Progress in understanding quantum spectrum of $AdS_5 \times S^5$ superstring

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R. Roiban, AT, arXiv:0906.4294, arXiv:1102.1209

M. Beccaria, S. Giombi, G. Macorini, R. Roiban, AT, arXiv:1203.5710

M. Beccaria, AT, arXiv:1205.3656

Aim I:

learn how to solve string theory in curved spaces

very few solvable examples known –
free fields, gauged WZW, plane waves, orbifolds of them, ...

RR background – extra complication

maximally supersymmetric case of AdS/CFT:
guidance from integrability and weak-coupling gauge theory side

Aim II:

learn how to solve non-trivial 4d quantum field theories

apparently simplest – supersymmetric CFT's
no solvable examples known

Maximally symmetric case of gauge-string duality:

planar $\mathcal{N}=4$ super Yang-Mills $\leftrightarrow$ free $AdS_5 \times S^5$ superstring

closed string states on $R \times S^1 \leftrightarrow$ gauge-inv. SYM states on $R \times S^3$

marginal str. vertex ops on $R^2 \leftrightarrow$ conf. primary SYM ops on R^4

correlators of $AdS_5 \times S^5$ string vertex operators

– analogs of S-matrix elements in flat 10d space —

are dual to correlators of

conformal operators of planar $\mathcal{N}=4$ SYM

In particular, relation of 2-point functions means that

spectrum of $AdS_5 \times S^5$ string energies

$\leftrightarrow$ spectrum of dimensions of SYM primary operators

Then spectrum of $\mathcal{N}=4$ SYM dimensions $\Delta(\lambda)$

should be described by 2d $AdS_5 \times S^5$ superstring sigma :

integrability in 4d has 2d origin

Integrability:

allows "in principle" to solve the problem of spectrum

enormous progress in the last 10 years

Some key inputs:

- SYM action + perturbation theory ($\lambda \ll 1$)
- $AdS_5 \times S^5$ GS superstring action + α'-expansion ($\sqrt{\lambda} \gg 1$)
- classical integrability of $AdS_5 \times S^5$ GS action
- perturbative integrability of SYM spectral problem:

(1-loop, 2-loop, ...) dilatation operator = spin chain Hamiltonian

[Minahan, Zarembo; Beisert, Staudacher, ...]

- guidance from large-charge limits: BMN, GKP, FT

Assume integrability extends to all orders on both sides

- construct interpolating Bethe ansatz guided by general principles, symmetries and data from both weak+strong coupling
- check consistency of its predictions

I. Spectrum of "long" operators / "semiclassical" string states
determined by Asymptotic Bethe Ansatz (2002-2007)

• its final [Beisert-Eden-Staudacher] form found by intricate superposition of data from $\lambda \ll 1$ gauge theory (spin chain, BA,...) and perturbative string theory (classical and 1-loop phase, BMN), symmetries (S-matrix), assumption of exact integrability

• consequences checked against available gauge and string data

Key example: cusp anomalous dimension – dim of $\mathrm{Tr}(\Phi D^S \Phi)$

$$\Delta = S + 2 + f(\lambda) \ln S + ... , \qquad S \gg 1$$

$$f_{\lambda \ll 1} = \frac{\lambda}{2\pi^2}\Big[1 - \frac{\lambda}{48} + \frac{11\lambda^2}{45 \cdot 2^8} - (\frac{73}{630} + \frac{4\zeta_3^2}{\pi^6})\frac{\lambda^3}{2^7} + ...\Big]$$

$$f_{\lambda \gg 1} = \frac{\sqrt{\lambda}}{\pi}\Big[1 - \frac{3\ln 2}{\sqrt{\lambda}} - \frac{K}{(\sqrt{\lambda})^2} - ...\Big] + O(e^{-\frac{1}{4}\sqrt{\lambda}})$$

$\zeta_k = \zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$, $K = \beta(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.915...$

from 2-loop string sigma-model integrals [Roiban,Tirziu,AT]

exact integral eq. [Basso, Korchemsky, Kotanski]: any order term

II. Spectrum of "short" operators = quantum string states

Thermodynamic Bethe Ansatz (2005-...)

• reconstructed from ABA using solely methods/intuition of 2-d integrable QFT, i.e. inspired by string-theory side

• highly non-trivial construction – lack of 2-d Lorentz invariance in standard BMN-vacuum-adapted l.c. gauge

• in few cases ABA "improved" by Luscher corrections is enough: 4- and 5-loop Konishi dim, 4-loop dim. of twist 2 operator

• complicated set of integral equations in need of simplification; so far predictions extracted only numerically starting from weak coupling and interpolating to larger λ

• need more data to check predictions at $\lambda \ll 1$ and $\lambda \gg 1$ – against perturbative gauge-theory and string-theory data

Key example:

dimension $\Delta = 2 + \gamma(\lambda)$ of Konishi operator $\mathrm{Tr}(\bar{\Phi}_i \Phi_i)$

$g^2 = \frac{\lambda}{(4\pi)^2} \ll 1$

$$
\begin{aligned}
\Delta = {} & 4 + 12g^2 - 48g^4 + 336g^6 \\
& +96\Big[-26 + 6\,\zeta_3 - 15\,\zeta_5\Big]g^8 \\
& -96\Big[-158 - 72\,\zeta_3 + 54\,\zeta_3^2 + 90\,\zeta_5 - 315\,\zeta_7\Big]g^{10} \\
& -48\Big[160 + 432\,\zeta_3^2 - 2340\,\zeta_5 \\
& \qquad -72\,\zeta_3(-76 + 45\,\zeta_5) - 1575\,\zeta_7 + 10206\,\zeta_9\Big]g^{12} \\
& +48\Big[-44480 - 8784\,\zeta_3^2 + 2592\,\zeta_3^3 - 4776\,\zeta_5 - 20700\,\zeta_5^2 \\
& \qquad +24\,\zeta_3(4540 + 357\,\zeta_5 - 1680\,\zeta_7) \\
& \qquad -26145\,\zeta_7 - 17406\,\zeta_9 + 152460\,\zeta_{11}\Big]g^{14} + \ldots
\end{aligned}
$$

all coefficients in γ are integer, divisible by 12 ....

new (multiple zeta?) numbers at 8 loops ? exact expression ?

