Schwarzschild Black Holes

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1 - \frac{2M}{r}}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- spherically symmetric

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- spherically symmetric

$$R = \partial_\varphi \qquad \to \text{easy to see: } \partial_\varphi g_{\mu\nu} = 0$$

$$S = \cos\varphi\, \partial_\theta - \cot\theta \sin\varphi\, \partial_\varphi$$

$$T = -\sin\varphi\, \partial_\theta - \cot\theta \cos\varphi\, \partial_\varphi$$

are Killing vector fields (KVF)

→ generate isometries

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- spherically symmetric

  $[R,S]=T \quad [S,T]=R \quad [T,R]=S$

  $SO(3)$ algebra

- ∂_t is a KVF $\Leftarrow \partial_t g_{\mu\nu} = 0$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- spherically symmetric

  $[R,S]=T \quad [S,T]=R \quad [T,R]=S$

  SO(3) algebra

- ∂_t is a KVF $\Leftarrow \partial_t g_{\mu\nu}=0$

$$\partial_t \cdot \partial_t = g_{00} = -\left(1-\frac{2M}{r}\right)$$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- spherically symmetric

  $[R,S]=T \quad [S,T]=R \quad [T,R]=S$

  SO(3) algebra

- ∂_t is a KVF $\Leftarrow \partial_t g_{\mu\nu} = 0$

$$\partial_t \cdot \partial_t = g_{00} = -\left(1-\frac{2M}{r}\right) \begin{cases} <0 & \text{for } r>2M \\ =0 & r=2M \\ >0 & r<2M \end{cases}$$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- spherically symmetric

  $[R,S]=T \qquad [S,T]=R \qquad [T,R]=S$

  $SO(3)$ algebra

- ∂_t is a KVF $\Leftarrow \partial_t g_{\mu\nu}=0$

$$\partial_t \cdot \partial_t = g_{00} = -\left(1-\frac{2M}{r}\right) \begin{cases} <0 & \text{for } r>2M \quad \text{timelike} \\ =0 & r=2M \\ >0 & r<2M \quad \text{spacelike} \end{cases}$$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$\boxed{\text{stationary!}}$

- spherically symmetric

$[R,S]=T \qquad [S,T]=R \qquad [T,R]=S$

$SO(3)$ algebra

- ∂_t is a KVF $\Leftarrow \partial_t g_{\mu\nu}=0$

$$\partial_t\cdot\partial_t = g_{00} = -\left(1-\frac{2M}{r}\right) \begin{cases} <0 & \text{for } r>2M \quad \text{timelike} \\ =0 & r=2M \\ >0 & r<2M \quad \text{spacelike} \end{cases}$$

$\Rightarrow \exists$ timelike KVF for big enough r

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$\boxed{\text{Static !}}$

- ∂_t is orthogonal to the $t=$ const hypersurfaces: $\partial_t \cdot \partial_r = \partial_t \cdot \partial_\theta = \partial_t \cdot \partial_\phi = 0$

- ∂_t is a KVF $\Leftarrow \partial_t g_{\mu\nu} = 0$

$$\partial_t \cdot \partial_t = g_{00} = -\left(1-\frac{2M}{r}\right) \begin{cases} <0 & \text{for } r>2M \quad \text{timelike} \\ =0 & r=2M \\ >0 & r<2M \quad \text{spacelike} \end{cases}$$

$\Rightarrow \exists$ timelike KVF for big enough r

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$\boxed{\text{Static !}}$

- ∂_t is orthogonal to the $t=$const hypersurfaces: $\partial_t \cdot \partial_r = \partial_t \cdot \partial_\theta = \partial_t \cdot \partial_\phi = 0$

Birkhoff's Theorem:

The Schwarzschild metric is the unique spherically symmetric solution of the vacuum Einstein field equations

$$G_{\mu\nu} = 0$$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

Static !

- ∂_t is orthogonal to the $t = \text{const}$ hypersurfaces: $\partial_t \cdot \partial_r = \partial_t \cdot \partial_\theta = \partial_t \cdot \partial_\phi = 0$

Birkhoff's Theorem:

The Schwarzschild metric is the unique spherically symmetric solution of the vacuum Einstein field equations

$$G_{\mu\nu} = 0$$

$\Rightarrow$ no "time dependent" spherically symmetric solutions

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- geometry outside a spherically symmetric collapsing star is Schwarzschild

---

Birkhoff's Theorem:

The Schwarzschild metric is the unique spherically symmetric solution of the vacuum Einstein field equations

$$G_{\mu\nu} = 0$$

$\Rightarrow$ no "time dependent" spherically symmetric solutions

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- geometry outside a spherically symmetric collapsing star is Schwarzschild
- the metric inside a spherical cavity is Minkowski (flat!)

Birkhoff's Theorem:

The Schwarzschild metric is the unique spherically symmetric solution of the vacuum Einstein field equations

$$G_{\mu\nu} = 0$$

$\Rightarrow$ no "time dependent" spherically symmetric solutions

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- $r = 2M$ is strange ...
  - $g_{\mu\nu}$ is singular in (t, r, θ, ϕ)

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- $r = 2M$ is strange ...
  - $g_{\mu\nu}$ is singular in (t, r, θ, ϕ)

	$r > 2M$	$r < 2M$
$-\ \partial_t$	timelike	space like

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- $r = 2M$ is strange ...
  - $g_{\mu\nu}$ is singular in (t, r, θ, ϕ)

	$r > 2M$	$r < 2M$
– ∂_t	timelike	space like
– ∂_r	spacelike	time like

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- $r = 2M$ is strange ...
  - $g_{\mu\nu}$ is singular in (t, r, θ, ϕ)

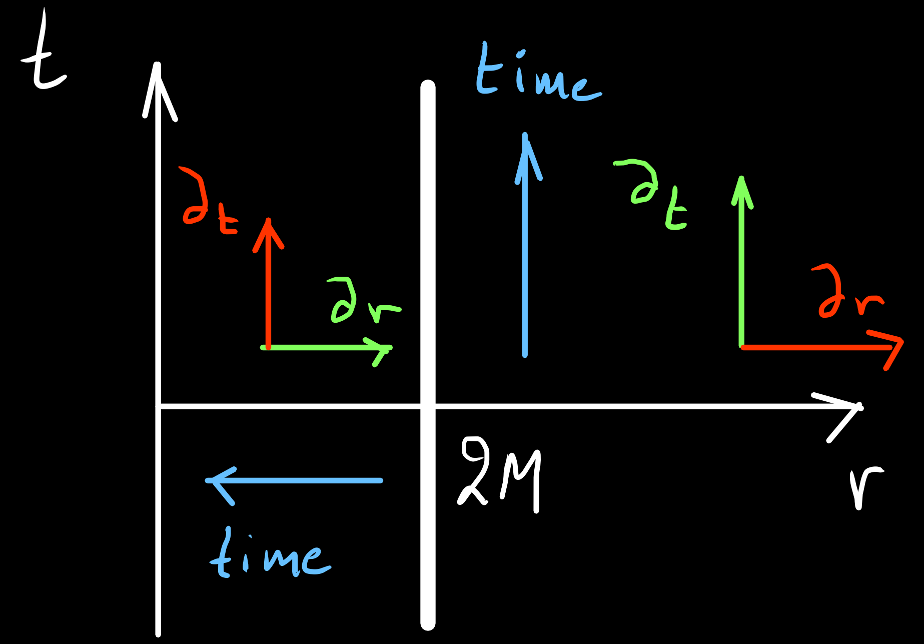
	$r > 2M$	$r < 2M$
– ∂_t	timelike	spacelike
– ∂_r	spacelike	timelike

time direction!

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- $r = 2M$ is strange ...
  - $g_{\mu\nu}$ is singular in (t, r, θ, ϕ)

	$r > 2M$	$r < 2M$
$-\ \partial_t$	timelike	space like
$-\ \partial_r$	space like	time like



$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1 - \frac{2M}{r}}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- $r = 2M$ is strange ...
  - $g_{\mu\nu}$ is singular in (t, r, θ, ϕ)

	$r > 2M$	$r < 2M$
– ∂_t	timelike	space like
– ∂_r	space like	time like

- causal geodesics starting at $r_0 > 2M$ need infinite t to approach $r = 2M$ – seem to never cross it

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- Radial null geodesics: $(d\theta = d\phi = 0)$

$$ds^2 = 0$$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- Radial null geodesics: $(d\theta = d\phi = 0)$

$$ds^2 = 0 \Rightarrow -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 = 0$$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- Radial null geodesics: $(d\theta = d\phi = 0)$

$$ds^2 = 0 \Rightarrow -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 = 0$$

$$\Rightarrow \frac{dt}{dr} = \pm\left(1-\frac{2M}{r}\right)^{-1} \quad = \text{slope in } r\text{-}t \text{ graph}$$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- Radial null geodesics: $(d\theta = d\phi = 0)$

$$ds^2 = 0 \Rightarrow -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 = 0$$

$$\Rightarrow \frac{dt}{dr} = \pm\left(1-\frac{2M}{r}\right)^{-1}$$ = slope in r-t graph

$$\Rightarrow \pm\int dt = \int\left(1-\frac{2M}{r}\right)^{-1}dr$$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- Radial null geodesics: $\quad (d\theta = d\phi = 0)$

$$ds^2 = 0 \Rightarrow -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 = 0$$

$$\Rightarrow \frac{dt}{dr} = \pm\left(1-\frac{2M}{r}\right)^{-1}$$ = slope in r-t graph

$$\Rightarrow \pm\int dt = \int\left(1-\frac{2M}{r}\right)^{-1}dr$$

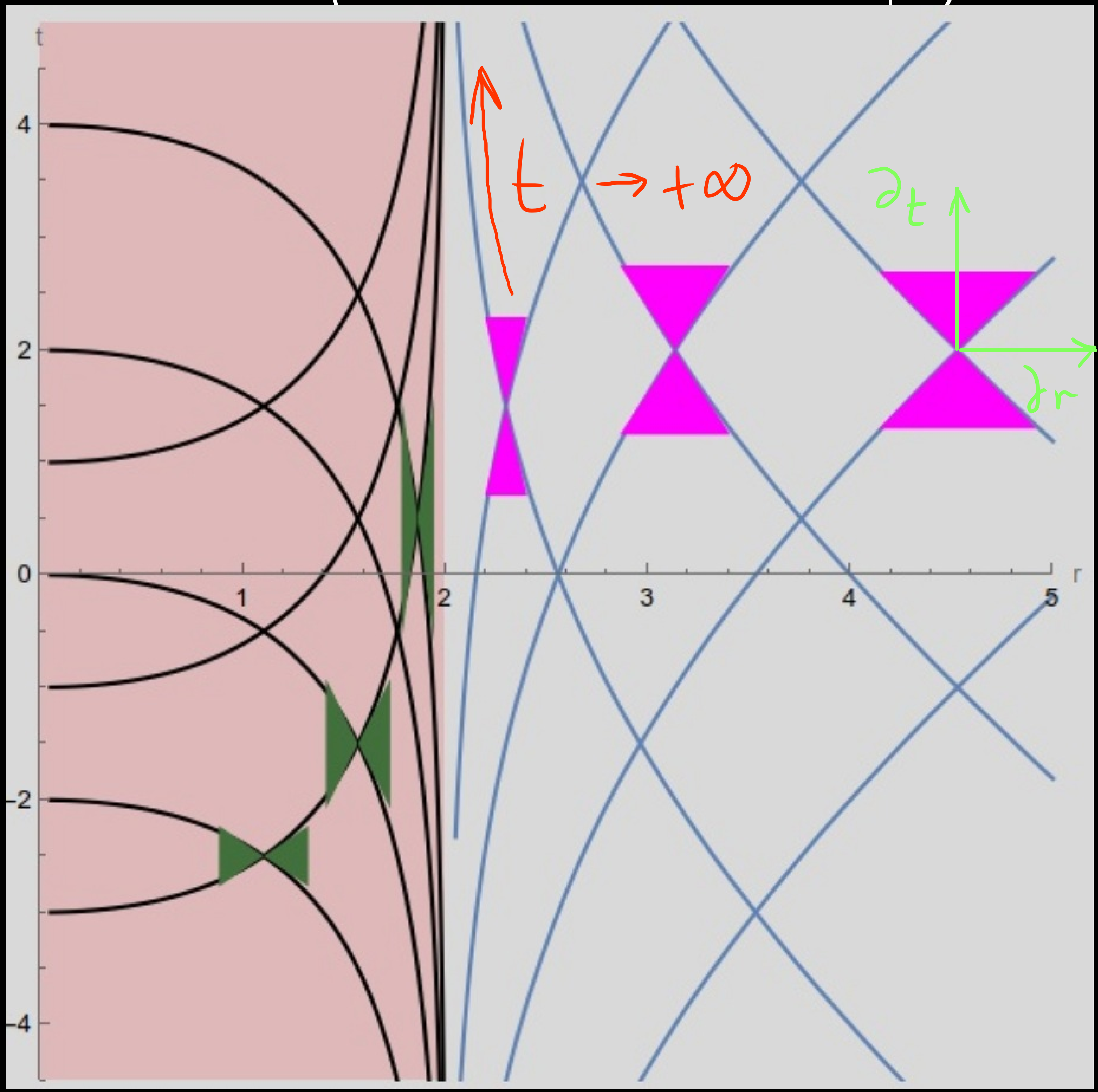
$$\Rightarrow t = \pm 2M\left[\frac{r}{2M} + \ln\left(\frac{r}{2M}-1\right)\right] + t_0$$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- Radial null geodesics:

$$\frac{dt}{dr} = \pm\left(1-\frac{2M}{r}\right)^{-1}$$

$$\frac{t}{2M} = \pm\left[\frac{r}{2M} + \ln\left(\frac{r}{2M}-1\right)\right] + c$$

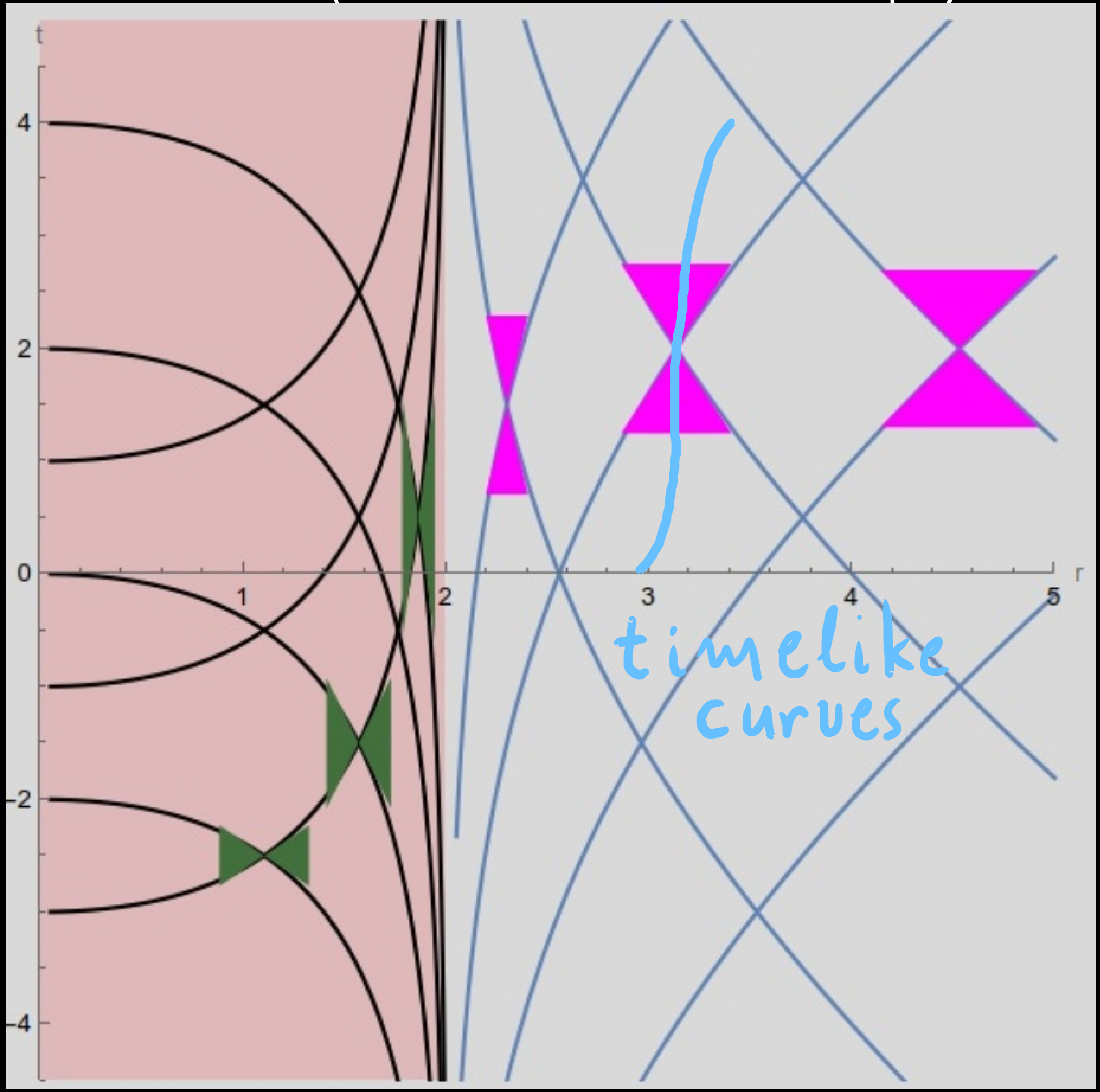


$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- Radial null geodesics:

$$\frac{dt}{dr} = \pm\left(1-\frac{2M}{r}\right)^{-1}$$

$$\frac{t}{2M} = \pm\left[\frac{r}{2M} + \ln\left(\frac{r}{2M}-1\right)\right] + c$$

