Double Field Theory

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Strings on a Torus

- States: momentum p, winding w
- String: Infinite set of fields $\psi(p, w)$
- Fourier transform to doubled space: $\psi(x, \tilde{x})$
- "Double Field Theory" from closed string field theory. Some non-locality in doubled space
- Subsector? e.g. $g_{ij}(x, \tilde{x}),\ b_{ij}(x, \tilde{x}),\ \phi(x, \tilde{x})$

Strings on Torus

D=n+d

Target space	$\mathbb{R}^{n-1,1} \times T^d$
Coordinates	$x^i = (x^\mu, x^a)$
Momenta	$p_i = (p_\mu, p_a)$
Winding	$w^i = (w^\mu, w^a)$
Dual coordinates (conjugate to winding)	$\tilde{x}_i = (\tilde{x}_\mu, \tilde{x}_a)$
Constant metric and B-field	$E_{ij} = G_{ij} + B_{ij}$

Compact dimensions

p_a, w^a discrete, in Narain lattice, $x^a, \tilde{x}_a$ periodic

Non-compact dimensions x^μ, p_μ continuous

Usually take $w^\mu = 0$ so $\dfrac{\partial}{\partial \tilde{x}_\mu} = 0$, fields $\psi(x^\mu, x^a, \tilde{x}_a)$

T-Duality

- Interchanges momentum and winding
- Equivalence of string theories on dual backgrounds with very different geometries
- String field theory symmetry, provided fields depend on both $x, \tilde{x}$ Kugo, Zwiebach
- For fields $\psi(x^\mu)$ not $\psi(x^\mu, x^a, \tilde{x}_a)$ Buscher
- Generalise to fields $\psi(x^\mu, x^a, \tilde{x}_a)$

Generalised T-duality Dabholkar & CMH

String Field Theory on Minkowski Space

Closed SFT: Zwiebach

String field $\Phi[X(\sigma), c(\sigma)]$

$$X^i(\sigma) \to x^i, \ \text{oscillators}$$

Expand to get infinite set of fields

$$g_{ij}(x), b_{ij}(x), \phi(x), \ldots, C_{ijk\ldots l}(x), \ldots$$

Integrating out massive fields gives field theory for

$$g_{ij}(x), b_{ij}(x), \phi(x)$$

String Field Theory on a torus

String field $\Phi[X(\sigma), c(\sigma)]$

$$X^i(\sigma) \to x^i, \tilde{x}_i, \text{ oscillators}$$

Expand to get infinite set of double fields

$$g_{ij}(x,\tilde{x}), b_{ij}(x,\tilde{x}), \phi(x,\tilde{x}), \ldots, C_{ijk\ldots l}(x,\tilde{x}), \ldots$$

Seek **double field theory** for

$$g_{ij}(x,\tilde{x}), b_{ij}(x,\tilde{x}), \phi(x,\tilde{x})$$

Free Field Equations (B=0)

$$L_0 + \bar{L}_0 = 2$$

$$p^2 + w^2 = N + \bar{N} - 2$$

$$L_0 - \bar{L}_0 = 0$$

$$p_i w^i = N - \bar{N}$$

Free Field Equations (B=0)

$$L_0 + \bar{L}_0 = 2$$

$$p^2 + w^2 = N + \bar{N} - 2$$

Treat as field equation, kinetic operator in doubled space

$$G^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + G_{ij}\frac{\partial^2}{\partial \tilde{x}_i \partial \tilde{x}_j}$$

$$L_0 - \bar{L}_0 = 0$$

$$p_i w^i = N - \bar{N}$$

Treat as constraint on double fields

$$\Delta \equiv \frac{\partial^2}{\partial x^i \partial \tilde{x}_i} \qquad (\Delta - \mu)\psi = 0$$

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$$G^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + G_{ij} \frac{\partial^2}{\partial \tilde{x}_i \partial \tilde{x}_j}$$

