On duality symmetry

in perturbative quantum field theory

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Recent advances in understanding S-matrix of supergravity:

- tree level – twistor methods
- loop corrections – use string-type (KLT) relations

to construct supergravity amplitudes from SYM $\times$ SYM

- $N = 8$ supergravity: no 3-loop R^4 in

in agreement with improved supersymmetry predictions
and E_7 global symmetry

- $N = 4$ supergravity: no 3-loop R^4

but so far no systematic explanation based on supersymmetry

role of anomaly [Marcus 85] of $SU(1,1)$ global symmetry ?
[Bossard,Howe, Stelle, Vanhove 11]
$f(\phi)R^4$ is allowed, divergences start from
5-point scalar– 4-graviton amplitude

Motivation to understand better
role of global symmetries of supergravities

Non-compact symmetries of supergravities:
involve duality rotations of 4d vector fields
These are on-shell symmetries – symmetries of eqs of motion
How such symmetries are realized in quantum theory –
in effective action, S-matrix ?
Their possible anomalies at quantum level?

history:

• use of $F \to F^*$ symmetry in Einstein-Maxwell (super) gravity
to restrict possible 1-loop on-shell counterterms:
only $T_{mn}T^{mn}$ are allowed [Deser, van Nieuwenhuizen et al 75]

• duality invariance of stress tensor $T_{mn} \to$
invariance of Hamiltonian $\to$ invariance of S-matrix
[Gaillard, Zumino 81]

Aim: study some simple 2d and 4d examples of models
with similar symmetries

General srtructure of scalar-vector sector of ($\mathcal{N} \geqslant 4$) supergravity

$$L = g_{IJ}(\Phi)\partial_m\Phi^I\partial^m\Phi^J + f_{rs}(\Phi)F^r_{mn}F^{smn} + h_{rs}(\Phi)F^{*r}_{mn}F^{smn}$$

$F^{*kl} \equiv \frac{1}{2}\epsilon^{klmn}F_{mn}, \qquad k,l,m,n = 0,1,2,3.$

g_{IJ} is metric of G/H space, eqs. of motion have G-covariance
or invariance if combine F_{mn} and G_{mn}, $G^* \equiv 2\frac{\partial L}{\partial F}$ in doublet

prototypical example: $\mathcal{N} = 4$ supergravity $\Phi = (\phi, \chi)$

$$S = -\frac{1}{2}\int d^4x\Big[(\partial_m\phi)^2 + e^{4\phi}(\partial_m\chi)^2 + \frac{1}{2}e^{-2\phi}F^2_{mn} + \frac{1}{2}\chi F^*_{mn}F^{mn}\Big]$$

scalar part is $SO(1,2)/SO(2)$ sigma model
its global invariance under $SL(2,R) \approx SO(1,2) \approx SU(1,1)$
is promoted to invariance of the full equations of motion

with vector-vector duality transformation

$$\begin{pmatrix} F \\ G \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}$$

$$\tau \to \frac{a\tau + b}{c\tau + d} , \qquad \tau = e^{-2\phi} + i\chi , \qquad ad - bc = 1$$

simplest example: $\chi = 0$

$$\phi \to -\phi, \;\; A_m \to \widetilde{A}_m , \qquad \widetilde{F}_{mn} \equiv G_{mn} = e^{-2\phi} F^*_{kl}$$

This symmetry is not a manifest local symmetry –
how it is realised in quantum theory?
If one integrates out the vector field – effective action Γ depending only on the scalars is expected to be $SL(2)$ invariant.
Performing vector-vector $A_m \to \widetilde{A}_m$ duality in path integral (via Lagrange multiplier term, etc.) gives same partition function with $\widetilde{A}_m$ coupled to $SL(2)$-transf. scalars;
integrating out vector should give invariant functional of scalars

[caveat: this is not automatically true in general on a curved
4d background one gets an "anomalous" local curvature coupling
$\int \phi R^* R^*$ as in dilaton shift under scalar-scalar duality in 2d case;
it can be removed by adding a local counterterm]

But what happens if one keeps both the scalars and vector in Γ
or as external states in the S-matrix?

- Natural expectation (ignoring issue of anomalies):
this on-shell duality symmetry should be present in
quantum effective action evaluated on equations of motion,
i.e. in on-shell S-matrix
Precise meaning of action of duality symmetry on S-matrix?
- far less clear possibility [Kallosh et al] : quantum eff. eq.
from off-shell effective action might be covariant under deformed
version of duality – if effective action is "self-dual", i.e. covariant
under "Legendre" transform from original to dual variables
But why this "self-duality" should apply and does it?

Aim: clarify these questions using

"doubled" or "phase-space" formulation

4d vector-vector duality (or p-form – p-form duality in $d = 2p+2$)

naturally acts on phase space: first-order action is duality-invariant

[Deser, Teitelboim, 76]

similar actions for chiral scalars or p-forms

[Floreanini, Jackiw 87; Henneaux, Teitelboim 88]

Replace momenta by spatial derivative of a new ("dual") field

$\rightarrow$ phase-space action as an action of "doubled" set of fields

Then duality acts locally – as manifest off-shell symmetry

but achieved at expense of standard Lorentz invariance

(standard Lorentz invariance is recovered on equations of motion)

Such manifestly duality invariant action first written in 2d:

duality there is $O(n,n)$ T-duality [AT, 90]

Similar construction later in 4d [Schwarz, Sen 93]

Recently applied to extended supergravity –
manifestly $E_{7(7)}$ invariant action of $\mathcal{N} = 8$ supergravity
[Hillmann, 09; Bossard, Hillmann, Nicolai, 10]

Action for “doubled” set of fields is describing same number
of d.o.f. as original action (and equivalent quantum theory)
more suitable for addressing question of
realization of duality at quantum level

BHN: explicitly confirmed vector contribution to
rigid $SU(8) \subset E_{7(7)}$ anomaly [Marcus, 85] and thus
cancellation of anomaly = preservation of $E_{7(7)}$ at quantum level
But issue of on-shell Lorentz invariance and realization of
$E_{7(7)}$ duality on vector-scalar S-matrix was not addressed