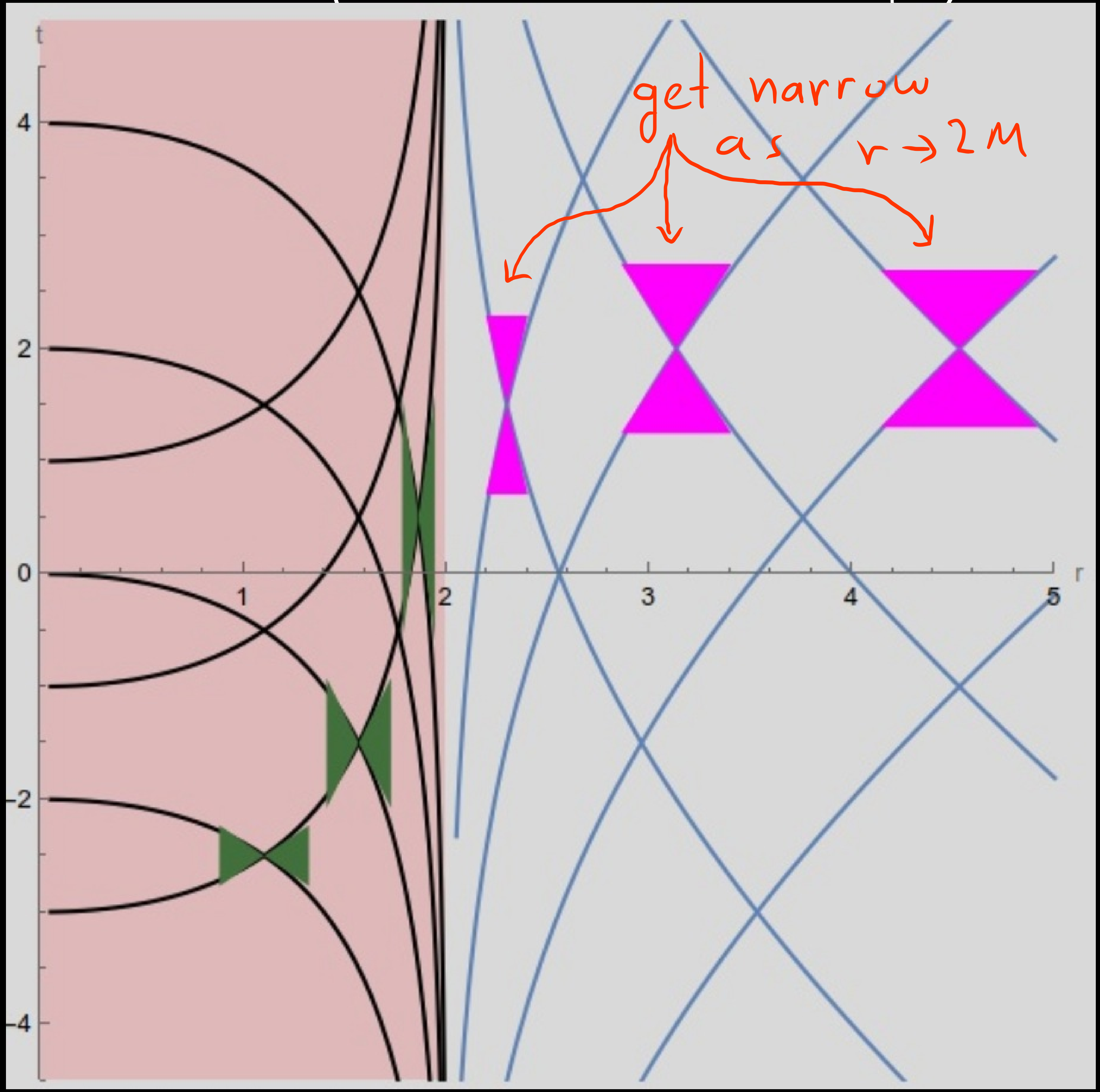


$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- Radial null geodesics:

$$\frac{dt}{dr} = \pm\left(1-\frac{2M}{r}\right)^{-1}$$

$$\frac{t}{2M} = \pm\left[\frac{r}{2M} + \ln\left(\frac{r}{2M}-1\right)\right] + c$$

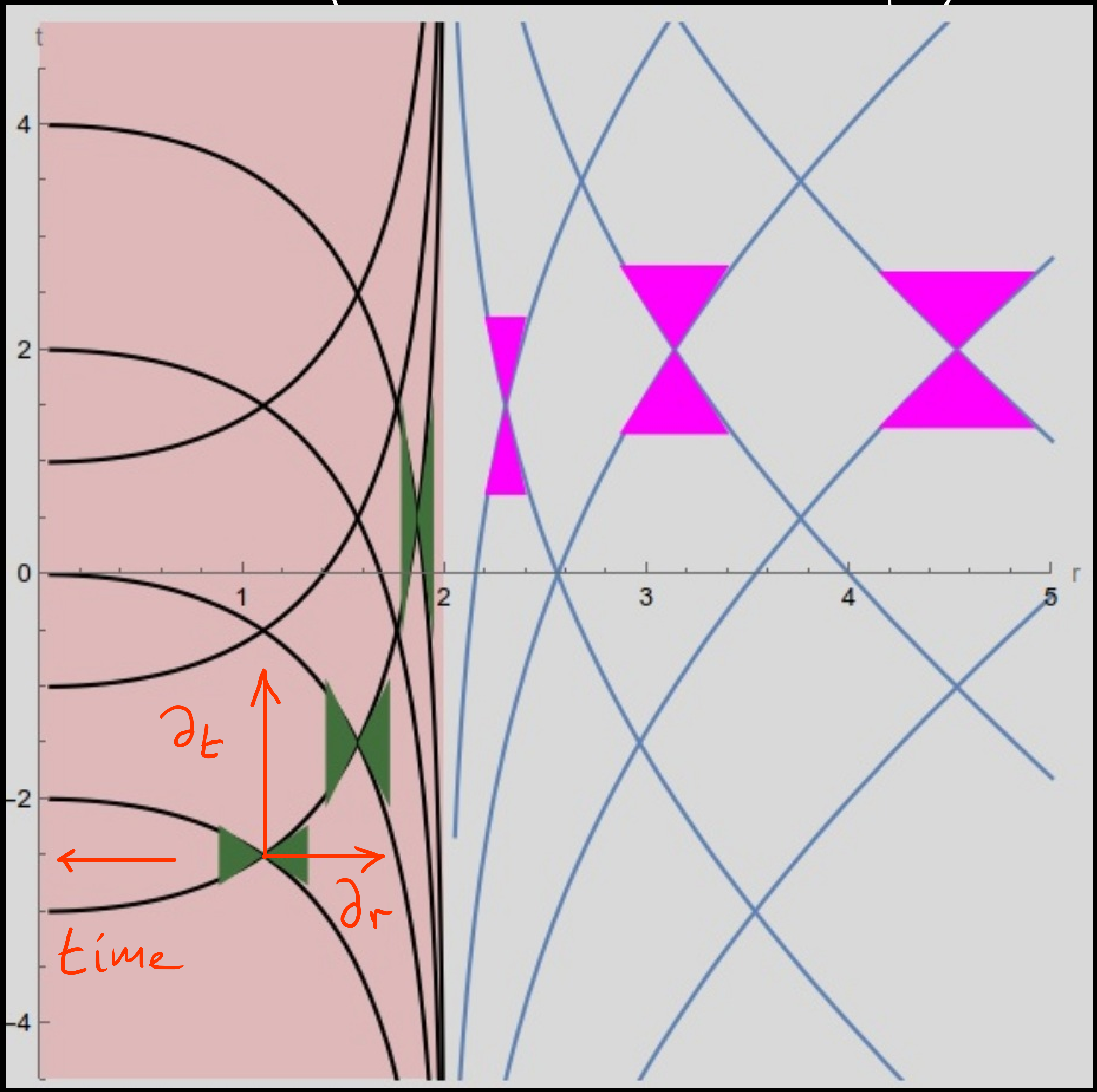


$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- Radial null geodesics:

$$\frac{dt}{dr} = \pm\left(1-\frac{2M}{r}\right)^{-1}$$

$$\frac{t}{2M} = \pm\left[\frac{r}{2M} + \ln\left(\frac{r}{2M}-1\right)\right] + c$$

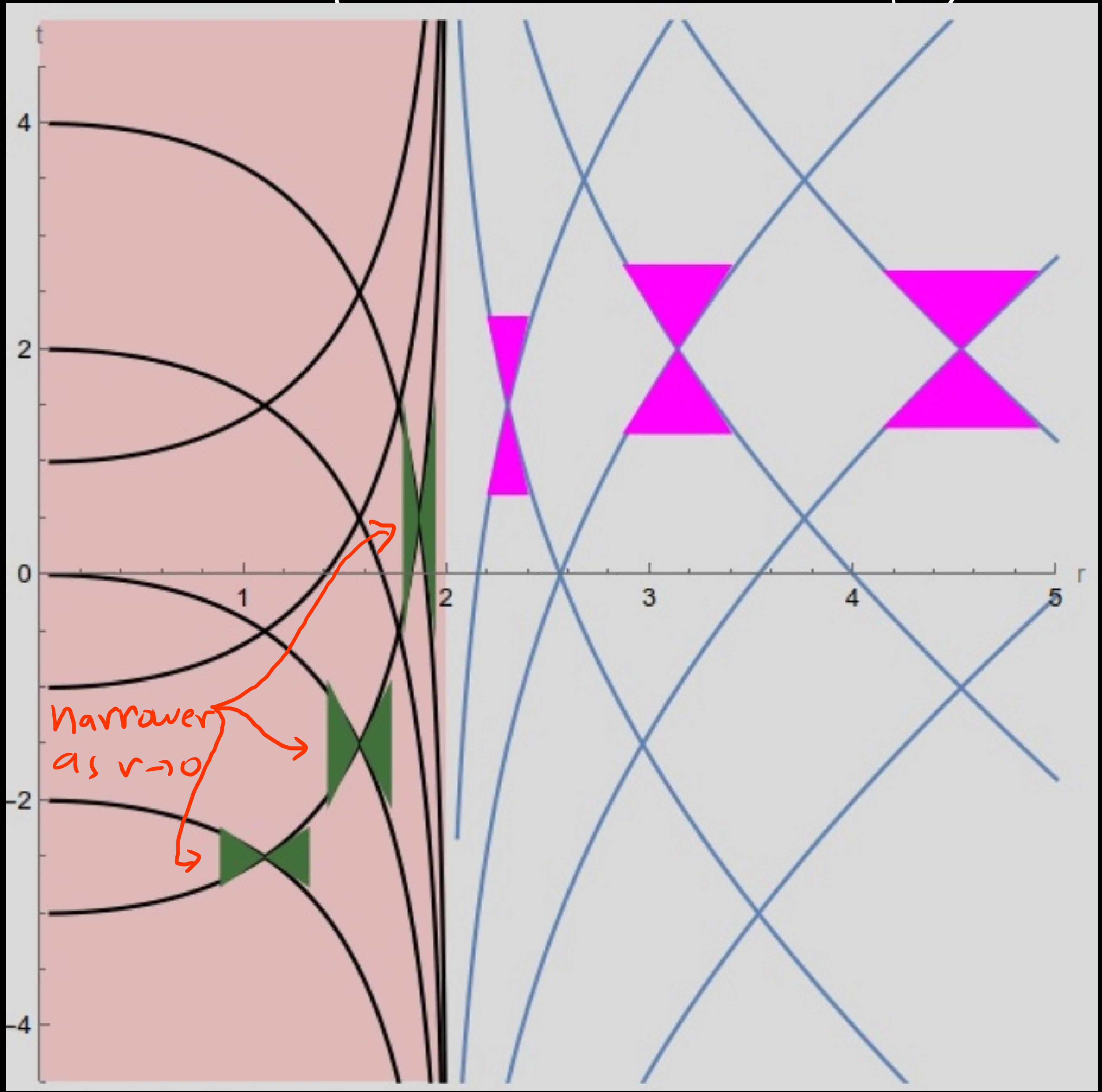


$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- Radial null geodesics:

$$\frac{dt}{dr} = \pm\left(1-\frac{2M}{r}\right)^{-1}$$

$$\frac{t}{2M} = \pm\left[\frac{r}{2M} + \ln\left(\frac{r}{2M}-1\right)\right] + c$$

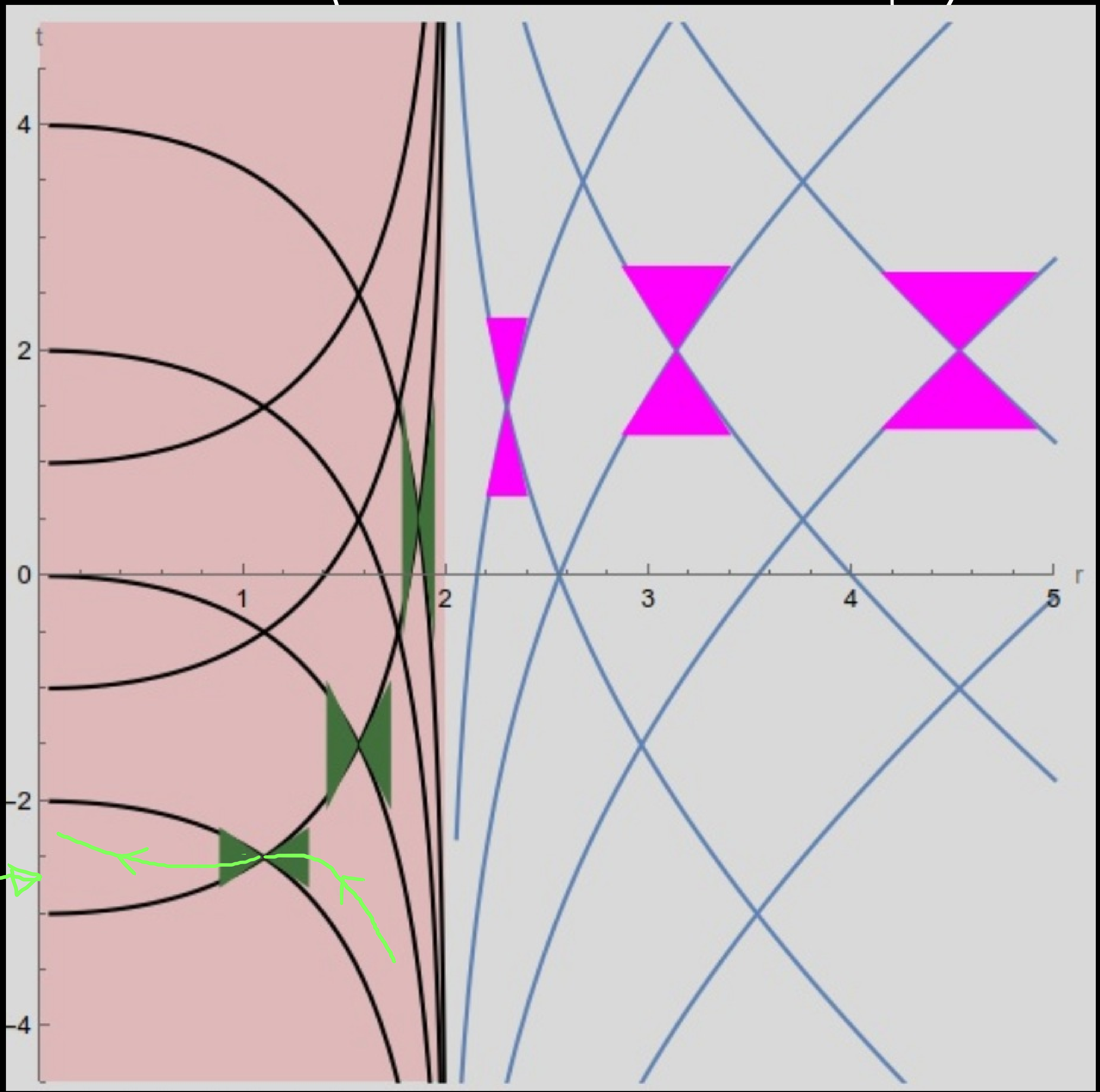


$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1 - \frac{2M}{r}}\, dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- Radial null geodesics:

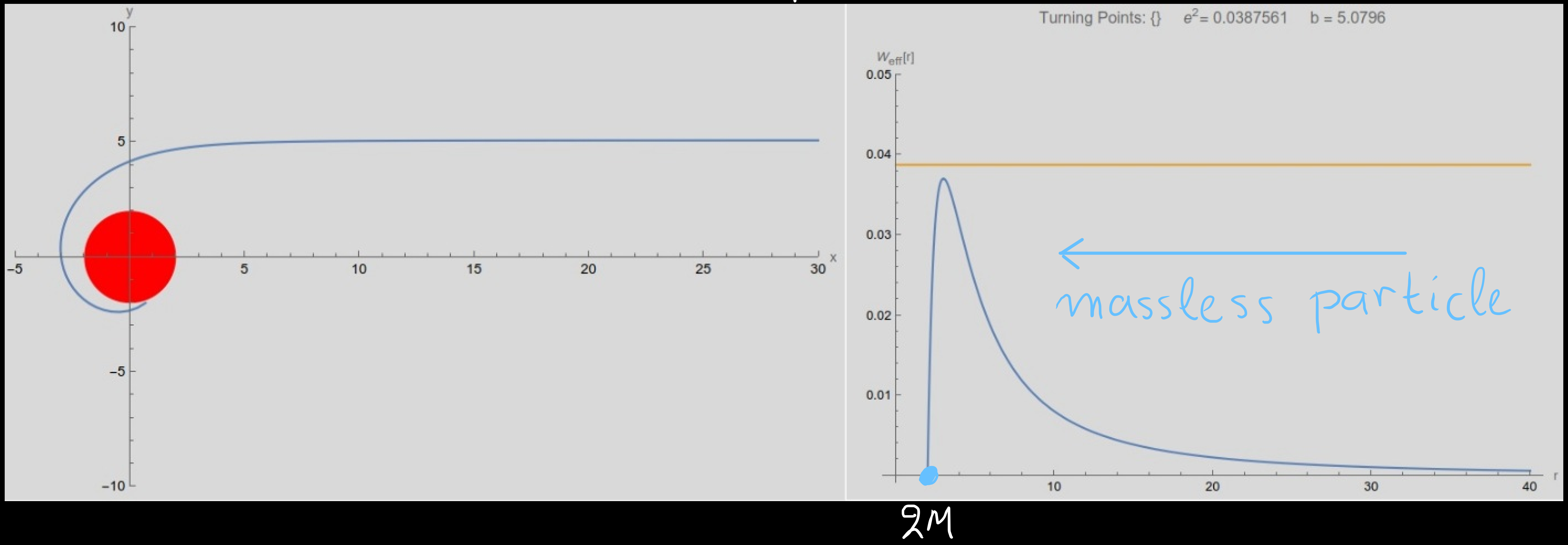
$$\frac{dt}{dr} = \pm\left(1 - \frac{2M}{r}\right)^{-1}$$

$$\frac{t}{2M} = \pm\left[\frac{r}{2M} + \ln\left(\frac{r}{2M} - 1\right)\right] + c$$



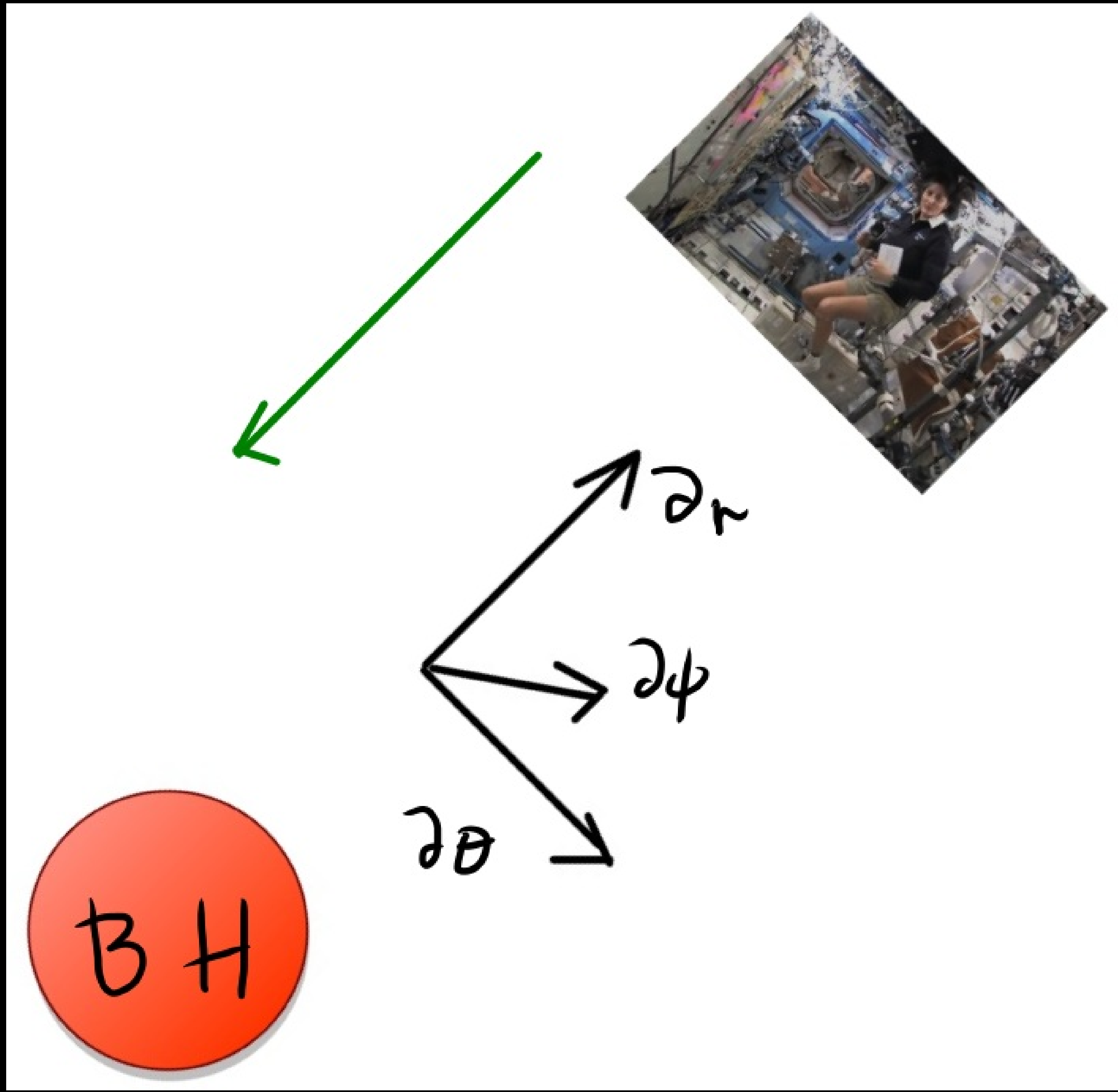
timelike curves must fall on $r = 0$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

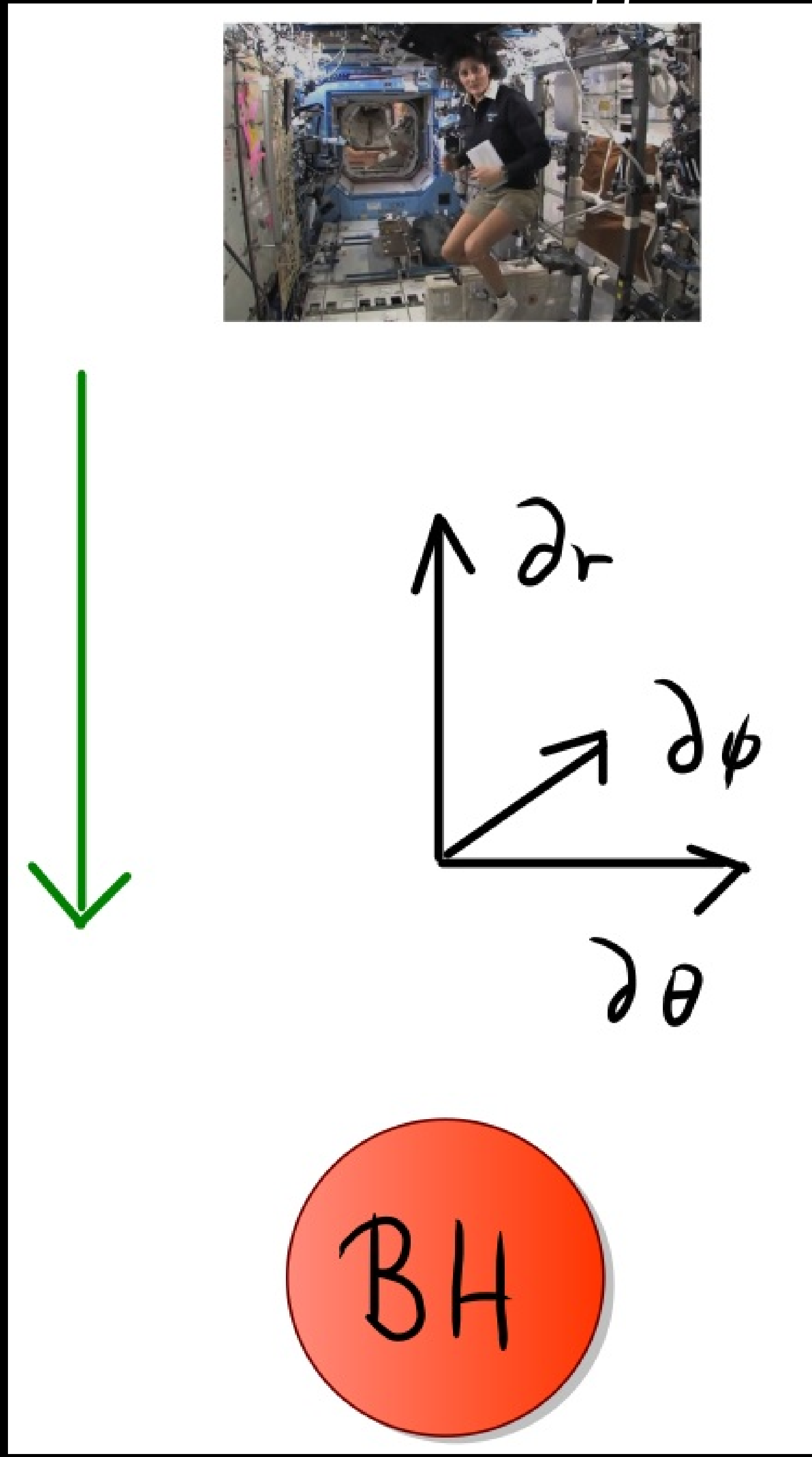


- we have seen particles (massless + massive) to get into the $r<2M$ region in finite affine parameter
  $\Rightarrow$ must be a problem of the coordinate's choice!

- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon?

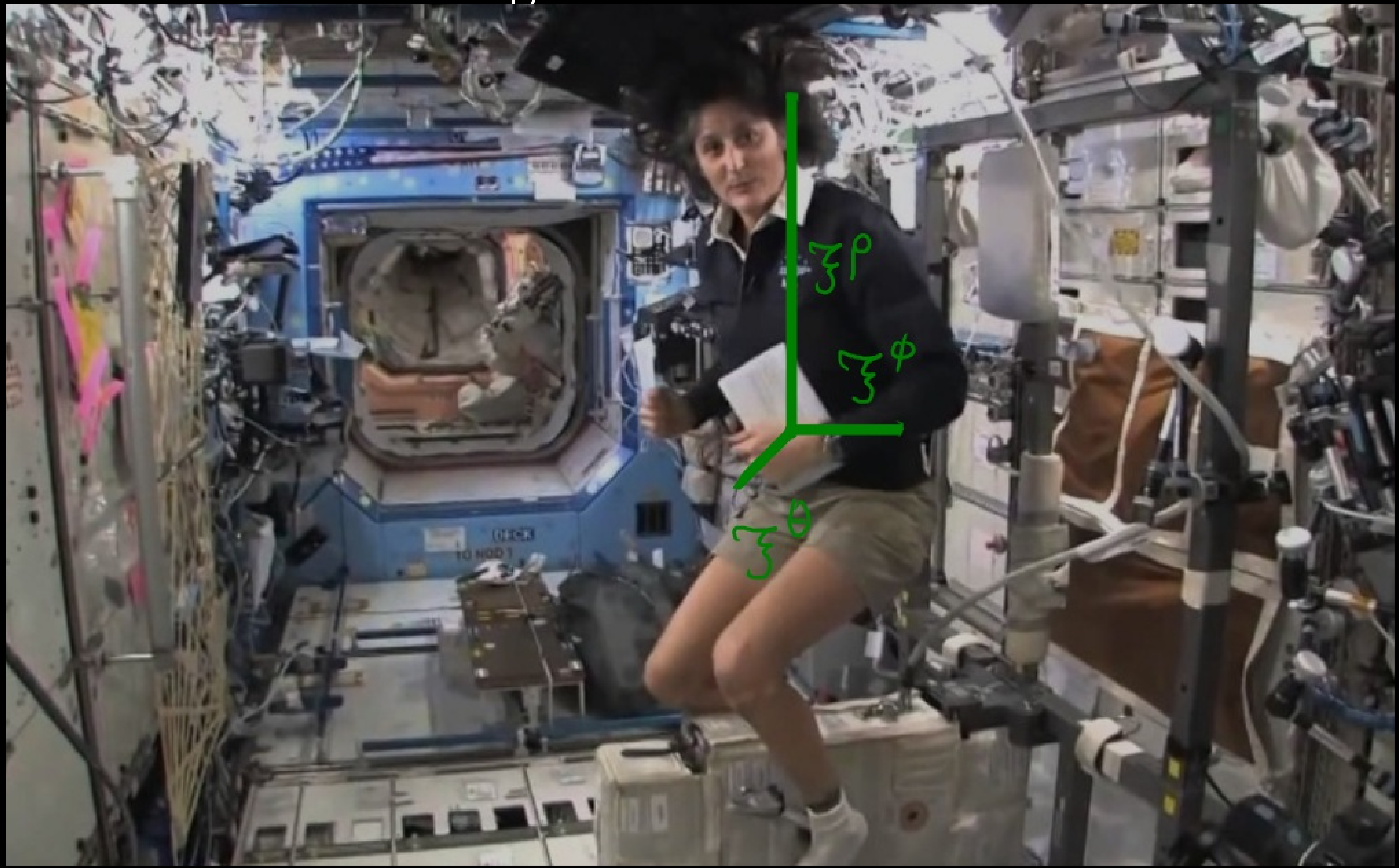


- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon?

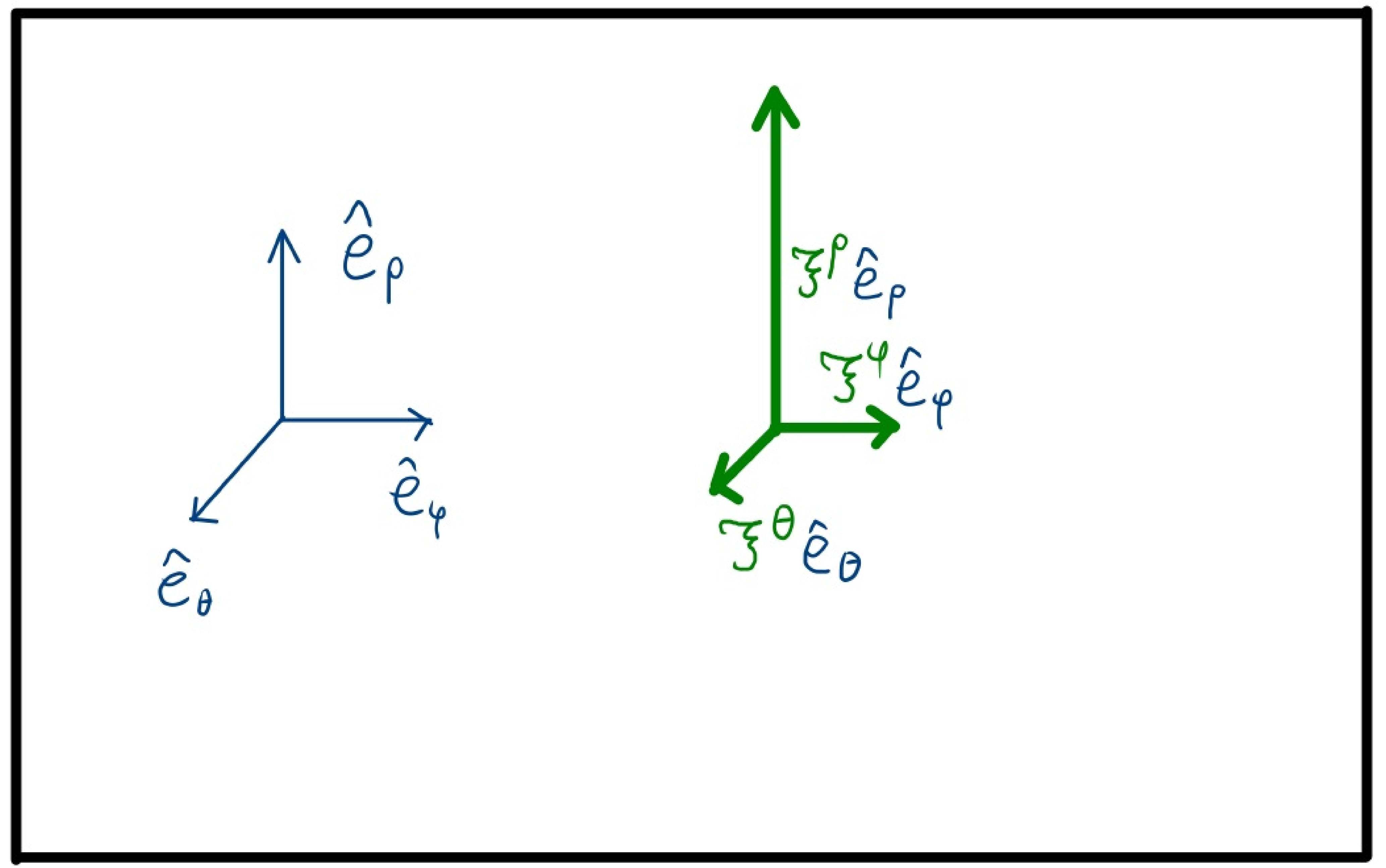


- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon?

- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon?



- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon?