Laplacian for metric

$$ds^2 = G_{ij} dx^i dx^j + G^{ij} d\tilde{x}_i d\tilde{x}_j$$

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Laplacian for metric

$$ds^2 = dx^i d\tilde{x}_i$$

$$g_{ij}(x, \tilde{x}), b_{ij}(x, \tilde{x}), \phi(x, \tilde{x})$$

$$N = \bar{N} = 1$$

$$p^2 + w^2 = 0$$

$$p \cdot w = 0$$

“Double Massless”

Constrained fields $\psi(x^\mu, x^a, \tilde{x}_a)$

$$(\Delta - \mu)\psi = 0$$

Momentum space $\psi(p_\mu, p_a, w^a)$ $\Delta = p_a w^a$

Momentum space: Dimension n+2d
Cone: $p_a w^a = 0$ or hyperboloid: $p_a w^a = \mu$
dimension n+2d-1

DFT: fields on cone or hyperboloid, with discrete p,w
Problem: naive product of fields on cone do not lie on cone. Vertices need projectors

Restricted fields: Fields that depend on d of 2d torus momenta, e.g. $\psi(p_\mu, p_a)$ or $\psi(p_\mu, w^a)$
Simple subsector, no projectors needed, no cocycles.

Torus Backgrounds

$$G_{ij} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & G_{ab} \end{pmatrix}, \quad B_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ab} \end{pmatrix} \qquad E_{ij} \equiv G_{ij} + B_{ij}$$

Fluctuations $e_{ij} = h_{ij} + b_{ij}$

Take $B_{ij} = 0$ $\qquad \tilde{\partial}_i \equiv G_{ik} \dfrac{\partial}{\partial \tilde{x}_k}$

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Fluctuations $\quad e_{ij} = h_{ij} + b_{ij}$

Take $\quad B_{ij} = 0 \qquad \tilde{\partial}_i \equiv G_{ik} \frac{\partial}{\partial \tilde{x}_k}$

Usual action $\quad \int dx \sqrt{-g} e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right]$

Quadratic part $\quad \int dx \; L[\, h, b, d; \partial \,]$

$e^{-2d} = e^{-2\phi} \sqrt{-g}$ (d invariant under usual T-duality)

Double Field Theory Action

$$S^{(2)} = \int [dx d\tilde{x}] \Big[\, L[\, h, b, d; \partial\,] + L[\, -h, -b, d; \tilde{\partial}\,]$$

$$+ (\partial_k h^{ik})(\tilde{\partial}^j b_{ij}) + (\tilde{\partial}^k h_{ik})(\partial_j b^{ij}) - 4\, d\, \partial^i \tilde{\partial}^j b_{ij} \;\Big]$$

Action + dual action + strange mixing terms

Double Field Theory Action

$$S^{(2)} = \int [dx d\tilde{x}] \Big[\; L[\,h, b, d; \partial\,] + L[-h, -b, d; \tilde{\partial}\,]$$

$$+ (\partial_k h^{ik})(\tilde{\partial}^j b_{ij}) + (\tilde{\partial}^k h_{ik})(\partial_j b^{ij}) - 4\, d\, \partial^i \tilde{\partial}^j b_{ij} \; \Big]$$

Action + dual action + strange mixing terms

$$\delta h_{ij} = \quad \partial_i \epsilon_j \, + \partial_j \epsilon_i \; + \; \tilde{\partial}_i \tilde{\epsilon}_j + \tilde{\partial}_j \tilde{\epsilon}_i \, ,$$

$$\delta b_{ij} = -(\tilde{\partial}_i \epsilon_j - \tilde{\partial}_j \epsilon_i) - (\partial_i \tilde{\epsilon}_j - \partial_j \tilde{\epsilon}_i) \, ,$$

$$\delta d \; = - \, \partial \cdot \epsilon + \, \tilde{\partial} \cdot \tilde{\epsilon} \, .$$

Invariance needs constraint

Diffeos and B-field transformations mixed.

