Some Aspects of $N = (2, 2)$ Non-Linear σ-Models

A short review

Alexander Sevrin
in collaboration with Wieland Staessens, Dimitri Terryn and Dan Thompson

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Vrije Universiteit Brussel

and

The International Solvay Institutes for Physics and Chemistry

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Motivation

- The NSR model $\equiv N = (2, 2)$ supersymmetric non-linear σ-models in d=2 gives a very explicit description of strings backgrounds with NSNS fluxes (including D-brane configurations interpolating between A & B type (AS, Staessens, Wijns '08, '09)).
- Deep connection with generalized Kähler & CY geometry (Gualtieri '03; ...); Ulf Lindström yesterday
- Doubled formalism $\leftrightarrow$ make contact with results by Hull, Hohm, Zwiebach, ...

The NSR approach

- $N \leq (1, 1)$ very simple and rather uninteresting. For type II string theory one has to study $N = (2, 2)$ non-linear σ-models in $d = 2$.
- The full off-shell description (in $N = (2, 2)$ superspace) is known $\Rightarrow$ good understanding of local, classical geometry. Elegant and natural generalization of Kähler geometry!
- Quantum calculations simplify considerably.

The NSR approach: geometric data

Geometric data: target manifold (Gates, Hull, Roček, '84)

- Target manifold $\mathcal{M}$, local coordinates x^a, $a \in \{1, \cdots, d\}$.
- Metric $g(X, Y)$.
- Closed 3-form H, $dH = 0$, locally $H = db$ and $b \simeq b + dk$.

Geometric data: worldsheet

- Worldsheet lightcone coordinates:

$$\sigma^{\mp\!\!\!\!+} = \tau + \sigma, \qquad \sigma^{=} = \tau - \sigma.$$

- Grassman coordinates: θ^+ and θ^-; derivatives:

$$D_+^2 = -\frac{i}{2}\,\partial_{\mp\!\!\!\!+}, \qquad D_-^2 = -\frac{i}{2}\,\partial_{=}\,, \qquad \{D_+, D_-\} = 0.$$

The NSR approach: the action

- The $N=(1,1)$ action in $N=(1,1)$ superspace:

$$\mathcal{S} = 4\int d^2\sigma\, d^2\theta\, \Big(D_+ x^a D_- x^b \,(g_{ab} + b_{ab})\Big)$$

- The $N=(1,1)$ action in ordinary space:

$$\begin{aligned}\mathcal{S} = 2\int d^2\sigma\, \big(&(g_{ab} + b_{ab})\partial_{\ddagger} x^a \partial_{=} x^b + \\ &2i\, g_{ab}\, \psi_+^a \nabla_{=}^{(+)} \psi_+^b + 2i\, g_{ab}\, \psi_-^a \nabla_{\ddagger}^{(-)} \psi_-^b \\ &+ R^{(-)}_{abcd} \psi_-^a \psi_-^b \psi_+^c \psi_+^d + \\ &2(F^a - i\Gamma_{(-)}{}^a{}_{cd} \psi_-^c \psi_+^d) g_{ab} (F^b - i\Gamma_{(-)}{}^b{}_{ef} \psi_-^e \psi_+^f)\big).\end{aligned}$$

- Note:

$$\Gamma^a_{\pm bc} \equiv \left\{ {a \atop bc} \right\} \pm \tfrac{1}{2} H^a{}_{bc}\,.$$

Additional supersymmetries

- Only possibility for extra susy transformations:

$$\delta x^a = \varepsilon^+ J_+{}^a{}_b(x)\, D_+ x^b + \varepsilon^- J_-{}^a{}_b(x)\, D_- x^b$$

- Closure of algebra $\Leftrightarrow$ J_+ and J_- are complex structures! So $d = 2n$.
- Note: generically **no** off-shell closure of the algebra!
- Invariance of the action $\Leftrightarrow$
  - Metric is hermitian: $g(J_\pm X, J_\pm Y) = g(X, Y)$.
  - Introduce $\omega_\pm(X, Y) = -g(X, J_\pm Y)$

$$d\omega_\pm(X, Y, Z) = \mp H(J_\pm X, J_\pm Y, J_\pm Z)$$

Solving the conditions (locally)

Can one solve:

- J_+ and J_- are complex structures!
- $g(J_\pm X, J_\pm Y) = g(X, Y)$.
- $d\omega_\pm(X, Y, Z) = \mp H(J_\pm X, J_\pm Y, J_\pm Z)$

Off-shell closure

- All off-shell non-closing terms are proportional to $[J_+, J_-]$!
  Note $\ker[J_+, J_-] = \ker(J_+ - J_-) \oplus \ker(J_+ + J_-)$.
- $T_{\mathcal{M}} = \ker(J_+ - J_-) \oplus \ker(J_+ + J_-) \oplus \mathrm{im}[J_+, J_-]g^{-1}$.
  - $\mathrm{im}[J_+, J_-]g^{-1} \Rightarrow$ semi-chiral $N = (2, 2)$ superfields.
    (Warning: type-changing!)
  - $\ker(J_+ + J_-) \Rightarrow$ twisted chiral $N = (2, 2)$ superfields.
  - $\ker(J_+ - J_-) \Rightarrow$ chiral $N = (2, 2)$ superfields.

