One-Loop Amplitudes as BPS state sums

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Based on work with
Carlo Angelantonj & *Boris Pioline*

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(Werner-Heisenberg-Institut)

Corfu, September 2012

MAX-PLANCK-GESELLSCHAFT

Closed String perturbation theory

Topological expansion over closed Riemann surfaces

$$\sum_{g=0}^{\infty} g_s^{2(g-1)} \int_{\text{moduli}} \int \mathcal{D}g_{ab} \mathcal{D}X \, \mathcal{D}\psi \, \ldots \mathcal{V}_i(z_i) \ldots \, e^{-S[X,\psi,g_{ab},\ldots]}$$

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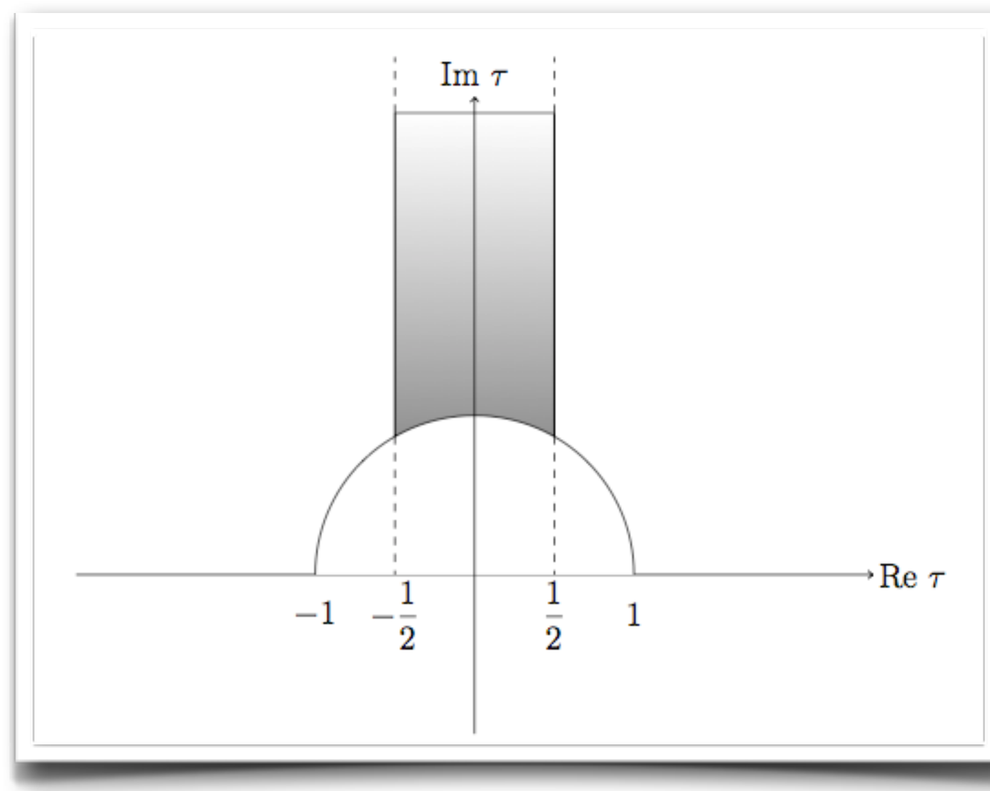

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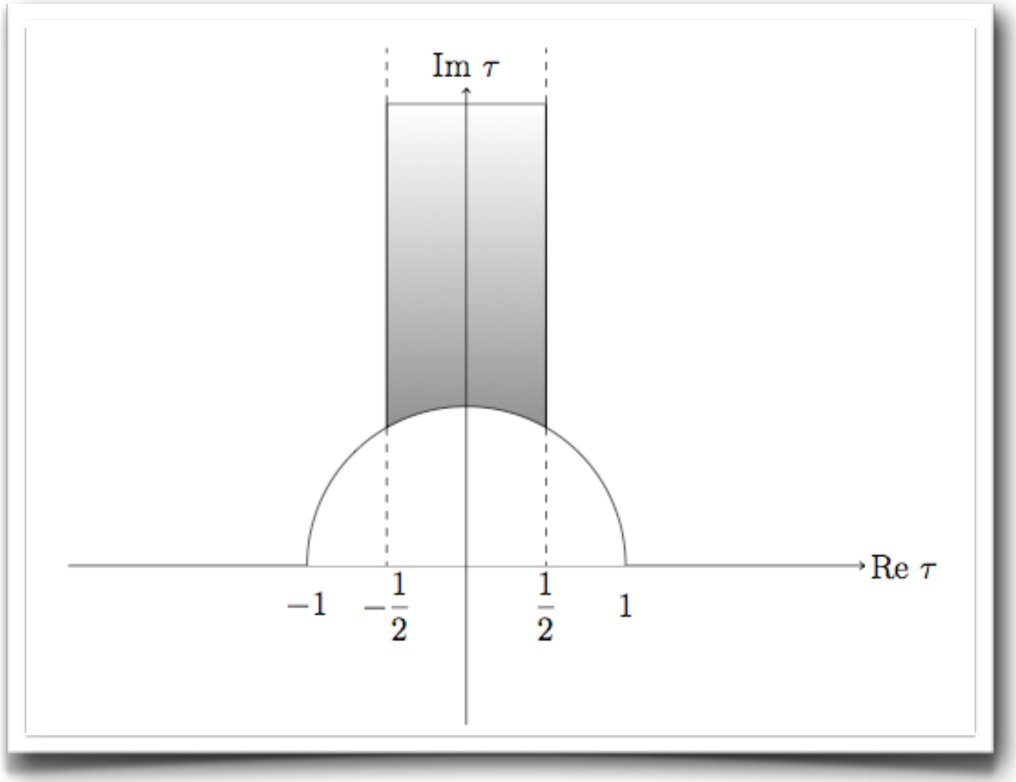

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- Gauge threshold corrections $R^2 F^{2h-2}$ in heterotic on $K3 \times T^2$
- F^4 couplings in heterotic on T^d
- R^4 couplings in type II on T^d
- R^2 couplings in type II on $K3 \times T^2$
- ...

Dixon, Kaplunovski, Louis ; Harvey, Moore
Bachas, Fabre, Kiritsis, Obers, Vanhove
Green, Vanhove, Kiritsis, Pioline
Gregori, Kiritsis, Kounnas, Obers, Petropoulos, Pioline

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Hagedorn phase transition	Initial Singularity

C. Angelantonj, M. Cardella, N. Irges 2006
I.F., C. Kounnas 2009
C. Angelantonj, M. Cardella, S. Elitzur, E. Rabinovici 2010
I.F., C. Kounnas, N. Toumbas 2010
I.F., C. Kounnas, H. Partouche, N. Toumbas 2010

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The known way out is a procedure that goes by the name "orbit method" or simply "unfolding"

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Absolute convergence: interchange integration with summation

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IR : "extra" massless modes (at T-self-dual point)

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Unfolding against the lattice is useful for extracting the large volume behaviour of the amplitude

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NONE of the individual pieces is invariant under T-duality
$SL(2;\mathbb{Z})_T \times SL(2;\mathbb{Z})_U \times \mathbb{Z}_2$

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A more complicated example:

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$$
\begin{aligned}
\int_{\mathcal{F}} d\mu\, \mathit{\Gamma}_{2,2}(T,U)\, \frac{\hat{E}_2\, E_4\, E_6}{\mathit{\Delta}} \simeq \mathrm{Re} \Bigg[& -24 \sum_{k>0} \left(11\, \mathrm{Li}_1(e^{2\pi i k T}) - \frac{30}{\pi T_2\, U_2} \mathcal{P}(kT) \right) \\
& - 24 \sum_{\ell>0} \left(11\, \mathrm{Li}_1(e^{2\pi i \ell U}) - \frac{30}{\pi T_2\, U_2} \mathcal{P}(\ell U) \right) \\
& + \sum_{k>0,\ell>0} \left(\tilde{c}(k\ell)\, \mathrm{Li}_1(e^{2\pi i (kT+\ell U)}) - \frac{3\, c(k\ell)}{\pi T_2\, U_2} \mathcal{P}(kT+\ell U) \right) \\
& + \mathrm{Li}_1(e^{2\pi i (T_1 - U_1 + i|T_2 - U_2|)}) - \frac{3}{\pi T_2\, U_2} \mathcal{P}\left(T_1 - U_1 + i|T_2 - U_2|\right) \Bigg] \\
& + \frac{60\,\zeta(3)}{\pi^2\, T_2 U_2} + 22 \log \left(\frac{8\pi e^{1-\gamma}}{\sqrt{27}}\, T_2 U_2 \right) \\
& + \left(\frac{4\pi}{3} \frac{U_2^2}{T_2} - \frac{22\pi}{3} U_2 - 4\pi T_2 \right) \mathit{\Theta}(T_2 - U_2) \\
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\end{aligned}
$$

where $\mathcal{P}(z) = y \mathrm{Li}_2(e^{2\pi i z}) + \frac{1}{2\pi} \mathrm{Li}_3(e^{2\pi i z})$

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Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)

I. Florakis, 2012

Traditional unfolding against the lattice

A more complicated example:

$$\int_{\mathcal{F}} d\mu\, \Gamma_{2,2}(T,U)\, \frac{\hat{E}_2\, E_4\, E_6}{\Delta} \simeq \mathrm{Re}\Bigg[-24 \sum_{k>0} \left(11\, \mathrm{Li}_1(e^{2\pi i k T}) - \frac{30}{\pi T_2 U_2} \mathcal{P}(kT) \right)$$

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(Werner-Heisenberg-Institut)
I. Florakis, 2012

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- Obscures singularities of the amplitude !
- Hard to check T-duality invariance !
- Useful for extracting asympotic behaviour in large volume limit

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I. Florakis, 2012

Idea : Let's unfold against something else !