Invariant cubic action found for full DFT of (h,b,d)

T-Duality Transformations of Background

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(d, d; \mathbb{Z})$$

T-duality

$$E' = (aE + b)(cE + d)^{-1}$$

$$X \equiv \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix}$$

transforms as a vector

$$X' = \begin{pmatrix} \tilde{x}' \\ x' \end{pmatrix} = gX = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{x} \\ x \end{pmatrix}$$

T-Duality is a Symmetry of the Action

Fields $e_{ij}(x,\tilde{x}),\ d(x,\tilde{x})$

Background E_{ij}

$$E' = (aE+b)(cE+d)^{-1}$$

$$X' = \begin{pmatrix} \tilde{x}' \\ x' \end{pmatrix} = gX = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{x} \\ x \end{pmatrix}$$

Action invariant if:

$$e_{ij}(X) = M_i{}^k\, \bar{M}_j{}^l\, e'_{kl}(X') \qquad M \equiv d^t - E\, c^t$$

$$d(X) = d'(X') \qquad \bar{M} \equiv d^t + E^t c^t$$

With general momentum and winding dependence!

Projectors and Cocycles

Naive product of constrained fields doesn't satisfy constraint

$$L_0^- \Psi_1 = 0, L_0^- \Psi_2 = 0 \quad \text{but} \quad L_0^- (\Psi_1 \Psi_2) \neq 0$$

$$\Delta A = 0, \Delta B = 0 \quad \text{but} \quad \Delta(AB) \neq 0$$

String product has explicit projection

Leads to a symmetry that is not a Lie algebra, but is a homotopy lie algebra.

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Double field theory requires projections.

SFT has non-local cocycles in vertices, DFT should too

Cocycles and projectors not needed in cubic action

General double fields $\psi(x, \tilde{x})$

Fields on Spacetime M $\psi(x)$

Restricted Fields on N, T-dual to M $\psi(x')$

M,N null wrt O(D,D) metric $ds^2 = dx^i d\tilde{x}_i$

Subsector with fields and parameters all restricted to M or N

- Constraint satisfied on all fields and products of fields
- No projectors or cocycles
- T-duality covariant: independent of choice of N
- Can find full non-linear form of gauge transformations
- Full gauge algebra, full non-linear action

Restricted DFT

Double fields restricted to null D-dimensional subspace N
T-duality "rotates" N to N'

O(D,D) Covariant Notation

$$X^M \equiv \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix} \qquad \partial_M \equiv \begin{pmatrix} \partial^i \\ \partial_i \end{pmatrix}$$

$$\eta_{MN} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad M = 1, ..., 2D$$

Constraint $\partial^M \partial_M A = 0$

Strong Constraint for restricted DFT

$$\partial^M \partial_M (AB) = 0 \qquad (\partial^M A)\,(\partial_M B) = 0$$

Background independent fields: g,b,d:

$$\mathcal{E} \equiv E + \left(1 - \frac{1}{2}\,e\right)^{-1} e$$

$$g_{ij} \equiv \mathcal{E}_{(ij)} \qquad b_{ij} \equiv \mathcal{E}_{[ij]}$$

defines metric and b-field with conventional actions and transformations

$$\mathcal{E}_{ij} = g_{ij} + b_{ij}$$

Generalised T-duality transformations:

$$X'^M \equiv \begin{pmatrix} \tilde{x}'_i \\ x'^i \end{pmatrix} = hX^M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix}$$

h in O(d,d;Z) acts on toroidal coordinates only

$$\mathcal{E}'(X') = (a\mathcal{E}(X) + b)(c\mathcal{E}(X) + d)^{-1}$$

$$d'(X') = d(X)$$

Buscher if fields independent of toroidal coordinates
Generalisation to case without isometries

O(D,D)

Non-compact dimensions x^μ, p_μ continuous

Strings: take $w^\mu = 0$ so $\frac{\partial}{\partial \tilde{x}_\mu} = 0$, fields $\psi(x^\mu, x^a, \tilde{x}_a)$