5-loop results first found using integrability
[Banjok, Janik 11]
confirmed later by more standard QFT methods
[Velizhanin; Eden et al 12]
very recent progress:
6-loop term: derivation from TBA [Leurent, Serban, Volin 12]
6- and 7-loop terms: from Luscher corrections approach
[Banjok, Janik 12]

Suppose one can sum up (convergent) $\lambda \ll 1$ expansion
and then re-expand at $\lambda \gg 1$

What one should expect to get for $\gamma(\lambda \gg 1)$?

Duality to string theory predicts the structure
of strong-coupling expansion:
leading term – near-flat-space expansion for fixed quant. numbers
[Gubser, Klebanov, Polyakov 98]

$$\Delta = \sqrt{2N\sqrt{\lambda}} + ...$$

Subleading terms: $\alpha' = \frac{1}{\sqrt{\lambda}}$ expansion of 2d anom. dimensions
of corresponding vertex operators [Roiban, AT 09] ($N = 2$)

$$\begin{aligned}
\gamma(\lambda \gg 1) &= 2\sqrt[4]{\lambda} + \frac{b_1}{\sqrt[4]{\lambda}} + \frac{b_2}{(\sqrt[4]{\lambda})^3} + \frac{b_3}{(\sqrt[4]{\lambda})^{5/2}} + ... \\
&= 2\sqrt[4]{\lambda}\Big[1 + \frac{b_1}{2\sqrt{\lambda}} + \frac{b_2}{2(\sqrt{\lambda})^2} + \frac{b_3}{2(\sqrt{\lambda})^3} + ...\Big]
\end{aligned}$$

Values of b_k from string theory? From TBA?

Dimensions of “short” SYM operators

= energies of quantum string states

find leading $\alpha' = \frac{1}{\sqrt{\lambda}}$ corrections to energy of

“lightest” massive string states on first massive string level

dual to operators in Konishi multiplet in SYM theory

– compare with predictions of TBA approach

important to check integrability-based approach

which involves subtle assumptions

directly against perturbative string sigma model

TBA results:

start at weak coupling for sl(2) Konishi descendant $\mathrm{Tr}(\Phi D^2 \Phi)$
use TBA to find $\Delta(\lambda)$ numerically;
match to expected form of strong-coupling expansion to extract b_k
[Gromov, Kazakov, Vieira 09; Frolov 10, 12]

$$b_1 \approx 1.988 \,, \qquad b_2 \approx -3.07$$

Compare to string theory:

One can find b_k using semiclassical "short string" expansion
[Roiban, AT 09, 11; Gromov, Serban, Shenderovich, Volin 11]

$$b_1 = 2 \,, \qquad b_2 = a - 3\zeta_3$$

rational a was found [Gromov, Valatka 11] using "2-loop"
coefficient in exact slope function $E^2 = h(\lambda) S$ [Basso 11]