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Goal: find some other way to unfold that does not spoil the manifest T-duality symmetries of the lattice

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$$\int_{\mathcal{F}_T} d\mu\ \Gamma_{d,d}(G,B;\tau,\bar{\tau})\, E^\star(s;\tau) \xrightarrow{\text{unfold}} \int_0^\infty d\tau_2\ \tau_2^{s-2} \int_{-1/2}^{1/2} d\tau_1\ \tau_2^{d/2} \sum_{m,n} e^{-2\pi\tau_2 \mathcal{M}^2} e^{2\pi i \tau_1 m^T n}$$

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(Werner-Heisenberg-Institut) I. Florakis, 2012

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New method required !

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What happens for integrands which are of rapid growth at the cusp ?
(unphysical tachyon)

New method required !

❷ $$\int_{\mathcal{F}} d\mu \; \Gamma_{d+k,d}(G,B,Y;\tau,\bar{\tau})\,\Phi(\tau)$$ unfold the elliptic genus !

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- Hyperbolic Laplacian Δ acts as Casimir operator and modular forms Φ can be organized into appropriate linear combinations of its eigenmodes

$$\Delta_w = 2\tau_2^2\,\partial_{\bar\tau}\left(\partial_\tau - \frac{iw}{2\tau_2}\right)$$

- Construct Φ by Poincaré representation such that the seed f is an eigenmode of Δ

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These conditions lead to the seed $\quad \varphi(\tau,\bar\tau) = \mathcal{M}_{s,w}(-\kappa\tau_2)\, e^{-2\pi i \kappa \tau_1}$

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- Hyperbolic Laplacian Δ acts as Casimir operator and modular forms Φ can be organized into appropriate linear combinations of its eigenmodes

$$\Delta_w = 2\tau_2^2\, \partial_{\bar\tau}\left(\partial_\tau - \frac{iw}{2\tau_2}\right)$$

- Construct Φ by Poincaré representation such that the seed f is an eigenmode of Δ

Impose order-κ pole at the cusp $\Phi \sim q^{-\kappa} + \dots$

The Poincaré series must be absolutely convergent (for w<0) to justify the unfolding

These conditions lead to the seed $\varphi(\tau,\bar\tau) = \mathcal{M}_{s,w}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1}$

$$\mathcal{M}_{s,w}(y) = |4\pi y|^{-w/2}\, M_{\frac{w}{2}\mathrm{sgn}(y),s-\frac{1}{2}}(4\pi|y|)$$

A new method: unfolding against the elliptic genus !

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Whittaker M-function

$$M_{\lambda,\mu}(z) = e^{-z/2}\, z^{\mu+\frac{1}{2}}\ {}_1F_1(\mu-\lambda+\tfrac{1}{2}; 1+2\mu; z)$$

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A new Poincaré series

This seed defines the Niebur-Poincaré series

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$$\mathcal{F}(s,\kappa,w) = \frac{1}{2}\sum_{\gamma\in SL(2;\mathbb{Z})/\Gamma_\infty} (c\tau+d)^{-w}\ \mathcal{M}_{s,w}(-\kappa\,\mathrm{Im}\,\gamma\cdot\tau)\ e^{-2\pi i\kappa \mathrm{Re}(\gamma\cdot\tau_1)}$$

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In string theory, the elliptic genera can have (at most) κ=1

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Weak quasi-holomorphic modular forms are eigenmodes of the Laplacian with eigenvalue $-w/2$

The N-P series has the same eigenvalue for $s=1-w/2$

In general, the N-P series with $s=1-w/2$ is a (weak) harmonic Maass form (Mock + Shadow)

However, by taking linear combinations of N-P series with definite coefficients, the Shadows cancel and the linear combination represents any weak holomorphic modular form !

Weak quasi-holomorphic modular forms can be formed from linear combinations of N-P series with $s=1-w/2+n$

The spectrum of modular forms as limits of the N-P series

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$w = 0$

$$\frac{\hat{E}_2 E_4 E_6}{\Delta} = \mathcal{F}(2,1,0) - 5\,\mathcal{F}(1,1,0) - 144$$

$$\frac{\hat{E}_2^2 E_4^2}{\Delta} = \tfrac{1}{5}\,\mathcal{F}(3,1,0) - 4\mathcal{F}(2,1,0) + 13\,\mathcal{F}(1,1,0) + 144$$

$$\frac{\hat{E}_2^3 E_6}{\Delta} = \tfrac{3}{175}\,\mathcal{F}(4,1,0) - \tfrac{3}{5}\,\mathcal{F}(3,1,0) + \tfrac{33}{5}\,\mathcal{F}(2,1,0) - 17\,\mathcal{F}(1,1,0) - 144$$

$$\frac{\hat{E}_2^4 E_4}{\Delta} = \tfrac{1}{1225}\,\mathcal{F}(5,1,0) - \tfrac{6}{175}\,\mathcal{F}(4,1,0) + \tfrac{18}{35}\,\mathcal{F}(3,1,0) - \tfrac{16}{5}\,\mathcal{F}(2,1,0) + \tfrac{29}{5}\,\mathcal{F}(1,1,0) + \tfrac{144}{5}$$

$$\frac{\hat{E}_2^6}{\Delta} = \tfrac{1}{1926925}\mathcal{F}(7,1,0) - \tfrac{3}{2695}\mathcal{F}(5,1,0) + \tfrac{6}{175}\mathcal{F}(4,1,0) - \tfrac{3}{7}\mathcal{F}(3,1,0) + \tfrac{12}{5}\mathcal{F}(2,1,0) - \tfrac{29}{7}\mathcal{F}(1,1,0) - \tfrac{144}{7}$$

$w = -2$

$$\frac{\hat{E}_2 E_4^2}{\Delta} = \tfrac{1}{40}\mathcal{F}(3,1,-2) - \tfrac{1}{3}\mathcal{F}(2,1,-2)$$

$$\frac{\hat{E}_2^2 E_6}{\Delta} = \tfrac{1}{525}\mathcal{F}(4,1,-2) - \tfrac{1}{20}\mathcal{F}(3,1,-2) + \tfrac{11}{30}\mathcal{F}(2,1,-2)$$

$$\frac{\hat{E}_2^3 E_4}{\Delta} = \tfrac{1}{11760}\mathcal{F}(5,1,-2) - \tfrac{1}{350}\mathcal{F}(4,1,-2) + \tfrac{9}{280}\mathcal{F}(3,1,-2) - \tfrac{2}{15}\mathcal{F}(2,1,-2)$$

$$\frac{\hat{E}_2^5}{\Delta} = \tfrac{1}{19819800}\mathcal{F}(7,1,-2) - \tfrac{1}{12936}\mathcal{F}(5,1,-2) + \tfrac{1}{525}\mathcal{F}(4,1,-2) - \tfrac{1}{56}\mathcal{F}(3,1,-2) + \tfrac{1}{15}\mathcal{F}(2,1,-2)$$

$w = -4$

$$\frac{\hat{E}_2 E_6}{\Delta} = \tfrac{1}{2520}\mathcal{F}(4,1,-4) - \tfrac{1}{120}\mathcal{F}(3,1,-4)$$

$$\frac{\hat{E}_2^2 E_4}{\Delta} = \tfrac{1}{70560}\mathcal{F}(5,1,-4) - \tfrac{1}{2520}\mathcal{F}(4,1,-4) + \tfrac{1}{280}\mathcal{F}(3,1,-4)$$

$$\frac{\hat{E}_2^4}{\Delta} = \tfrac{1}{148648500}\mathcal{F}(7,1,-4) - \tfrac{1}{129360}\mathcal{F}(5,1,-4) + \tfrac{1}{6300}\mathcal{F}(4,1,-4) - \tfrac{1}{840}\mathcal{F}(3,1,-4)$$

