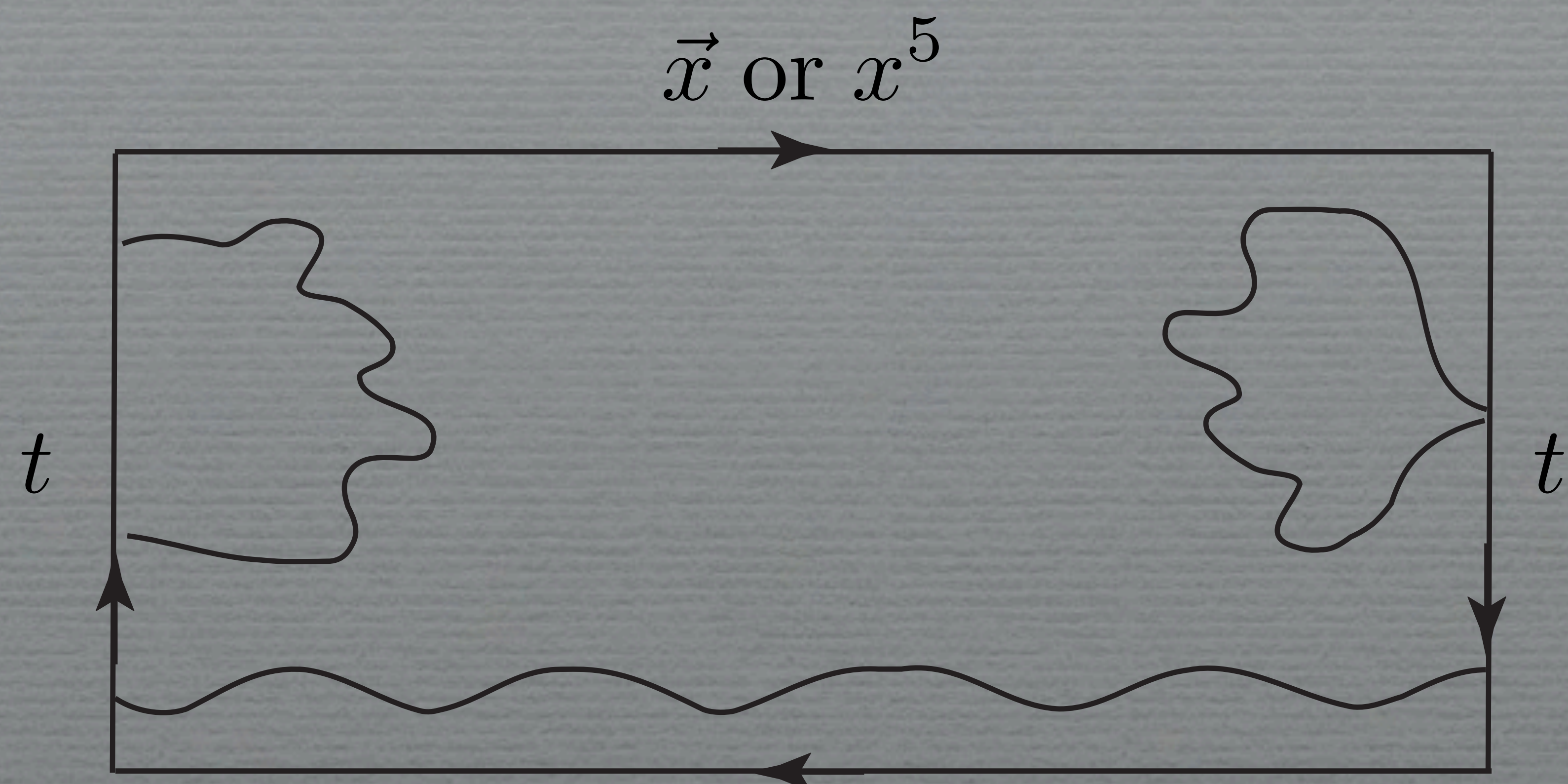
$$\overline{K}^{-1}((p'_0,\vec{p}'),5,\alpha) = \sum_{p'_5,p''_5}\tilde{\Delta}^{(N_5)}(p'_5)\tilde{\Delta}^{(N_5)}(-p''_5)K^{-1}(p'',5,\alpha;p',5,\alpha)$$

$$\overline{\overline{K}}^{-1}(t,\vec{p}',\alpha) = \sum_{p'_0} e^{ip'_0 t}\overline{K}^{-1}((p'_0,\vec{p}'),5,\alpha)$$

The free energy

$$F^{(1)} = F^{(0)} + \frac{1}{2\mathcal{N}}\sum_p \ln\left[\det\left(-\mathbf{1} + \tilde{K}^{(hh)}_{\alpha'=0}\tilde{K}^{(vv)}_{\alpha'=0}\right)\det\left(-\mathbf{1} + \tilde{K}^{(hh)}_{\alpha'\neq 0}\tilde{K}^{(vv)}_{\alpha'\neq 0}\right)^3\Delta_{\rm FP}^{-2}\right]$$