$$b_2 = \frac{1}{2} - 3\zeta_3 \ \approx -3.106...$$

Remarkable agreement with TBA - check of quantum integrability

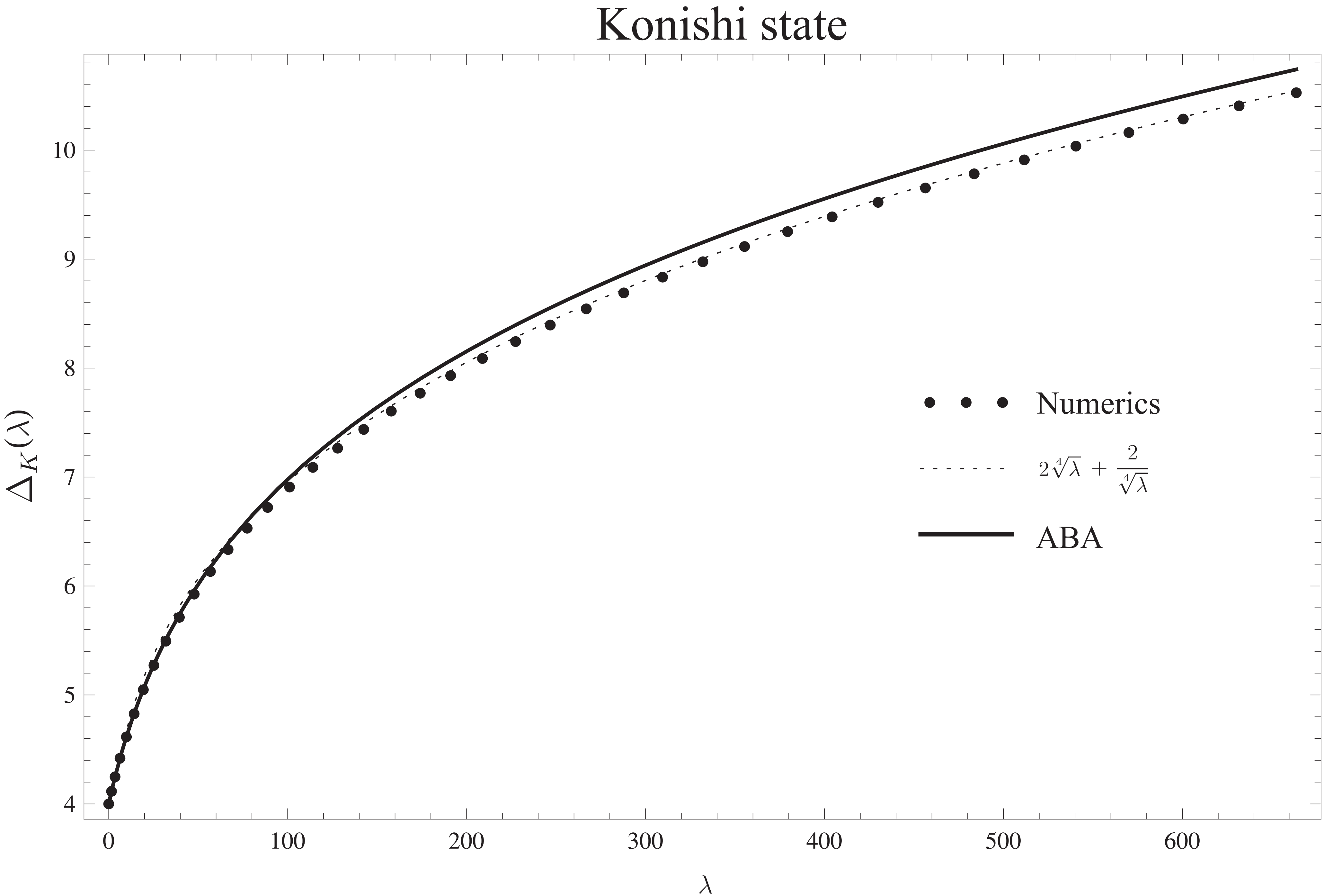

Figure 1: Plot from Gromov, Kazakov, Vieira [09]

Recent work on string side: [BGMRT 12; BT12]

• highest transcendentality terms in b_k
are $\sim \zeta_{2k-1}$ and have 1-loop origin, e.g.,

$$b_3 = a_1 + a_2\zeta_3 + a_3\zeta_5$$

rational a_1 receives contribution from 3 loops; a_2 from
2-loops, etc.; $b_4 \sim \zeta_7 + ...$, etc.

• supermultiplet structure: universality of coefficients in E
for string states with spins in different $AdS_5 \times S^5$ directions:
dual operators from Konishi multiplet have same energy
(up to constant shift depending on position in the multiplet)

• states on leading Regge trajectory:
general structure of dependence of energy on
string tension $\sqrt{\lambda}$, string level (spin) and S^5 orbital momentum J

Some open questions:

• Analytic form of strong-coupling expansion from TBA?

• only ζ_k coefficients in $\Delta(\lambda)$ in both
weak and strong coupling expansions
or other transcendental constants appear?
(cf. cusp anomalous dimension)
[2-loop string computation may shed light on this ...]

• Asymptotic form of strong coupling expansion:
$e^{-k\sqrt{\lambda}}$ corrections to cusp dimension
absent for short strings / operators like Konishi?
[no such corrections in slope function; no massless S^5 modes]

• Energies of other quantum states: general structure of spectrum?

Konishi multiplet:

long multiplet related to singlet $[0,0,0]_{(0,0)}$ by susy

$[J_2 - J_3, J_1 - J_2, J_2 + J_3]_{(s_L, s_R)}$
$s_{L,R} = \frac{1}{2}(S_1 \pm S_2)$

$SO(6)$ (J_1, J_2, J_3) and $SO(4)$ (S_1, S_2) labels
of $SO(2,4) \times SO(6)$ global symmetry

$\Delta = \Delta_0 + \gamma(\lambda), \qquad \Delta_0 = 2, \frac{5}{2}, 3, ..., 10$
same anomalous dimension γ for all members

singlet eigen-state of anom. dim. matrix with lowest eigenvalue

Examples of gauge-theory operators in Konishi multiplet:

$[0,0,0]_{(0,0)}$:
$\mathrm{Tr}(\bar{\Phi}_i \Phi_i), \quad i = 1,2,3, \qquad\qquad \Delta_0 = 2$

$[2,0,2]_{(0,0)}$:
$\mathrm{Tr}([\Phi_1, \Phi_2]^2)$ in $su(2)$ sector, $\Delta_0 = 4$

$[0,2,0]_{(1,1)}$:
$\mathrm{Tr}(\Phi_1 D^2 \Phi_1)$ in $sl(2)$ sector, $\Delta_0 = 4$