$w = -6$

$$\frac{\hat{E}_2 E_4}{\Delta} = \tfrac{1}{241920}\mathcal{F}(5,1,-6) - \tfrac{1}{10080}\mathcal{F}(4,1,-6)$$

$$\frac{\hat{E}_2^3}{\Delta} = \tfrac{1}{792792000}\mathcal{F}(7,1,-6) - \tfrac{1}{887040}\mathcal{F}(5,1,-6) + \tfrac{1}{50400}\mathcal{F}(4,1,-6)$$

$w = -8$

$$\frac{\hat{E}_2^2}{\Delta} = \tfrac{1}{2854051200}\mathcal{F}(7,1,-8) - \tfrac{1}{3991680}\mathcal{F}(5,1,-8)$$

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Unfolding against the N-P series gives a BPS sum

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$$= \int_0^\infty d\tau_2 \; \tau_2^{d/2-2} \; \mathcal{M}_{s,-\frac{k}{2}}(-\kappa\tau_2) \sum_{\text{BPS}} e^{-\pi\tau_2 \, (P_L^2+P_R^2)/2}$$

for generic values of $s \neq \frac{d}{2}+\frac{k}{4}$

$$I = (4\pi\kappa)^{1-\frac{d}{2}} \, \Gamma(s+\tfrac{d}{2}+\tfrac{k}{4}-1) \times \sum_{\text{BPS}} {}_2F_1\left(s-\frac{k}{4}, s+\frac{d}{2}+\frac{k}{4}-1; 2s; \frac{4\kappa}{P_L^2}\right) \left(\frac{P_L^2}{4\kappa}\right)^{1-s-\frac{d}{2}-\frac{k}{4}}$$

- For Re(s)>$d/2+k/4$, sum converges absolutely, with a simple pole at $s=d/2+k/4$
- Manifestly T-duality invariant
- Chamber independent

BPS state sums & Singularity Structure

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One-dimensional lattice

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$$\int_{\mathcal{F}} d\mu \; \Gamma_{(1,1)}(R) \; \mathcal{F}(1+n,1,0) = 2^{2+2n} \sqrt{\pi}\, \Gamma(n+\tfrac{1}{2}) \left(R^{1+2n} + \frac{1}{R^{1+2n}} - \left| R^{1+2n} - \frac{1}{R^{1+2n}} \right| \right)$$

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General result for $n>d/2-1$ or for **odd-dimension** (independently of n):

$$I_1 = (4\pi\kappa)^{1-\frac{d}{2}} \frac{\Gamma(2n+2+\frac{k}{2})\,\Gamma(n+\frac{d+k}{2})}{n!} \sum_{m=0}^{d/2-2} \binom{n}{m} \frac{(-)^m}{\Gamma(n-m+\frac{d+k}{2})}$$

$$\times \sum_{\mathrm{BPS}} \left(\frac{P_L^2}{4\kappa}\right)^{n-m} \left[\Gamma(\tfrac{d}{2}-m-1) \left(\frac{P_R^2}{4\kappa}\right)^{m+1-\frac{d}{2}} - \sum_{\ell=0}^{2n+k/2} \frac{\Gamma(\frac{d}{2}-m-1+\ell)}{\ell!} \left(\frac{P_L^2}{4\kappa}\right)^{1+m-\frac{d}{2}-\ell} \right]$$

General result for **even-dimension and $n \leq d/2-1$** is given by **adding I_1+I_2**, where:

$$I_2 = (4\pi\kappa)^{1-\frac{d}{2}} \frac{\Gamma(2n+2+\frac{k}{2})\,\Gamma(n+\frac{d+k}{2})}{n!} \sum_{\mathrm{BPS}} \sum_{m=d/2-1}^{n} \binom{n}{m} \frac{(-)^m}{\Gamma(n-m+\frac{d+k}{2})} \left(\frac{P_L^2}{4\kappa}\right)^{n-m}$$

$$\times \Bigg\{ -\sum_{\ell=m+2-d/2}^{2n+k/2} \frac{\Gamma(\frac{d}{2}-m-1+\ell)}{\ell!} \left(\frac{P_L^2}{4\kappa}\right)^{1+m-\frac{d}{2}-\ell} + \frac{(-)^{m+1-\frac{d}{2}}}{\Gamma(m+2-\frac{d}{2})} \left(\frac{P_R^2}{4\kappa}\right)^{m+1-\frac{d}{2}}$$

$$\times \left[H_{m+1-\frac{d}{2}} - \log\left(\frac{P_R^2}{P_L^2}\right) \right] - \frac{1}{\Gamma(m+2-\frac{d}{2})} \sum_{\ell=0}^{m+1-d/2} \binom{m+1-\frac{d}{2}}{\ell} \left(-\frac{P_L^2}{4\kappa}\right)^{m+1-\frac{d}{2}-\ell} H_{m+1-\frac{d}{2}-\ell} \Bigg\}$$

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Universal singularity behaviour in 2d

$$I_{2,2}(s=1+n,\kappa=1) \;\sim\; -\frac{(2n+1)!}{n!}\,\log|j(T)-j(U)|^4$$

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Amplitudes involving linear combinations of modular forms, such that the unphysical tachyon pole is cancelled are regular at any point in Narain moduli space

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$\mathcal{N} = 2$ heterotic vacuum at the orbifold point $T^2 \times T^4/\mathbb{Z}_2$

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$$\tfrac{1}{4}P_L^2 - \tfrac{1}{4}P_R^2 = 1 \quad \leftrightarrow \quad m_i\, n^i = 1$$

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Results regular at any point in moduli space and in any chamber !

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One-loop BPS amplitudes with momentum insertions

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Consider modular integrals with insertions of left/right- moving lattice momenta:

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Modular form of weight ($\lambda+d+k/2$,0) provided that $\rho(x,y)$ satisfies:

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and that $\rho(x,y)\, e^{-\pi(x^2+y^2)}$ decays sufficiently fast at infinity

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Further simplifications possible, when $\rho(x,y)$ is polynomial

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An example from non-compact heterotic vacua

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Non-trivial integrals without moduli dependence

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Unfold à la Niebur:

$$\frac{\hat{E}_2^2 E_4^2}{\Delta} - 2\frac{\hat{E}_2^2 E_4 E_6}{\Delta} = \tfrac{1}{5}\mathcal{F}(3,1,0) - 6\mathcal{F}(2,1,0) + 23j + 984$$

An example from non-compact heterotic vacua

$$\Gamma = \int_F d\mu \, \left(\sqrt{\tau_2}\, \eta\, \bar{\eta}\right)^3 \, \frac{\hat{E}_2^2\, E_8 - 2\, \hat{E}_2\, E_{10}}{\Delta} \; = \; ?$$

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An example from non-compact heterotic vacua

$$\Gamma = \int_F d\mu \left(\sqrt{\tau_2}\,\eta\,\bar{\eta}\right)^3 \frac{\hat{E}_2^2\,E_8 - 2\,\hat{E}_2\,E_{10}}{\Delta} \;=\; -20\sqrt{2}$$

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$\Delta p \cdot \Delta q \geq \frac{1}{2}\hbar$
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1	$\int_{\mathcal{F}} d\mu\ \Phi(\tau)$	Stokes' theorem
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4	$\int_{\mathcal{F}} d\mu\ \mathcal{Z}(\tau,\bar{\tau})$	No general approach... yet !

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- ☑ Even in the absence of the lattice itself !

Conclusions & Outlook

- [x] Generalization for modular forms of congruence subgroups of SL(2;Z) (freely-acting orbifolds)

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☑ Effective potential of strings at finite temperature (String Cosmology)

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Thank you !

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Backup Slides

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$$\beta_{12} = 1 + 2\pi \sum_{c>0} \frac{S(1,1;c)}{c} J_{11}(4\pi/c) \approx 2.84029...$$

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Similar methods should be applicable for integrals of the full (non-holomorphic) partition function

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Idea : Let's unfold against something else !