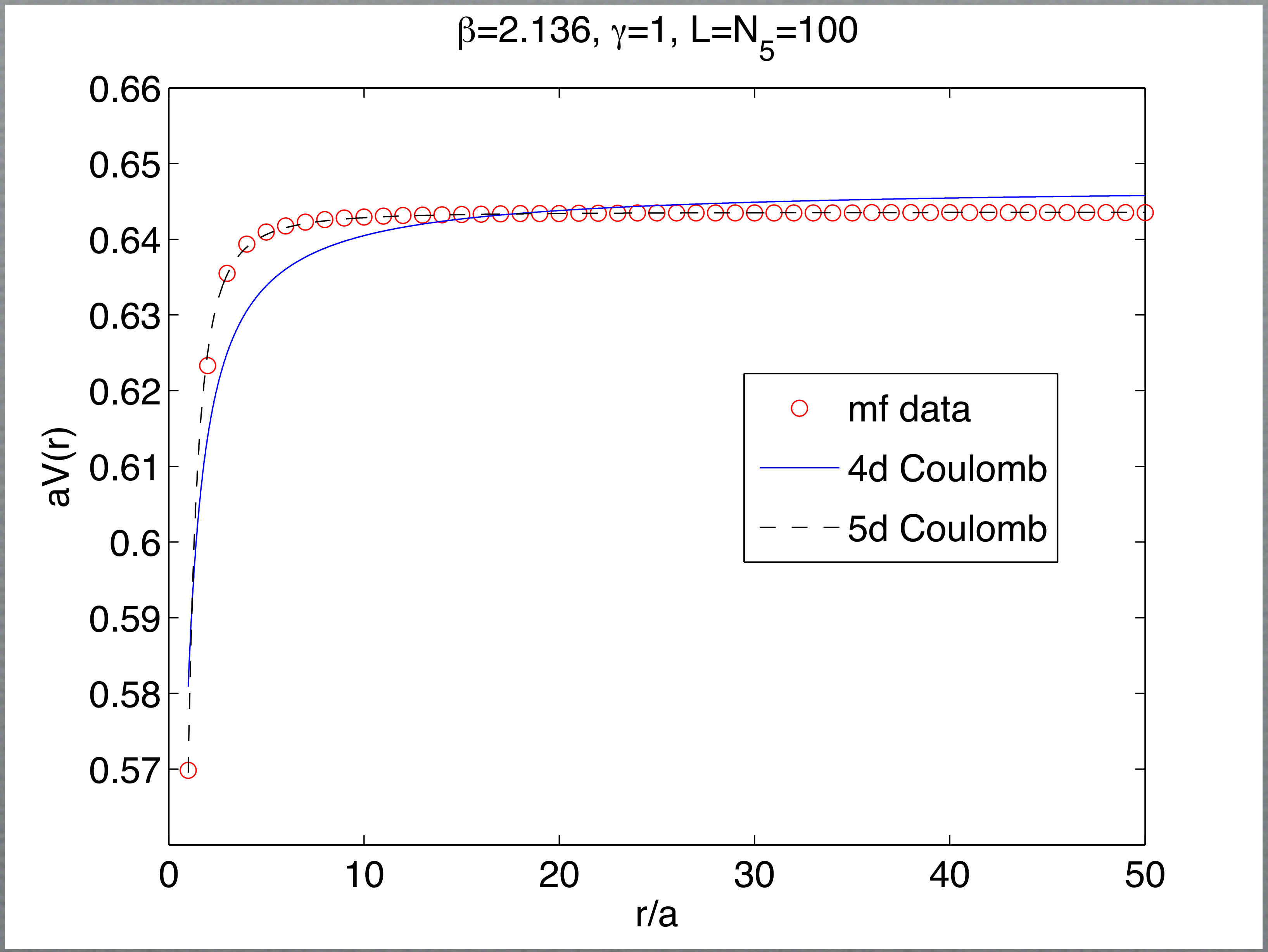
The Wilson loop



$$\mathrm{t} \to \infty : \qquad \mathrm{e}^{-V\,t} \simeq \, < \mathcal{O}_W >$$

The SU(2) static potential on the isotropic lattice (L=N5):

$$V = const. - \frac{c_2}{r^2} \quad \text{or} \quad aV = \mu - \frac{\bar{c}_2}{(r/a)^2} \qquad \bar{c}_2 = c_2/a$$



computed @:

$$q_{\mathrm{LMF}} = \frac{a_4 m_V}{a_4 m_S} = \frac{4\pi}{L(a_4 m_S)}$$

$q_{\mathrm{LMF}} = N_5/L$ if $am_S = 4\pi/N_5$

i.e., if it is a K-K state

$\mathrm{SU}(2), \beta_c = 1.676201676, \;\; @ \;\; q_{\mathrm{LMF}} = 1$

$L = 24: \quad \bar{c}_2 = 0.1689 \quad \xrightarrow{am_S = 0.5236} \quad w_2(1.6764598) = 0.0708$

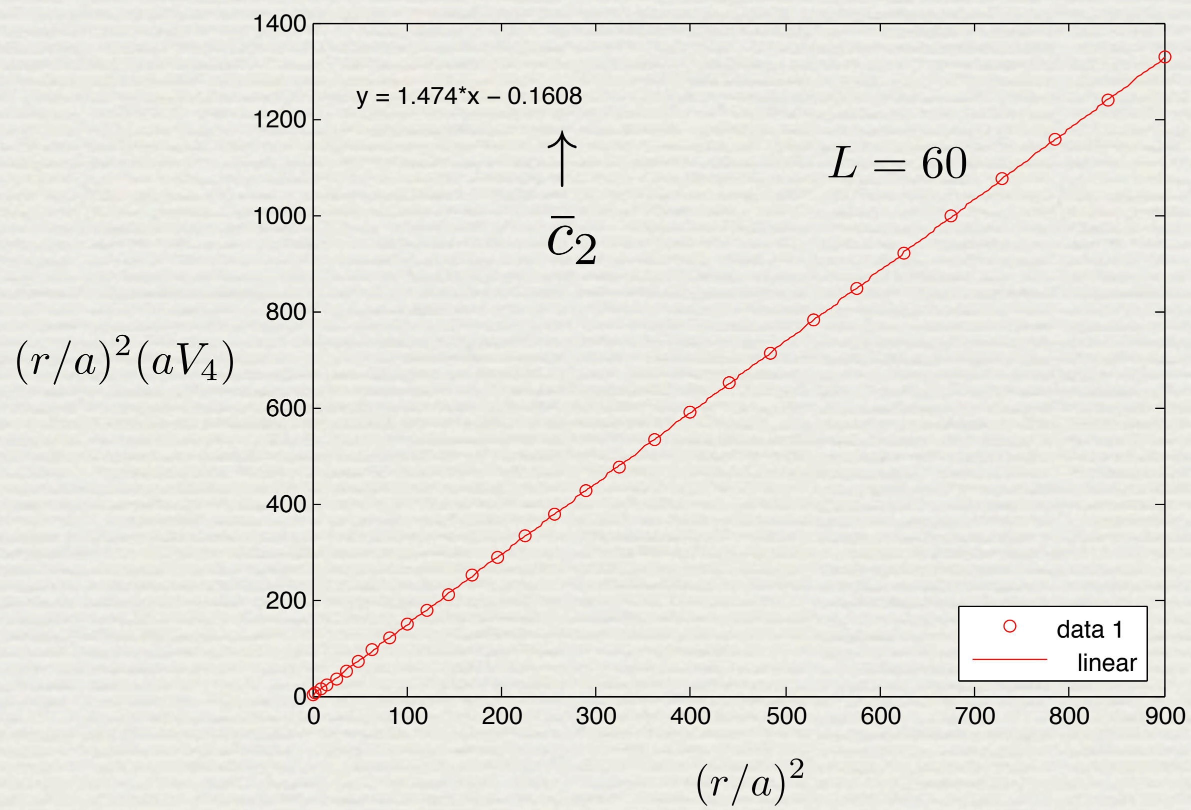
$L = 36: \quad \bar{c}_2 = 0.1653 \quad \xrightarrow{am_S = 0.3490} \quad w_2(1.67625254) = 0.0693$