General issues with quantum realization of duality

are same in any $d = 2p + 2$ of dimensions –

concentrate on $d = 2$ instead of $d = 4$

$$S = -\frac{1}{2} \int d^2\sigma \Big[(\partial_a \phi)^2 + e^{4\phi} (\partial_a \chi)^2 + e^{-2\phi} (\partial_a x_s)^2 + \epsilon^{ab} \epsilon^{rs} \chi \partial_a x_r \partial_b x_s \Big]$$

$a, b = 0, 1, \quad r, s = 1, 2$

$SL(2)$ symmetry of the (ϕ, χ) sector extended to the full set

of e.o.m. when combined with 2d duality on scalars x_s

need at least $n = 2$ scalars x_s to have the $O(n, n)$ duality group

(acting on x_s and their "momenta")

big enough to contain $SL(2)$ acting on (ϕ, χ)

[this sigma model is not conformal – has 3-form $H_{\chi rs} = \epsilon_{rs}$

and target metric has only one component of R_{mn}: $R_{\phi\phi} = -\frac{3}{2}$]

If integrate out x_s get $SL(2)$ invariant quantum theory for (ϕ, χ)
but realization of duality on full set (ϕ, χ, x_s) at quantum level?

In "doubled" formulation duality in x_s sector and thus $SL(2)$ of
full model is manifest; useful to split x_s into its chiral parts

Will consider discrete subgroup of duality ($\phi \to -\phi$ for $\chi = 0$):
duality of the S-matrix translates into a symmetry under
flipping sign of anti-chiral part x_s^- and sign of ϕ

Similar transformation will apply to higher-dimensional models:
in 4d one is to flip the sign of the anti-chiral part of the vector field
Starting with duality symmetric "doubled" formulation will check
2d Lorentz inv of quantum on-shell effective action or S-matrix

AdS sigma model: duality-invariant theory in 2d

sigma-model based on euclidean AdS_{n+1} metric ($s = 1, ..., n$)

$$ds^2 = d\phi^2 + e^{-2\phi} dx_s dx_s$$

2d duality in all x_s maps it into itself if combined with
coordinate transformation $\phi \to -\phi$ [Kallosh, AT 98]
Note: this transformation interchanges manifest (Noether) charges
with equivalent subset of hidden charges (conserved due to
integrability of the model) [Ricci, Wolf, AT 07]
strong-coupling origin of "dual conformal symmetry"
[Berkovits, Maldacena 08; Beisert, Ricci, Wolf, AT 08]

Classical Theory: sigma model action in "first-order" form

$$S(\phi, x) = \frac{1}{2} \int d^2\sigma \Big[-(\partial_a \phi)^2 - e^{-2\phi} (\partial_a x_s)^2 \Big] \ \to$$

$$S(\phi, p, x) = \frac{1}{2} \int d^2\sigma \Big[-(\partial_a \phi)^2 + 2 p_s \dot{x}_s - e^{-2\phi} x_s'^2 - e^{2\phi} p_s^2 \Big]$$

introduce new field $\widetilde{x}_s$ by $p_s = \widetilde{x}'_s$

gives duality-invariant action [AT 90] $(I, J = 1, ..., 2n)$

$$\widehat{S}(\phi, x, \widetilde{x}) = -\int d^2\sigma \Big[(\partial_a \phi)^2 - \dot{x}_s \widetilde{x}'_s - \dot{\widetilde{x}}_s x'_s + e^{-2\phi} x'^2_s + e^{2\phi} \widetilde{x}'^2_s \Big]$$

$$= -\int d^2\sigma \Big[(\partial_a \phi)^2 - \Omega_{IJ} \dot{X}^I X'^J + M_{IJ} X'^I X'^J \Big]$$

$$X = \begin{pmatrix} x \\ \widetilde{x} \end{pmatrix} , \qquad \Omega = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} , \qquad M = \begin{pmatrix} e^{-2\phi} & 0 \\ 0 & e^{2\phi} \end{pmatrix}$$

duality symmetry $X \to \Omega X, \ M \to \Omega M \Omega,$ i.e.

$$x_s \to \widetilde{x}_s \ , \ \ \widetilde{x}_s \to x_s \ , \ \ \phi \to -\phi$$

$O(n, n)$ transf.: $X \to \Lambda X, \ \ \Lambda^T \Omega \Lambda = \Omega,$

preserves the structure of the action if also $M \to \Lambda^{-T} M \Lambda^{-1}$

but this change of M cannot be in general compensated

by a redefinition of ϕ

Doubled action for full model with χ:

$$\widehat{S} = -\int d^2\sigma \Big[(\partial_a \phi)^2 + e^{4\phi} (\partial_a \chi)^2 - \Omega_{IJ} \dot{X}^I X'^J + M_{IJ} X'^I X'^J \Big]$$

$$X = \begin{pmatrix} x \\ \widetilde{x} \end{pmatrix} \ , \ \ \Omega = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \ , \quad M = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}$$

$$(G - BG^{-1}B)_{rs} = (e^{-2\phi} + 4\chi^2 e^{2\phi}) \delta_{rs}$$
$$(BG^{-1})_{rs} = 2\chi e^{2\phi} \epsilon_{rs} \ , \qquad G^{-1}_{rs} = e^{2\phi} \delta_{rs}$$

symmetry is $SO(1,2)$ subgroup of $O(2,2)$ on M that can be compensated by $SL(2)$ transf. on (ϕ, χ)

Classical eqs for x_s and $\widetilde{x}_s$

$$(\dot{x}_s - e^{2\phi} \widetilde{x}'_s)' = 0 \ , \qquad (\dot{\widetilde{x}}_s - e^{-2\phi} x'_s)' = 0$$
$$\rightarrow \ \ \dot{x}_s - e^{2\phi} \widetilde{x}'_s = 0 \ , \qquad \dot{\widetilde{x}}_s - e^{-2\phi} x'_s = 0 \ ,$$