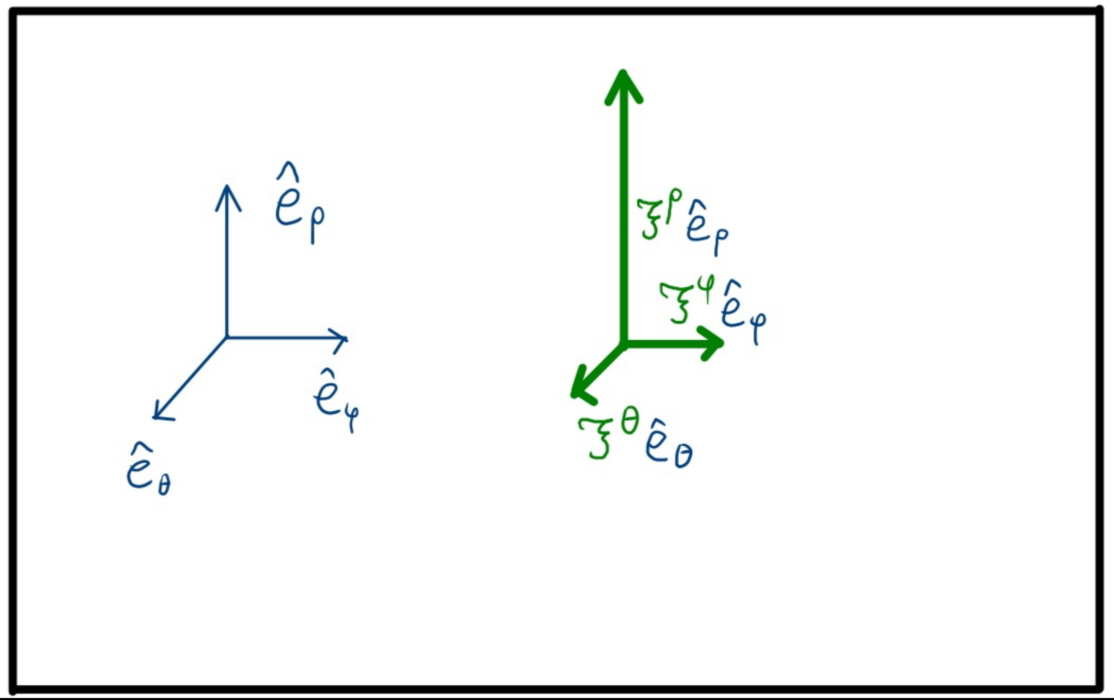
- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon?

- Get into the ship:

  - orthonormal basis

$$\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$$

$$\hat{e}_\mu \cdot \hat{e}_\nu = \eta_{\mu\nu}$$



- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon?

- Get into the ship:

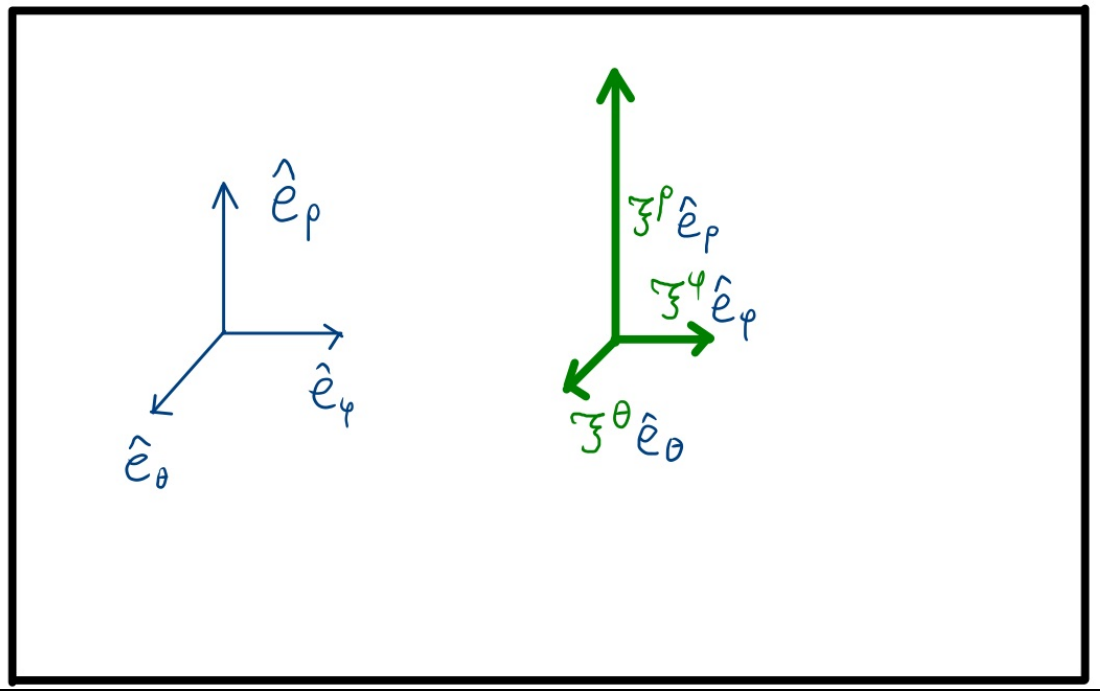
  - orthonormal basis

$$\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$$

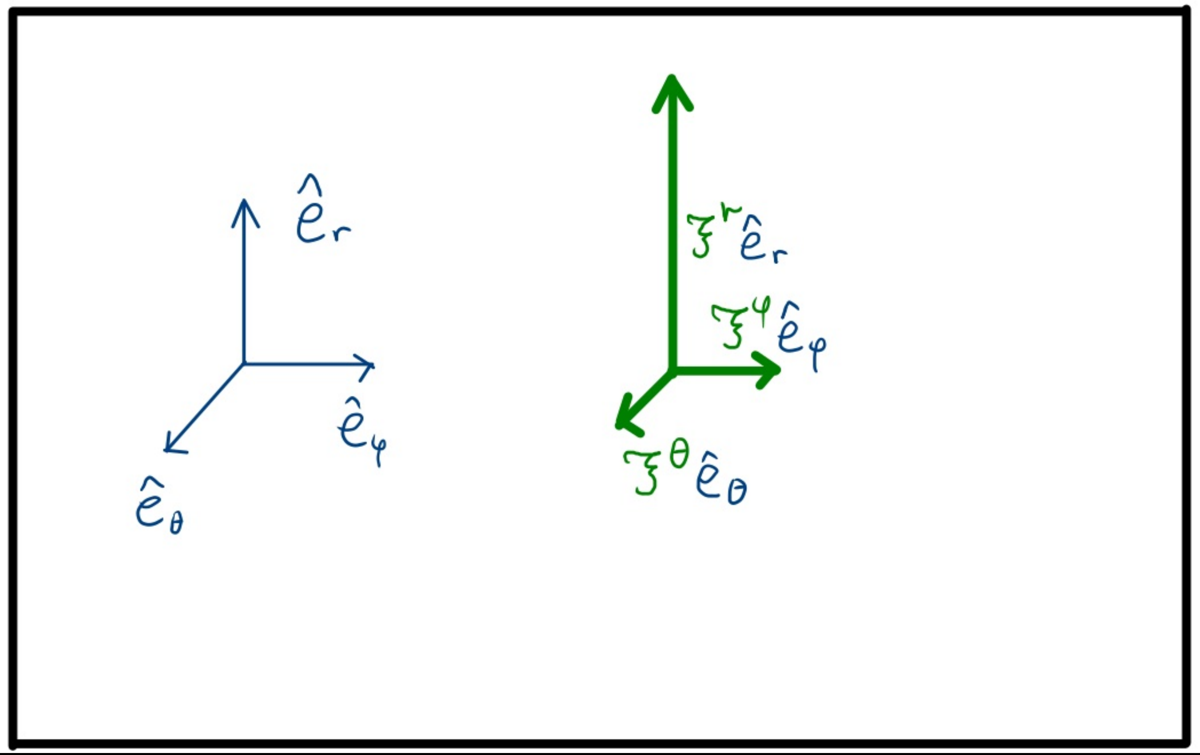
$$\hat{e}_\tau \cdot \hat{e}_\tau = \eta_{00} = -1$$

$$\hat{e}_\rho \cdot \hat{e}_\rho = \hat{e}_\theta \cdot \hat{e}_\theta = \hat{e}_\phi \cdot \hat{e}_\phi = \eta_{ii} = +1$$

$$\hat{e}_i \cdot \hat{e}_j = \hat{e}_\tau \cdot \hat{e}_j = 0$$



- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon?

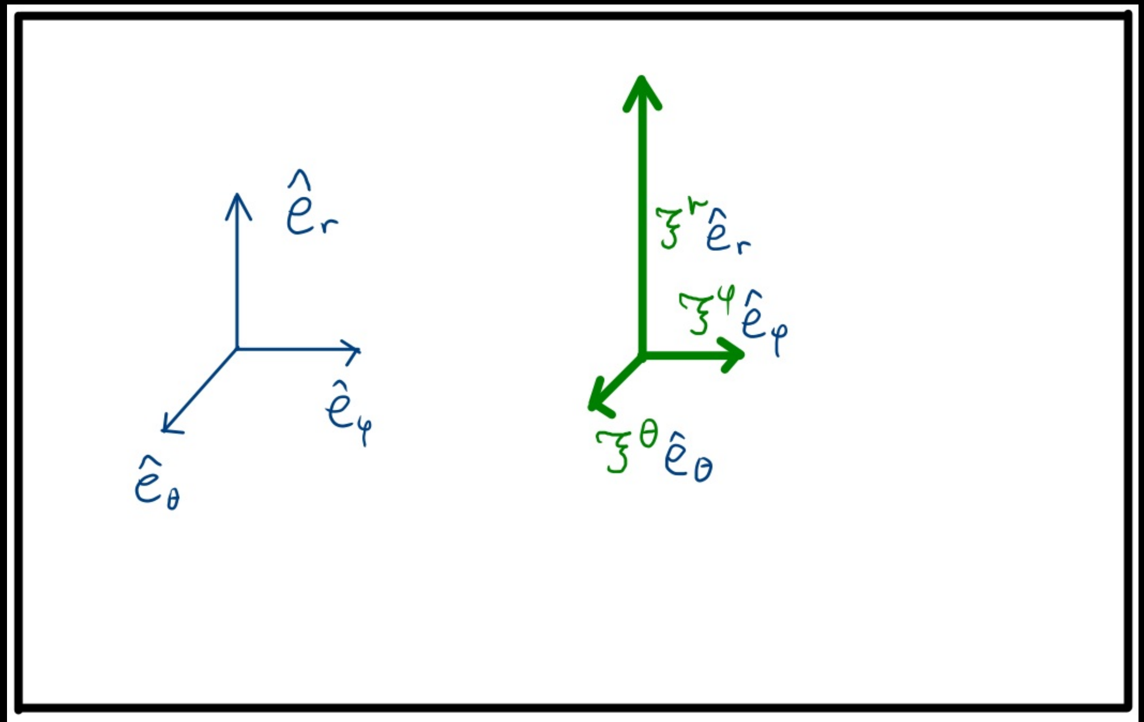


- Stationary Observer (first)
  - orthonormal basis $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi\}$
  - 4-velocity

$$u^\mu = [u^t, 0, 0, 0]$$

$$u^\mu u_\mu = g_{\mu\nu} u^\mu u^\nu = g_{00} u^t u^t = -\left(1 - \frac{2M}{r}\right)(u^t)^2$$

- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon?



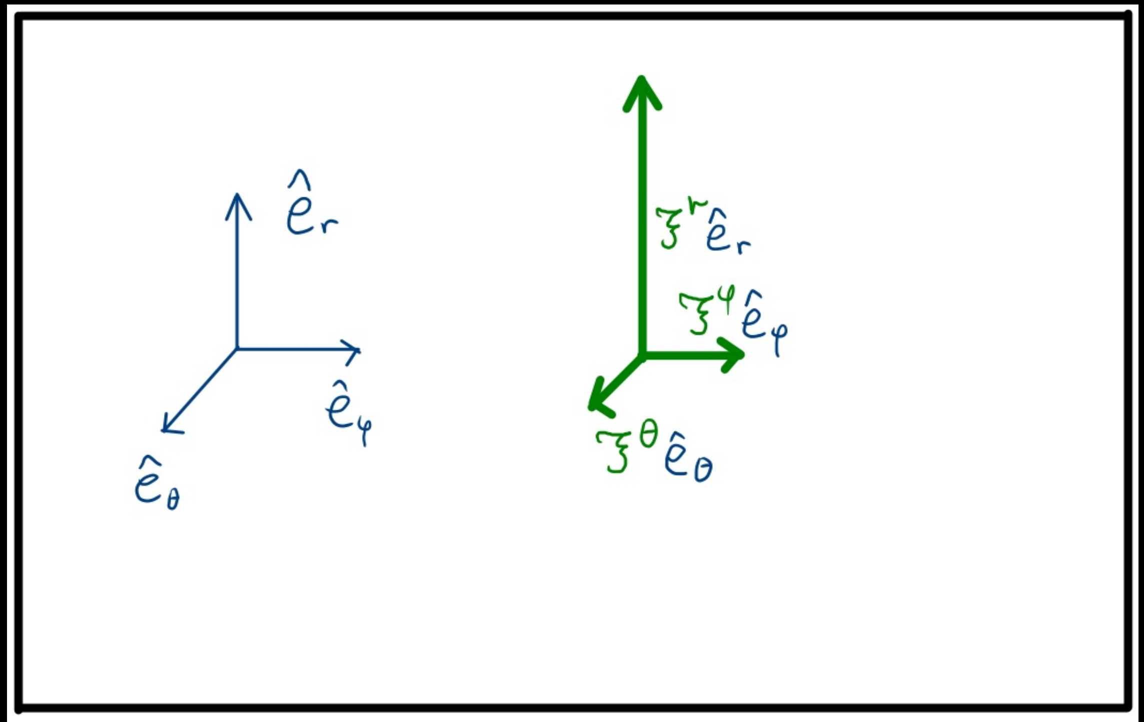
- Stationary Observer
  - orthonormal basis $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi\}$
  - 4-velocity

$$u^\mu = [u^t, 0, 0, 0]$$

$$u^\mu u_\mu = g_{\mu\nu} u^\mu u^\nu = g_{00} u^t u^t = -\left(1 - \frac{2M}{r}\right)(u^t)^2$$

$$u^\mu u_\mu = -1 \Rightarrow u^t = \left(1 - \frac{2M}{r}\right)^{-1/2}$$

- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon?



- Stationary Observer

  - orthonormal basis $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi\}$

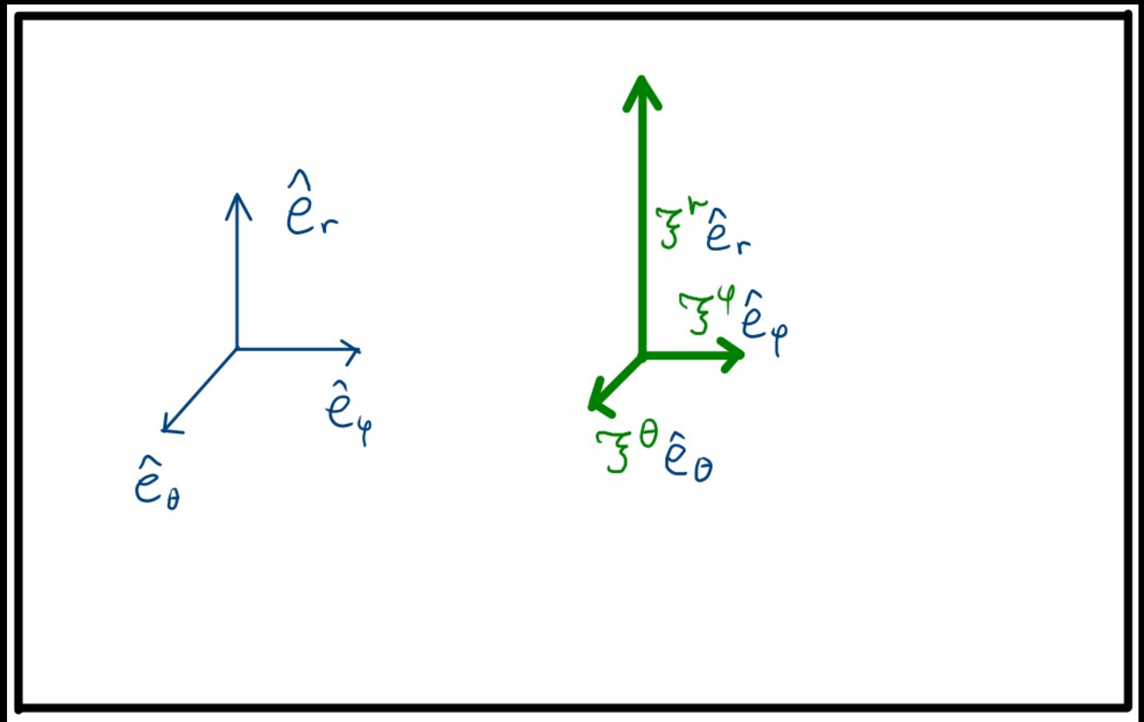
  - 4-velocity

$$u^\mu = [u^t, 0, 0, 0] = \left[\left(1-\frac{2M}{r}\right)^{-1/2}, 0, 0, 0\right] = \left(1-\frac{2M}{r}\right)^{-1/2}\partial_t$$

$$u^\mu u_\mu = g_{\mu\nu} u^\mu u^\nu = g_{00} u^t u^t = -\left(1-\frac{2M}{r}\right)(u^t)^2$$

$$u^\mu u_\mu = -1 \Rightarrow u^t = \left(1-\frac{2M}{r}\right)^{-1/2}$$

- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon?