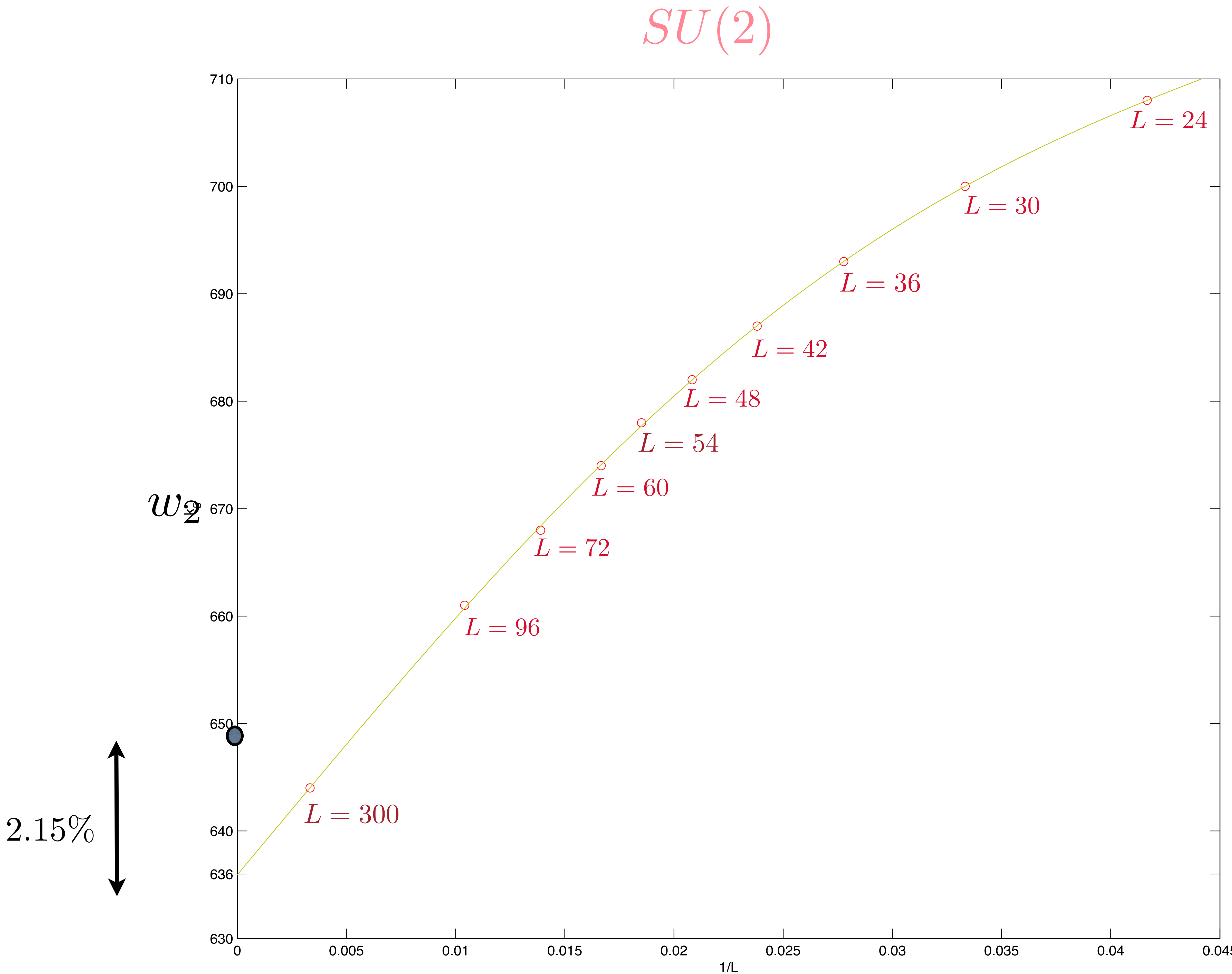
$L = 48: \quad \bar{c}_2 = 0.1627 \quad \xrightarrow{am_S = 0.2618} \quad w_2(1.67621776) = 0.0682$

$L = 60: \quad \bar{c}_2 = 0.1608 \quad \xrightarrow{am_S = 0.2094} \quad w_2(1.67620825) = 0.0674$