[dropped τ-dependent integration functions: if none at $\sigma \rightarrow \pm\infty$]

equivalent form in terms of "chiral-scalar" combinations

$$\widehat{S}(\phi, x^+, x^-) = -\int d^2\sigma \Big[\tfrac{1}{2}(\partial_a\phi)^2 + x_s^{+\prime}\partial_- x_s^+ + x_s^{-\prime}\partial_+ x_s^- + f_1(\phi)\,(x_s^{+\prime 2} + x_s^{-\prime 2}) - 2 f_2(\phi)\, x_s^{+\prime} x_s^{-\prime} \Big]$$

$$x_s = x_s^+ + x_s^- \ , \qquad \widetilde{x}_s = x_s^+ - x_s^- \ , \qquad x_s^\pm = \tfrac{1}{2}(x_s \pm \widetilde{x}_s)$$

$$f_1 = 2\sinh^2\phi \ , \qquad f_2 = \sinh 2\phi$$

duality symmetry:

$$\phi \to -\phi \ , \qquad x_s^+ \to x_s^+ \ , \qquad x_s^- \to -x_s^-$$

Lorentz symmetry on shell for given f_1, f_2 only

Quantum theory

original and doubled theory are to be quantum-equivalent
for common observables, e.g. scattering amplitudes of x fields
where $\widetilde{x}$ fields enter only through loops:
integrating out $\widetilde{x}$ gives back original action

But doubled theory has larger set of observables, e.g.
scattering amplitudes of both x and $\widetilde{x}$
with duality acting as standard symmetry
one expects this symmetry in effective action

$$\Gamma[\phi, x, \widetilde{x}] = \Gamma[-\phi, \widetilde{x}, x]\ , \quad \text{i.e.}\ \ \Gamma[\phi, x^+, x^-] = \Gamma[-\phi, x^+, -x^-]$$

need maintain symmetry at quantum level by proper choice
of quantization (regularization / path integral measure)

Γ like the classical action Lorentz-invariant on-shell,
i.e. is S-matrix Lorentz-invariant?
On-shell invariance may have two different interpretations:
(I) $\Gamma[\phi, x, \widetilde{x}]$ should be Lorentz-invariant once evaluated
on a solution of quantum e.o.m.
[quantum S-matrix generating functional $\widehat{\mathrm{S}}[\varphi_{in}] = \Gamma[\varphi(\varphi_{in})]$
$\varphi(\varphi_{in})$ is the solution of the quantum e.o.m. $\frac{\delta\Gamma}{\delta\varphi} = 0$
with "in" b.c. $\varphi = \varphi_{in} + \ldots$, $(\partial^2 + m^2)\varphi_{in} = 0$

Γ evaluated on classical solution may differ from
Γ on solution of $\frac{\delta\Gamma}{\delta\varphi} = 0$ starting with 2-loop order]

(II) quantum equations of motion following from $\Gamma[\phi, x, \widetilde{x}]$
should be Lorentz-invariant

(I) is expected given that classical e.o.m. are Lorentz-invariant
and that integrating out $\widetilde{x}$ leads back to Lorentz-invariant action;

(II) is less clear: should one expect
(some deformed version of) Lorentz invariance
to apply to full quantum equations of motion;
essentially equivalent to assumption that quantum equations
of motion derived from original Lorentz-covariant action
should admit an analog of the duality symmetry

As $\Gamma[\phi, x, \widetilde{x}]$ on "in" solution is generating functional for S-matrix
(I) is equivalent to Lorentz invariance of the S-matrix
for $\{\phi, x^+_s, x^-_s\}$ (in addition to duality invariance)

Key fact: on-shell conditions for chiral scalars are Lorentz-invariant

$$\partial_- x_s^+ = 0 \ , \qquad \partial_+ x_s^- = 0$$

Can then demonstrate Lorentz invariance of S-matrix
using that tree-level Green's functions with on-shell $x^\pm$
and off-shell ϕ's are Lorentz invariant
and that determinant of
$x^\pm$ -quadratic fluctuation operator
depending on an off-shell ϕ is Lorentz invariant

Compute 1-loop S-matrix elements explicitly

check (i) duality invariance and (ii) Lorentz invariance

simplest on-shell matrix elements are Lorentz-invariant

$$A(\phi(p_1), x_s^+(p_2), x_s^+(p_3)) = A(\phi(p_1), x_s^-(p_2), x_s^-(p_3)) = 0$$

$$A(\phi(p_1), x_s^-(p_2), x_s^+(p_3)) \sim p_{2-}p_{3+} \ln \Lambda + \text{finite}$$

Λ= UV cutoff

no 4-point scattering amplitudes with odd number of x_s^-;

for even (e.g. 4) of x_s^- get Lorentz-invariant results

$$A(x_s^+(p_1), x_s^+(p_2), x_s^+(p_3), x_s^+(p_4)) =$$
$$\int d^2l \Big[\frac{(p_1+l)_-}{(p_1+l)_+} + \frac{(p_2+l)_-}{(p_2+l)_+}\Big]\Big[\frac{(p_3-l)_-}{(p_3-l)_+} + \frac{(p_4-l)_-}{(p_4-l)_+}\Big]\frac{p_{1+}p_{2+}p_{3+}p_{4+}}{l^2(l+p_1+p_2)^2}$$

$$A(x_s^+(p_1), x_s^+(p_2), x_s^-(p_3), x_s^-(p_4)) =$$
$$\int d^2l \Big[\frac{(p_1+l)_-}{(p_1+l)_+} + \frac{(p_2+l)_-}{(p_2+l)_+}\Big]\Big[\frac{(p_3-l)_+}{(p_3-l)_-} + \frac{(p_4-l)_+}{(p_4-l)_-}\Big]\frac{p_{1+}p_{2+}p_{3-}p_{4-}}{l^2(l+p_1+p_2)^2}$$

$$A(x_s^-(p_1), x_s^-(p_2), x_s^-(p_3), x_s^-(p_4)) =$$
$$\int d^2l \Big[\frac{(p_1+l)_+}{(p_1+l)_-} + \frac{(p_2+l)_+}{(p_2+l)_-}\Big]\Big[\frac{(p_3-l)_+}{(p_3-l)_-} + \frac{(p_4-l)_+}{(p_4-l)_-}\Big]\frac{p_{1-}p_{2-}p_{3-}p_{4-}}{l^2(l+p_1+p_2)^2}$$

Example of non-linear 2d scalar action

scalar theory depending on $(\partial x)^2$ with classical duality symmetry

$$S = \int d^2\sigma \; L(x) \; , \qquad\qquad L(x) = -\sqrt{1 + (\partial_a x)^2}$$

duality symmetry of equations of motion $x \to \widetilde{x}$

with $\epsilon^{ab}\partial_b \widetilde{x} = [1 + (\partial_a x)^2]^{-1/2}\partial^a x$

duality at the quantum level?