For DFT, if we allow dependence on $\tilde{x}_\mu$
DFT invariant under

$$O(n,n) \times O(d,d;\mathbb{Z})$$

Subgroup of O(D,D) preserving periodicities

O(D,D) is symmetry if all directions non-compact:
theory has formal O(D,D) covariance

Generalised Metric Formulation

$$\mathcal{H}_{MN} \quad = \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix} \, .$$

2 Metrics on double space $\quad \mathcal{H}_{MN}, \; \eta_{MN}$

Generalised Metric Formulation

$$\mathcal{H}_{MN} \;=\; \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix} .$$

2 Metrics on double space $\mathcal{H}_{MN},\ \eta_{MN}$

$$\mathcal{H}^{MN} \equiv \eta^{MP}\mathcal{H}_{PQ}\eta^{QN}$$

Constrained metric $\mathcal{H}^{MP}\mathcal{H}_{PN} \;=\; \delta^{M}{}_{N}$

Generalised Metric Formulation

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$$\mathcal{H}^{MN} \equiv \eta^{MP}\mathcal{H}_{PQ}\eta^{QN}$$

Constrained metric $\mathcal{H}^{MP}\mathcal{H}_{PN} = \delta^M{}_N$

Covariant Transformation

$$h^P{}_M h^Q{}_N \mathcal{H}'_{PQ}(X') = \mathcal{H}_{MN}(X)$$

$$X' = hX \qquad h \in O(D,D)$$

O(D,D) covariant action

$$S = \int dx d\tilde{x}\, e^{-2d}\, L$$

$$L = \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\,\partial_L\mathcal{H}_{MK}$$
$$- 2\,\partial_M d\,\partial_N\mathcal{H}^{MN} + 4\mathcal{H}^{MN}\,\partial_M d\,\partial_N d$$

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L cubic! Indices raised and lowered with η

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$$- 2\,\partial_M d\,\partial_N\mathcal{H}^{MN} + 4\mathcal{H}^{MN}\,\partial_M d\,\partial_N d$$

Gauge Transformation

$$\delta_\xi\mathcal{H}^{MN} = \xi^P\partial_P\mathcal{H}^{MN}$$
$$+ (\partial^M\xi_P - \partial_P\xi^M)\,\mathcal{H}^{PN} + (\partial^N\xi_P - \partial_P\xi^N)\,\mathcal{H}^{MP}$$

O(D,D) covariant action

$$S = \int dx d\tilde{x}\, e^{-2d}\ L$$

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$$+ (\partial^M\xi_P - \partial_P\xi^M)\,\mathcal{H}^{PN} + (\partial^N\xi_P - \partial_P\xi^N)\,\mathcal{H}^{MP}$$

Rewrite as “Generalised Lie Derivative”

$$\delta_\xi\mathcal{H}^{MN} = \widehat{\mathcal{L}}_\xi\mathcal{H}^{MN}$$

Generalised Lie Derivative

$$A_{N_1 \dots}^{M_1 \dots}$$

$$\widehat{\mathcal{L}}_\xi A_M{}^N \equiv \xi^P \partial_P A_M{}^N$$

$$+ (\partial_M \xi^P - \partial^P \xi_M) A_P{}^N + (\partial^N \xi_P - \partial_P \xi^N) A_M{}^P$$

Generalised Lie Derivative

$$A_{N_1\dots}^{M_1\dots}$$

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$$+(\partial_M \xi^P - \partial^P \xi_M) A_P{}^N + (\partial^N \xi_P - \partial_P \xi^N) A_M{}^P$$

$$\widehat{\mathcal{L}}_\xi A_M{}^N = \mathcal{L}_\xi A_M{}^N - \eta^{PQ}\eta_{MR}\, \partial_Q \xi^R\, A_P{}^N$$
$$+ \eta_{PQ}\eta^{NR}\, \partial_R \xi^Q\, A_M{}^P$$

Generalized scalar curvature

$$\mathcal{R} \equiv \; 4\,\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN}$$
$$-\,4\,\mathcal{H}^{MN}\partial_M d\,\partial_N d + 4\partial_M\mathcal{H}^{MN}\,\partial_N d$$
$$+\,\frac{1}{8}\,\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_K\mathcal{H}_{NL}$$