Δ_0	
2	$[0,0,0]_{(0,0)}$
$\frac{5}{2}$	$[0,0,1]_{(0,\frac{1}{2})} + [1,0,0]_{(\frac{1}{2},0)}$
3	$[0,0,0]_{(\frac{1}{2},\frac{1}{2})} + [0,0,2]_{(0,0)} + [0,1,0]_{(0,1)+(1,0)} + [1,0,1]_{(\frac{1}{2},\frac{1}{2})} + [2,0,0]_{(0,0)}$
$\frac{7}{2}$	$[0,0,1]_{(\frac{1}{2},0)+(\frac{1}{2},1)+(\frac{3}{2},0)} + [0,1,1]_{(0,\frac{1}{2})+(1,\frac{1}{2})} + [1,0,0]_{(0,\frac{1}{2})+(0,\frac{3}{2})+(1,\frac{1}{2})} + [1,0,2]_{(\frac{1}{2},0)}$ $+[1,1,0]_{(\frac{1}{2},0)+(\frac{1}{2},1)} + [2,0,1]_{(0,\frac{1}{2})}$
4	$[0,0,0]_{(0,0)+(0,2)+(1,1)+(2,0)} + [0,0,2]_{(\frac{1}{2},\frac{1}{2})+(\frac{3}{2},\frac{1}{2})} + [0,1,0]_{2(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})} + [2,0,2]_{(0,0)}$ $+[0,1,2]_{(1,0)} + [0,2,0]_{2(0,0)+(1,1)} + [1,0,1]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [1,1,1]_{2(\frac{1}{2},\frac{1}{2})} + [2,0,0]_{(\frac{1}{2}}$
6	$[0,0,0]_{3(0,0)+3(1,1)+(2,2)} + [0,0,2]_{3(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},\frac{3}{2})} + [0,1,0]_{4(\frac{1}{2},\frac{1}{2})+2(\frac{1}{2},\frac{3}{2})+2(\frac{3}{2},\frac{1}{2})+}$ $+[0,1,2]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [0,2,0]_{3(0,0)+(0,1)+(0,2)+(1,0)+3(1,1)+(2,0)} + [0,2,2]_{(\frac{1}{2},\frac{1}{2})}$ $+[0,3,0]_{2(\frac{1}{2},\frac{1}{2})} + [0,4,0]_{(0,0)} + [1,0,1]_{(0,0)+3(0,1)+3(1,0)+4(1,1)+(1,2)+(2,1)} + [1,0,3]_{(\frac{1}{2},\frac{1}{2})} + [0,$ $+[1,1,1]_{4(\frac{1}{2},\frac{1}{2})+2(\frac{1}{2},\frac{3}{2})+2(\frac{3}{2},\frac{1}{2})} + [1,2,1]_{(0,0)+(0,1)+(1,0)} + [2,0,0]_{3(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},\frac{3}{2})}$ $+[2,0,2]_{(0,0)+(1,1)} + [2,1,0]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [2,2,0]_{(\frac{1}{2},\frac{1}{2})} + [3,0,1]_{(\frac{1}{2},\frac{1}{2})} + [4,0,0]_{(0,0}$
$\frac{17}{2}$	$[0,0,1]_{(0,\frac{1}{2})+(0,\frac{3}{2})+(1,\frac{1}{2})} + [0,1,1]_{(\frac{1}{2},0)+(\frac{1}{2},1)} + [1,0,0]_{(\frac{1}{2},0)+(\frac{1}{2},1)+(\frac{3}{2},0)} + [1,0,2]_{(0,\frac{1}{2})}$ $+[1,1,0]_{(0,\frac{1}{2})+(1,\frac{1}{2})} + [2,0,1]_{(\frac{1}{2},0)}$
9	$[0,0,0]_{(\frac{1}{2},\frac{1}{2})} + [0,0,2]_{(0,0)} + [0,1,0]_{(0,1)+(1,0)} + [1,0,1]_{(\frac{1}{2},\frac{1}{2})} + [2,0,0]_{(0,0)}$
$\frac{19}{2}$	$[0,0,1]_{(\frac{1}{2},0)} + [1,0,0]_{(0,\frac{1}{2})}$
10	$[0,0,0]_{(0,0)}$

Table 1: Long Konishi multiplet (part of it)

Comparison between gauge and string theory states:

• $\lambda \ll 1$: gauge-theory operators built out of free fields, canonical dim. Δ_0 determines operators that can mix

• $\lambda \gg 1$: in near-flat-space expansion string states built out of free oscillators, level N determines states that can mix

(i) relate states with same global charges

(ii) assume direct interpolation (no "level crossing") for states with same quantum numbers as λ changes from small to large values

• Konishi operator dual to
"lightest" among massive $AdS_5 \times S^5$ string states

• large $\sqrt{\lambda} = \frac{\mathrm{R}^2}{\alpha'}$:
"short" strings probe near-flat limit of $AdS_5 \times S^5$

• members of supermultiplet:
strings with spins/oscillators in different $AdS_5 \times S^5$ directions

String spectrum in $AdS_5 \times S^5$:

long multiplets of $PSU(2,2|4)$

highest weight states:

$[J_2 - J_3, J_1 - J_2, J_2 + J_3]_{(s_1, s_2)}$

$s_{1,2} = \frac{1}{2}(S_1 \pm S_2)$

Flat-space string spectrum can be re-organized

in multiplets of $SO(2,4) \times SO(6) \subset PSU(2,2|4)$

[Bianchi, Morales, Samtleben 03; Beisert et al 03]

$SO(4) \times SO(5) \subset SO(9)$ rep.

lifted to $SO(4) \times SO(6)$ rep. of $SO(2,4) \times SO(6)$

Konishi multiplet:

$\mathcal{K} = (1 + Q + Q \wedge Q + ...)[0,0,0]_{(0,0)}$

determines the “floor” of 1-st excited string level

$\sum_{J=0}^{\infty} [0, J, 0]_{(0,0)} \times \mathcal{K}$

Spins: S_1, S_2 in AdS_5; (J_1, J_2) in S^5
orbital momentum $J = J_3$ in S^5

Examples:

• folded string with spin S_1 and momentum J:

$S_1 = J = 2 \quad \rightarrow \quad [0, 2, 0]_{(1,1)}, \quad \Delta_0 = 4$

• folded string with spin J_1 and momentum J:

$J_1 = J = 2 \quad \rightarrow \quad [2, 0, 2]_{(0,0)}, \quad \Delta_0 = 4$

• circular string with spins $J_1 = J_2$ and momentum J:

$J_1 = J_2 = 1, J = 2 \quad \rightarrow \quad [0, 1, 2]_{(0,0)}, \quad \Delta_0 = 6$

• circular string with spins $S_1 = S_2$ and momentum J:

$S_1 = S_2 = 1, J = 2 \quad \rightarrow \quad [0, 2, 0]_{(0,1)}, \quad \Delta_0 = 6$

• circular string with spins $S_1 = J_1$ and momentum J:

$S_1 = J_1 = 1, J = 2 \quad \rightarrow \quad [1, 1, 1]_{(\frac{1}{2},\frac{1}{2})}, \quad \Delta_0 = 6$

Direct approaches to computation of quantum string energies:

(i) vertex operator approach:

use $AdS_5 \times S^5$ string sigma model perturbation theory to find leading terms in 2d anomalous dimension of corresponding vertex operators and impose marginality condition [Polyakov 01; AT 03]

(ii) "light-cone" gauge approach:

start with AdS light-cone gauge $AdS_5 \times S^5$ string action and compute corrections to energy of corresponding flat-space oscillator string state [Metsaev, Thorn, AT 00]

both approaches yet to be developed in detail;
here will be guided by vertex operator approach
but use indirect "semiclassical" approach:
"short string" limit of semiclassical expansion
[Tirziu, AT 08; Roiban, AT 09, 11]

Massive string states in curved background:

$$\int d^D x \sqrt{g}\, \left[\Phi_{...}(-D^2 + m^2 + X)\Phi_{...} + ...\right]$$
$$m^2 = \tfrac{2N}{\alpha'}\ , \qquad X = R_{....} + O(\alpha')$$

case of $AdS_5 \times S^5$ background

$$R_{mn} - \tfrac{1}{96}(F_5 F_5)_{mn} = 0, \qquad R = 0\ , \qquad F_5^2 = 0$$

Find leading-order term in X ...

leading α' correction to scalar string state mass is 0 (?!)

$$[-D^2 + m^2 + O(\tfrac{1}{\sqrt{\lambda}})]\Phi \ = 0$$
$$\Delta = 2 + \sqrt{2N + 4 + O(\tfrac{1}{\sqrt{\lambda}})}$$
$$\Delta_{_{N=2}} = 2 + 2\sqrt[4]{\lambda}\, \left[1 + \tfrac{1}{2\sqrt{\lambda}} + O(\tfrac{1}{(\sqrt{\lambda})^2})\right]$$

Too naive: $SO(6)$ scalar, not 10d scalar, mixing,...

What is found for non-singlet (susy descendant) Konishi states?

Vertex operator approach

calculate 2d anomalous dimensions from "first principles"– superstring theory in $AdS_5 \times S^5$:

$$I = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \Big[\partial Y_p \bar{\partial} Y^p + \partial X_k \bar{\partial} X_k + \text{fermions} \Big]$$

$$-Y_0^2 - Y_5^2 + Y_1^2 + ... + Y_4^2 = -1 \, , \qquad X_1^2 + ... + X_6^2 = 1$$

construct marginal (1,1) operators in terms of Y_p and X_k

e.g. vertex operator for dilaton (in NSR framework)

$$\text{V}_J = \ (Y_+)^{-\Delta} \, (X_x)^J \, \Big[\partial Y_p \bar{\partial} Y^p + \partial X_k \bar{\partial} X_k + \text{fermions} \Big]$$

$$Y_+ \equiv Y_0 + iY_5 = z + z^{-1} x_m x_m \sim e^{it}$$

$$X_x \equiv X_1 + iX_2 \sim e^{i\varphi}$$

$$2 = 2 + \frac{1}{2\sqrt{\lambda}} [\Delta(\Delta - 4) - J(J+4)] + O(\tfrac{1}{(\sqrt{\lambda})^2})$$

i.e. $\Delta = 4 + J$ (BPS)

Vertex operators = eigenstates of 2d anomalous dimension matrix

particular linear combinations like

$$V = f_{k_1...k_\ell m_1...m_{2s}} X_{k_1}...X_{k_\ell} \partial X_{m_1} \bar{\partial} X_{m_2} ... \partial X_{m_{2s-1}} \bar{\partial} X_{m_{2s}}$$

their renormalization studied in $O(n)$ sigma model [Wegner 90]

simplest case: $f_{k_1...k_\ell} X_{k_1}...X_{k_\ell}$ with traceless $f_{k_1...k_\ell}$

h.-w. rep. $V_J = (X_x)^J, \qquad \widehat{\gamma} = 2 - \frac{1}{2\sqrt{\lambda}} J(J+4) + ...$

$AdS_5 \times S^5$: candidates for operators on leading Regge trajectory:

$$\mathrm{V}_J = (Y_+)^{-\Delta} \big(\partial X_x \bar{\partial} X_x\big)^{J/2} , \qquad X_x \equiv X_1 + iX_2$$

$$\mathrm{V}_S = (Y_+)^{-\Delta} \big(\partial Y_u \bar{\partial} Y_u\big)^{S/2} , \qquad Y_u \equiv Y_1 + iY_2$$

+ fermionic terms

+ $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ terms from diagonalization of anom. dim. op.

– mixing with ops with same charges and dimension

Example of higher-level scalar/singlet operator:

$$Y_+^{-\Delta}\Big[(\partial X_k \bar{\partial} X_k)^r + ...\Big]\ , \qquad\qquad N = 2(r-1)$$

Marginality condition:

[cf. Kravtsov, Lerner, Yudson 89; Castilla, Chakravarty 96]

$$0 = 2(r-1) - \frac{1}{2\sqrt{\lambda}}\Big[\Delta(\Delta-4) + 2r(r-1)\Big] - \frac{1}{(\sqrt{\lambda})^2}\Big[\tfrac{2}{3}r(r-1)(r-\tfrac{7}{2}) + 4r\Big] + ...$$

$r=1$: ground level– fermions should make $r=1$ zero of $\widehat{\gamma}$

$r=2$: excited level – analog of singlet Konishi state $\Delta_0 = 2$

$$\Delta(\Delta-4) = 4\sqrt{\lambda} - 4 + O(\tfrac{1}{\sqrt{\lambda}})\ ,$$
$$\Delta - \Delta_0 = 2\sqrt[4]{\lambda}\Big[1 + 0 \times \tfrac{1}{\sqrt{\lambda}} + O(\tfrac{1}{(\sqrt{\lambda})^2})\Big]$$

fermionic contributions change subleading coefficients

How to take fermionic contributions into account?