phase-space or "doubled" theory: set momentum $p \equiv \widetilde{x}'$

– get manifestly duality-invariant action

$$\widehat{L}(x, \widetilde{x}) = \widetilde{x}'\dot{x} - \sqrt{1 + x'^2}\sqrt{1 + \widetilde{x}'^2}$$

classically equivalent but integral over $\widetilde{x}$ (momentum) is non-gaussian:

quantum theories for $L(x)$ and $L(x, \widetilde{x})$ are not a priori equivalent

(but are in leading semiclassical approximation of integral over $\widetilde{x}$)

Semiclassically $L(x)$ equivalent to

$$L(x,\phi) = -\tfrac{1}{2}\Big[G(\partial_a x)^2 + G + G^{-1}\Big]\ , \qquad\qquad G \equiv e^{-2\phi}$$

G or ϕ – auxiliary 2d field (cf. Nambu vs Polyakov action)
corresponding "doubled" action

$$\widehat{L}(x,\widetilde{x},\phi) = \widetilde{x}'\dot{x} - \tfrac{1}{2}G(1+x'^2) - \tfrac{1}{2}G^{-1}(1+\widetilde{x}'^2)$$

has manifest duality symmetry

$$x \to \widetilde{x}\ , \qquad \widetilde{x} \to x\ , \qquad\quad G \to G^{-1}$$

solving for G leads back to $L(x,\widetilde{x})$; integrating out $\widetilde{x}$ gives $L(x)$
[this and earlier action $(\partial_a\phi)^2 + e^{-2\phi}(\partial_a x)^2$ are special cases of

$$\widehat{L} = -\tfrac{1}{2}(\partial_a\phi)^2 - \tfrac{1}{2}e^{-2\phi}(\partial_a x)^2 - \cosh 2\phi$$

also invariant under $x \leftrightarrow \widetilde{x},\ \phi \to -\phi$]

Define quantum theory by path integral with action $L(x,\phi)$
or equivalent "doubled" action $L(x,\widetilde{x},\phi)$
If start with $L(x,\phi)$ and integrate out x get $\Gamma(\phi)=\Gamma(-\phi)$.
If keep background for x and evaluate effective action on shell
can show that that get duality-symmetric result.
Classical solution $x_{(0)}, G_{(0)}$

$$G_{(0)} n^a = \epsilon^{ab} \widetilde{n}_b \,, \qquad n_a \equiv \partial_a x_{(0)}$$
$$G_{(0)} = (1+n^2)^{-1/2} = (1+\widetilde{n}^2)^{1/2} = \widetilde{G}_{(0)}^{-1}$$

$\widetilde{n}_a, \widetilde{G}_{(0)}$ is classical solution for dual action

$$\widetilde{L}(\widetilde{x},G) = -\tfrac{1}{2}\Big[G^{-1}(\partial_a \widetilde{x})^2 + G + G^{-1}\Big]$$

Expand near classical solution $x = x_{(0)} + \eta, \;\; G = G_{(0)}(1+\xi)$
effective action $\Gamma_1(x_{(0)}, G_{(0)})$
inv under $n_a = \partial_a x_{(0)} \to \widetilde{n}_a = \partial_a \widetilde{x}_{(0)}, G_{(0)} \to \widetilde{G}_{(0)} = G_{(0)}^{-1}$

Special case of $n_a = \partial_a x_{(0)}$ =const:

1-loop on-shell effective action

$$\Gamma_1 = \frac{1}{2} \ln \det K \ , \qquad K = G^{-1}_{(0)} \partial^a \partial_a - G_{(0)} (n^a \partial_a)^2$$

Γ_1 is duality invariant under $x \to \widetilde{x},\ G \to G^{-1}$

Classical plus one-loop effective action for const $n_a = \partial_a x$

$$\Gamma = \int d^2\sigma \Big[- \sqrt{1 + (\partial_a x)^2} + \Lambda^2 F\big(\sqrt{1 + (\partial_a x)^2}\ \big)\Big]$$

duality symmetry: $F(y) = F(G^{-1}_{(0)}),\ G_{(0)} = (1 + n^2)^{-1/2}$=const

$$F(y) = \ln[\tfrac{1}{2}(y^{1/2} + y^{-1/2})] \ , \qquad F(y) = F(y^{-1})$$

What if start with $L(x) = -\sqrt{1+(\partial_a x)^2}$:
tree-level action plus ∂x-dep. part of 1-loop eff action

$$\Gamma(\partial x) \quad = \quad -\int d^2\sigma \sqrt{1+(\partial_a x)^2} + \hbar\,\Gamma_1(\partial x) + \mathcal{O}(\hbar^2)$$

leads to duality-covariant quantum equations of motion?
Γ_1 is same as found by starting from $L(x,G)$
depends on $\partial_a x$ only through $G_{(0)}^{-1} = \sqrt{1+(\partial_a x)^2}$
To check duality do "Legendre" transform to dual variable:
replace $\partial_a x$ by independent field strength n_a
and introduce dual $\widetilde{x}$ via Lagrange multiplier term

$$\begin{aligned}\widehat{\Gamma}(n,\partial\widetilde{x}) = -\int d^2\sigma \sqrt{1+n_a^2} + \hbar\Gamma_1(n) + \mathcal{O}(\hbar^2) \\ + \int d^2\sigma\ \epsilon^{ab} n_a \partial_b \widetilde{x}\end{aligned}$$

Solve effective equation for n_a perturbatively in $\hbar$
As Γ_1 is invariant under classical duality $\Gamma_1(\partial x) = \Gamma_1(\partial\widetilde{x})$

$\widetilde{\Gamma}(\partial\widetilde{x})$ has same form as $\Gamma(\partial x)$ up to $\mathcal{O}(\hbar^2)$ terms

$$\widetilde{\Gamma}(\partial\widetilde{x}) = -\int d^2\sigma\sqrt{1+(\partial_a\widetilde{x})^2} + \hbar\Gamma_1\Big|_{n_a\to\partial_a\widetilde{x}} + \mathcal{O}(\hbar^2)$$

If leading quantum correction to classically "self-dual" action is duality-invariant, resulting effective action is "self-dual" up to higher-order corrections (relation of original and dual fields receives loop corrections).