We want to find some other way to unfold that does not spoil the manifest T-duality symmetries of the lattice

In particular, we are looking for the representation that captures the behaviour around T-self-dual points

Such a method is known in the mathematics literature as the Rankin-Selberg method, later extended by Zagier

Start with the modular integral $\displaystyle\int_{\mathcal{F}} d\mu\ F(\tau,\bar\tau)$

Assume we are only dealing with functions of moderate growth at the cusp & impose a hard IR cut-off $\mathcal{F}_T = \mathcal{F}\cap\{\tau_2 \leq T\}$

Consider instead the integral $\displaystyle\int_{\mathcal{F}_T} d\mu\ F(\tau,\bar\tau)\ E^\star(\tau;s)$

$$E^\star(\tau;s) \equiv \tfrac{1}{2}\zeta^\star(2s)\sum_{(c,d)=1}\frac{\tau_2^s}{|c\tau+d|^{2s}} = \zeta^\star(2s)\sum_{\gamma\in SL(2;\mathbb{Z})/\Gamma_\infty}\Big[\mathrm{Im}(\gamma\cdot\tau)\Big]^s$$

$$\zeta^\star(s)\equiv \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

$E^\star(\tau;s)$ is a meromorphic function in s, with simple poles at $s=0,1$

$$E^\star(\tau;s) = \frac{1}{2(s-1)} + \tfrac{1}{2}\left(\gamma - \log(4\pi\tau_2|\eta(\tau)|^4)\right) + \mathcal{O}(s-1)$$

The whole trick is based on the fact that the residue is independent of τ

$$2\,\mathrm{Res}_{s=1}\int_{\mathcal{F}_T} d\mu\ F(\tau,\bar\tau)\ E^\star(\tau;s) = \int_{\mathcal{F}_T} d\mu\ F(\tau,\bar\tau)$$

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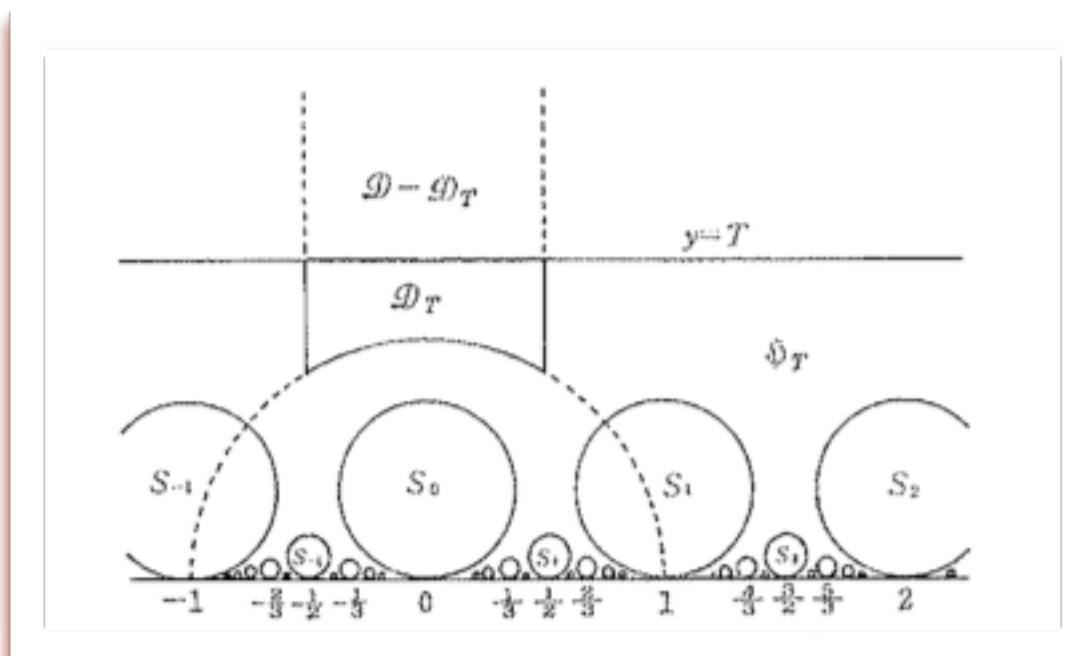
The Rankin-Selberg-Zagier method

Now we are ready to unfold the Eisenstein series, modulo a little subtlety

$SL(2;\mathbb{Z})$-transformations do not simply map $\mathcal{F}_T$ to the "naive" truncated Poincaré upper half-plane

$$\mathcal{H}_T \equiv \mathcal{H} \cap \{\tau_2 \leq T\} - \bigcup_{c\geq 1, (a,c)=1} S_{a/c}$$

but one has to subtract an infinite number of disks $S_{a/c}$, of radius $1/(2c^2T)$ and tangent to the real axis at a/c



The Rankin-Selberg-Zagier method

Unfolding and taking the residue eventually gives

Renormalized Integral

$$2\,\mathrm{Res}_{s=1}\left[\zeta^\star(2s)\int_0^\infty d\tau_2\ \tau_2^{s-2}F_0(\tau_2)\right] = \int_{\mathcal{F}_T} d\mu F(\tau,\bar\tau) + \int_{\mathcal{F}-\mathcal{F}_T} d\mu\ \left(F(\tau,\bar\tau)-\phi(\tau_2)\right)$$

removes the power divergence →

$$-\,2\,\mathrm{Res}_{s=1}\Big[\zeta^\star(2s)h_T(s)+\zeta^\star(2s-1)h_T(1-s)\Big]$$

For functions F of "rapid decay" at the cusp, $\phi(\tau_2)\sim\tau_2^\alpha$, $\mathrm{Re}(\alpha)<1$, the renormalized integral reduces to the usual integral

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The Rankin-Selberg-Zagier method

Let us apply this to the case of a *d*-dimensional lattice

$$I = 2\,\mathrm{Res}_{s=1}\left[\zeta^\star(2s)\int_0^\infty d\tau_2\;\tau_2^{s+d/2-2}\sum_{m^T n=0}{}^{'}\, e^{-\pi\tau_2\mathcal{M}^2}\right] = 2\,\mathrm{Res}_{s=1}\left[\frac{\zeta^\star(2s)\Gamma(s+\frac{d}{2}-1)}{\pi^{s+d/2-1}}\,\mathcal{E}^d_{\mathbb{V}}(G,B;s+\tfrac{d}{2}-1)\right]$$

$$\mathcal{E}^d_{\mathbb{V}}(G,B;s) \equiv \sum_{m^T n=0}{}^{'}\,\frac{1}{\mathcal{M}^{2s}}$$

is the constrained Epstein zeta series in the vectorial representation of *O(d,d)*

For the one-dimensional lattice, it is easy to recover the well-known closed-form expression

$$I_{d=1} = 2\,\mathrm{Res}_{s=1}\left[\zeta^\star(2s)\zeta^\star(2s-1)\left(R^{1-2s}+R^{2s-1}\right)\right] = \frac{\pi}{3}\left(R+\frac{1}{R}\right)$$

The Rankin-Selberg-Zagier method

Now consider the integral of the two-dimensional lattice, parametrized by the complex structure and Kähler moduli, U and T

$$\begin{cases} P_L = \Big(m_1 + U m_2 + \bar{T}(n^2 - U n^1)\Big)/\sqrt{2T_2U_2} \\ P_R = \Big(m_1 + U m_2 + T(n^2 - U n^1)\Big)/\sqrt{2T_2U_2} \end{cases}$$

To proceed we need to solve the Diophantine constraint

$$m_1 n^1 + m_2 n^2 = 0 \quad , \quad m_i, n^i \in \mathbb{Z}$$

The general solution has two contributions

$$S_1 \ : \ \{(m_1, m_2, 0, 0) \ | \ m_1, m_2 \in \mathbb{Z}\}$$

$$S_2 \ : \ \{(c\tilde{m}_1, c\tilde{m}_2, -d\tilde{m}_2, d\tilde{m}_1) \ | \ (\tilde{m}_1, \tilde{m}_2) = 1 \quad , \quad d \geq 1\}$$

$$\mathcal{E}^{2\star}_{\mathbb{V}}(T, U; s) = 2\, E^\star(T; s)\, E^\star(U; s)$$

The two contributions combine into a simple expression manifestly reflecting the group isomorphism

$$O(2,2;\mathbb{Z}) \sim SL(2;\mathbb{Z})_T \times SL(2;\mathbb{Z})_U \ltimes \mathbb{Z}_2$$

The Rankin-Selberg-Zagier method

$\mathcal{E}^{2\star}_{\mathbb{V}}(T,U;s) = 2\,E^\star(T;s)\,E^\star(U;s)$ ⟵ has a double pole at $s=0$ and $s=1$

The residue can be computed by using Kronecker limit fomula

$$I_{d=2} = 2\,\mathrm{Res}_{s=1}\left(\frac{1}{2(s-1)^2} + \frac{1}{s-1}\left[\gamma - \tfrac{1}{2}\log\left(16\pi^2\, T_2 U_2\, |\eta(T)\eta(U)|^4\right)\right]\right)$$

...and one immediately recovers the well-known result

$$I_{d=2} \equiv R.N. \int_{\mathcal{F}} d\mu\; \Gamma_{(2,2)}(T,U) = -\log\Big(4\pi e^{-\gamma}\, T_2 U_2\, |\eta(T)\eta(U)|^4\Big)$$

- The derivation is remarkably simpler
- No need for additional regularization of the degenerate orbit
- T-duality manifest at every step (“dimensional regularization”)
- Additive constant depends on the renormalization scheme

L. Dixon, V. Kaplunovsky, J. Louis 1991
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What happens for integrals of the type

$$\int_{\mathcal{F}} d\mu\;\Gamma_{(d+k,d)}\;\Phi(\tau)$$

where the integrand is now a function of rapid growth

?