- Stationary Observer
  - orthonormal basis $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi\}$
  - 4-velocity

$$u^\mu = [u^t, 0, 0, 0] = \left[\left(1-\frac{2M}{r}\right)^{-1/2}, 0, 0, 0\right] = \left(1-\frac{2M}{r}\right)^{-1/2} \partial_t$$

$$\hat{e}_t = u = \left(1-\frac{2M}{r}\right)^{-1/2} \partial_t = \frac{1}{\sqrt{|g_{tt}|}} \partial_t \qquad r > 2M$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

will provide the orthonormal frame of the stationary observer

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis

$$e_t = \partial_t \quad \Rightarrow \quad e_t{}^{\mu} = [1, 0, 0, 0]$$

$$e_r = \partial_r \quad \Rightarrow \quad e_r{}^{\mu} = [0, 1, 0, 0]$$

$$e_\theta = \partial_\theta \quad \Rightarrow \quad e_\theta{}^{\mu} = [0, 0, 1, 0]$$

$$e_\phi = \partial_\phi \quad \Rightarrow \quad e_\phi{}^{\mu} = [0, 0, 0, 1]$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$

- orthonormal basis: $(r > 2M)$

$$\hat{e}_\mu = \alpha\, e_\mu$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$

- orthonormal basis: $(r > 2M)$

$$\hat{e}_\mu = \alpha\, e_\mu \Rightarrow \hat{e}_\mu \cdot \hat{e}_\mu = \alpha^2\, e_\mu \cdot e_\mu$$

no summation if both indices are downstairs or upstairs

– Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $(r > 2M)$

$$\hat{e}_\mu = \alpha\, e_\mu \Rightarrow \eta_{\mu\mu} = \alpha^2\, e_\mu \cdot e_\mu = \alpha^2\, g_{\mu\mu}$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $(r > 2M)$

$$\hat{e}_\mu = \alpha\, e_\mu \Rightarrow \eta_{\mu\mu} = \alpha^2\, e_\mu \cdot e_\mu = \alpha^2\, g_{\mu\mu} \Rightarrow \alpha = |g_{\mu\mu}|^{-1/2}$$

same sign

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$

- orthonormal basis: $(r > 2M)$

$$\hat{e}_\mu = \underbrace{|g_{\mu\mu}|^{-1/2}}_{\text{no-summation!}} e_\mu$$

– Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $(r > 2M)$

$$\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} e_\mu$$

$$\hat{e}_t = [\,|g_{tt}|^{-1/2}, 0, 0, 0\,] = [\,|1 - \tfrac{2M}{r}|^{-1/2}, 0, 0, 0\,]$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $(r > 2M)$

$$\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} e_\mu$$

$$\hat{e}_t = [\,|g_{tt}|^{-1/2}, 0, 0, 0\,] = [\,|1 - \tfrac{2M}{r}|^{-1/2}, 0, 0, 0\,]$$

$$\hat{e}_r = [\,0, |g_{rr}|^{-1/2}, 0, 0\,] = [\,0, |1 - \tfrac{2M}{r}|^{1/2}, 0, 0\,]$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$

- orthonormal basis:

$$\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} e_\mu$$

$$\hat{e}_t = [\,|g_{tt}|^{-1/2}, 0, 0, 0\,] = [\,|1-\tfrac{2M}{r}|^{-1/2}, 0, 0, 0\,]$$

$$\hat{e}_r = [\,0, |g_{rr}|^{-1/2}, 0, 0\,] = [\,0, |1-\tfrac{2M}{r}|^{1/2}, 0, 0\,]$$

$$\hat{e}_\theta = [\,0, 0, |g_{\theta\theta}|^{-1/2}, 0\,] = [\,0, 0, r^{-1}, 0\,]$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{ e_t, e_r, e_\theta, e_\phi \}$

- orthonormal basis:

$$\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} e_\mu$$

$$\hat{e}_t = [\,|g_{tt}|^{-1/2}, 0, 0, 0\,] = [\,|1-\tfrac{2M}{r}|^{-1/2}, 0, 0, 0\,]$$

$$\hat{e}_r = [\,0, |g_{rr}|^{-1/2}, 0, 0\,] = [\,0, |1-\tfrac{2M}{r}|^{1/2}, 0, 0\,]$$

$$\hat{e}_\theta = [\,0, 0, |g_{\theta\theta}|^{-1/2}, 0\,] = [\,0, 0, r^{-1}, 0\,]$$

$$\hat{e}_\phi = [\,0, 0, 0, |g_{\phi\phi}|^{-1/2}\,] = [\,0, 0, 0, (r\sin\theta)^{-1}\,]$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2}\, e_\mu$

vector: $v = v^\mu e_\mu = v^{\hat{\mu}}\, \hat{e}_\mu$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2}\, e_\mu$

vector: $v = v^\mu e_\mu = v^{\hat{\mu}}\, \hat{e}_\mu \Rightarrow$

$$|g_{\mu\mu}|^{1/2} v^\mu\, \hat{e}_\mu = v^{\hat{\mu}}\, \hat{e}_\mu$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2}\, e_\mu$

vector: $v = v^\mu e_\mu = v^{\hat{\mu}}\, \hat{e}_\mu \Rightarrow$

$$|g_{\mu\mu}|^{1/2} v^\mu\, \hat{e}_\mu = v^{\hat{\mu}}\, \hat{e}_\mu \Rightarrow v^{\hat{\mu}} = |g_{\mu\mu}|^{1/2} v^\mu$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2}\, e_\mu$

$$v^{\hat{\mu}} = \underbrace{|g_{\mu\mu}|^{1/2}\, v^\mu}_{\text{no summation!}}$$

$$v^{\hat{t}} = |g_{tt}|^{1/2}\, v^t$$

$$v^{\hat{r}} = |g_{rr}|^{1/2}\, v^r$$

etc ...

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2}\, e_\mu$

$$v^{\hat{\mu}} = |g_{\mu\mu}|^{1/2}\, v^\mu$$

dual basis: $\hat{e}^\mu = |g_{\mu\mu}|^{1/2} e^\mu = |g_{\mu\mu}|^{1/2} dx^\mu$

indeed $\hat{e}^\mu(\hat{e}_\nu) = \hat{e}^\mu(|g_{\nu\nu}|^{-1/2} e_\nu) = |g_{\mu\mu}|^{1/2} |g_{\nu\nu}|^{-1/2} e^\mu(e_\nu)$

$$= |g_{\mu\mu}|^{1/2} |g_{\nu\nu}|^{-1/2} \delta^\mu{}_\nu = |g_{\mu\mu}|^{1/2} |g_{\nu\nu}|^{-1/2} \delta^\mu{}_\nu = \delta^\mu{}_\nu$$

Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} e_\mu$ $\quad\quad \hat{e}^\mu = |g_{\mu\mu}|^{1/2} e^\mu$

$$v^{\hat{\mu}} = |g_{\mu\mu}|^{1/2} v^\mu$$

one forms: $\omega = \omega_\mu e^\mu = \omega_{\hat{\mu}} \hat{e}^\mu$

_Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} e_\mu$ $\qquad \hat{e}^\mu = |g_{\mu\mu}|^{1/2} e^\mu$

$$v^{\hat{\mu}} = |g_{\mu\mu}|^{1/2} v^\mu$$

one forms: $\omega = \omega_\mu e^\mu = \omega_{\hat{\mu}} \hat{e}^\mu =$

$$|g_{\mu\mu}|^{-1/2} \omega_\mu \hat{e}^\mu = \omega_{\hat{\mu}} \hat{e}^\mu$$

_ Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\, dt^2 + g_{rr}\, dr^2 + g_{\theta\theta}\, d\theta^2 + g_{\phi\phi}\, d\phi^2$$

- coordinate basis $\{ e_t, e_r, e_\theta, e_\phi \}$
- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} e_\mu$ $\qquad \hat{e}^\mu = |g_{\mu\mu}|^{1/2} e^\mu$

$$v^{\hat{\mu}} = |g_{\mu\mu}|^{1/2} v^\mu$$

one forms: $\omega = \omega_\mu e^\mu = \omega_{\hat{\mu}} \hat{e}^\mu =$

$$|g_{\mu\mu}|^{-1/2} \omega_\mu \hat{e}^\mu = \omega_{\hat{\mu}} \hat{e}^\mu \quad \Rightarrow \quad \omega_{\hat{\mu}} = |g_{\mu\mu}|^{-1/2} \omega_\mu$$

_Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\, dt^2 + g_{rr}\, dr^2 + g_{\theta\theta}\, d\theta^2 + g_{\phi\phi}\, d\phi^2$$

- coordinate basis $\{ e_t, e_r, e_\theta, e_\phi \}$
- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} e_\mu \qquad \hat{e}^\mu = |g_{\mu\mu}|^{1/2} e^\mu$

  $v^{\hat{\mu}} = |g_{\mu\mu}|^{1/2} v^\mu \qquad \omega_{\hat{\mu}} = |g_{\mu\mu}|^{-1/2} \omega_\mu$

Tensors: $R^{\hat{\mu}}{}_{\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}|^{1/2} |g_{\nu\nu}|^{-1/2} |g_{\rho\rho}|^{-1/2} |g_{\lambda\lambda}|^{-1/2} R^\mu{}_{\nu\rho\lambda}$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} e_\mu \qquad \hat{e}^\mu = |g_{\mu\mu}|^{1/2} e^\mu$

$$v^{\hat{\mu}} = |g_{\mu\mu}|^{1/2} v^\mu \qquad \omega_{\hat{\mu}} = |g_{\mu\mu}|^{-1/2} \omega_\mu$$

Tensors: $R^{\hat{\mu}}{}_{\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}|^{1/2} |g_{\nu\nu}|^{-1/2} |g_{\rho\rho}|^{-1/2} |g_{\lambda\lambda}|^{-1/2} R^\mu{}_{\nu\rho\lambda}$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = \eta_{\hat{\mu}\hat{\sigma}} R^{\hat{\sigma}}{}_{\hat{\nu}\hat{\rho}\hat{\lambda}}$$

$$R_{0101} = R_{trtr} = -\frac{2M}{r^3}$$

$$R_{0202} = R_{t\theta t\theta} = \frac{M}{r}\left(1-\frac{2M}{r}\right)$$

$$R_{1212} = R_{r\theta r\theta} = -\frac{M}{r}\left(1-\frac{2M}{r}\right)^{-1}$$

$$R_{0303} = R_{t\phi t\phi} = \frac{M}{r}\left(1-\frac{2M}{r}\right)\sin^2\theta$$

$$R_{1313} = R_{r\phi r\phi} = -\frac{M}{r}\left(1-\frac{2M}{r}\right)^{-1}\sin^2\theta$$

$$R_{2323} = R_{\theta\phi\theta\phi} = 2M\, r\sin^2\theta$$

$$R_{0101} = R_{trtr} = -\frac{2M}{r^3}$$

$$R_{0202} = R_{t\theta t\theta} = \frac{M}{r}\left(1-\frac{2M}{r}\right)$$

$$R_{1212} = R_{r\theta r\theta} = -\frac{M}{r}\left(1-\frac{2M}{r}\right)^{-1}$$

$$R_{0303} = R_{t\phi t\phi} = \frac{M}{r}\left(1-\frac{2M}{r}\right)\sin^2\theta$$

$$R_{1313} = R_{r\phi r\phi} = -\frac{M}{r}\left(1-\frac{2M}{r}\right)^{-1}\sin^2\theta$$

$$R_{2323} = R_{\theta\phi\theta\phi} = 2M\, r \sin^2\theta$$

Notice singular behavior as $r \to 2M$!

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2}\, R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = |g_{tt}|^{-1}\, |g_{rr}|^{-1}\, R_{rtrt} = \left|1 - \frac{2M}{r}\right|^{-1} \left|1 - \frac{2M}{r}\right| \left(-\frac{2M}{r^3}\right)$$

$$= -\frac{2M}{r^3}$$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -\frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = |g_{tt}|^{-1} |g_{\theta\theta}|^{-1} R_{\theta t\theta t} = \left|1-\frac{2M}{r}\right|^{-1} (r^2)^{-1} \frac{M(r-2M)}{r^2}$$

$$= \left(1-\frac{2M}{r}\right)^{-1} \frac{1}{r^2} \frac{Mr}{r^2} \left(1-\frac{2M}{r}\right)$$

Notice $\left|1-\frac{2M}{r}\right|^{-1} = \left(1-\frac{2M}{r}\right)^{-1}$

for $r > 2M$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -\frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = |g_{tt}|^{-1} |g_{\theta\theta}|^{-1} R_{\theta t \theta t} = \left|1 - \frac{2M}{r}\right|^{-1} (r^2)^{-1} \frac{M(r-2M)}{r^2}$$

$$= \left(1 - \frac{2M}{r}\right)^{-1} \frac{1}{r^2} \frac{Mr}{r^2} \left(1 - \frac{2M}{r}\right)$$

$$= + \frac{M}{r^3}$$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -\frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = +\frac{M}{r^3}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = |g_{\theta\theta}|^{-1} |g_{\phi\phi}|^{-1} R_{\theta\phi\theta\phi} = r^{-2}\, r^{-2} \sin^{-2}\theta \; 2Mr \sin^2\theta$$

$$= +\frac{2M}{r^3}$$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = | g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda} |^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -\frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = +\frac{M}{r^3}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = +\frac{2M}{r^3}$$

$$R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = | g_{tt} |^{-1} | g_{\phi\phi} |^{-1} R_{\phi t \phi t} = \left(1 - \frac{2M}{r}\right)^{-1} r^{-2} \sin^{-2}\theta \, \frac{M(r-2M)}{r^2} \sin^2\theta$$

$$= +\frac{M}{r^3}$$

$$R_{\hat\mu\hat\nu\hat\rho\hat\lambda} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat r \hat t \hat r \hat t} = -\frac{2M}{r^3}$$

$$R_{\hat\theta\hat t\hat\theta\hat t} = +\frac{M}{r^3}$$

$$R_{\hat\theta\hat\phi\hat\theta\hat\phi} = +\frac{2M}{r^3}$$

$$R_{\hat\phi\hat t\hat\phi\hat t} = +\frac{M}{r^3}$$

$$R_{\hat r\hat\theta\hat r\hat\theta} = |g_{rr}|^{-1} |g_{\theta\theta}|^{-1} R_{r\theta r\theta} = \left(1 - \frac{2M}{r}\right) r^{-2} \frac{M}{2M-r} = -\frac{M}{r^3}$$

$$R_{\hat\mu\hat\nu\hat\rho\hat\lambda} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat r\hat t\hat r\hat t} = -\frac{2M}{r^3}$$

$$R_{\hat\theta\hat t\hat\theta\hat t} = +\frac{M}{r^3}$$

$$R_{\hat\theta\hat\phi\hat\theta\hat\phi} = +\frac{2M}{r^3}$$

$$R_{\hat\phi\hat t\hat\phi\hat t} = +\frac{M}{r^3}$$

$$R_{\hat r\hat\theta\hat r\hat\theta} = -\frac{M}{r^3}$$

$$R_{\hat r\hat\phi\hat r\hat\phi} = |g_{rr}|^{-1}|g_{\phi\phi}|^{-1} R_{r\phi r\phi} = \left(1-\frac{2M}{r}\right) r^{-2}\sin^{-2}\theta\, \frac{M\sin^2\theta}{2M-r} = -\frac{M}{r^3}$$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -\frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = +\frac{M}{r^3}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = +\frac{2M}{r^3}$$

$$R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = +\frac{M}{r^3}$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = -\frac{M}{r^3}$$

$$R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = -\frac{M}{r^3}$$

$$R_{\hat\mu\hat\nu\hat\rho\hat\lambda} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat r \hat t \hat r \hat t} = -\frac{2M}{r^3}$$

$$R_{\hat\theta\hat t\hat\theta\hat t} = +\frac{M}{r^3}$$

$$R_{\hat\theta\hat\phi\hat\theta\hat\phi} = +\frac{2M}{r^3}$$

$$R_{\hat\phi\hat t\hat\phi\hat t} = +\frac{M}{r^3}$$

$$R_{\hat r\hat\theta\hat r\hat\theta} = -\frac{M}{r^3}$$

$$R_{\hat r\hat\phi\hat r\hat\phi} = -\frac{M}{r^3}$$

For $r < 2M$ $\quad e_t = \partial_t$ spacelike

$e_r = \partial_r$ timelike

so

$$\hat e_t = -|g_{rr}|^{-1/2} e_r$$

$$\hat e_r = |g_{tt}|^{-1/2} e_t \quad ,$$

and , e.g.

$$R_{\hat\theta\hat t\hat\theta\hat t} = |g_{\theta\theta}|^{-1} |g_{rr}|^{-1} R_{\theta r\theta r} \;!$$

<u>Exercise:</u> compute $R_{\hat\mu\hat\nu\hat\rho\hat\lambda}$, show they are the same!

$$R_{\hat\mu\hat\nu\hat\rho\hat\lambda} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat r\hat t\hat r\hat t} = -\frac{2M}{r^3}$$

$$R_{\hat\theta\hat t\hat\theta\hat t} = +\frac{M}{r^3}$$

$$R_{\hat\theta\hat\phi\hat\theta\hat\phi} = +\frac{2M}{r^3}$$

$$R_{\hat\phi\hat t\hat\phi\hat t} = +\frac{M}{r^3}$$

$$R_{\hat r\hat\theta\hat r\hat\theta} = -\frac{M}{r^3}$$

$$R_{\hat r\hat\phi\hat r\hat\phi} = -\frac{M}{r^3}$$

For $r < 2M$: $e_t = \partial_t$ spacelike

$e_r = \partial_r$ timelike

so

$$\hat e_t = -|g_{rr}|^{-1/2}\, e_r$$

time flows in $-e_r$ direction

$$\hat e_r = |g_{tt}|^{-1/2}\, e_t \;,$$

and, e.g.