Higher-loop corrections Γ_n must satisfy constraints for Γ to be "self-dual" at higher order e.g. 2-loop effective action should be solution of

$$\Gamma_2(\partial\widetilde{x}) = \Gamma_2(n_{(0)}) + \int d^2\sigma \frac{1}{(1+n_{(0)}^2)^{1/2}}\Big[n_{(1)}^2 - \frac{(n_{(0)}\cdot n_{(1)})^2}{1+n_{(0)}^2}\Big]$$

It is not a priori clear why Γ_2 should obey this constraint, i.e why effective action should be invariant under modified duality

Duality in 4d vector models

"doubled" formalism:

- duality symmetry is manifest off-shell symmetry
- action has a symmetry becoming standard Lorentz inv on shell
- main features same for discrete or continuous duality
- "doubled" action duality inv $\to$ effective action duality inv
- on-shell S-matrix should have duality and Lorentz symmetry

4d vector case is very similar to 2d scalar case

start with first-order phase-space action for

$$S = -\tfrac{1}{2}\int d^4x \Big[(\partial_m\phi)^2 + e^{4\phi}(\partial_m\chi)^2 + \tfrac{1}{2}e^{-2\phi}F_{mn}^2 + \tfrac{1}{2}\chi F^*_{mn}F^{mn}\Big]$$

fixing $A_0 = 0$ and introducing $\widetilde{A}_i$: $\partial_0 A_i = \epsilon_{ijk}\partial_j\widetilde{A}_k$ $(i = 1, 2, 3)$

$$\widehat{S} = -\tfrac{1}{2}\int d^4x \Big[(\partial_a\phi)^2 + e^{4\phi}(\partial_a\chi)^2 - \widehat{L}(A, \widetilde{A}; \phi, \chi)\Big]$$

$$\widehat{L} = \mathrm{E}_i^T \widehat{\Omega} \mathrm{B}_i - \mathrm{B}_i^T M \mathrm{B}_i$$

$$\mathrm{E}_i = \partial_0 \mathrm{A}_i \ , \qquad \mathrm{B}_i = \epsilon_{ijk} \partial_j \mathrm{A}_k \ , \qquad \mathrm{A}_i = \begin{pmatrix} A_i \\ \widetilde{A}_i \end{pmatrix}$$

$$\widehat{\Omega} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \qquad M = \begin{pmatrix} e^{-2\phi} + 4\chi^2 e^{2\phi} & -2\chi e^{2\phi} \\ -2\chi e^{2\phi} & e^{2\phi} \end{pmatrix}$$

for $\chi = 0$ invariant under Z_2 duality: $\mathrm{A}_i' = \widehat{\Omega} \mathrm{A}_i \quad M' = \widehat{\Omega}^T M \widehat{\Omega}$

$$A_i' = \widetilde{A}_i, \qquad \widetilde{A}_i' = -A_i, \qquad \phi' = -\phi$$

equations of motion: $E_i - e^{2\phi} \widetilde{B}_i = 0, \ \ \widetilde{E}_i + e^{-2\phi} B_i = 0$

$\widehat{S}$ has modified Lorentz-type symmetry [Schwarz, Sen 93]

which becomes standard Lorentz symmetry on e.o.m.

as in 2d case expect S-matrix to have duality and Lorentz inv

as $\widehat{\Omega}^2 = -I$ introduce

$$\mathrm{A}_i^{\pm} \equiv A_i \pm \mathrm{i} \widetilde{A}_i \ , \qquad \bar{\mathrm{A}}_i^{+} = \mathrm{A}_i^{-}$$

which transform under duality as

$$(\mathrm{A}_i^{\pm})' = \mp \mathrm{i} \mathrm{A}_i^{\pm} \ , \qquad \phi' = -\phi$$

classical equations in terms of derivatives of $\mathrm{A}_i^{\pm}$

$$\mathrm{E}^{+} + \mathrm{i}(\mathrm{B}^{+}\cosh 2\phi - \mathrm{B}^{-}\sinh 2\phi) = 0$$
$$\mathrm{E}^{-} - \mathrm{i}(\mathrm{B}^{-}\cosh 2\phi - \mathrm{B}^{+}\sinh 2\phi) = 0$$

if $\phi = 0$ become (anti)self-duality conditions: $\mathrm{F}^{\pm}_{mn} = \pm\mathrm{i}\epsilon_{mn}{}^{kl}\mathrm{F}^{\pm}_{kl}$
$\mathrm{A}_i^{\pm}$ thus describe on shell photons of definite helicity

Lagrangian $\widehat{L}$ in terms of $\mathrm{A}_i^{\pm}$

$$\widehat{L} = \mathrm{i}(\mathrm{E}_i^{+}\mathrm{B}_i^{-} - \mathrm{E}_i^{-}\mathrm{B}_i^{+})$$
$$-2\cosh 2\phi\,\mathrm{B}_i^{+}\mathrm{B}_i^{-} - \sinh 2\phi\,(\mathrm{B}_i^{+}\mathrm{B}_i^{+} + \mathrm{B}_i^{-}\mathrm{B}_i^{-})$$

dulity $(\mathrm{A}_i^{\pm})' = \mp\mathrm{i}\mathrm{A}_i^{\pm},\ \ \phi' = -\phi$ implies:
S-matrix elements without external ϕ lines $(\mathrm{A}^{+})^{n_+}(\mathrm{A}^{-})^{n_-}$
transform by phase $i^{-n_+ + n_-}$ but must be invariant
so are nonvanishing only if $n_+ - n_- = 4k$

Similar discussion for 4d Born-Infeld theory ($\phi, \chi = 0$)

$$L(A) = -\sqrt{1 + \frac{1}{2} F^{mn} F_{mn} - \frac{1}{16} (F^{mn} F^*_{mn})^2}$$

semiclassically equivalent to action with real U, V [Rocek, AT 98]

$$L(A; U, V) = \frac{1}{2} (V F^{mn} F_{mn} - U F^{mn} F^*_{mn}) + V + V^{-1} + V^{-1} U^2$$