A new Poincaré series

One is then lead to define the following Niebur-Poincaré series

D. Niebur 1973
D. Hejhal 1983
J. Bruinier 2002

$$\mathcal{F}(s,\kappa,w) = \frac{1}{2} \sum_{\gamma\in SL(2;\mathbb{Z})/\Gamma_\infty} (c\tau+d)^{-w}\ \mathcal{M}_{s,w}(-\kappa\,\mathrm{Im}\,\gamma\cdot\tau)\ e^{-2\pi i\kappa \mathrm{Re}(\gamma\cdot\tau_1)}$$

$$= \frac{1}{2} \sum_{(c,d)=1} (c\tau+d)^{-w}\ \mathcal{M}_{s,w}\left(-\frac{\kappa\tau_2}{|c\tau+d|^2}\right)\ \exp\left\{-2\pi i\kappa\left(\frac{a}{c}-\frac{c\tau_1+d}{c|c\tau+d|^2}\right)\right\}$$

- Converges absolutely for Re(s)>1, independently of $\varkappa$ and w
- For $\varkappa>0$, the behaviour at the cusp is $$\mathcal{M}_{s,w}(-\kappa\tau_2)e^{-2\pi i\kappa\tau_1} \sim \frac{\Gamma(2s)}{\Gamma(s+\frac{w}{2})}\, q^{-\kappa}$$
- By construction, it is an eigenmode of the hyperbolic Laplacian $$\left[\Delta_w + \frac{s(1-s)}{2} + \frac{w(w+2)}{8}\right]\mathcal{F}(s,\kappa,w) = 0$$

One may define raising and lowering operators that raise / lower the modular weight by 2 units

$$D_w = \frac{i}{\pi}\left(\partial_\tau - \frac{iw}{2\tau_2}\right) \qquad D_w\cdot\mathcal{F}(s,\kappa,w) = 2\kappa(s+\tfrac{w}{2})\,\mathcal{F}(s,\kappa,w+2)$$

$$\bar{D}_w = -i\pi\tau_2^2\,\partial_{\bar\tau} \qquad \bar{D}_w\cdot\mathcal{F}(s,\kappa,w) = \frac{1}{8\kappa}(s-\tfrac{w}{2})\,\mathcal{F}(s,\kappa,w-2)$$

The elliptic genera encountered in string theory have (at most) $\varkappa$=1

One may generate the N-P series at arbitrary $\varkappa$, by considering the action of Hecke operators

$$T_\kappa\cdot\mathcal{F}(s,1,w) = \mathcal{F}(s,\kappa,w)$$

$$(T_\kappa\cdot\Phi)(\tau) = \sum_{d|\kappa} d^{-w} \sum_{b\in\mathbb{Z}_d} \Phi\left(\frac{\kappa}{d^2}\tau+\frac{b}{d}\right)$$

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Fourier expansion of Niebur-Poincaré series

In order to extract the Fourier expansion one separates out the contribution $c=0, d=1$ and then sets $d=d'+mc$ with $m\in\mathbb{Z}$ and $d'\in(\mathbb{Z}/c\mathbb{Z})^*$. Poisson re-summing over m and using the properties of Kloostermann sums we can turn the "Fourier" integral into a contour integral defining the (modified) Bessel functions

$$\mathcal{F}(s,\kappa,w)=\mathcal{M}_{s,w}(-\kappa\tau_2)e^{-2\pi i\kappa\tau_1}+\sum_{m\in\mathbb{Z}}\tilde{\mathcal{F}}_m(s,\kappa,w)\,e^{2\pi i m\tau_1}$$

modes

$$\tilde{\mathcal{F}}_0(s,\kappa,w)=\frac{2^{2-w}i^{-w}\pi^{1+s-\frac{w}{2}}\kappa^{s-\frac{w}{2}}\Gamma(2s-1)\sigma_{1-2s}(\kappa)}{\Gamma(s-\frac{w}{2})\Gamma(s+\frac{w}{2})\zeta(2s)}\;\tau_2^{2-s-\frac{w}{2}}$$

$$\tilde{\mathcal{F}}_m(s,\kappa,w)=\frac{4\pi\kappa i^{-w}\,\Gamma(2s)}{\Gamma(s+\frac{w}{2}\mathrm{sgn}(m))}\left|\frac{m}{\kappa}\right|^{\frac{w}{2}}\mathcal{W}_{s,w}(m\tau_2)\,\mathcal{Z}_s(m,-\kappa)$$

$$S(a,b;c)=\sum_{d\in(\mathbb{Z}/c\mathbb{Z})^*}\exp\left[\frac{2\pi i}{c}\left(ad+\frac{b}{d}\right)\right]$$

Kloostermann sum

$$\mathcal{W}_{s,w}(y)=|4\pi y|^{-w/2}\,W_{\frac{w}{2}\mathrm{sgn}(y),s-\frac{1}{2}}(4\pi|y|)$$

Whittaker W-function

$$W_{\lambda,\mu}(z)=\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)}M_{\lambda,\mu}(z)+\frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)}M_{\lambda,-\mu}(z)$$

Kloostermann-Selberg zeta function

$$\mathcal{Z}_s(a,b)=\frac{1}{2\sqrt{|ab|}}\sum_{c>0}\frac{S(a,b;c)}{c}\times\begin{cases}J_{2s-1}\left(\frac{4\pi}{c}\sqrt{ab}\right) & ,\ ab>0\\ I_{2s-1}\left(\frac{4\pi}{c}\sqrt{-ab}\right) & ,\ ab<0\end{cases}$$

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Harmonic Maass Forms from the Laplacian

Weak almost holomorphic modular forms are eigenmodes of Δ_w with eigenvalue $-w/2$

$$\left[\Delta_w + \frac{s(1-s)}{2} + \frac{w(w+2)}{8}\right]\mathcal{F}(s,\kappa,w) = 0$$

This is the case for the N-P series $\mathcal{F}(s,\kappa,w)$ with $s = 1-\frac{w}{2}$ and $s = \frac{w}{2}$

However, weak almost holomorphic modular forms are not the only eigenmodes of Δ_w with this eigenvalue

Weak harmonic Maass forms

- Transform like modular forms
- Eigenmodes of the Laplacian with eigenvalue $-w/2$
- Not holomorphic in general : infinite tower of negative frequency modes

Fourier expansion

$$\Phi = \sum_{m\le -1} (-m)^{w-1}\,\bar{b}_{-m}\,\Gamma(1-w,-4\pi m\tau_2)\,q^m + \frac{\bar{b}_0(4\pi\tau_2)^{1-w}}{w-1} + \sum_{m\ge -\kappa} a_m\, q^m$$

non-holomorphic

holomorphic "Mock modular"

$$\bar{D}_w \cdot \Phi = -2^{1-2w}\,(\pi\tau_2)^{2-w}\,\bar{\Psi} \qquad \Psi(\tau) = \sum_{m=0}^{\infty} b_m\, q^m$$

Annihilates the holomorphic part of Φ and produces the complex conjugate of a holomorphic modular form Ψ of weight 2-w

"Shadow"

$$D_w^{1-w}\cdot\Phi = \left(\frac{i}{\pi}\,\partial_\tau\right)^{1-w}\cdot\Phi = \Xi \qquad \Xi(\tau) = \sum_{m\ge-\kappa} (-2m)^{1-w}\, a_m\, q^m$$

"Farey transform" annihilates the non-holomorphic part of Φ and produces a weak holomorphic modular form Ξ of weight 2-w

"Ghost"

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Harmonic Maass Forms from the N-P series

Observe that the N-P series $\mathcal{F}(s,\kappa,w)$ with $s = 1-\frac{w}{2}$ is by construction a weak harmonic Maass form

(for $w < 0$, within the convergence domain)

$$a_{-\kappa} = \Gamma(2-w)$$
$$a_{-\kappa<m<0} = 0$$
$$a_0 = \frac{4\pi^2\kappa}{(2\pi i)^w}\,\frac{\sigma_{w-1}(\kappa)}{\zeta(2-w)}$$
$$a_{m>0} = 4\pi i^{-w}\kappa\,\Gamma(2-w)\left(\frac{m}{\kappa}\right)^{w/2}\mathcal{Z}_{1-\frac{w}{2}}(m,-\kappa)$$
$$b_0 = 0$$
$$b_{m>0} = (1-w)\kappa^{1-w}\delta_{m,\kappa} + 4\pi i^w(1-w)(m\kappa)^{1-w/2}\mathcal{Z}_{1-\frac{w}{2}}(m,\kappa)$$