2(−) signs = (+)

$$R_{\hat\theta\hat t\hat\theta\hat t} = |g_{\theta\theta}|^{-1}\, |g_{rr}|^{-1}\, R_{\theta r\theta r} \;!$$

Exercise: compute $R_{\hat\mu\hat\nu\hat\rho\hat\lambda}$, show they are the same!

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2}\, R_{\mu\nu\rho\lambda}$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -\frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = +\frac{M}{r^3}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = +\frac{2M}{r^3}$$

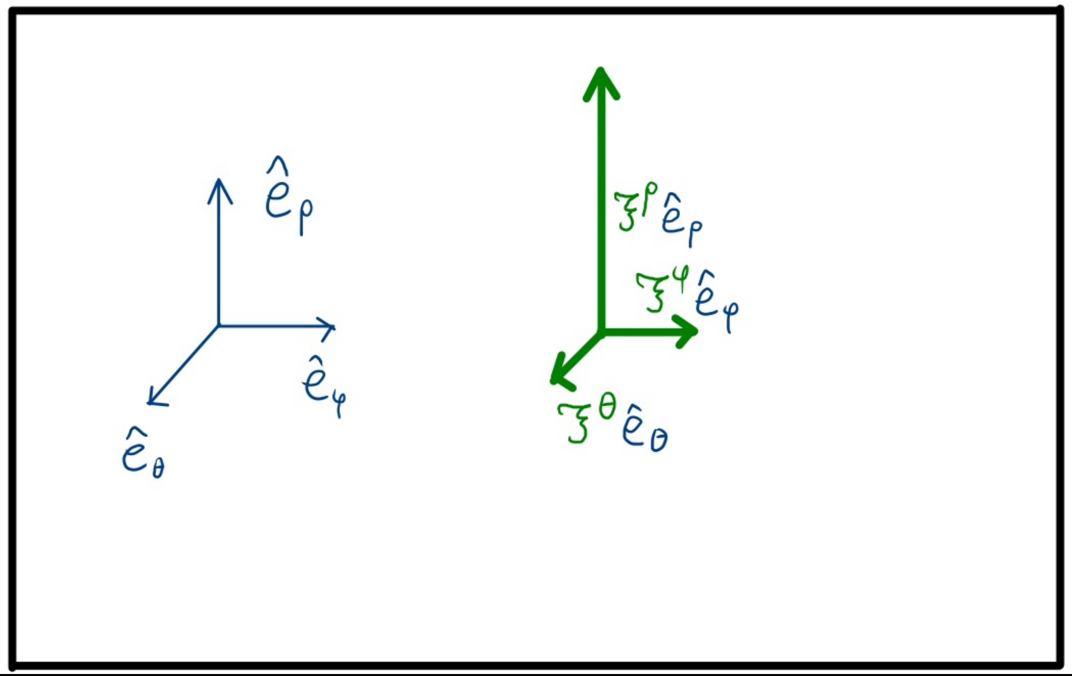
$$R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = +\frac{M}{r^3}$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = -\frac{M}{r^3}$$

$$R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = -\frac{M}{r^3}$$

There is no singular behavior of $R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}}$ at $r = 2M$!

- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon? Misner et al §32.6 p 860



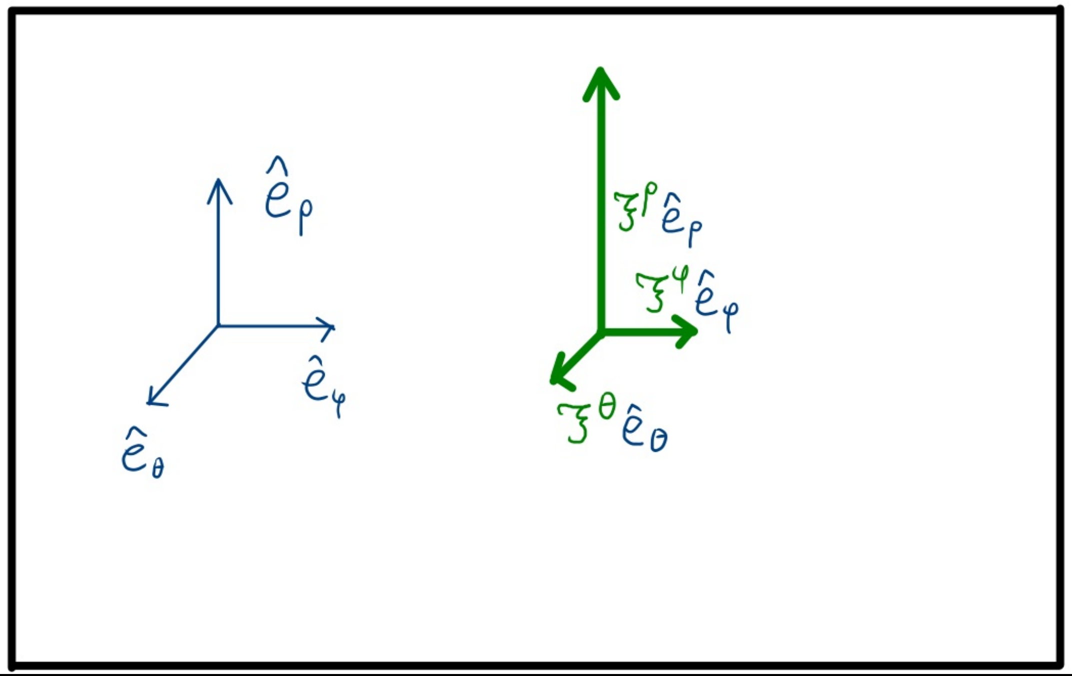
- Back to the ship:
  - orthonormal basis $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\varphi\}$
  - 4-velocity $\quad u = \hat{e}_\tau = [1,0,0,0]$

↳ not the same as stationary observer.

Now

$$u = u^t \partial_t + \underbrace{u^r}_{\text{nonzero}} \partial_r$$

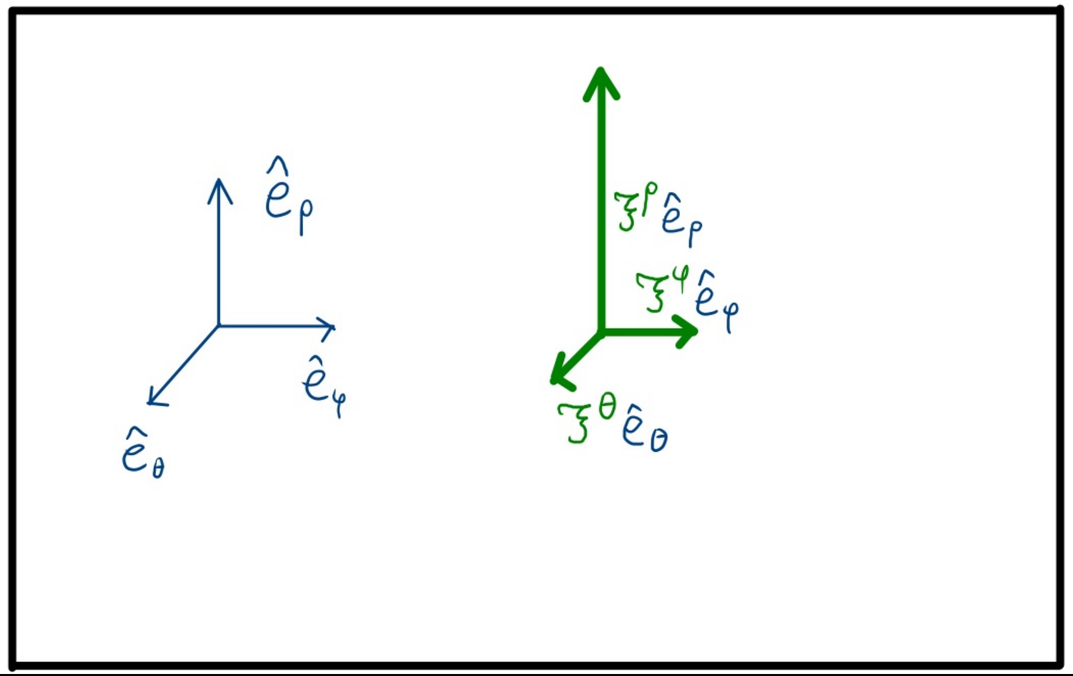
- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon? Misner et al §32.6 p 860



- Back to the ship:
  - orthonormal basis $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\varphi\}$
  - 4-velocity $u = \hat{e}_\tau = [1,0,0,0]$

  All particles momentarily at rest in the ship's frame have the same 4 velocity

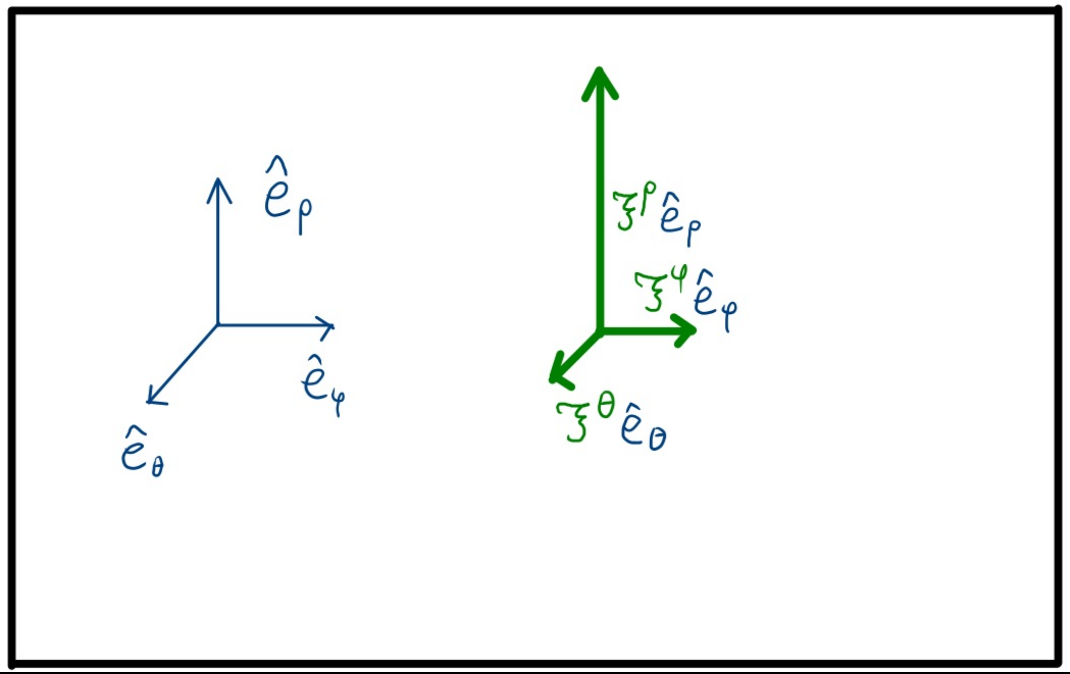
- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon? Misner et al §32.6 p 860



- Back to the ship:
  - orthonormal basis $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\varphi\}$
  - 4-velocity $\quad u = \hat{e}_\tau = [1,0,0,0]$
  - Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}} u^{\hat{\nu}} u^{\hat{\lambda}} \xi^{\hat{\sigma}}$$

- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon?

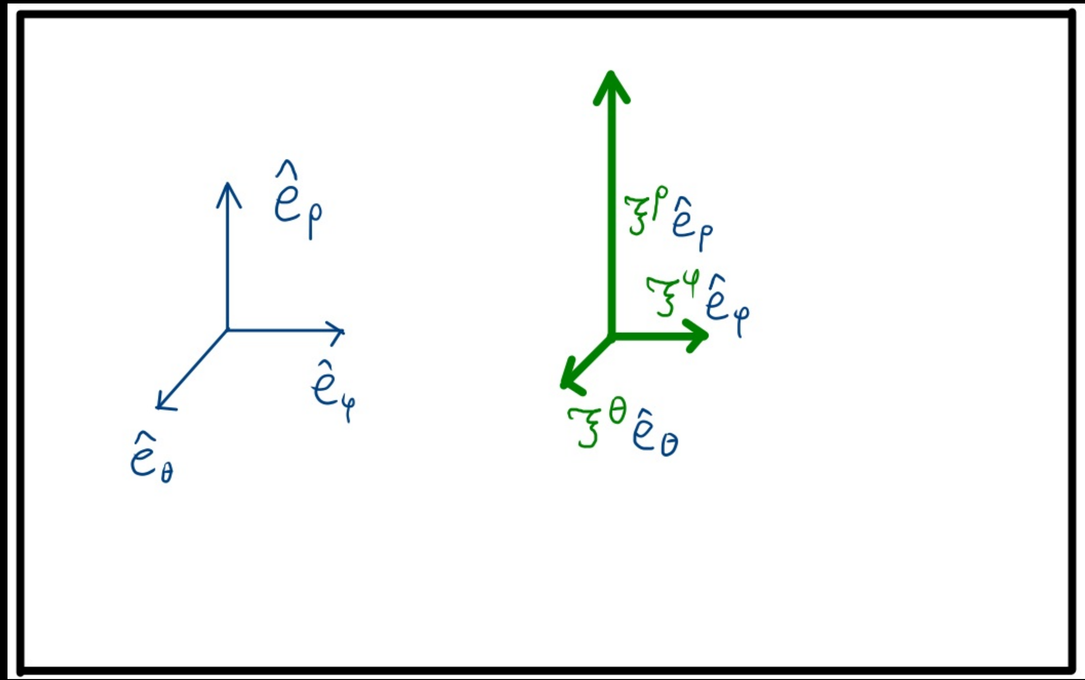


- Back to the ship:
  - orthonormal basis $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\varphi\}$
  - 4-velocity $\quad u = \hat{e}_\tau = [1,0,0,0]$
  - Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}} u^{\hat{\nu}} u^{\hat{\lambda}} \xi^{\hat{\sigma}}$$

only $\hat{\nu} = \hat{\lambda} = \hat{\tau}$ survive – only $u^{\hat{\tau}}$ is nonzero

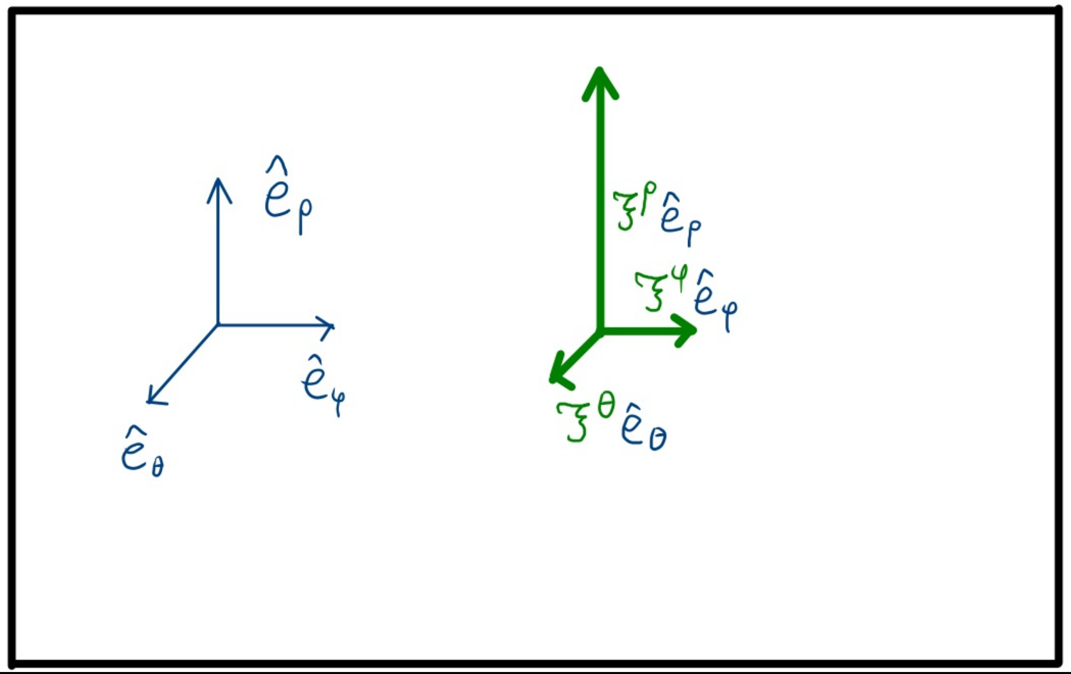
- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon?



- Back to the ship:
  - orthonormal basis $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\varphi\}$
  - 4-velocity $u = \hat{e}_\tau = [1,0,0,0]$
  - Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}}\, u^{\hat{\nu}} u^{\hat{\lambda}} \xi^{\hat{\sigma}} = R^{\hat{\mu}}{}_{\hat{\tau}\hat{\tau}\hat{\sigma}}\, \xi^{\hat{\sigma}}$$

- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon?