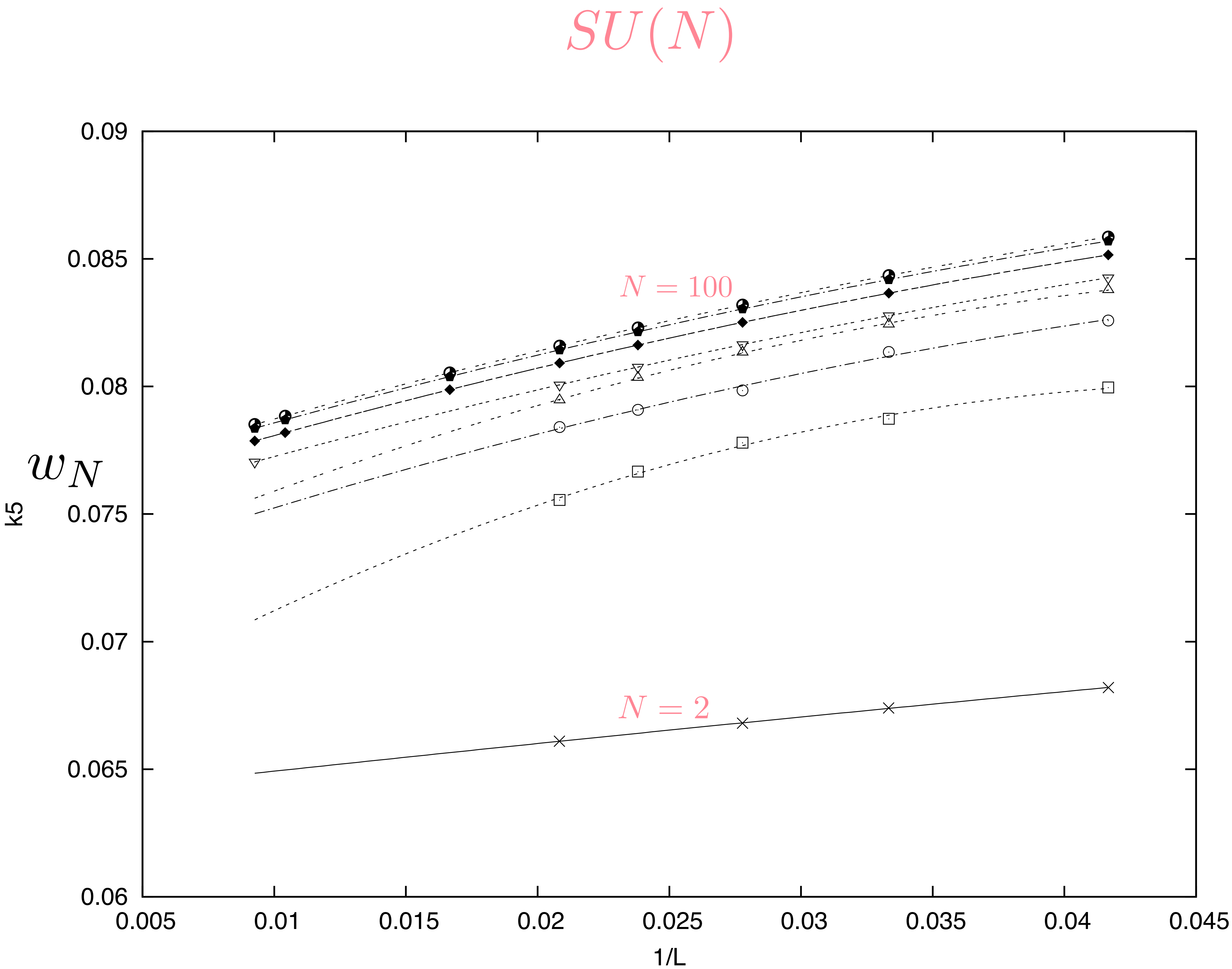
$L = 96: \quad \bar{c}_2 = 0.1577 \quad \xrightarrow{am_S = 0.1309} \quad w_2(1.676202674) = 0.0661$

$L = 300: \quad \bar{c}_2 = 0.1536 \quad \xrightarrow{am_S = 0.0419} \quad w_2(1.6762016769) = 0.0644$



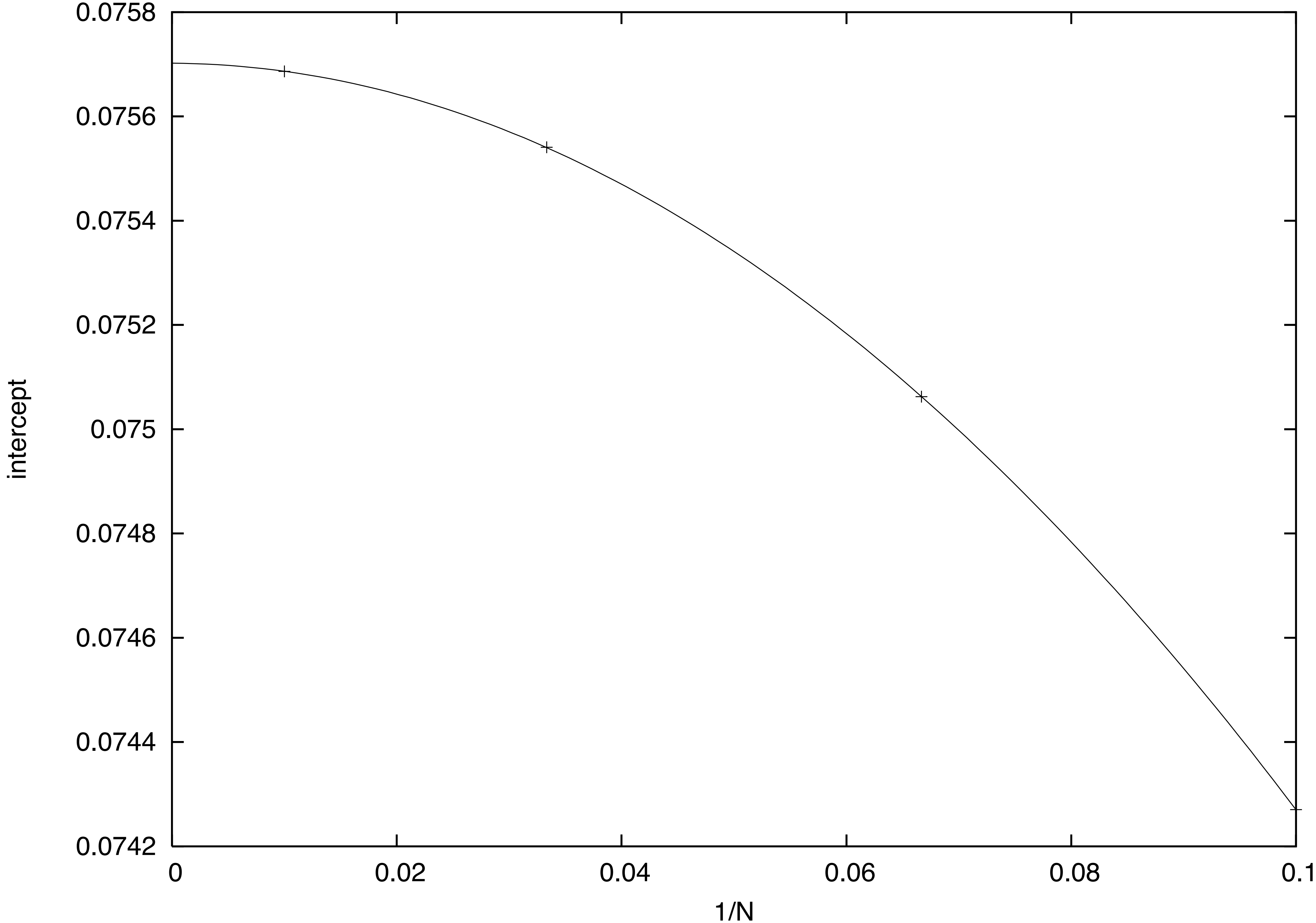


$SU(2)$
w_2
2.15%
710
700
690
680
670
660
650
640
636
630
0
0.005
0.01
0.015
0.02
0.025
0.03
0.035
0.04
0.045
1/L
$L = 24$
$L = 30$
$L = 36$
$L = 42$
$L = 48$
$L = 54$
$L = 60$
$L = 72$
$L = 96$
$L = 300$