(Lindström, Roček, von Unge, Zabzine '05) (AS, Troost '96)

$N = (2, 2)$ superspace

- Coordinates: $\sigma^{\ddagger}$, $\sigma^{=}$, θ^{+}, θ^{-}, $\hat{\theta}^{+}$, $\hat{\theta}^{-}$ (and corresponding derivatives).
- Action:

$$\mathcal{S} = 4 \int d^2\sigma \, d^2\theta \, d^2\hat{\theta} \, \mathcal{V}(X).$$

- $\mathcal{V}$ can only be some function of the scalar superfields $\Rightarrow$ constraints needed!

$N = (2, 2)$ superfields

- Simplest choice:

$$\hat{D}_{\pm} X^a = J_{\pm}{}^a{}_b(X)\, D_{\pm} X^b.$$

- But:

$$\hat{D}_{+}^2 = D_{+}^2 = -\tfrac{i}{2}\partial_{\mp}, \qquad \hat{D}_{-}^2 = D_{-}^2 = -\tfrac{i}{2}\partial_{=}$$

  and all other (anti-)commutators zero.

- Integrability conditions $\Rightarrow J_+$ and J_- are commuting complex structures!

$N = (2, 2)$ superfields

- $\Rightarrow$ they can be simultaneously diagonalized with eigenvalues $\pm i$.
- $\Rightarrow$ two cases:
  1. They have the same eigenvalue $\leftrightarrow$ chiral superfields $\leftrightarrow$ $\ker\left(J_+ - J_-\right)$. We call them z and $\bar{z}$: $\hat{D}_+ z = +i\, D_+ z$, $\hat{D}_- z = +i\, D_- z$.
  2. They have the opposite eigenvalue $\leftrightarrow$ twisted chiral superfields $\leftrightarrow$ $\ker\left(J_+ + J_-\right)$. We call them w and $\bar{w}$: $\hat{D}_+ w = +i\, D_+ w$, $\hat{D}_- w = -i\, D_- w$. (Gates, Hull, Roček, '84)
- Twisted chiral and chiral $N = (2, 2)$ superfields have the same number of components as $N = (1, 1)$ superfields $\Rightarrow$ no new auxiliary dof's are introduced.

$N = (2,2)$ superfields

- Only other possibility: chiral constraints $\Rightarrow$ these $N = (2,2)$ superfields have twice as many components compared to $N = (1,1)$, half auxiliary?
- Auxiliary fields $\Leftrightarrow$ must come in complex pairs.
- These are semi-chiral superfields, $\mathrm{im}[J_+, J_-]g^{-1}$. We call them l, $\bar{l}$, r and $\bar{r}$. (Buscher, Lindström, Roček, '88)

$$\hat{D}_+ l = i\, D_+ l, \quad \hat{D}_- r = i\, D_- r,$$

$\hat{D}_- l$ and $\hat{D}_+ r$ are auxiliary.

Note: other superfields possible

Besides semi-chiral, twisted chiral and chiral superfields other matter superfields are possible. Complex and real linear and twisted linear superfields exist as well: they are defined by constraints quadratic in the superspace derivatives. They provide a dual description to models in terms of chiral and twisted chiral superfields.

There are also gauge superfields, they are unconstrained real or complex superfields.

Action

- The action is simply
$$\mathcal{S} = 4 \int d^2\sigma \, d^2\theta \, d^2\hat{\theta} \, V(X),$$
where V is an (arbitrary) real function of the semi-chiral, twisted chiral and chiral superfields.
- Integrating over $\hat{\theta}^+$ and $\hat{\theta}^-$ and eliminating the auxiliary fields yields explicit expressions for J_+, J_-, g and b. Generically they are non-linear expressions of derivatives of the generalized Kähler potential. Elegant expressions available.