"doubled" action $\widehat{L}(A, \widetilde{A}; U, V)$ is quantum-equivalent to $L(A; U, V)$
"doubled" action for original BI theory from phase-space action
written in terms of derivatives of $\mathrm{A}_i^{\pm} = A_i + \mathrm{i} \widetilde{A}_i$:

$$\widehat{L}(A, \widetilde{A}) = \frac{1}{2} (E_i \widetilde{B}_i - \widetilde{E}_i B_i) - \sqrt{1 + B_i^2 + \widetilde{B}_i^2 + B_i^2 \widetilde{B}_k^2 - (B_i \widetilde{B}_i)^2}$$

$$= \frac{1}{4} \mathrm{i} (\mathrm{E}_i^+ \mathrm{B}_i^- - \mathrm{E}_i^- \mathrm{B}_i^+) - \sqrt{1 + \mathrm{B}_i^+ \mathrm{B}_i^- + \frac{1}{4} (\mathrm{B}_i^+ \mathrm{B}_i^-)^2 - \frac{1}{4} (\mathrm{B}_i^+ \mathrm{B}_i^+)(\mathrm{B}_k^- \mathrm{B}_k^-)}$$

• as in non-linear scalar theory quantum equivalence to $\widehat{L}(A, \widetilde{A}; U, V)$
only semiclassically (at tree and 1-loop level):
integral over $\widetilde{A}_i$ (or $\widetilde{B}_i$) is non-gaussian

• $\widehat{L}(A,\widetilde{A})$ invariant under same duality $A_i' = \widetilde{A}_i, \ \ \widetilde{A}_i' = -A_i$
[on-shell relation of dual and original field strengths is modified; in this sense BI e.o.m. are covariant with respect to "deformed" Maxwell duality; this distinction absent in "doubled" description]

• consequence of duality for scattering amplitudes is also same: difference between no. of positive and negative helicity photons must be $n_+ - n_- = 4k, \ k = 0, 1, 2, ...$

• S-matrix corresponding to $\widehat{L}(A,\widetilde{A})$ is actually helicity-conserving $n_+ = n_- \ \ (k = 0)$
[earlier demonstrations of this for BI theory:
Rosly, Selivanov 02; Boels, Larsen, Obers, Vonk 08]

• S-matrix is more constrained than required by duality: helicity conservation reflects special property of BI action: apart from discrete duality $\widehat{L}(A,\widetilde{A})$ has continuous symmetry:
$(\mathrm{A}_i^{\pm})' = e^{\pm \mathrm{i}\alpha} \mathrm{A}_i^{\pm}$; as a consequence $n_+ - n_- = 0$

Comments on duality in extended supergravities

$\mathcal{N} = 8$ supergravity may be obtained
from IIB 10d supergravity on a 6-torus
$O(6, 6)$ symmetry is part of $E_{7(7)}$ duality group
realized on scalars + duality rotations of 28 vectors
[Cremmer, Julia, Scherk 77]
$E_{7(7)}$ duality may be viewed as closure of 6 commuting
Z_2 subgroups of $O(6, 6)$ together with $SL(2, R)$ of IIB SG
and global $SU(8)$ symmetry acting on physical states

Discussion of realisation of duality in quantum theory
applies to full duality symmetry of $\mathcal{N} = 8$ SG;
suggests that S-matrix and thus on-shell counterterms
computed in perturbative loop expansion
should be invariant under $E_{7(7)}$ on scalars
together with duality acting on vectors on-shell

Duality on vectors manifest in the “doubled” formulation:
action is not invariant under the standard
(tangent-space) Lorentz symmetry but
on-shell effective action or S-matrix
should have this symmetry along with the duality symmetry
– as discussed above on simple examples

No need to consider deformation of duality
when looking for leading UV counterterms

Anomalies of duality?

Local non-invariant terms:

depend on quantization prescription,

may be cancelled by local counterterms

Chiral anomaly:

non-invariant non-local term in 1-loop Γ

• scalars couple to fermions via chiral currents –

possible global $SU(8) \subset E_{7(7)}$ anomaly in $\mathcal{N} = 8$ case

and $U(1) \subset SU(1,1)$ anomaly in $\mathcal{N} = 4$ case

[Girardello, Di Vecchia, Ferrara 84]

• vectors also contribute to anomaly [Marcus, 85]

as transform chirally, e.g. $(F + iG) \to e^{i\alpha}(F + iG)$, $G \sim F^*$

seen explicitly in doubled approach [Bossard,Hillmann,Nicolai 11]

• anomaly cancels in $\mathcal{N} = 8$ case but survives in $\mathcal{N} = 4$ case

Local "anomaly" of duality on curved background

anomalies depend on definition of theory:

– which symmetries are expected to be preserved

– reflected in definition of regularization/measure/counterterms

2d scalar case

Integrating $x, \widetilde{x}$ out in $\widehat{S}(\phi, x, \widetilde{x})$

expect to find $\phi \to -\phi$ symmetry in Γ

not automatic if other fields/symmetries present:

depends on quantization prescription

2d scalar x in external scalar ϕ and metric g_{ab}

$$e^{-\Gamma[\phi, g_{ab}]} = \int [dx] \exp\Big[-\tfrac{1}{2} \int d^2\sigma \sqrt{g} g^{ab} e^{-2\phi} \partial_a x \partial_b x \Big]$$

$G \equiv e^{-2\phi}$ as target space metric in direction x

2d on-shell duality: $G \to G^{-1}$, $x \to \widetilde{x}$, $G\sqrt{g}g^{ab}\partial_a x = i\epsilon^{ab}\partial_b \widetilde{x}$

$$\Gamma[\phi, g_{ab}] - \Gamma[-\phi, g_{ab}] = \frac{1}{8\pi} \int d^2\sigma \sqrt{g}\, \phi R$$