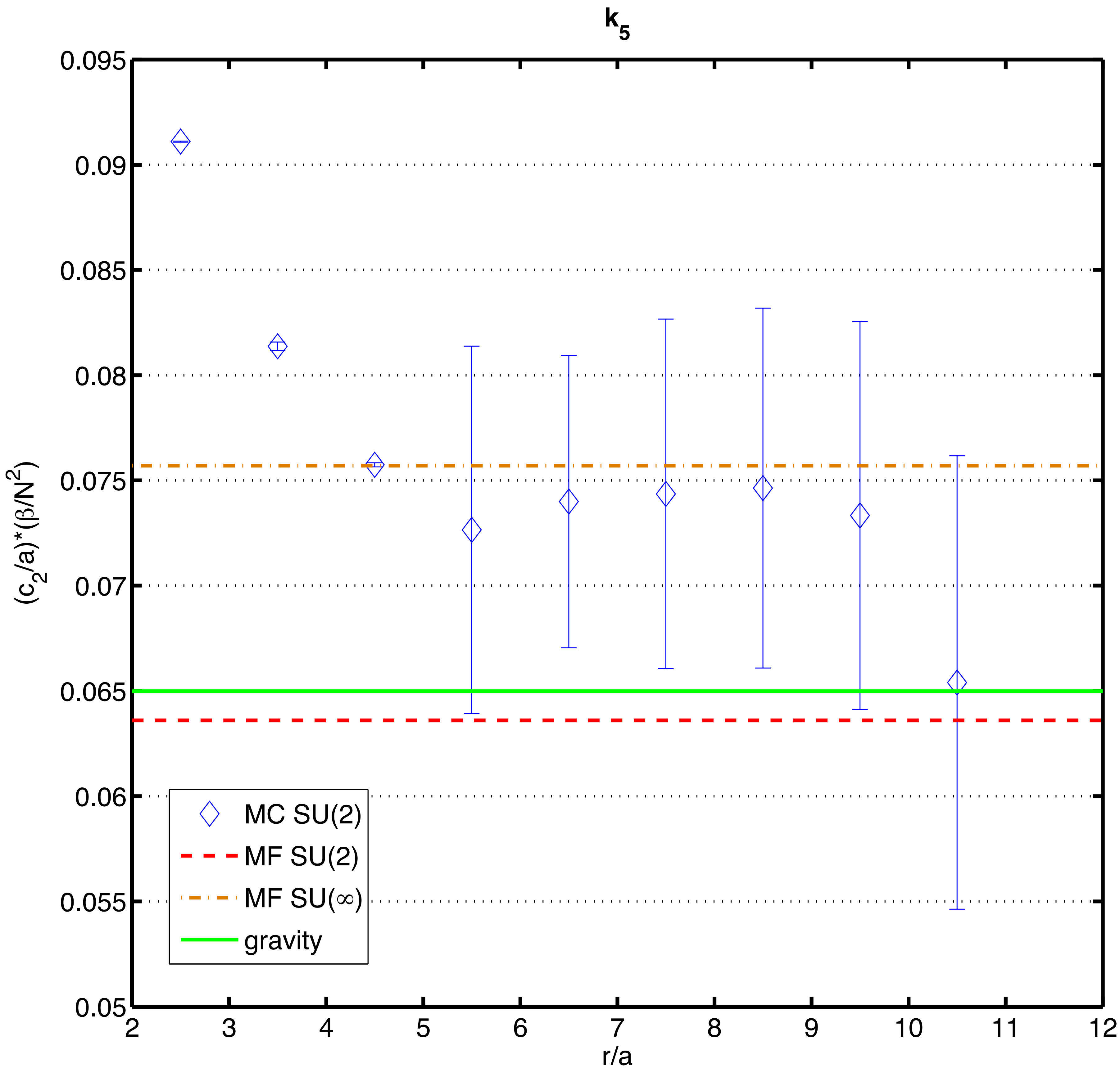
$SU(N)$
$N = 100$
$N = 2$
w_N
k5
0.09
0.085
0.08
0.075
0.07
0.065
0.06
0.005
0.01
0.015
0.02
0.025
0.03
0.035
0.04
0.045
1/L

$$N = \infty : w_N = 0.0757(\sim 17\%)$$



Monte Carlo - SU(2)

by F. Knechtli & P. Dziennik (U.Wuppertal)



The lattice size is 48x24^4, beta=1.7. 2000 independent measurements.

c_2(r+1/2) = 0.5*(r+1/2)^3*F(r+1/2)

F(r+1/2) = V(r+1)-V(r)

V(r) is extracted using the generalized eigenvalue technique from a 5x5 matrix-correlation of Wilson loops (they smear the space-like links of the loops using up to 40 iterations of spatial HYP smearing; for the time-like links the one-link integral is used).

Conclusions

1. The non-perturbative regime of 5d gauge theories can be probed analytically by the mean-field expansion.

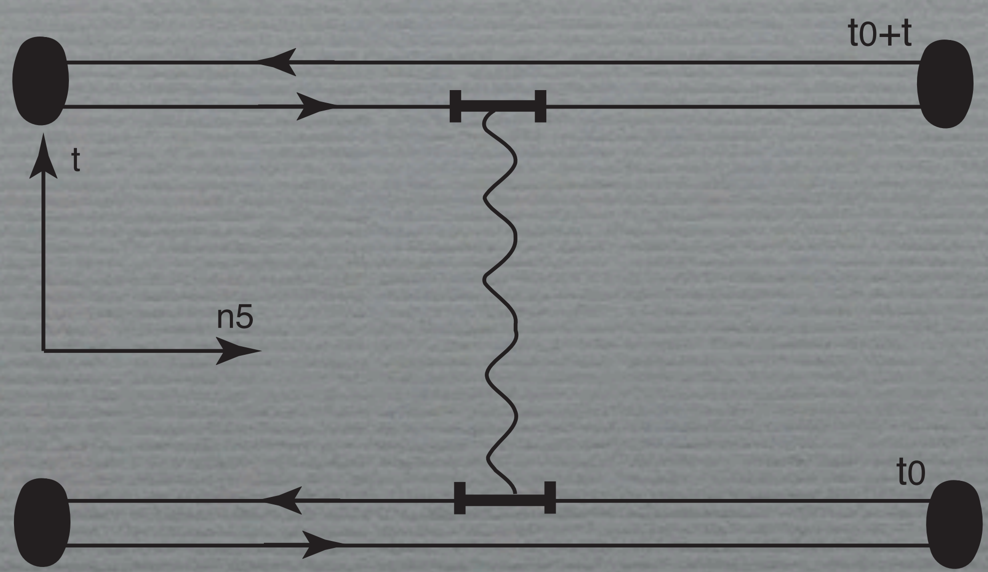
2. Gauge Theory-Gravity duality is tested using the 5d Coulomb constant which is only weakly N-dependent.