Uniqueness

- $l \to l'(l, w, z)$, $r \to r'(r, \bar{w}, z)$, $w \to w'(w)$, $z \to z'(z)$.
- The potential is determined modulo a generalized Kähler transformation
  $$V \to V + F(l, w, z) + \bar{F}(\bar{l}, \bar{w}, \bar{z}) + G(r, \bar{w}, z) + \bar{G}(\bar{r}, w, \bar{z}).$$
- Semi-chiral $\leftrightarrow$ Legendre transformations:
  $$\hat{V}(\hat{l}, \bar{\hat{l}}, \hat{r}, \bar{\hat{r}}) = V(l, \bar{l}, r, \bar{r}) - F(l, \hat{l}) - \bar{F}(\bar{l}, \bar{\hat{l}}) + G(r, \hat{r}) + \bar{G}(\bar{r}, \bar{\hat{r}})$$
- The local "mirror" transformation is
  $$V(l, \bar{l}, r, \bar{r}, w, \bar{w}, z, \bar{z}) \to -V(l, \bar{l}, \bar{r}, r, z, \bar{z}, w, \bar{w}).$$

UV properties

One-loop β-function: necessary condition for $N=(2,2)$ superconformal invariance @ quantum level. Counterterm: (Grisaru, Massar, AS, Troost, '99)

$$\mathcal{S}_{1-\text{loop}} \propto \frac{1}{\varepsilon} \int d^2\sigma\, d^2\theta\, d^2\hat{\theta}\, \ln \frac{\det\left(N_+\right)}{\det\left(N_-\right)}$$

with

$$N_+ = \begin{pmatrix} V_{l\bar{l}} & V_{lr} & V_{l\bar{w}} \\ V_{\bar{r}\bar{l}} & V_{\bar{r}r} & V_{\bar{r}\bar{w}} \\ V_{w\bar{l}} & V_{wr} & V_{w\bar{w}} \end{pmatrix}, \quad N_- = \begin{pmatrix} V_{l\bar{l}} & V_{l\bar{r}} & V_{l\bar{z}} \\ V_{r\bar{l}} & V_{r\bar{r}} & V_{r\bar{z}} \\ V_{z\bar{l}} & V_{z\bar{r}} & V_{z\bar{z}} \end{pmatrix}.$$

and vanishes $\Leftrightarrow$

$$\frac{\det\left(N_+\right)}{\det\left(N_-\right)} = \pm|f_+(l,w,z)|^2|f_-(r,\bar{w},z)|^2$$

Generalized CY $\Leftrightarrow$ $\frac{\det\left(N_+\right)}{\det\left(N_-\right)} =$ constant . E. g. (Hull, Lindström, Roček, von Unge, Zabzine '12)

Examples

Simple example: $SU(2) \times U(1) = S^3 \times S^1$.

- Parameterization:

$$g = e^{i\rho} \begin{pmatrix} \cos\psi\, e^{i\varphi_1} & \sin\psi\, e^{i\varphi_2} \\ -\sin\psi\, e^{-i\varphi_2} & \cos\psi\, e^{-i\varphi_1} \end{pmatrix},$$

  and

$$\varphi_1,\ \varphi_2,\ \rho \in \mathbb{R} \bmod 2\pi \text{ and } \psi \in [0, \pi/2]$$

Type $(1,1)$: 1 twisted chiral + 1 chiral

Superfields: (Roček, Schoutens, AS '91)

$$w = \cos\psi \, e^{-\rho - i\varphi_1}, \qquad z = \sin\psi \, e^{-\rho + i\varphi_2}$$

Generalized Kähler potential:

$$V_{\psi \neq \frac{\pi}{2}} = \int^{\frac{z\bar{z}}{w\bar{w}}} \frac{dq}{q} \ln(1+q) - \frac{1}{2} (\ln w\bar{w})^2$$

or

$$V_{\psi \neq 0} = -\int^{\frac{w\bar{w}}{z\bar{z}}} \frac{dq}{q} \ln(1+q) + \frac{1}{2} (\ln z\bar{z})^2$$

and

$$V_{\psi \neq \frac{\pi}{2}} - V_{\psi \neq 0} = -\ln\left(z\bar{z}\right) \ln\left(w\bar{w}\right)$$

Note: has in fact $N = (4,4)$ susy, can be lifted to projective superspace (see U. Lindström yesterday):

$$V \propto \int_{\mathcal{C}} \frac{d\zeta^+}{\zeta^+} \int_{\mathcal{C}'} \frac{d\zeta^-}{\zeta^-} \ln \Upsilon$$