Shadow is a cusp form of weight $2 - w > 2$

$$\bar{D}_w\cdot\mathcal{F}(1-\tfrac{w}{2},\kappa,w) = \frac{1-w}{8\kappa}\,\mathcal{F}(1-\tfrac{w}{2},\kappa,w-2)\ \sim\ \tau_2^{2-w}\,\overline{P(-\kappa,2-w)}$$

$$D_w^{1-w}\cdot\mathcal{F}(1-\tfrac{w}{2},\kappa,w) = (2\kappa)^{1-w}\,\Gamma(2-w)\,\mathcal{F}(1-\tfrac{w}{2},\kappa,2-w)$$
$$\sim\ \mathcal{F}(\tfrac{w'}{2},\kappa,w') = P(\kappa,w')$$

$w' = 2 - w > 2$ (within the convergence domain)

For special values of w

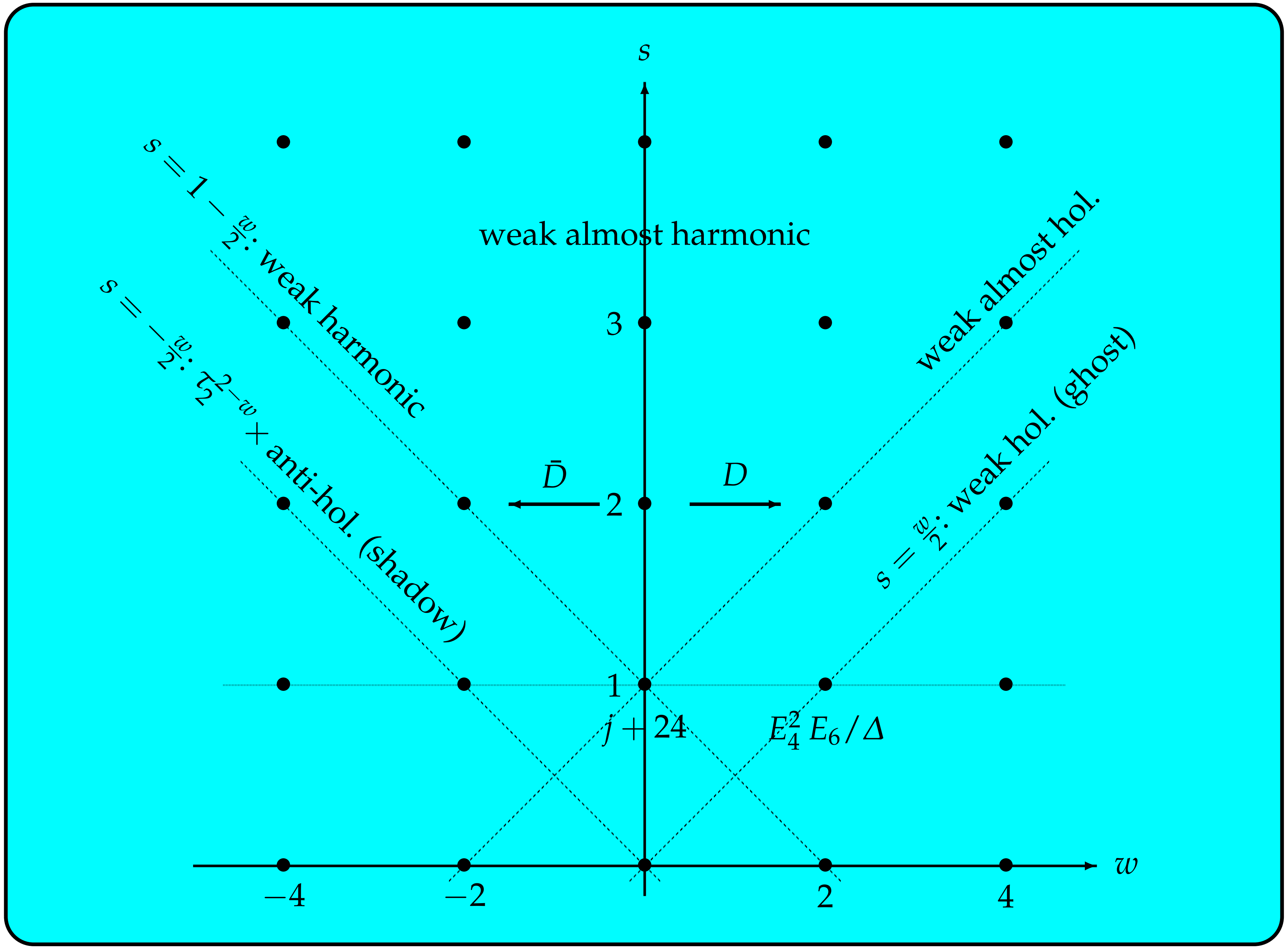
$$w\in\{-2,-4,-6,-8,-12\}$$

The space of cusp forms of weight $2 - w$ is empty !

The shadow vanishes & the Maass form is actually a weak holomorphic modular form !

!

The spectrum of modular forms as limits of the N-P series



$$\bar{D}_w \cdot \mathcal{F}(1-\tfrac{w}{2},\kappa,w) = \frac{1-w}{8\kappa}\,\mathcal{F}(1-\tfrac{w}{2},\kappa,w-2) \;\sim\; \tau_2^{2-w}\,\overline{P(-\kappa,2-w)}$$

$$D_w^{1-w}\cdot \mathcal{F}(1-\tfrac{w}{2},\kappa,w) = (2\kappa)^{1-w}\,\Gamma(2-w)\,\mathcal{F}(1-\tfrac{w}{2},\kappa,2-w)$$

$$\sim\; \mathcal{F}(\tfrac{w'}{2},\kappa,w') = P(\kappa,w')$$

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The spectrum of modular forms as limits of the N-P series

For these special values of w , $\mathcal{F}(1-\frac{w}{2},1,w)$ can be recognized as an element of the ring of weak holomorphic modular forms by matching the principal part of the expansions

For values of $w<0$ outside this list the space of cusp forms of weight $2-w$ is not empty and $\mathcal{F}(1-\frac{w}{2},1,w)$ is a genuine harmonic Maass form with non-vanishing shadow

w	$\mathcal{F}(1-\frac{w}{2},1,w)$	$\mathcal{F}(1-\frac{w}{2},1,2-w)$
0	$j+24$	$E_4^2\,E_6\,\Delta^{-1}$
-2	$3!\,E_4E_6\Delta^{-1}$	$E_4\,(j-240)$
-4	$5!\,E_4^2\,\Delta^{-1}$	$E_6\,(j+204)$
-6	$7!\,E_6\,\Delta^{-1}$	$E_4^2\,(j-480)$
-8	$9!\,E_4\,\Delta^{-1}$	$E_4E_6\,(j+264)$
-12	$13!\,\Delta^{-1}$	$E_4^2E_6\,(j+24)$

"Ghost"

However, the linear combination

$$\mathcal{G}(s,w)=\frac{1}{\Gamma(2-w)}\sum_{-\kappa\leq m<0}a_m\,\mathcal{F}(s,m,w)$$

with coefficients determined by the principal part of any weak holomorphic modular form Φ

$$\Phi_w^-=\sum_{-\kappa\leq m<0}a_m q^{-m}$$

of negative weight w, reduces in the limit $s=1-\frac{w}{2}$ to the holomorphic modular form Φ itself !

The shadows of the weak Maass forms cancel in the linear combination !

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The spectrum of modular forms as limits of the N-P series

What about weak almost holomorphic modular forms ?

$$D^n_w = \left(\frac{i}{\pi}\right)^n \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(w+n)}{\Gamma(w+k)} (2i\tau_2)^{k-n}\, \partial^k_\tau$$

They can be obtained from the ordinary holomorphic modular forms by the action of the modular derivatives $D^n\Phi$

$$D\hat{E}_2 = \tfrac{1}{6}(E_4 - \hat{E}_2^2)$$
$$DE_4 = \tfrac{2}{3}(E_6 - \hat{E}_2 E_4)$$
$$DE_6 = E_4^2 - \hat{E}_2 E_6$$
$$D(\Delta^{-1}) = 2\hat{E}_2\,\Delta^{-1}$$

$$\mathcal{F}(1-\tfrac{w}{2}+n,\kappa,w) = \frac{1}{(2\kappa)^n n!}\, D^n \mathcal{F}(1-\tfrac{w}{2}+n,\kappa,w-2n)$$

Harmonic Maass form

Hence, we can produce a weak almost holomorphic modular form from the linear combination

$$\mathcal{G}(1-\tfrac{w}{2}+n,w) = \frac{1}{\Gamma(2-w)} \sum_{-\kappa\leq m<0} a_m\, \mathcal{F}(1-\tfrac{w}{2}+n,m,w)$$

where the coefficients form the principal part of a weak holomorphic modular form of weight $w-2n$

$$\Phi^-_{w-2n} = \sum_{-\kappa\leq m<0} \frac{a_m}{(2m)^n n!}\, q^m$$

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The spectrum of modular forms as limits of the N-P series