Generalized scalar curvature

$$
\begin{aligned}
\mathcal{R} \equiv\ & 4\,\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} \\
& - 4\,\mathcal{H}^{MN}\partial_M d\,\partial_N d + 4\partial_M\mathcal{H}^{MN}\,\partial_N d \\
& + \frac{1}{8}\,\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_K\mathcal{H}_{NL}
\end{aligned}
$$

$$
S = \int dx\,d\tilde{x}\,e^{-2d}\,\mathcal{R}
$$

Generalized scalar curvature

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$$- \,4\,\mathcal{H}^{MN}\partial_M d\,\partial_N d + 4\partial_M\mathcal{H}^{MN}\,\partial_N d$$
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$$S = \int dx\, d\tilde{x}\, e^{-2d}\,\mathcal{R}$$

Gauge Symmetry

$$\delta_\xi\mathcal{R} \; = \widehat{\mathcal{L}}_\xi\mathcal{R} = \; \xi^M\partial_M\mathcal{R}$$
$$\delta_\xi\, e^{-2d} = \partial_M(\xi^M e^{-2d})$$

Generalized scalar curvature

$$\mathcal{R} \equiv 4\,\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN}$$
$$- 4\,\mathcal{H}^{MN}\partial_M d\,\partial_N d + 4\partial_M\mathcal{H}^{MN}\,\partial_N d$$
$$+ \frac{1}{8}\,\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_K\mathcal{H}_{NL}$$

$$S = \int dx\, d\tilde{x}\, e^{-2d}\,\mathcal{R}$$

Gauge Symmetry

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$$\delta_\xi\, e^{-2d} = \partial_M(\xi^M e^{-2d})$$

Field equations give gen. Ricci tensor

2-derivative action

$$S = S^{(0)}(\partial, \partial) + S^{(1)}(\partial, \tilde{\partial}) + S^{(2)}(\tilde{\partial}, \tilde{\partial})$$

Write $S^{(0)}$ in terms of usual fields

Gives usual action (+ surface term)

$$\int dx \sqrt{-g} e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right]$$

$$S^{(0)} = S(\mathcal{E}, d, \partial)$$

2-derivative action

$$S \;=\; S^{(0)}(\partial,\partial) + S^{(1)}(\partial,\tilde{\partial}) + S^{(2)}(\tilde{\partial},\tilde{\partial})$$

Write $S^{(0)}$ in terms of usual fields

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$$\int dx\sqrt{-g}e^{-2\phi}\left[R + 4(\partial\phi)^2 - \frac{1}{12}H^2\right]$$

$$S^{(0)} = S(\mathcal{E}, d, \partial)$$

$$S^{(2)} = S(\mathcal{E}^{-1}, d, \tilde{\partial})$$ T-dual!

$S^{(1)}$ strange mixed terms

- Restricted DFT:
  fields independent of half the coordinates
- If independent of $\tilde{x}$, equivalent to usual action
- Duality covariant: duality changes which half of coordinates theory is independent of
- Equivalent to Siegel's formulation Hohm & Kwak
- Good for non-geometric backgrounds

Gauge Algebra

Parameters $(\epsilon^i, \tilde{\epsilon}_i) \to \Sigma^M$

Gauge Algebra $[\delta_{\Sigma_1}, \delta_{\Sigma_2}] = \delta_{[\Sigma_1, \Sigma_2]_C}$

$$[\,\widehat{\mathcal{L}}_{\xi_1}, \widehat{\mathcal{L}}_{\xi_2}\,] = -\widehat{\mathcal{L}}_{[\xi_1, \xi_2]_{\rm C}}$$

C-Bracket:

$$[\Sigma_1, \Sigma_2]_C \equiv [\Sigma_1, \Sigma_2] - \frac{1}{2}\,\eta^{MN}\eta_{PQ}\,\Sigma^P_{[1}\,\partial_N\,\Sigma^Q_{2]}$$