R = curvature of g_{ab} [Schwarz, AT 92]

i.e. under T-duality $G \to G^{-1}$ target-space dilaton

gets shifted by $\phi = -\frac{1}{2} \ln G$ [Buscher 88]

may interpret this as anomaly of $\phi \to -\phi$ duality

present in curved 2d background

not a genuine non-local anomaly but rather

a finite local counterterm required for preservation of

other symmetry – target space reparametrization covariance:

in 2d sigma model context target space covariance

is assumed in definition of path integral measure

breaking of $\phi \to -\phi$ symmetry means

2d duality and target space rep. inv. cannot be both manifest

4d vector case

corresponding 4d example on curved 4d background

$$e^{-\Gamma[\phi,g_{mn}]} = \int [dA] \exp\Big[- \int d^4x \sqrt{g} g^{mn} g^{pq}\, e^{-2\phi} F_{mp} F_{nq} \Big]$$

classical equations of motion invariant under
$A \to \widetilde{A}, \;\; \phi \to -\phi$ with $e^{-2\phi}(dA)^* = d\widetilde{A}$

symmetry of effective action $\Gamma[\phi, g_{mn}] = \Gamma[-\phi, g_{mn}]$?
expected from formal path integral transformation argument:
Γ should depend only on $\partial\phi$ and only even powers of ϕ
[true, e.g. for UV div. and Weyl-anomalous part of Γ, Osborn 03]
If follow same steps as in 2d example:
set of 2nd order operators appearing in duality transf.
is 4d elliptic complex (scalar, vector and 2-tensor operators)
$\frac{1}{2} \sum_{n=0}^{d} (-1)^n (n+1) \ln\det \Delta_n$
given by combination of Seeley coefficients $b_4 \sim R^2_{mnkl} + ...$

Under same assumption as in 2d case that
all measure factors are same for all operators in the complex
get direct analog of 2d "dilaton shift" wih
2d Euler density $\rightarrow$ 4d Euler density [Gilkey et al 02]

$$\Gamma[\phi, g_{mn}] - \Gamma[-\phi, g_{mn}] = -\frac{1}{32\pi^2} \int d^2x \sqrt{g}\, \phi\, R^* R^*$$

$$R^* R^* = R^2_{mnkl} - 4R^2_{mn} + R^2 = \partial_n w^n$$

local term – interpret its presence as consequence of preservation
some other symmetry at expense of duality $\phi \rightarrow -\phi$

In contrast to 2d sigma model or string path integral
in 4d vector case do not have "target space" diffs on vectors
Here may insist on preservation of the duality:
cancel this "anomaly" by local counterterm

Chiral anomaly of $SL(2)$ duality in $\mathcal{N}=4$ SG

scalar sector of $\mathcal{N}=4$ conformal or Poincare supergravity

or type IIB supergravity: $SU(1,1)/U(1)$ coset

may describe by 2 complex scalars Φ_α with spurious local $U(1)$

$\eta^{\alpha\beta}\Phi_\alpha\Phi^*_\beta = \Phi_1\Phi^*_1 - \Phi_2\Phi^*_2 = \Phi^\alpha\Phi_\alpha = 1$

Φ_α transform under global $SU(1,1)$ and local $U(1)$

$\Phi'_\alpha = e^{-i\gamma(x)}U^\beta_\alpha\Phi_\beta$

$$L = D_m\Phi^\alpha D^m\Phi_\alpha\ ,\quad D_m\Phi_\alpha = \partial_m\Phi_\alpha + ia_m\Phi_\alpha\ ,\quad a_m = i\Phi^\alpha\partial_m\Phi_\alpha$$

a_m is $SU(1,1)$ invariant, $a_m \to a_m + \partial_m\gamma$ under $U(1)$

fermions couple to scalars via a_m – have chiral weights

In physical gauge, e.g., $\Phi_1 = \Phi^*_1$

$$\Phi_1 = (1-|C|^2)^{-1/2}\ , \qquad \Phi_2 = C(1-|C|^2)^{-1/2}$$
$$a_m = \frac{i}{2}(1-|C|^2)^{-1}(C^*\partial_m C - C\partial_m C^*)$$

$SU(1,1)$ acts non-linearly on C and by gradient shift on a_m

Chiral anomaly of local $U(1)$ translates into rigid $SU(1,1)$ anomaly:

gravitational anomaly of rigid $U(1) \subset SU(1,1)$ from fermions

and self-dual tensors coupled to a_m and gravitational connection

It can be found from local $U(1)$ anomaly (not fixing gauge)

adding a local counterterm to cancel it but breaking $SU(1,1)$

if $\psi' = e^{i\gamma(x)}\psi$ then

$$\partial_m j^m = -\frac{1}{24(4\pi)^2} RR^*$$

as $j^m = \frac{\delta\Gamma}{\delta a_m}$ get corresponding term in 1-loop effective action

split $a_m = a_m^{||} + a_m^{\perp}, \quad \nabla^m a_m^{\perp} = 0$

integrating $U(1)$ anomaly find effective action for a_m and g_{mn}

$$\Gamma[a_m; g_{mn}] = \Gamma_{anom}[a_m^{||}; g_{mn}] + \Gamma_{inv}[a_m^{\perp}; g_{mn}]$$

$$\Gamma_{anom}[a^{||}, g] = k \int RR^* \nabla^{-2} \nabla^m a_m$$

$a_m^{||}$ and $a_m^{\perp}$ are separately $SU(1,1)$ invariant same applies to Γ_{anom} and Γ_{inv}

parametrize scalars as

$$\Phi_1 = \sqrt{1+r^2}\; e^{i(a-b)} \;, \qquad \Phi_2 = r\; e^{i(a+b)}$$

r, a, b are real; a transforms under local $U(1)$ by shift

all 3 fields transform under $SU(1,1)$

$$a_m = -\partial_m a + (1+2r^2)\partial_m b$$
$$L = |D_m \Phi_1|^2 - |D_m \Phi_2|^2 = -\frac{(\partial_m r)^2}{1+r^2} - 4(1+r^2) r^2 (\partial_m b)^2$$

$SU(1,1)$ and local $U(1)$ invariant – does not depend on a

$$\Gamma = \Gamma_{anom}[a, b, r; g] + \Gamma_{inv}[b, r; g] \;,$$
$$\Gamma_{anom} = -k \int RR^* a + k \int RR^* \nabla^{-2} \nabla^m \big[(1+2r^2)\; \partial_m b \,\big]$$

Γ is $SU(1,1)$ invariant, but not invariant under local $U(1)$.

But anomalous term is local – can cancell it by local counterterm.