(i) compute energies of semiclassical string states in $\frac{1}{\sqrt{\lambda}}$ expansion
using full $AdS_5 \times S^5$ Green-Schwarz action

(ii) compare to structure of $E = \Delta$
expected from marginality condition

(iii) determine unknown coefficients in E expanded in $\frac{1}{\sqrt{\lambda}}$

General structure of dimension/energy $\Delta = E$

marginality condition – condition on quantum numbers Q_i

$Q = (E(\lambda), S_1, S_2; J_1, J_2, J_3; ...)$; $N = \sum_i a_i Q_i$ = level

$$0 = 2N + \frac{1}{\sqrt{\lambda}}\Big(\sum_{i,j} c_{ij} Q_i Q_j + \sum_i c_i Q_i\Big)$$
$$+ \frac{1}{(\sqrt{\lambda})^2}\Big(\sum_{i,j,k} c_{ijk} Q_i Q_j Q_k + \sum_{i,j} c'_{ij} Q_i Q_j + \sum_i c'_i Q_i\Big) + ...$$

States on "leading Regge trajectory": (max spin for given E)

marginality condition: $Q = (E, J; N)$, N= spin

$$0 = 2N + \frac{1}{\sqrt{\lambda}}\Big(- E^2 + J^2 + n_{02} N^2 + n_{11} N\Big)$$
$$+ \frac{1}{(\sqrt{\lambda})^2}\Big(n_{01} J^2 N + n_{03} N^3 + n_{12} N^2 + n_{21} N\Big) + ...$$

solution for E^2 takes form [Roiban, AT 09, 11; BGMRT 12]

$$E^2 = 2\sqrt{\lambda}N + J^2 + n_{02}N^2 + n_{11}N + \frac{1}{\sqrt{\lambda}}\big(n_{01}J^2N + n_{03}N^3 + n_{12}N^2 + n_{21}N\big) + \frac{1}{(\sqrt{\lambda})^2}\big(\widetilde{n}_{11}J^2N + \widetilde{n}_{02}J^2N^2 + n_{04}N^4 + n_{13}N^3 + n_{22}N^2 + n_{31}N\big) + ...$$

Expanding in large $\sqrt{\lambda}$ for fixed N, J

$$E = \sqrt{2\sqrt{\lambda}N}\Big[1 + \frac{A_1}{\sqrt{\lambda}} + \frac{A_2}{(\sqrt{\lambda})^2} + O(\frac{1}{(\sqrt{\lambda})^3})\Big]$$

$$A_1 = \frac{1}{4N}J^2 + \frac{1}{4}(n_{02}N + n_{11})$$

$$A_2 = -\frac{1}{2}A_1^2 + \frac{1}{4}(n_{01}J^2 + n_{03}N^2 + n_{12}N + n_{21})$$

Gives strong-coupling dimension of dual SYM operator

States on 1-st excited superstring level: $N = 2$

Konishi multiplet states: $N = 2$, $J = 2$

$$E = \sqrt[4]{\lambda}\Big[2 + \frac{b_1}{\sqrt{\lambda}} + \frac{b_2}{(\sqrt{\lambda})^2} + O(\frac{1}{(\sqrt{\lambda})^3})\Big]$$
$$b_1 = 1 + n_{02} + \tfrac{1}{2}n_{11}$$
$$b_2 = -4b_1^2 + 2n_{01} + 2n_{03} + n_{12} + \tfrac{1}{2}n_{21}$$

coefficients n_{km} =? – use semiclassical "short string" expansion:

• start with solitonic string carrying same charges

as vertex operator representing particular quantum string state

• perform semiclassical expansion: $\sqrt{\lambda} \gg 1$

for fixed classical parameters $\mathcal{N} = \frac{1}{\sqrt{\lambda}}N, \quad \mathcal{J} = \frac{1}{\sqrt{\lambda}}J$

• expand E in small values of $\mathcal{N}, \mathcal{J}$

• re-interpret the resulting E in terms of N, J: get n_{km}

Key point: limit $\mathcal{N} = \frac{N}{\sqrt{\lambda}} \to 0, \quad \mathcal{J} = \frac{J}{\sqrt{\lambda}} \to 0$

corresponds to $\sqrt{\lambda} \gg 1$ for fixed values of quantum charges N, J

Digression: **Slope function**

Semicl. expansion of E^2 organized as expansion in small $\mathcal{N}$ or N

$$E^2 = J^2 + h_1(\lambda, J)\, N + h_2(\lambda, J)\, N^2 + h_3(\lambda, J)\, N^3 + \ldots$$

$$h_1 = 2\sqrt{\lambda} + n_{11} + \frac{n_{21}}{\sqrt{\lambda}} + \frac{n_{31}}{(\sqrt{\lambda})^2} + \ldots + J^2\Big(\frac{n_{01}}{\sqrt{\lambda}} + \frac{\widetilde{n}_{11}}{(\sqrt{\lambda})^2} + \ldots\Big) + \ldots$$

$$h_2 = n_{02} + \tfrac{n_{12}}{\sqrt{\lambda}} + \ldots\,, \qquad h_3 = \tfrac{n_{03}}{\sqrt{\lambda}} + \ldots$$

exact "slope" h_1 for sl(2) sector operator $\mathrm{Tr}(D^S \Phi^J)$

dual to AdS_5 folded spinning string ($N = S$)

from BA (I_J - modif. Bessel of 1st type) [Basso 11,12;Gromov 12]

$$h_1(\lambda, J) = 2J + 2\sqrt{\lambda}\, \frac{I_{J+1}(\sqrt{\lambda})}{I_J(\sqrt{\lambda})}$$

$$= 2\sqrt{\lambda}\sqrt{1+\mathcal{J}^2} - \frac{1}{1+\mathcal{J}^2} - \frac{\frac{1}{4} - \mathcal{J}^2}{\sqrt{\lambda}(1+\mathcal{J}^2)^{5/2}} + \ldots$$

$$= 2\sqrt{\lambda + J^2} - \frac{\lambda}{\lambda + J^2} - \frac{\lambda(\frac{1}{4}\lambda - J^2)}{(\lambda + J^2)^{5/2}} + \ldots$$

h_1: does not depend on wrappings or dressing phase corrections

[h_1 from direct summation of 4d or 2d graphs or localization ?]