Lie bracket + metric term

Parameters $\Sigma^M(X)$ restricted to N
Decompose into vector + 1-form on N
C-bracket reduces to Courant bracket on N

Same covariant form of gauge algebra found in similar context by Siegel

Jacobi Identities not satisfied!

$$J(\Sigma_1, \Sigma_2, \Sigma_3) \equiv [\,[\Sigma_1, \Sigma_2]\,, \Sigma_3\,] + \text{cyclic} \neq 0$$

for both C-bracket and Courant-bracket

How can bracket be realised as a symmetry algebra?

$$[\,[\delta_{\Sigma_1}, \delta_{\Sigma_2}]\,, \delta_{\Sigma_3}\,] + \text{cyclic} = \delta_{J(\Sigma_1, \Sigma_2, \Sigma_3)}$$

Symmetry is Reducible

Parameters of the form $\Sigma^M = \eta^{MN}\partial_N\chi$
do not act

Gauge algebra determined up to such transformations

cf 2-form gauge field $\delta B = d\alpha$
Parameters of the form $\alpha = d\beta$
do not act

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cf 2-form gauge field $\delta B = d\alpha$
Parameters of the form $\alpha = d\beta$
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Resolution:

$$J(\Sigma_1, \Sigma_2, \Sigma_3)^M = \eta^{MN}\partial_N \chi$$

$\delta_{J(\Sigma_1,\Sigma_2,\Sigma_3)}$ does not act on fields

D-Bracket

$$[\,A,B\,]_{\mathrm{D}} \equiv \widehat{\mathcal{L}}_A B$$

$$[\,A,B\,]_{\mathrm{D}}^{M} = [\,A,B\,]_{\mathrm{C}}^{M} + \frac{1}{2}\partial^{M}\left(B^{N}A_{N}\right)$$

Not skew, but satisfies Jacobi-like identity

$$[A,[B,C]_{\mathrm{D}}]_{\mathrm{D}} = [[A,B]_{\mathrm{D}}],C]_{\mathrm{D}} + [B,[A,C]_{\mathrm{D}}]_{\mathrm{D}}$$

On restricting to null subspace N

C-bracket → Courant bracket

D-bracket → Dorfman bracket

Gen Lie Derivative → GLD of Grana, Minasian, Petrini and Waldram

Large gauge transformations

Not diffeomorphisms of doubled space, as algebra given by C-bracket, not Lie bracket.

What do you get by exponentiating infinitesimal transformations?

Hohm, Zwiebach

cf exponentiating usual Lie derivative

$$A'_m(x) \;=\; e^{\mathcal{L}_\xi}\, A_m(x)$$

gives transformations induced by diffeomorphism

$$x'^m \;=\; e^{-\xi^k \partial_k}\, x^m$$

Finite transformations for DFT can be written in form

$$X \to X' = f(X)$$

with generalised vectors transforming as

$$A'_M(X') \;=\; \mathcal{F}_M{}^N A_N(X)$$

$$\mathcal{F}_M{}^N \;\equiv\; \frac{1}{2}\Big(\frac{\partial X^P}{\partial X'^M}\,\frac{\partial X'_P}{\partial X_N} + \frac{\partial X'_M}{\partial X_P}\,\frac{\partial X^N}{\partial X'^P}\Big)$$

For conventional diffeos, would have

$$\mathcal{F}_M{}^N = \frac{\partial X^N}{\partial X'^M}$$

Important property: η_{MN} invariant!

Looks like a conventional geometry.
But there's a catch....
Exponentiating gen. Lie derivative

$$A'_M(X) \;=\; e^{\widehat{\mathcal{L}}_\xi}\, A_M(X)\,,$$

gives transformations of fields that form a group
(violation of Jacobi's doesn't act on fields)

These induce transformations of coordinates

$$X'^M \;=\; e^{-\Theta^K(\xi)\partial_K}\, X^M \qquad\qquad \Theta^K(\xi) \;\equiv\; \xi^K + \mathcal{O}(\xi^3)\,,$$

Not a group. Strange composition law.
Non-associative geometry?