Important difference compared to standard gauge theory where
a_m is fundamental field and anomalous term is non-local:
here variables in path integral are scalars a, b, r not a_m

$$\Gamma' = \Gamma'_{anom}[b, r; g] + \Gamma_{inv}[b, r; g] \ ,$$
$$\Gamma'_{anom} = \Gamma_{anom} + S_{c.t.} \ , \qquad S_{c.t.} = k \int RR^* \left[a + f(b, r)\right]$$

$S_{c.t.}$ restores local $U(1)$ – independence of a
$\int RR^* a = \frac{1}{4} \int RR^* \ln \frac{\Phi_1 \Phi_2}{\Phi_1^* \Phi_2^*}$: $SU(1, 1)$ non-invariant
$f(b, r)$ parametrizes ambiguity in local counterterm
but no $SU(1, 1)$ invariant from algebraic functions of a, b, r
– cannot restore $SU(1, 1)$
non-local Γ' cannot be eliminated by local c.t.;
since $S_{c.t.}$ is not $SU(1, 1)$ invariant, same is for Γ'
Illustration of general "compensator" mechanism
[Grisaru, de Wit 85]

removable $U(1)$ anomaly means:
from classical theory in two different $U(1)$ gauges
get two effective actions but differing only by local term

may then specify to particular $U(1)$ gauge from the start
and interterpret Γ_{anom} as $SU(1,1)$ anomalous part of eff. action
$a = b$ gauge: real, $\Phi_1,\ \Phi_2 = C\Phi_1, C = \frac{r}{\sqrt{1+r^2}} e^{2ib}$
$a = 0$ gauge: $\Phi_1 = \sqrt{1+r^2} e^{-ib},\ \Phi_2 = re^{ib}$

$$(\Gamma_{anom})_{a=b} = k \int RR^* \nabla^{-2} \nabla^m \big(2r^2\ \partial_m b \ \big)\ ,$$
$$(\Gamma_{anom})_{a=0} = k \int RR^* b + (\Gamma_{anom})_{a=b}$$

anomalous terms seen in graviton-graviton-scalars amplitudes

$U(1) \subset SU(1,1)$ gravitational anomaly in $\mathcal{N} = 4$ Poincare SG

anomaly does not cancel [Marcus 85]

implies $SU(1,1)$ breaking in some 1-loop amplitudes

from triangular graph with 2 graviton and $U(1)$ current legs

• graviton-scalar sector: leading is 2-graviton– 4-scalar amplitude

from $\int RR^* \partial^{-2}(C^2 \partial_m C^* \partial^m C^* - C^{*2} \partial_m C \partial^m C)$

• S-matrix elements with vectors:

$SO(4)$ invariant formulation of $\mathcal{N} = 4$ SG

$$L = -\frac{1}{4}R - \frac{1}{2}\frac{\partial^m C \partial_m C^*}{(1-|C|^2)^2}$$
$$-\frac{1}{8}\Big(\big[h_1(C)\delta_{ij}\delta_{kl} - h_2(C)\epsilon_{ijkl}\big]F^{+ik}_{mn}F^{+jlmn} + c.c.\Big)$$
$$h_1 = \frac{1+C^2}{1-C^2}\,, \qquad h_2 = \frac{C}{1-C^2}\,, \qquad i, j, , l = 1, ..., 4$$

$\partial^m a_m = \frac{i}{2}(C\partial^2 C^* - C^*\partial^2 C) + ...$ on "in" soln for C

$$\partial^2 C^* + ... = -\frac{1}{8}\epsilon_{ijkl}F^{+ik}_{mn}F^{+jlmn} + CF^{+ik}_{mn}F^{+ikmn} + ...$$

$$\partial^m a_m = -\frac{i}{16}\left(C\epsilon_{ijkl}F^{+ik}_{mn}F^{+jlmn} - C^*\epsilon_{ijkl}F^{-ik}_{mn}F^{-jlmn}\right) + ...$$

anomalous term contributes to 5-point amplitude with 2 gravitons,

one complex scalar and two chiral $SO(4)$ vectors

helicity structure consistent with global susy?

cf. $2iRR^* = (R^+)^2 - (R^-)^2$

yes, there are other parity-even terms

in full effective action – e.g. $R^*R^*\phi$

[unpublished discussions with R. Kallosh and R. Roiban]

$SU(4)$ invariant formulation of $\mathcal{N}=4$ SG:

related by local field redefs and on-shell duality rotation of vectors

effective actions equivalent modulo local counterterm

non-local part of $SU(1,1)$ anomalous term in eff action remains

relevant bosonic terms ($I=1,...,6$)

$$L = -\frac{1}{4}R - \frac{1}{2}(\partial^m\phi\partial_m\phi + e^{4\phi}\partial^m\chi\partial_m\chi)$$
$$-\frac{1}{4}e^{-2\phi}F^I_{mn}F^{Imn} - \frac{1}{2}\chi F^I_{mn}F^{*Imn}$$

ϕ and χ correspond to Poincare coordinates of Euclidean AdS_2:

$$C = C_1 + iC_2 = \frac{1-e^{-2\phi}+2i\chi}{1+e^{-2\phi}-2i\chi} \approx \phi + i\chi + ...$$
$$a_m = -\phi\partial_m\chi + \chi\partial_m\phi - (\chi^2+2\phi^2)\partial_m\chi + ...$$
$$\partial^m a_m = \frac{1}{2}\chi F^I_{mn}F^{Imn} + \frac{1}{2}\phi F^I_{mn}F^{*Imn} + ...$$

anomalous term starts contributing from 5-point amplitude:

2 gravitons, 2 vectors and 1 scalar

Comments

• such “anomalous” amplitudes are present in Bern et al construction of $\mathcal{N} = 4$ SG S-matrix by “doubling” SYM $\otimes$ YM S-matrix

– checked recently for 3-point $hh\chi$ functions [Bern, Dixon]

• relation to absence of R^4 3-loop counterterm? [cf. Bossard,Howe, Stelle, Vanhove 11, and to appear]

Conclusions

• doubled approach natural framework for understanding duality

• on-shell effective action and S-matrix in doubled formalism are Lorentz-invariant

• duality acts in simple (tree-level) way on on-shell S-matrix

• not clear if quantum effective equations remain covariant under deformed duality (but not needed for study of leading divergences)

• detailed consequences of possible duality anomalies remain to be clarified further