3. It is computed for N=2 on the lattice via a MF expansion and also via a Monte Carlo simulation. For large N it is computed via holography and via the MF expansion.

4. Between the gravity and lattice calculations we observe a 17 % agreement.

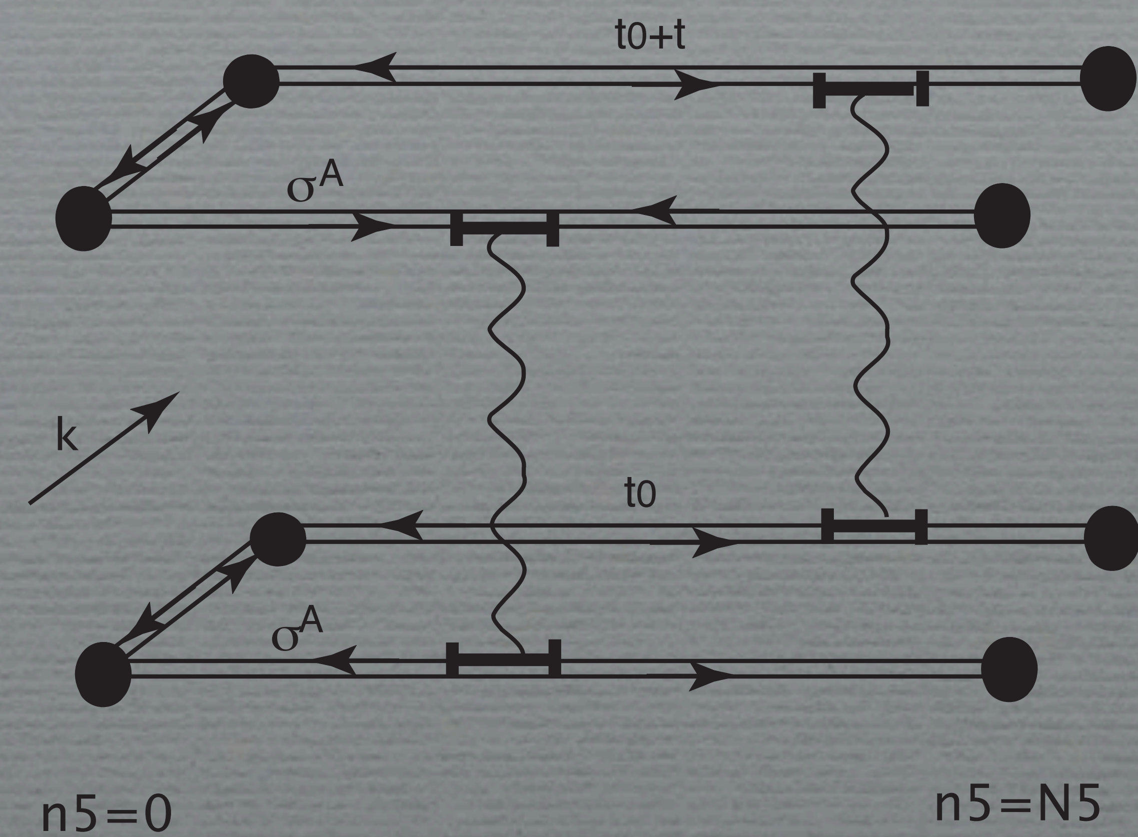
Lattice Observables
(Polyakov loops)

The scalar



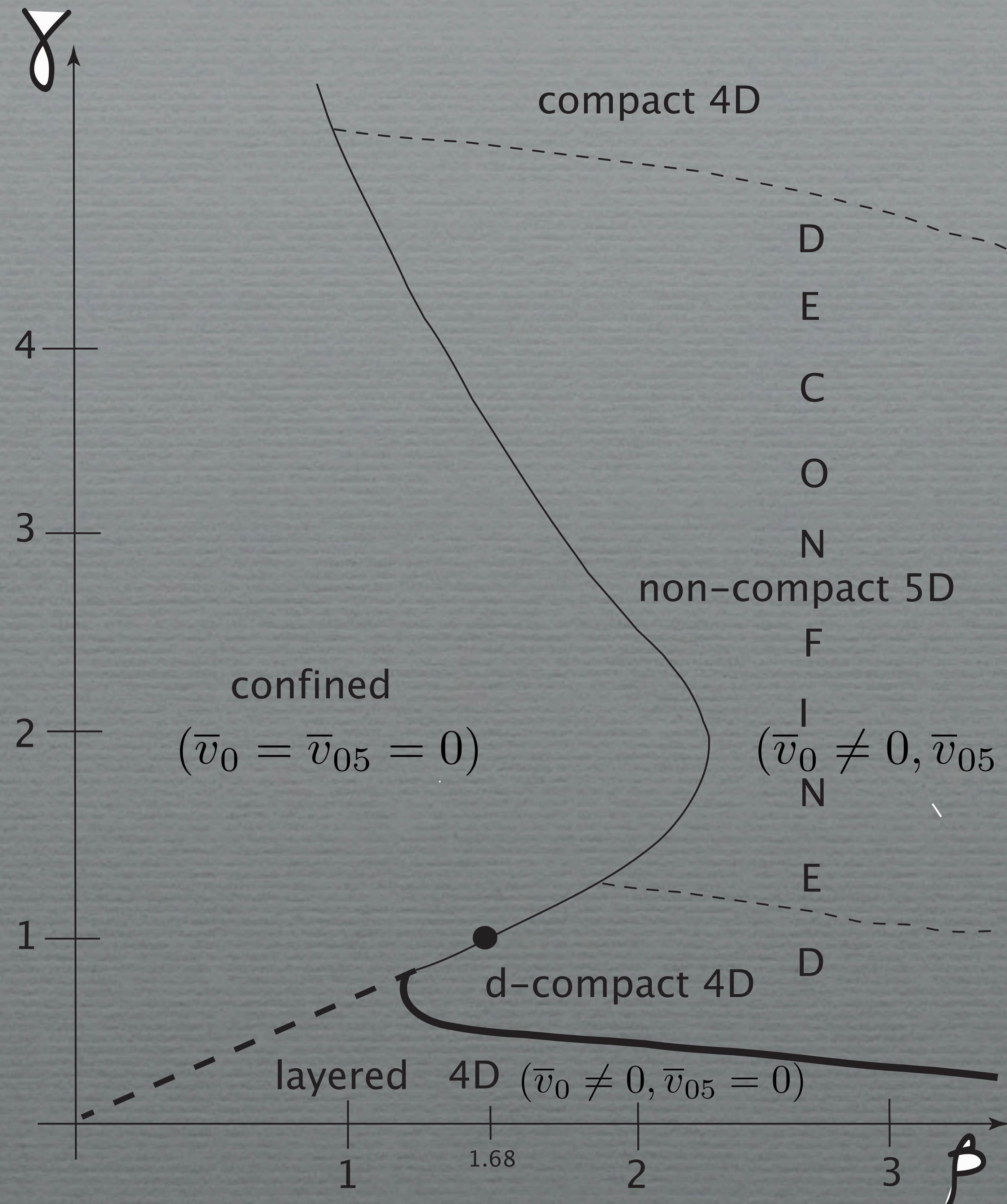
$$m = \lim_{t\to\infty} \ln \frac{C^{(1)}(t)}{C^{(1)}(t-1)}$$