Frames for Doubled Space

$$e^M{}_A$$

Basis, labelled by A=1,...,2D

$$e^M{}_A \to e^M{}_B \Lambda^B{}_A \,, \quad \Lambda(X) \in GL(2D, \mathbb{R})$$

$$\mathcal{H}_{AB} \equiv e^M{}_A e^N{}_B \mathcal{H}_{MN} \qquad \hat{\eta}_{AB} \equiv e^M{}_A e^N{}_B \eta_{MN}$$

e.g. Orthonormal Frame

$$e^M{}_A e^N{}_B \eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \to \quad \hat{\eta}_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Reduces tangent space group to O(D,D)

Generalized Connection

Coimbra, Strickland-Constable & Waldram

$$D_M W^A = \partial_M W^A + \tilde{\Omega}_M{}^A{}_B W^B$$

$$\tilde{\Omega}^A{}_B W^B \in GL(2D, \mathbb{R})$$

Generalised Lie Derivative

$$\widehat{\mathcal{L}}_\xi A^N \equiv \xi^P \partial_P A^N + (\partial^N \xi_P - \partial_P \xi^N) A^P$$

Covariantised Lie Derivative

$$\widehat{\mathcal{L}}_\xi^D A^N \equiv \xi^P D_P A^N + (D^N \xi_P - D_P \xi^N) A^P$$

Difference is Covariant, defines TORSION

$$\widehat{\mathcal{L}}_\xi^D A^N - \widehat{\mathcal{L}}_\xi A^N = T^N_{MP} \xi^M A^P$$

Generalized Curvature

$$R(U,V,W) = [D_U D_V]W - D_{[U,V]_C} W$$

Scale by functions:

$$U \to fU,\ V \to gV,\ W \to hW$$

$$R(fU,gV,hW) = fgh\, R(U,V,W) - \frac{1}{2} h\, \eta(U,V)\, D_{(f dg - g df)} W$$

Non-tensorial!

But tensorial for vectors with $\eta(U,V) = 0$
U,V tangent to null subspace

The Connection

Coimbra, Strickland-Constable & Waldram

Choose conformal frames $e^{M}{}_{A} e^{N}{}_{B}\, \eta_{MN} = \Phi^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$GL(2D, \mathbb{R}) \to O(D) \times O(D) \times \mathbb{R}^{+}$$

Seek torsion-Free Connection $T^{P}_{MN} = 0$

$$D\mathcal{H} = 0, \quad D\Phi = 0$$

Find general connection NOT UNIQUE!
Determined up to tensor A
Curvature depends on A. But A drops out of Ricci tensor, scalar curvature and Dirac equation
Field equations given by Ricci tensor, indep of A
Susy variations independent of A

$$\mathcal{H}^{MP}\mathcal{H}_{PN} \;=\; \delta^M{}_N$$

$S^M{}_N \equiv \eta^{MP}\mathcal{H}_{PN}$ satisfies $S^2 = 1$

Split basis:

$$e^M{}_a \quad e^M{}_{\bar{a}} \qquad\qquad a, \bar{a} = 1, \ldots, D$$

$$Se_a = -e_a, \quad Se_{\bar{a}} = e_{\bar{a}}$$

This form of basis preserved by $GL(D, \mathbb{R}) \times GL(D, \mathbb{R})$

$$\mathcal{H}_{AB} = 2\begin{pmatrix} g_{\bar{a}\bar{b}} & 0 \\ 0 & g_{ab} \end{pmatrix}, \qquad \hat{\eta}_{AB} = 2\begin{pmatrix} g_{\bar{a}\bar{b}} & 0 \\ 0 & -g_{ab} \end{pmatrix}$$

Partially fix gauge

$$GL(D, \mathbb{R}) \times GL(D, \mathbb{R}) \to O(D) \times O(D) \times \mathbb{R}^+$$

$$g_{ab} = \Phi^2\delta_{ab}, \quad g_{\bar{a}\bar{b}} = \Phi^2\delta_{\bar{a}\bar{b}}$$