Type $(0,0)$: 1 semi-chiral

Superfields: (Troost, AS '96)

$$l = w,\ r = \frac{\bar{w}}{z}$$

Generalized Kähler potential:

$$V_{\psi \neq 0} = \ln \frac{l}{\bar{r}} \ln \frac{\bar{l}}{r} - \int^{r\bar{r}} \frac{dq}{q} \ln\left(1+q\right)$$

or

$$V_{\psi \neq \frac{\pi}{2}} = -\ln \frac{l}{r} \ln \frac{\bar{l}}{\bar{r}} + \int^{r\bar{r}} \frac{dq}{q} \ln\left(1+q\right)$$

and

$$V_{\psi \neq \frac{\pi}{2}}(l', \bar{l}', r', \bar{r}') - \frac{1}{2}\left(\ln r'\right)^2 - \frac{1}{2}\left(\ln \bar{r}'\right)^2 =$$
$$V_{\psi \neq 0}(l, \bar{l}, r = r'^{-1}, \bar{r} = \bar{r}'^{-1}) - \ln l \ln l' - \ln \bar{l} \ln \bar{l}'$$

with

$$l' = \frac{\bar{l}}{r}, \qquad r' = \frac{1}{r}$$

- Both descriptions are T-dual, dualize along the S^1 of $S^1 \times S^3$.
- Note: type-changing occurs: at $\psi = \frac{\pi}{2}$: $J_- = -J_+$; at $\psi = 0$: $J_- = +J_+$.

Outlook

T-duality and doubled formalism

Intimate relation relation between $N=(2,2)$ σ-models and generalized CY-geometry. Hohm-Hull-Zwiebach: doubled formalism suggests an intricate generalized geometrical structure.

T-duality in $N=(2,2)$ superspace:

- Chiral $\leftrightarrow$ twisted chiral (Gates, Hull, Roček, '84)

$$V(w+\bar{w},\cdots)\leftrightarrow\hat{V}(z+\bar{z},\cdots)$$

- Chiral + twisted chiral $\leftrightarrow$ semi-chiral (Grisaru, Massar, AS, Troost, '98)

$$V(z+\bar{z},w+\bar{w},i(z-\bar{z}-w+\bar{w}),\cdots)\leftrightarrow$$
$$\hat{V}(l+\bar{l},r+\bar{r},i(l-\bar{l}-r+\bar{r}),\cdots)$$

Doubled formalism: include both original and dual fields + "chirality constraint" in doubled space.

Outlook

Simple example

- The potentials,
$$V = \tfrac{1}{2}\left(z + \bar{z}\right)^2$$
and
$$\hat{V} = -\tfrac{1}{2}\left(w + \bar{w}\right)^2$$
are T-dual.
- In the doubled space with coordinates z, $\bar{z}$, w and $\bar{w}$, the constraint,
$$w + \bar{w} = z + \bar{z}$$
holds, *i.e.* a kind of coisotropic brane is singled out, eliminating the “overdoubled” coordinates.

Outlook

- The Hull constraints ($dX \propto {}^\star d\tilde{X}$) follow:
  $$\hat{D}_\pm(w + \bar{w}) = \hat{D}_\pm(z + \bar{z}) \Rightarrow D_\pm(w - \bar{w}) = \pm D_\pm(z - \bar{z}) \,.$$

Notation

- Introduce $\mathbb{D}_\pm = \hat{D}_\pm + i\,D_\pm$, $\bar{\mathbb{D}}_\pm = \hat{D}_\pm - i\,D_\pm$.
  $\{\mathbb{D}_+, \bar{\mathbb{D}}_+\} = \partial_{\ddagger}, \quad \{\mathbb{D}_-, \bar{\mathbb{D}}_-\} = \partial_{=}$,
  all other (anti-)commutators vanish.
- Chiral field z: $\bar{\mathbb{D}}_+ z = \bar{\mathbb{D}}_- z = 0$ (also $\mathbb{D}_+ \bar{z} = \mathbb{D}_- \bar{z} = 0$).
- Twisted chiral field w: $\bar{\mathbb{D}}_+ w = \mathbb{D}_- w = 0$ (also $\mathbb{D}_+ \bar{w} = \bar{\mathbb{D}}_- \bar{w} = 0$).

Outlook

- Implies a theory of chiral bosons
  $$\mathbb{D}_\pm(w+\bar{w}) = \mathbb{D}_\pm(z+\bar{z}) \Rightarrow \mathbb{D}_+(z-w) = \mathbb{D}_-(z-\bar{w}) = 0\,.$$
  But also,
  $$\bar{\mathbb{D}}_+(z-w) = \bar{\mathbb{D}}_-(z-\bar{w}) = 0\,,$$
  implying,
  $$\partial_{\ddagger}(z-w) = \partial_{=}(z-\bar{w}) = 0\,.$$
  So a kind of "chiral" semi-chiral multiplet.
- Extend PST to $N = (2,2)$ superspace: subtle but feasible.
- Study the doubled formulation (classical & quantum).

To be continued...