The vector

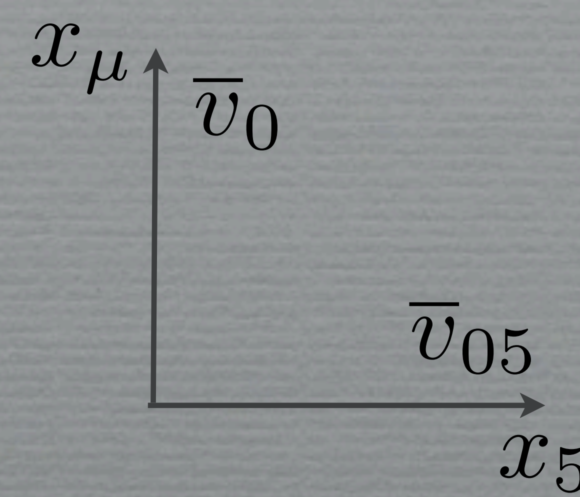


$$m = \lim_{t\to\infty} \ln \frac{C^{(2)}(t)}{C^{(2)}(t-1)}$$

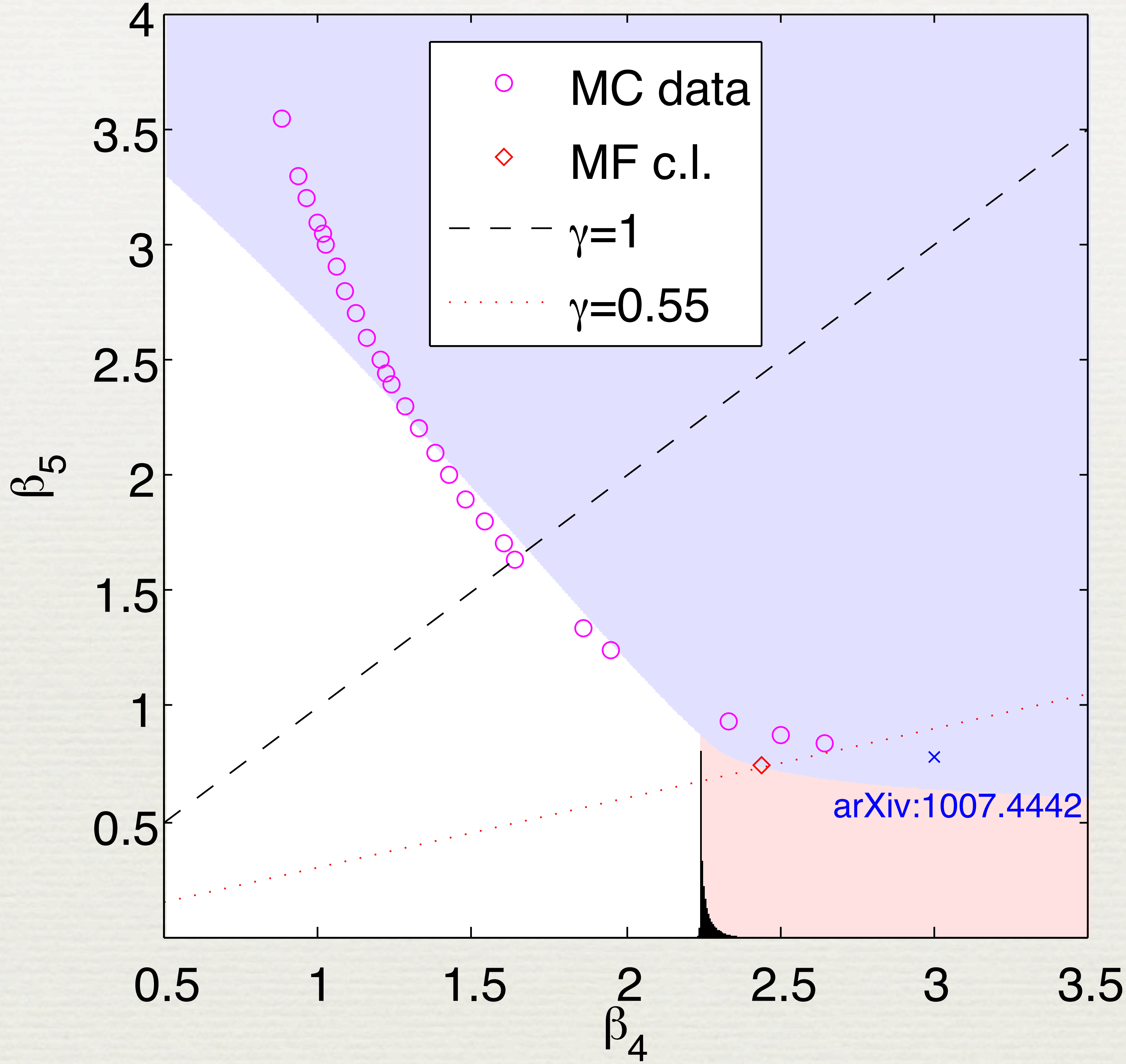
The MF phase diagram



the mean-field background

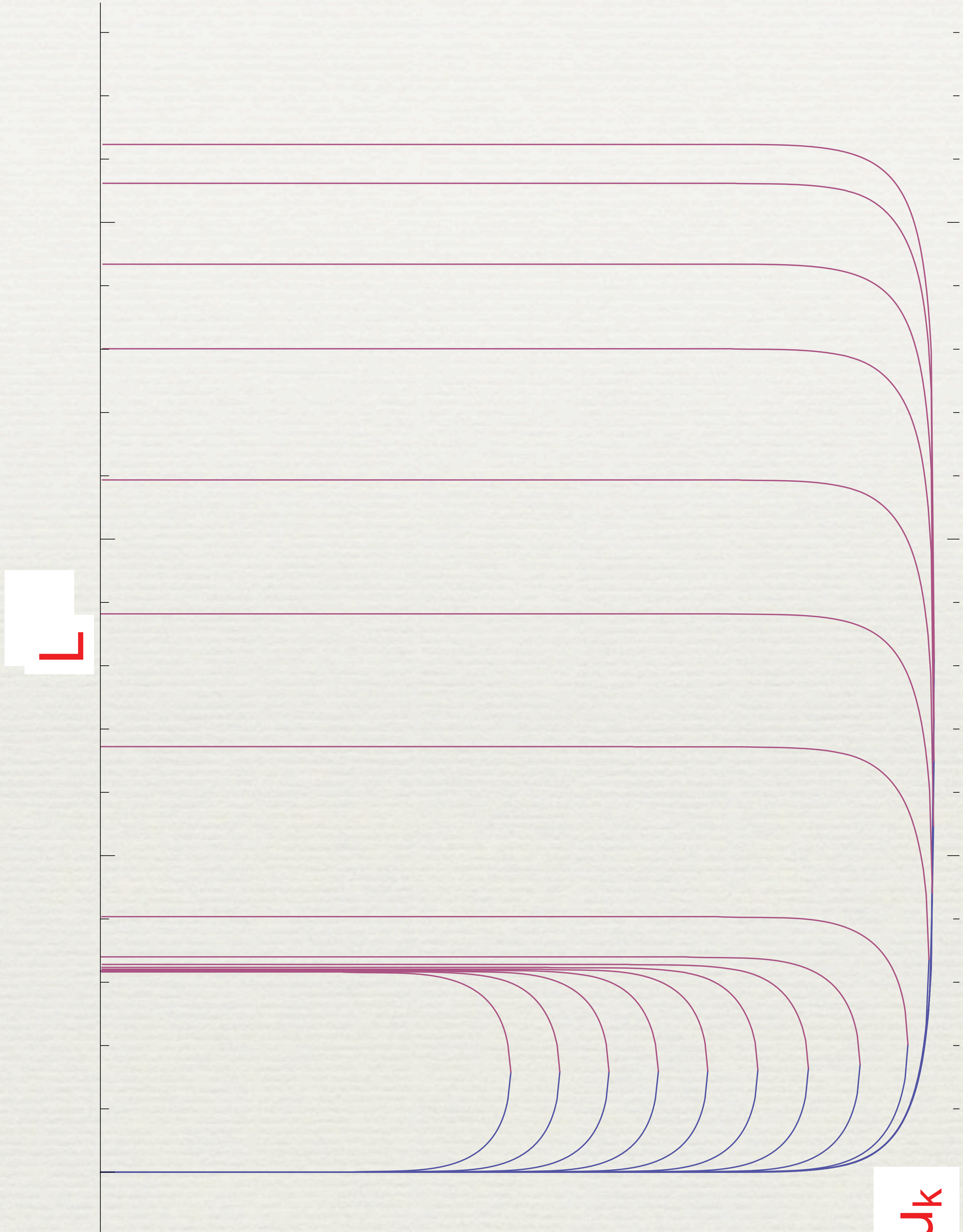


MC vs MF



MC: F. Knechtli, M. Luz and A. Rago, NPB 856 (2012) *74* and N.I. & F. Knechtli

MF: N.I . & F. Knechtli, Nucl. Phys. B822 (2009) 1, Phys. Lett. B685 (2010) 86



Courtesy of D. Giataganas

To 2nd order

$$S_{\rm eff} = S_{\rm eff}[\overline{V},\overline{H}] + \frac{1}{2}\left(\frac{\delta^2 S_{\rm eff}}{\delta H^2}h^2 + 2\frac{\delta^2 S_{\rm eff}}{\delta H \delta V}hv + \frac{\delta^2 S_{\rm eff}}{\delta V^2}v^2\right)$$

$$+\frac{1}{6}\left(\frac{\delta^3 S_{\rm eff}}{\delta H^3}h^3 + \frac{\delta^3 S_{\rm eff}}{\delta V^3}v^3\right) + \frac{1}{24}\left(\frac{\delta^4 S_{\rm eff}}{\delta H^4}h^4 + \frac{\delta^4 S_{\rm eff}}{\delta V^4}v^4\right) + \cdots$$