Coimbra, Strickland-Constable & Waldram

$O(D)\times O(D)\times\mathbb{R}^+$ Torsion-Free Connection

$$D\mathcal{H}=0,\quad D\Phi=0 \qquad T^{P}_{MN}=0$$

Take $\Phi=e^{-2\phi}\sqrt{-g}=e^{-2d}$

Gives non-unique connection

$$D_a w^b=\nabla_a w^b-\frac{1}{6}H_a{}^b{}_c w^c-\frac{2}{9}\big(\delta_a{}^b\partial_c\phi-\eta_{ac}\partial^b\phi\big)w^c+A^{+}_a{}^b{}_c w^c,$$

$$D_{\bar a} w^b=\nabla_{\bar a} w^b-\frac{1}{2}H_{\bar a}{}^b{}_c w^c,$$

$$D_a w^{\bar b}=\nabla_a w^{\bar b}+\frac{1}{2}H_a{}^{\bar b}{}_{\bar c} w^{\bar c},$$

$$D_{\bar a} w^{\bar b}=\nabla_{\bar a} w^{\bar b}+\frac{1}{6}H_{\bar a}{}^{\bar b}{}_{\bar c} w^{\bar c}-\frac{2}{9}\big(\delta_{\bar a}{}^{\bar b}\partial_{\bar c}\phi-\eta_{\bar a\bar c}\partial^{\bar b}\phi\big)w^{\bar c}+A^{-}_{\bar a}{}^{\bar b}{}_{\bar c} w^{\bar c},$$

Ambiguity: A terms arbitrary

- Gualtieri: $O(D) \times O(D)$ connection
- Waldram et al: similar $O(D) \times O(D) \times \mathbb{R}^+$ connection, with dilaton.
- Jeon, Lee Park: similar connection, with A=0
- Siegel: similar connection, but curvatures etc constructed without geometry
- Hohm & Kwak; Hohm & Zwiebach: more on Siegel construction

Generalised Geometry, M-Theory

- Generalised Geometry doubles Tangent space, Metric + B-field, action of O(D,D)

Hitchin; Gualtieri

- DFT doubles space, doubles coordinates.
- Extended geometry: extends tangent space, metric and 3-form gauge field, action of exceptional U-duality group

Hull; Pacheco & Waldram

- 10-d type II sugra action in terms of extended geometry

Coimbra, Strickland-Constable & Waldram

Hillman

- 11-d sugra action in terms of extended geometry

Berman, Perry et al

Coimbra, Strickland-Constable & Waldram

Double Field Theory

- Constructed cubic action, quartic has new stringy features
- T-duality symmetry, cocycles, symmetry a homotopy Lie algebra, constraints
- Restricted DFT: have non-linear background independent theory, duality covariant
- Courant bracket gauge algebra

- Stringy issues in simpler setting than SFT
- Geometry. Meaning of curvature.
- Use for non-geometric backgrounds
- General spaces, not tori?
- Full theory without restriction? Does it close on a geometric action with just these fields?
- Doubled geometry *physical* and *dynamical*

- Early work on strings and doubled space: Duff, Tseytlin...
- Sigma model for doubled geometry: Hull; Hull & Reid-Edwards,...
- Doubled sigma model: Quantization Hull; Hackett-Jones & Moutsopoulos; Beta functions: Copland; Berman, Copland & Thompson
- D-Branes: Hull; Lawrence, Schulz, Wecht; Bergshoeff & Riccioni
- Geometry with projectors; YM DFT: Jeon, Lee, Park
- O(10,10) from E_{11} ; West
- Non-geometry: Andriot, Larfors, Lust, Patalong; Ditetto, Fernandez-Melgarejo, Marques & Roest
- Twisted torus: Grana & Marques; Chatzistavrakidis & Jonke
- Heterotic DFT: Hohm & Kwak
- Type II DFT: Thompson; Hull; Hohm, Kwak, Zwiebach