Niebur-Poincaré series for various values of (s,w)

$s\backslash w$	-10	-8	-6	-4	-2	0	2	4	6	8	10
5	0	$9!\frac{E_4}{\Delta}$	$\frac{9!}{2}D\frac{E_4}{\Delta}$	$\frac{9!}{8}D^2\frac{E_4}{\Delta}$	$\frac{9!}{2^3\,3!}D^3\frac{E_4}{\Delta}$	$\frac{9!}{2^4\,4!}D^4\frac{E_4}{\Delta}$	$\frac{9!}{2^5\,5!}D^5\frac{E_4}{\Delta}$	$\frac{9!}{2^6\,6!}D^6\frac{E_4}{\Delta}$	$\frac{9!}{2^7\,7!}D^7\frac{E_4}{\Delta}$	$\frac{9!}{2^7\,8!}D^8\frac{E_4}{\Delta}$	$E_4E_6(j+264)$
4	0	0	$7!\frac{E_6}{\Delta}$	$\frac{7!}{2}D\frac{E_6}{\Delta}$	$\frac{7!}{8}D^2\frac{E_6}{\Delta}$	$\frac{7!}{2^3\,3!}D^3\frac{E_6}{\Delta}$	$\frac{7!}{2^4\,4!}D^4\frac{E_6}{\Delta}$	$\frac{7!}{2^5\,5!}D^5\frac{E_6}{\Delta}$	$\frac{7!}{2^6\,6!}D^6\frac{E_6}{\Delta}$	$E_4^2(j-480)$	$\frac{7!}{2^8\,8!}D^8\frac{E_6}{\Delta}$
3	0	0	0	$5!\frac{E_4^2}{\Delta}$	$\frac{5!}{2}D\frac{E_4^2}{\Delta}$	$\frac{5!}{8}D^2\frac{E_4^2}{\Delta}$	$\frac{5!}{2^3 3!}D^3\frac{E_4^2}{\Delta}$	$\frac{5!}{2^4 4!}D^4\frac{E_4^2}{\Delta}$	$E_6(j+504)$	$\frac{5!}{2^6 6!}D^6\frac{E_4^2}{\Delta}$	$\frac{5!}{2^7 7!}D^7\frac{E_4^2}{\Delta}$
2	0	0	0	0	$3!\frac{E_4E_6}{\Delta}$	$3D\frac{E_4E_6}{\Delta}$	$\frac{3}{4}D^2\frac{E_4E_6}{\Delta}$	$E_4(j-240)$	$\frac{3!}{2^4 4!}D^4\frac{E_4E_6}{\Delta}$	$\frac{3!}{2^5 5!}D^5\frac{E_4E_6}{\Delta}$	$\frac{3!}{2^6 6!}D^6\frac{E_4E_6}{\Delta}$
1	0	0	0	0	0	$j+24$	$\frac{E_4^2E_6}{\Delta}$	$\frac{1}{2^2 2!}D^2 j$	$\frac{1}{2^3 3!}D^3 j$	$\frac{1}{2^4 4!}D^4 j$	$\frac{1}{2^5 5!}D^5 j$

$$D\hat{E}_2 = \frac{1}{6}(E_4 - \hat{E}_2^2)$$
$$DE_4 = \frac{2}{3}(E_6 - \hat{E}_2 E_4)$$
$$DE_6 = E_4^2 - \hat{E}_2 E_6$$
$$D(\Delta^{-1}) = 2\hat{E}_2\,\Delta^{-1}$$

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Unfolding against the N-P series

Now we can return to our original goal : $I_{d+k,d}(s,\kappa;T) = \int_{\mathcal{F}_T} d\mu\ \Gamma_{(d+k,d)}(G,B,Y)\ \mathcal{F}(s,\kappa,-\frac{k}{2})$

IR cutoff

$(w=-k/2<0)$

Unfold against the Niebur-Poincaré series :

$$I_{d+k,d}(s,\kappa;T) = \int_0^T \frac{d\tau_2}{\tau_2^2} \int_{-1/2}^{1/2} d\tau_1\ \Gamma_{(d+k,d)}\ \mathcal{M}_{s,-\frac{k}{2}}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1}$$

$$- \int_{\mathcal{F}-\mathcal{F}_T} d\mu\ \Gamma_{(d+k,d)} \left(\mathcal{F}(s,\kappa,-\tfrac{k}{2}) - \mathcal{M}_{s,-\frac{k}{2}}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1}\right)$$

BPS state sum

half-BPS states

$P_L^2 - P_R^2 = m^T n = 4\kappa$

$$= \int_0^\infty \frac{d\tau_2}{\tau_2^2}\ \mathcal{M}_{s,-\frac{k}{2}}(-\kappa\tau_2)\ \tau_2^{d/2} \sum_{\rm BPS} e^{-\pi\tau_2(P_L^2+P_R^2)/2}$$

$$- \int_{\mathcal{F}-\mathcal{F}_T} d\mu\ \Gamma_{(d+k,d)} \left(\mathcal{F}(s,\kappa,-\tfrac{k}{2}) - \mathcal{M}_{s,-\frac{k}{2}}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1} - f_0(s)\tau_2^{1-s+\frac{k}{4}}\right)$$

$$- \int_{\mathcal{F}-\mathcal{F}_T} d\mu\ \left(\Gamma_{(d+k,d)} - \tau_2^{d/2}\right) \left(\mathcal{M}_{s,-\frac{k}{2}}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1} + f_0(s)\tau_2^{1-s+\frac{k}{4}}\right)$$

$$- \int_{\mathcal{F}-\mathcal{F}_T} d\mu\ \tau_2^{d/2} \left(\mathcal{M}_{s,-\frac{k}{2}}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1} + f_0(s)\tau_2^{1-s+\frac{k}{4}}\right)$$

exponentially suppressed at the cusp, away from extended symmetry points

$$f_0(s)\, \frac{T^{\frac{d}{2}+\frac{k}{4}-s}}{s-\frac{d}{2}-\frac{k}{4}}$$

$$f_0(s) = \frac{(4\pi)^{1+\frac{k}{4}}\ \pi^s\ i^{\frac{k}{2}}\ \kappa^{s+\frac{k}{4}}\ \Gamma(2s-1)\,\sigma_{1-2s}(\kappa)}{\Gamma(s+\frac{k}{4})\,\Gamma(s-\frac{k}{4})\,\zeta(2s)}$$

BPS sum is analytic except for simple pole in s

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An Example of Gauge Threshold calculation

$\mathcal{N} = 2$ heterotic vacuum in the orbifold point $T^2 \times T^4/\mathbb{Z}_2$

In the absence of Wilson lines
$E_8 \times E_8 \to E_8 \times E_7 \times SU(2)$

Genus-one correction to 2-point function of two gauge bosons

$$\langle e_1^\mu A_\mu^a(p_1)\ e_2^\nu A_\nu^a(p_2)\rangle = \int d^2z \langle \mathcal{V}^a(z,\bar{z};p)\mathcal{V}^a(0;p)\rangle$$

$$\mathcal{V}^a(z,\bar{z};p) = ie_\mu\left(\partial X^\mu + ip\cdot\psi\,\psi^\mu\right)(z)\bar{J}^a(\bar{z})\ e^{ip\cdot X(z,\bar{z})}$$

$$-\frac{e_1^\mu e_2^\nu}{2(2\pi\sqrt{2})^4}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\int d^2z\ \langle(\partial X^\mu + ip_1\cdot\psi\ \psi^\mu)\ \bar{J}^a\ e^{ip_1\cdot X}(z,\bar{z})\ (\partial X^\nu + ip_2\cdot\psi\ \psi^\nu)\ \bar{J}^b\ e^{ip_2 X}(0)\rangle$$

Performing the contractions and keeping quadratic terms in *p* relevant for $\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}$

$$\frac{(p_1\cdot p_2)(e_1\cdot e_2)-(e_1\cdot p_2)(e_2\cdot p_1)}{2(2\pi\sqrt{2})^4}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\int d^2z\left(\langle X\partial X\rangle^2 - \langle\psi\psi\rangle^2\right)\langle\bar{J}^a\bar{J}^a\rangle$$

$$\langle\bar{J}^a(\bar{z})\bar{J}^b(0)\rangle = \frac{k}{4\pi^2}\,\bar{\partial}^2\log\bar{\theta}_1(\bar{z}) + \mathrm{Tr}\,Q^2$$

$$\langle\psi(z)\psi(0)\rangle^2 = S^2[^a_b](z) = \left(\frac{\theta[^a_b](z)\theta_1'(0)}{\theta[^a_b](0)\theta_1(z)}\right)^2 = \mathcal{P}(z) + 4\pi i\,\partial_\tau\log\frac{\theta[^a_b]}{\eta}$$

for even spin structures

Szegö kernel

$$\mathcal{P}(z) = 4\pi i\,\partial_\tau\log\eta - \partial_z^2\log\theta_1(z)$$

Weierstrass function

$$\langle X(z,\bar{z})X(0)\rangle = -\log\theta_1(z)\bar{\theta}_1(\bar{z}) + 2\pi\frac{[\mathrm{Im}(z)]^2}{\tau_2}$$

bosonic 2-point function on the torus

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An Example of Gauge Threshold calculation