h_1 in large J expansion:

$$h_1 = 2J + \sum_{n=1}^{\infty} \frac{c_n(\lambda)}{J^n}$$

$c_n = a_1 \lambda^k + ... + a_k \lambda + a_{k+1}$ – same finite polynomials for

$\lambda \ll 1, \ J \gg 1$ and $\sqrt{\lambda} \gg 1, \ \mathcal{J} = \frac{J}{\sqrt{\lambda}} \gg 1$

same coefficients "seen" in opposite string and gauge expansions:

an extension of known "non-renormalization" relations

[Beisert, Minahan, Staudacher, Zarembo; Frolov, AT03;...]

Slope function in su(2) sector [Beccaria, AT 12; Gromov 12]

state in su(2) sector $\mathrm{Tr}(Z^J \Phi^{J'})$ dual to folded string in S^5

Relation between folded string in AdS_5 $(\mathcal{E}, \mathcal{S}; \mathcal{J})$ and in S^5:$(\mathcal{E}; \mathcal{J}', \mathcal{J})$

analytic continuation [Beisert,Frolov,Staudacher,AT 03]

$\mathcal{E} \to -\mathcal{J}, \quad \mathcal{J} \to -\mathcal{E}, \quad \mathcal{J}' \to \mathcal{S}, \qquad \sqrt{\lambda} \to -\sqrt{\lambda}$

su(2) slope function $\widetilde{h}_1$ is then related to sl(2) one

$$E^2 = J^2 + \widetilde{h}_1(J,\lambda)J' + ... \,, \qquad \widetilde{h}_1(J,\lambda) = -h_1(-J,-\sqrt{\lambda})$$

$$\widetilde{h}_1(\lambda,J) = 2J + 2\sqrt{\lambda}\,\frac{K_{J-1}(\sqrt{\lambda})}{K_J(\sqrt{\lambda})}$$

K_J= modified Bessel function of 2nd type

regular $\lambda \gg 1$ expansion but singularities at $\lambda \ll 1$ at fixed J

$$h_1(J,\lambda) = 2\,J+\frac{\lambda}{J+1}-\frac{\lambda^2}{4\,(J+1)^2\,(J+2)}+\frac{\lambda^3}{8\,(J+1)^3\,(J+2)\,(J+3)}+\cdots\,,$$

$$\widetilde{h}_1(J,\lambda) = 2\,J+\frac{\lambda}{J-1}-\frac{\lambda^2}{4\,(J-1)^2\,(J-2)}+\frac{\lambda^3}{8\,(J-1)^3\,(J-2)\,(J-3)}+\cdots\,.$$

resummation reflected in $\lambda^n \ln^k \lambda$ terms

$$J=2: \quad \widetilde{h}_1 = 4 + \lambda + \frac{1}{4}\lambda^2\big(\ln\tfrac{\lambda}{4} + 2\gamma_{\rm E}\big) + ...$$

$$J=3: \quad \widetilde{h}_1 = 6 + \frac{\lambda}{2} - \frac{\lambda^2}{16} - \frac{\lambda^3}{128}\big(2\ln\tfrac{\lambda}{4} + 4\gamma_{\rm E} - 1\big) + \ldots,$$

$$J=4: \quad \widetilde{h}_1 = 8 + \frac{\lambda}{3} - \frac{\lambda^2}{72} + \frac{\lambda^3}{432} + \frac{\lambda^4}{20736}\big(9\ln\tfrac{\lambda}{4} + 18\,\gamma_{\rm E} - 8\big) + \ldots,$$

$$J=5: \quad \widetilde{h}_1 = 10 + \frac{\lambda}{4} - \frac{\lambda^2}{192} + \frac{\lambda^3}{3072} - \frac{7\lambda^4}{147456} - \frac{\lambda^5}{2359296}\big(16\ln\tfrac{\lambda}{4} + 32\gamma_{\rm E} - 19\big)$$

meaning of $\lambda^J \ln \lambda$ terms ?

• defn of slope function at finite J is non-trivial:

requires analytic continuation to small values of spin

in $\mathfrak{su}(2)$ sector J' is bounded by the fixed length of spin chain $L = J' + J$

$\mathfrak{su}(2)$ slope is defined only in the large J limit ?

• in contrast to $\mathfrak{sl}(2)$ slope, the $\mathfrak{su}(2)$ slope may (?) receive

wrapping contributions which also start at $(\lambda^L)_{J' \to 0} \sim \lambda^J$ order

starting with a TBA generalization of ABA may (?) lead to

cancellation of $\lambda^J \ln \lambda$ terms

• may be $\lambda^J \ln \lambda$ terms have physical meaning:

non-perturbative terms from resummation of λ^n expansion

analogous to $\lambda^n \ln \lambda + ...$ terms appearing in (ladder-diagram) IR-resummed

perturbation theory for Wilson loop for q-$\bar{q}$ potential

[Erickson,Semenoff,Szabo,Zarembo 00; Correa,Maldacena,Sever 12]

$< W(\phi, \lambda) >$ for cusp is described by an integrable TBA system

analogy between expectation value of the cusp Wilson loop

at small ϕ and $\mathfrak{sl}(2)$ slope function h_1 at $J = 1$

suggests that q-$\bar{q}$ potential ($\phi \to \pi$) is related to the $\mathfrak{su}(2)$ slope $\widetilde{h}_1$?