$$\mathcal{O}[V] = \mathcal{O}[\overline{V}] + \frac{\delta \mathcal{O}}{\delta V}v + \frac{1}{2}\frac{\delta^2 \mathcal{O}}{\delta V^2}v^2 + \frac{1}{6}\frac{\delta^3 \mathcal{O}}{\delta V^3}v^3 + + \frac{1}{24}\frac{\delta^4 \mathcal{O}}{\delta V^4}v^4 + \cdots$$

2nd order master formula

$$< \mathcal{O} >= \mathcal{O}[\overline{V}] + \frac{1}{2}\left(\frac{\delta^2 \mathcal{O}}{\delta V^2}\right)_{ij}(K^{-1})_{ij}$$

$$+\frac{1}{24}\sum_{i,j,l,m}\left(\frac{\delta^4 \mathcal{O}}{\delta V^4}\right)_{ijlm}\left((K^{-1})_{ij}(K^{-1})_{lm} + (K^{-1})_{il}(K^{-1})_{jm} + (K^{-1})_{im}(K^{-1})_{jl}\right)$$

$$C^{(2)}(t) = \frac{1}{24}\sum_{i,j,l,m}\left(\frac{\delta^4 \mathcal{O}^c(t)}{\delta v^4}\right)_{ijlm}\left((K^{-1})_{ij}(K^{-1})_{lm} + (K^{-1})_{il}(K^{-1})_{jm} + (K^{-1})_{im}(K^{-1})_{jl}\right)$$