Putting everything together, we perform integral over the location of the vertex operator insertion over the torus

$$\int d^2z\ \left(S^2\left[{a \atop b}\right](z) - \langle X\partial X\rangle^2\right)\left(\frac{k}{4\pi^2}\,\bar\partial^2\log\bar\theta_1(\bar z)+\mathrm{Tr}\,Q^2\right)$$

$$= \int d^2z \left[\mathcal{P}(z)+4\pi i\partial_\tau\log\frac{\theta\left[{a \atop b}\right]}{\eta} - \left(\partial_z\log\theta_1(z)+2\pi i\frac{\mathrm{Im}(z)}{\tau_2}\right)^2\right]\left[\frac{k}{4\pi^2}\,\bar\partial^2\log\bar\theta_1(\bar z)+\mathrm{Tr}\,Q^2\right]$$

$$= 4\pi i\,\tau_2\,\partial_\tau\log\frac{\theta\left[{a \atop b}\right]}{\eta}\left(\mathrm{Tr}\,Q^2-\frac{k}{4\pi\tau_2}\right)\ .$$

Finally, perform the sum over all even spin structures and fix the overall normalization

$$\left.\frac{16\pi^2}{g^2}\right|_{1-\mathrm{loop}} = \frac{i}{2\pi}\int_{\mathcal F}\frac{d^2\tau}{\tau_2}\,\frac{1}{\eta^2\bar\eta^2}\sum_{(a,b)\neq(1,1)}\partial_\tau\left(\frac{\theta\left[{a \atop b}\right]}{\eta}\right)\ \mathrm{Tr}\left[\left(Q^2-\frac{k}{4\pi\tau_2}\right)q^{L_0-c/24}\,\bar q^{\bar L_0-\bar c/24}\right]$$

One-loop correction to the gauge coupling associated to a gauge group factor G

In our particular model, the sum over the even spin structures contributes

$$I=\frac{1}{2}\sum_{a,b}(-)^{a+b+ab}\,\frac{\theta^2\left[{a \atop b}\right]\theta\left[{a+h \atop b+g}\right]\theta\left[{a-h \atop b-g}\right]}{\eta^4}\ 4\pi i\partial_\tau\log\frac{\theta\left[{a \atop b}\right]}{\eta} = 4\pi^2\eta^2\ \theta\left[{1-h \atop 1-g}\right]\theta\left[{1+h \atop 1+g}\right]$$

Using this together with the contribution of the twisted lattice

$$\frac{i}{2\pi\eta^2\bar\eta^2}\,\frac{1}{2}\sum_{(a,b)\neq(1,1)}(-)^{a+b+ab}\partial_\tau\left(\frac{\theta\left[{a \atop b}\right]}{\eta}\right)\frac{\theta\left[{a \atop b}\right]\theta\left[{a+h \atop b+g}\right]\theta\left[{a-h \atop b-g}\right]}{\eta^3}\ \frac{\Gamma_{(4,4)}\left[{h \atop g}\right]}{\eta^4\bar\eta^4} = \frac{8\eta^2}{\bar\theta\left[{1+h \atop 1+g}\right]\bar\theta\left[{1-h \atop 1-g}\right]}$$

An Example of Gauge Threshold calculation

The final ingredient is the group trace over, say the E8 group factor

$$\left(\frac{1}{(2\pi i)^2}\,\partial_v^2 - \frac{1}{4\pi\tau_2}\right)\frac{1}{2}\sum_{\rho,\sigma}\frac{\bar\theta[^\rho_\sigma]^7\bar\theta[^\rho_\sigma](v)}{\bar\eta^8}\Bigg|_{v=0} = \frac{1}{2}\sum_{\rho,\sigma}\frac{\bar\theta[^\rho_\sigma]^7}{\bar\eta^8}\left(\frac{i}{\pi}\partial_{\bar\tau} - \frac{1}{4\pi\tau_2}\right)\bar\theta[^\rho_\sigma]$$

$$= \frac{1}{2}\sum_{\rho,\sigma}\frac{\bar\theta[^\rho_\sigma]^8}{\bar\eta^8}\left(\frac{i}{\pi}\partial_{\bar\tau}\log\bar\theta[^\rho_\sigma] - \frac{1}{4\pi\tau_2}\right) = \frac{1}{12}\,\frac{\hat{\bar E}_2\bar E_4 - \bar E_6}{\bar\eta^8}$$

Putting everything together we are left with

$$\frac{16\pi^2}{g_{E_8}^2} = \int_{\mathcal F}\frac{d^2\tau}{\tau_2}\,\frac{1}{2}\sum_{(h,g)\neq(0,0)}\frac{1}{2}\sum_{\gamma,\delta}\frac{8\eta^2\Gamma_{(2,2)}(T,U)}{\bar\theta[^{1+h}_{1+g}]\bar\theta[^{1-h}_{1-g}]}\,\frac{1}{12}\,\frac{\hat{\bar E}_2\bar E_4 - \bar E_6}{\bar\eta^8}\,\frac{\bar\theta[^\gamma_\delta]^6\bar\theta[^{\gamma+h}_{\delta+g}]\bar\theta[^{\gamma-h}_{\delta-g}]}{\bar\eta^8}$$

The final result is a modular integral of the (2,2) lattice times a modular function

$$\frac{16\pi^2}{g_{E_8}^2}\Bigg|_{1-\text{loop}} = -\frac{1}{12}\int_{\mathcal F}\frac{d^2\tau}{\tau_2}\,\Gamma_{(2,2)}(T,U)\,\frac{\hat{\bar E}_2\bar E_4\bar E_6 - \bar E_6^2}{\bar\eta^{24}}$$

$$\frac{\hat E_2 E_4 E_6 - E_6^2}{\Delta} = \mathcal F(2,1,0) - 6j + 720$$

We now unfold the Niebur-Poincaré series and obtain

$$\frac{16\pi^2}{g_{E_8}^2}\Bigg|_{1-\text{loop}} = 72\log\left(T_2U_2|\eta(T)\eta(U)|^4\right) + \sum_{m^Tn=1}\left[1+\tfrac{1}{4}P_R^2\log\left(\frac{P_R^2}{P_L^2}\right)\right]$$

Non-singular because the unphysical tachyon is neutral

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An Example of Gauge Threshold calculation

$$\frac{16\pi^2}{g^2_{E_8}}\bigg|_{1-\text{loop}} = 72 \log\left(T_2 U_2 |\eta(T)\eta(U)|^4\right) + \sum_{m^T n=1}\left[1+\tfrac{1}{4}P_R^2 \log\left(\frac{P_R^2}{P_L^2}\right)\right]$$

Take the limit where the 2-torus decompactifies into a circle

$$T = iR_1 R_2 \quad , \quad U = iR_2/R_1 \quad , \quad R_2 \to \infty \quad , \quad R_1 = \text{fixed}$$

The dominant dependence in the circle radius is

$$\frac{16\pi^2}{g^2_{E_8}}\bigg|_{1-\text{loop}} \sim 72 \times \left[-\frac{\pi}{3}\left(R_1 + \frac{1}{R_1}\right)\right] \sim -24\pi R_1$$

We will now compare this with the result we would have obtained if we had considered the decompactification limit from the very beginning

$$\frac{16\pi^2}{g^2_{E_8}}\bigg|_{1-\text{loop}} = -\frac{1}{12}\int_{\mathcal{F}} d\mu\ \Gamma_{(1,1)}(R)\ \left(\mathcal{F}(2,1,0) - 6j + 720\right)$$

$$= -\tfrac{2\pi}{3}\left(R^3 + \frac{1}{R^3} - \left|R^3 - \frac{1}{R^3}\right|\right) - 2\pi\left(R + \frac{1}{R} + \left|R - \frac{1}{R}\right|\right) - 20\pi\left(R + \frac{1}{R}\right)$$

$$= \begin{cases} -\frac{4\pi}{3R^3} - 4\pi R - 20\pi(R + \frac{1}{R}) & , \quad R > 1 \\ -\frac{4\pi R^3}{3} - \frac{4\pi}{R} - 20\pi(R + \frac{1}{R}) & , \quad R < 1 \end{cases}$$

The dominant behaviour matches in both cases, as expected

There is no conical singularity, despite the presence of the two conical terms