Back to spectrum problem:

To find E for quantum states one need coefficients in higher "slopes" $h_2, h_3, ...$ which already depend on wrapping corrections

Strategy: consider examples of "small" semiclassical string states corresponding to quantum string states with angular momentum J and few oscillator (spin-carrying) modes excited

- start with classical string solutions in flat space representing states on leading Regge trajectory
- find the corresponding solutions in $AdS_5 \times S^5$
- find 1-loop correction to their energy E
- expand E in $\mathcal{N} = \frac{N}{\sqrt{\lambda}} \to 0$ – interpolate result to finite N
- find the coefficients n_{km}
- check universality of E for $N = 2$ (implied by susy)

Examples studied: folded strings with $S_1 = J = 2$; with $J_1 = J = 2$; circular strings with $J_1 = J_2 = 1, J = 2$; with $S_1 = S_2 = 1, J = 2$; with $S_1 = J_1 = 1, J = 2$

Results: for several states on leading Regge trajectory

$$E^2 = 2\sqrt{\lambda}N + J^2 + n_{02}N^2 + n_{11}N + \frac{1}{\sqrt{\lambda}}\left(n_{01}J^2N + n_{03}N^3 + n_{12}N^2 + n_{21}N\right) + \frac{1}{(\sqrt{\lambda})^2}\left(\widetilde{n}_{11}J^2N + \widetilde{n}_{02}J^2N^2 + n_{04}N^4 + n_{13}N^3 + n_{22}N^2 + n_{31}N\right) + \frac{1}{(\sqrt{\lambda})^3}\left(\widetilde{n}_{01}J^4N + \widetilde{n}_{21}J^2N + \widetilde{n}_{12}J^2N^2 + n_{05}N^5 + ...\right) + ...$$

- $n_{01} = 1,\ \widetilde{n}_{01} = -\frac{1}{4}, ...$ from near-BMN expansion ($J \ll \sqrt{\lambda}$)

$E^2 = J^2 + 2N\sqrt{\lambda + J^2} + ... = J^2 + N(2\sqrt{\lambda} + \frac{J^2}{\sqrt{\lambda}} + ...)$

- "tree-level" coeffs $n_{02}, n_{03}, n_{04}, ...$ are all rational
- leading 1-loop n_{11} is rational [Roiban, AT 09; Gromov et al 11]
- $\widetilde{n}_{11} = -n_{11}$, i.e. in general [BGMRT 12]

$$h_1 = 2\sqrt{\lambda}\sqrt{1+\mathcal{J}^2} + \frac{n_{11}}{1+\mathcal{J}^2} + \frac{1}{\sqrt{\lambda}}\left(n_{21} + \widetilde{n}_{21}\mathcal{J}^2 + ...\right) + ...$$

$$h_2 = \frac{n_{02}+\mathcal{J}^2}{1+\mathcal{J}^2} + \frac{1}{\sqrt{\lambda}}\left(n_{12} + \widetilde{n}_{12}\mathcal{J}^2 + ...\right) + ...$$

- $n_{12} = n'_{12} - 3\zeta_3, \qquad n'_{12} = -\frac{3}{8} - 2n_{03}$ is rational

[Tirziu, AT 08; Roiban, AT 09; Gromov-Valatka 11]

ζ_3 term is universal for states on leading Regge trajectory

• $\widetilde{n}_{12} = \widetilde{n}'_{12} + 3\zeta_3 + \frac{15}{4}\zeta_5, \quad \widetilde{n}'_{12}$ rational

• n_{1k} contains universal ζ_{2k-1} (universal UV $n \gg 1$ asymptotics)

e.g. $n_{13} = \widetilde{n}'_{12} + \widetilde{n}''_{1k}\zeta_3 + \frac{15}{4}\zeta_5$

• leading 2-loop coefficient n_{21} is universal: $n_{21} = -\frac{1}{4}$

for folded string state [Basso]; evidence from universality [BGMRT]

of the Konishi state energy ($J = N = 2$)

$$E_{N=J=2} = \sqrt[4]{\lambda}\Big[2 + \frac{b_1}{\sqrt{\lambda}} + \frac{b_2}{(\sqrt{\lambda})^2} + \frac{b_3}{(\sqrt{\lambda})^3} + \ldots\Big]$$

$$b_1 = 1 + n_{02} + \frac{1}{2}n_{11} = 2$$

$$b_2 = -\frac{1}{4}b_1^2 + 2n_{01} + 2n_{03} + n_{12} + \frac{1}{2}n_{21} = \frac{1}{2} - 3\zeta_3$$

$$b_3 = a_1 + a_2\zeta_3 + \frac{15}{2}\zeta_5\ , \qquad \ldots$$

b_1, b_2: match TBA predictions interpolated to $\lambda \gg 1$

• need 2-loop string sigma model computation

to confirm universality of n_{21}, fix $n_{22} \to$ determine b_3

Conclusions

• progress in understanding of $AdS_5 \times S^5$ string spectrum
or spectrum of conformal $\mathcal{N} = 4$ SYM operators

• agreement with numerical results from TBA:
non-trivial check of quantum string integrability

• prediction of transcendental structure of leading coefficients:
reproduce them by an analytic solution of TBA at strong coupling?

• evidence of universality of some coefficients in strong coupling
expansion of dimensions of states on leading Regge trajectory

• exact results for leading "slope" functions

• need systematic study of quantum string theory in $AdS_5 \times S^5$
in near-flat-space expansion

• still need first-principles solution for
spectrum of $AdS_5 \times S^5$ superstring = spectrum of $\mathcal{N} = 4$ SYM
based on integrability
... it now seems within reach...