- Back to the ship:
  - orthonormal basis $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\varphi\}$
  - 4-velocity $u = \hat{e}_\tau = [1,0,0,0]$
  - Relative accelerations given by geodesic deviation equation:

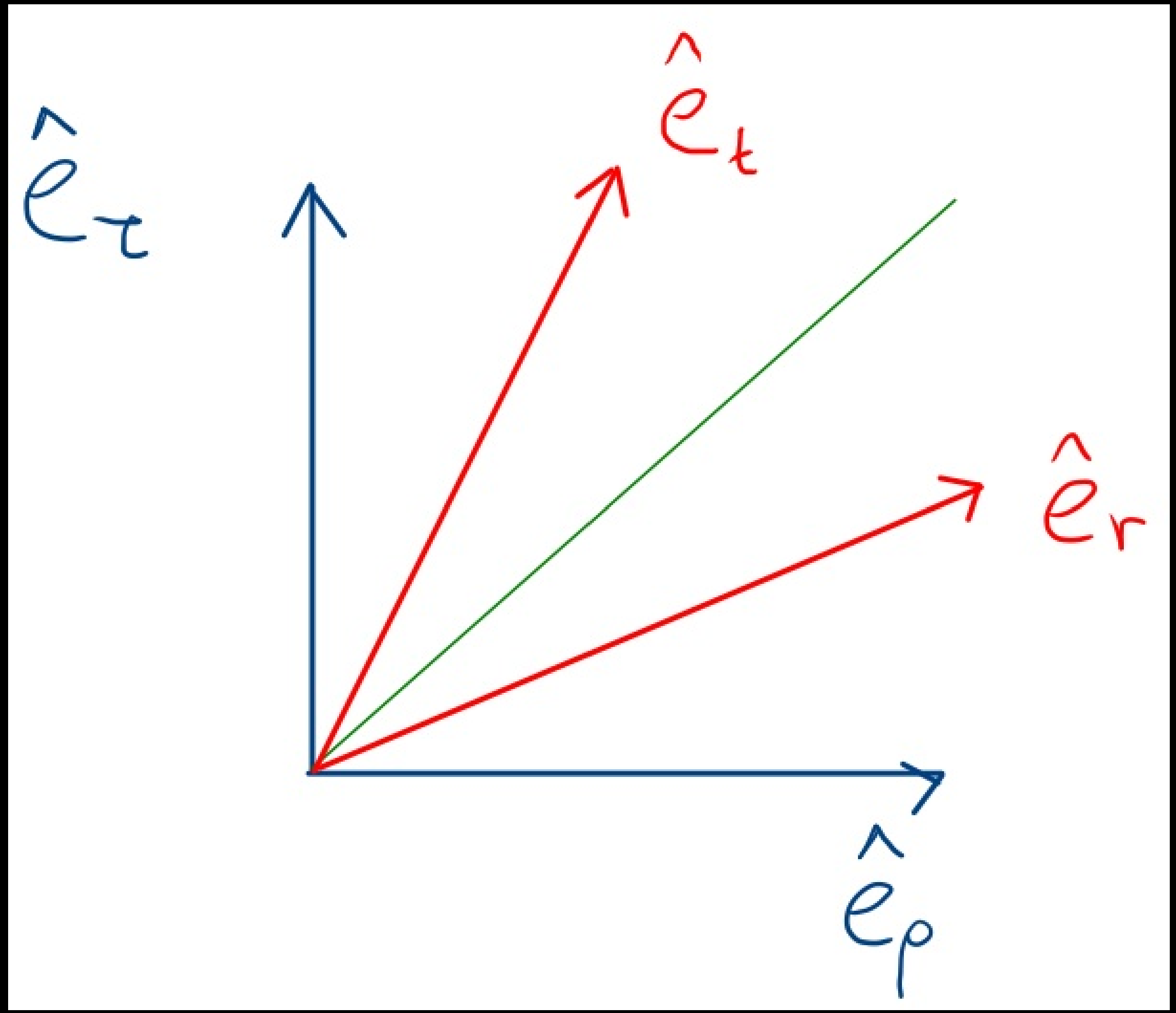
$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}} u^{\hat{\nu}} u^{\hat{\lambda}} \xi^{\hat{\sigma}} = R^{\hat{\mu}}{}_{\hat{\tau}\hat{\tau}\hat{\sigma}} \xi^{\hat{\sigma}}$$

$\hookrightarrow$ sum over $\hat{\sigma}$

The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:

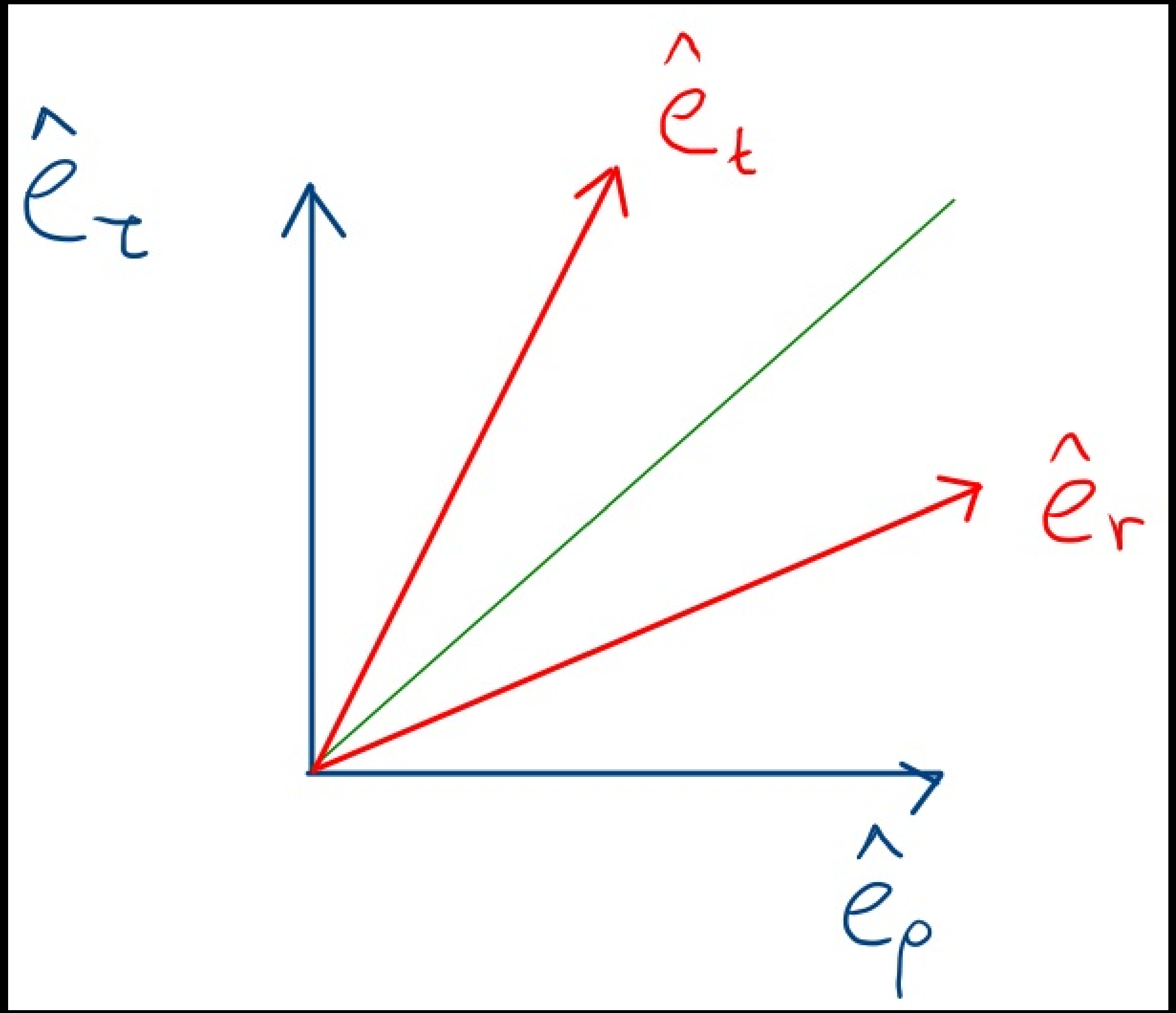
$$\hat{e}_\tau = \cosh\beta \; \hat{e}_t + \sinh\beta \; \hat{e}_r$$

$$\hat{e}_\rho = \sinh\beta \; \hat{e}_t + \cosh\beta \; \hat{e}_r$$



The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:

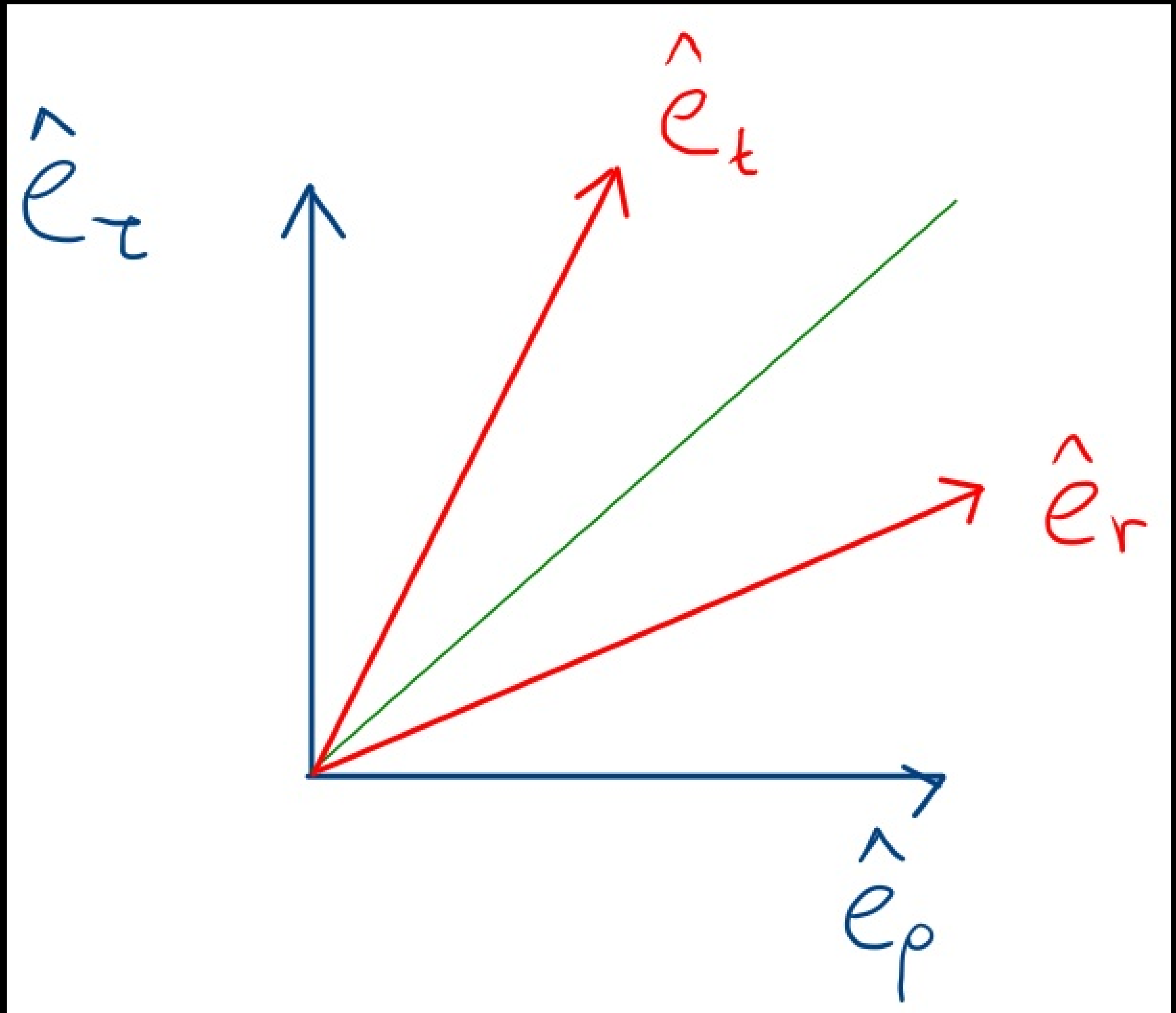
$$\hat{e}_\tau = \cosh\beta \ \hat{e}_t + \sinh\beta \ \hat{e}_r$$
$$\hat{e}_\rho = \sinh\beta \ \hat{e}_t + \cosh\beta \ \hat{e}_r$$



Indeed, the $\hat{e}_\tau = u = [u^t, u^r, 0, 0]$ (radial fall)

↳ components in (t, r, θ, ϕ)

The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:

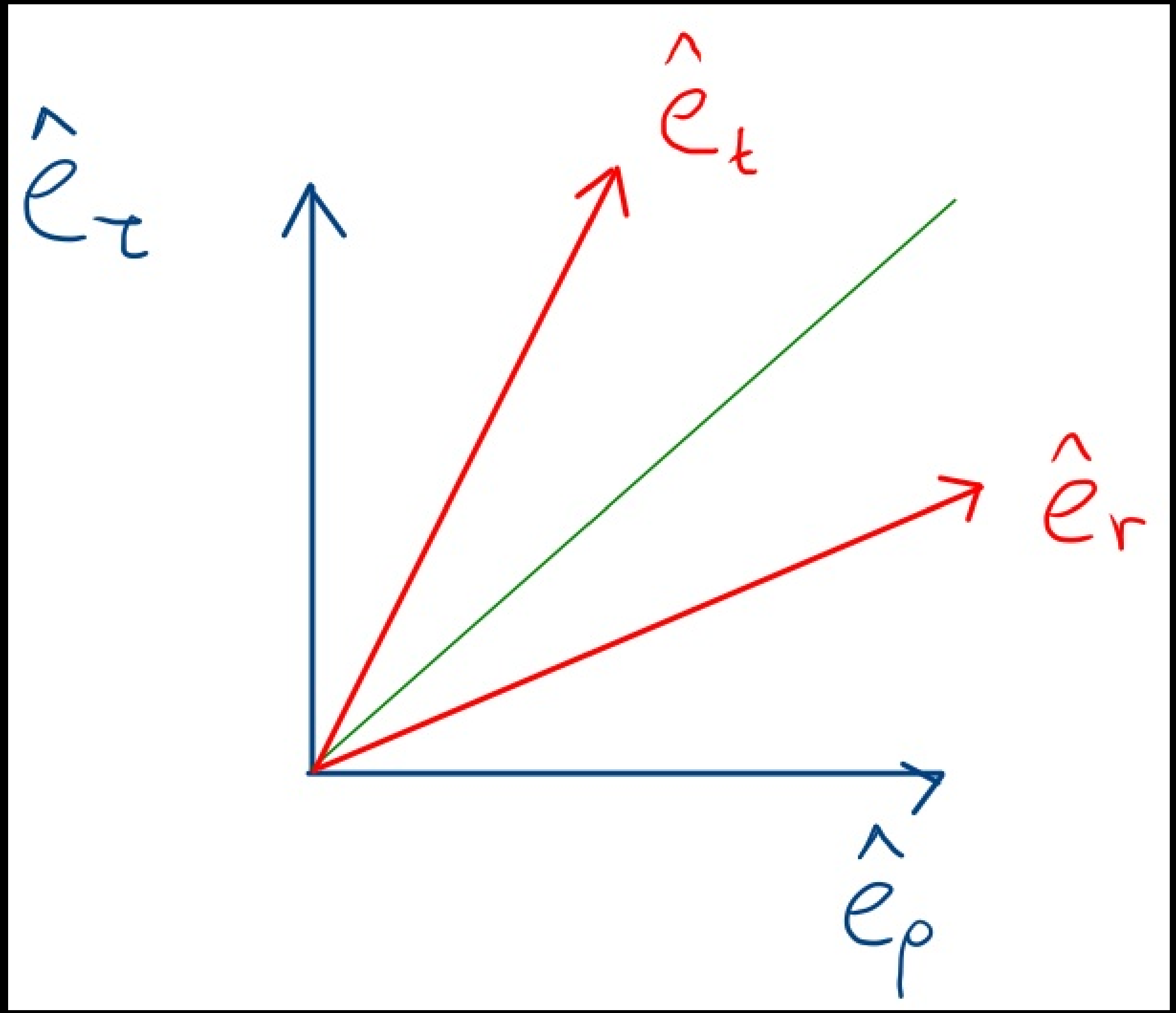


$$\hat{e}_\tau = \cosh\beta \; \hat{e}_t + \sinh\beta \; \hat{e}_r$$
$$\hat{e}_\rho = \sinh\beta \; \hat{e}_t + \cosh\beta \; \hat{e}_r$$

Indeed, the $\hat{e}_\tau = u = [u^t, u^r, 0, 0]$ (radial fall) $\Rightarrow$

$$u = u^t \partial_t + u^r \partial_r$$

The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:



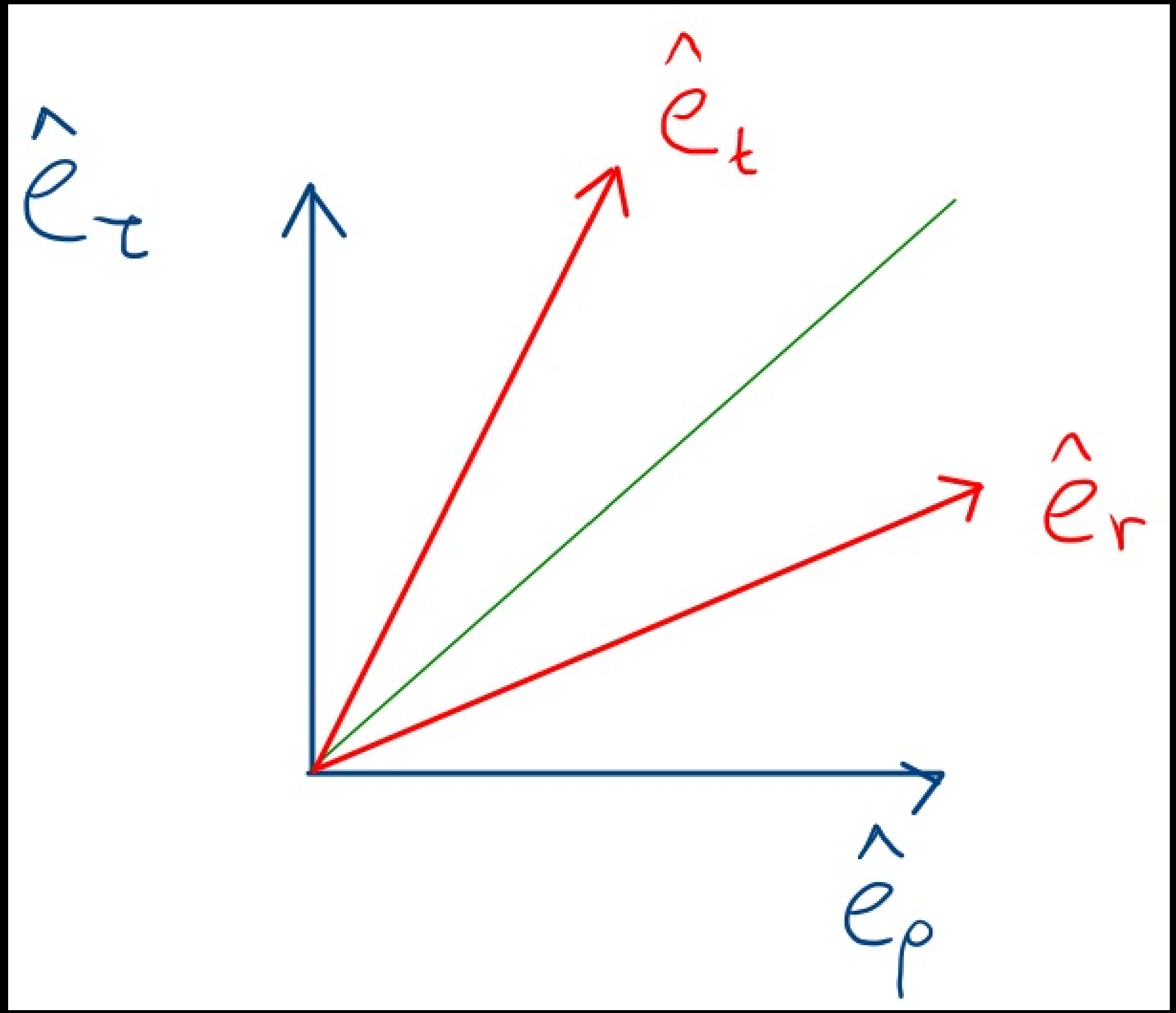
$$\hat{e}_\tau = \cosh\beta \ \hat{e}_t + \sinh\beta \ \hat{e}_r$$

$$\hat{e}_\rho = \sinh\beta \ \hat{e}_t + \cosh\beta \ \hat{e}_r$$

Indeed, the $\hat{e}_\tau = u = [u^t, u^r, 0, 0]$ (radial fall) $\Rightarrow$

$$u = u^t \partial_t + u^r \partial_r = \sqrt{|g_{tt}|}\, u^t \, \hat{e}_t + \sqrt{g_{rr}}\, u^r \, \hat{e}_r$$

The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:

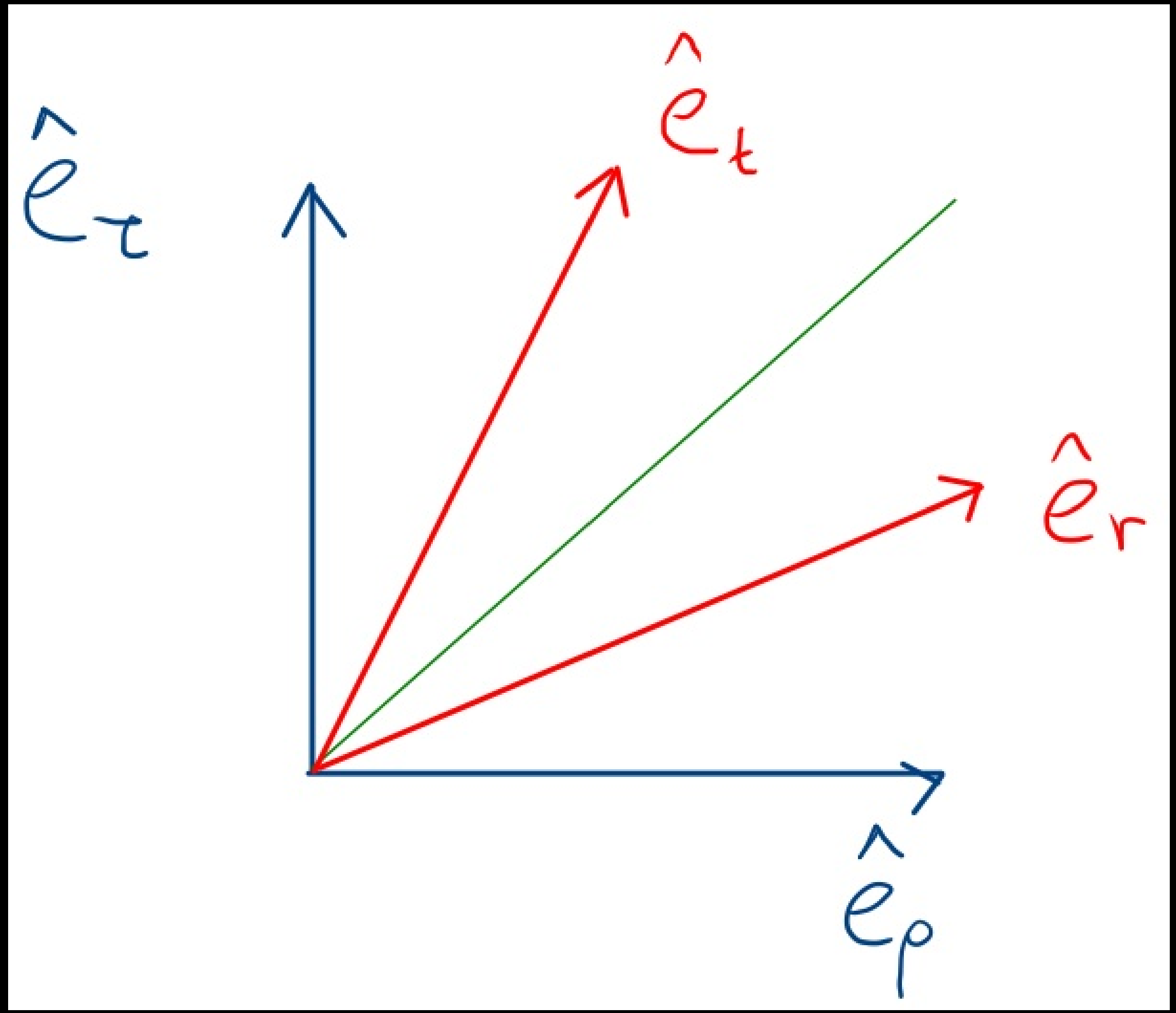


$$\hat{e}_\tau = \cosh\beta \;\hat{e}_t + \sinh\beta\;\hat{e}_r$$
$$\hat{e}_\rho = \sinh\beta\;\hat{e}_t + \cosh\beta\;\hat{e}_r$$

Indeed, the $\hat{e}_\tau = u = [u^t, u^r, 0, 0]$ (radial fall) $\Rightarrow$

$$u = u^t \partial_t + u^r \partial_r = \sqrt{|g_{tt}|}\, u^t\, \hat{e}_t + \sqrt{g_{rr}}\, u^r\, \hat{e}_r$$
$$= u^{\hat{t}}\, \hat{e}_t + u^{\hat{r}}\, \hat{e}_r$$

The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:



$$\hat{e}_\tau = \cosh\beta \; \hat{e}_t + \sinh\beta \; \hat{e}_r$$
$$\hat{e}_\rho = \sinh\beta \; \hat{e}_t + \cosh\beta \; \hat{e}_r$$

Indeed, the $\hat{e}_\tau = u = [u^t, u^r, 0, 0]$ (radial fall) $\Rightarrow$

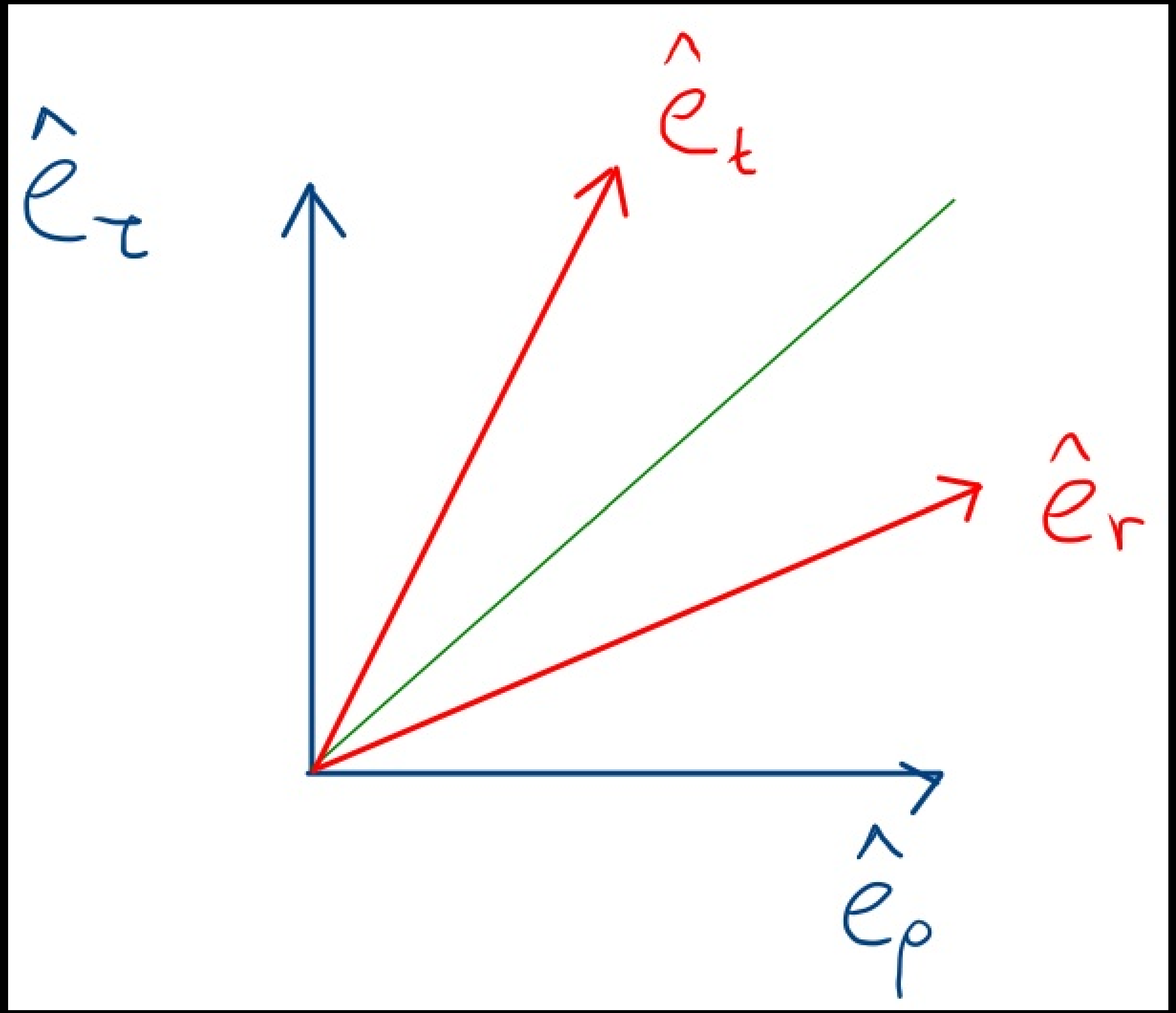
$$u = u^t \partial_t + u^r \partial_r = \sqrt{|g_{tt}|}\, u^t \hat{e}_t + \sqrt{|g_{rr}}\, u^r \hat{e}_r$$
$$= u^{\hat{t}} \hat{e}_t + u^{\hat{r}} \hat{e}_r$$

Therefore $u^{\hat{t}} = \cosh\beta$ $u^{\hat{r}} = \sinh\beta$, since

$$-1 = u \cdot u = (u^{\hat{t}})^2 \hat{e}_t \cdot \hat{e}_t + (u^{\hat{r}})^2 \hat{e}_r \cdot \hat{e}_r = -(u^{\hat{t}})^2 + (u^{\hat{r}}) = -\cosh^2\beta + \sinh^2\beta$$

The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:

$$\begin{pmatrix} \hat{e}_\tau \\ \hat{e}_\rho \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} \hat{e}_t \\ \hat{e}_r \end{pmatrix}$$

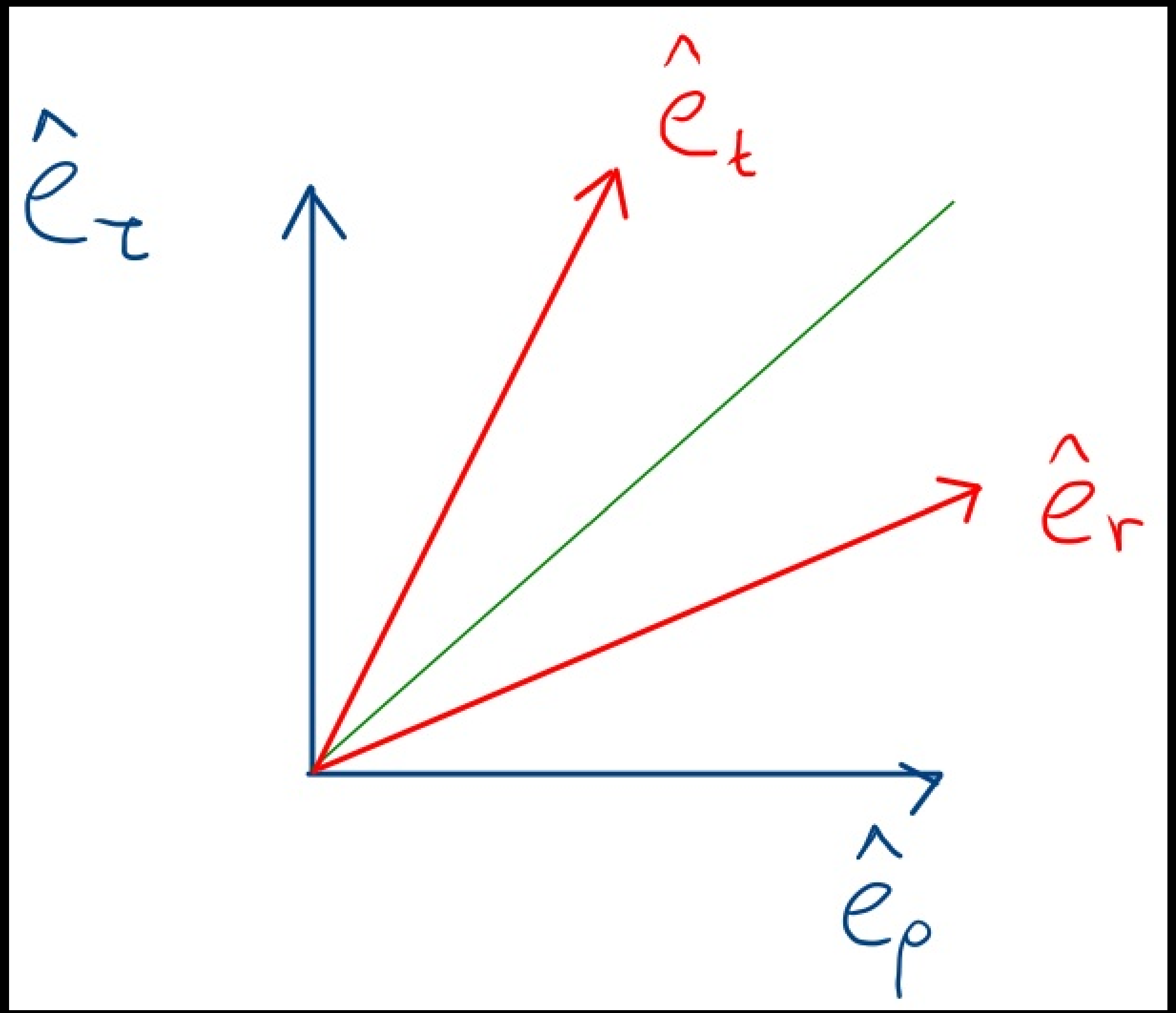


The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:

$$\begin{pmatrix} \hat{e}_\tau \\ \hat{e}_\rho \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} \hat{e}_t \\ \hat{e}_r \end{pmatrix}$$

$$\begin{pmatrix} V^{\hat{\tau}} \\ V^{\hat{\rho}} \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix}^{-1} \begin{pmatrix} V^{\hat{t}} \\ V^{\hat{r}} \end{pmatrix}$$

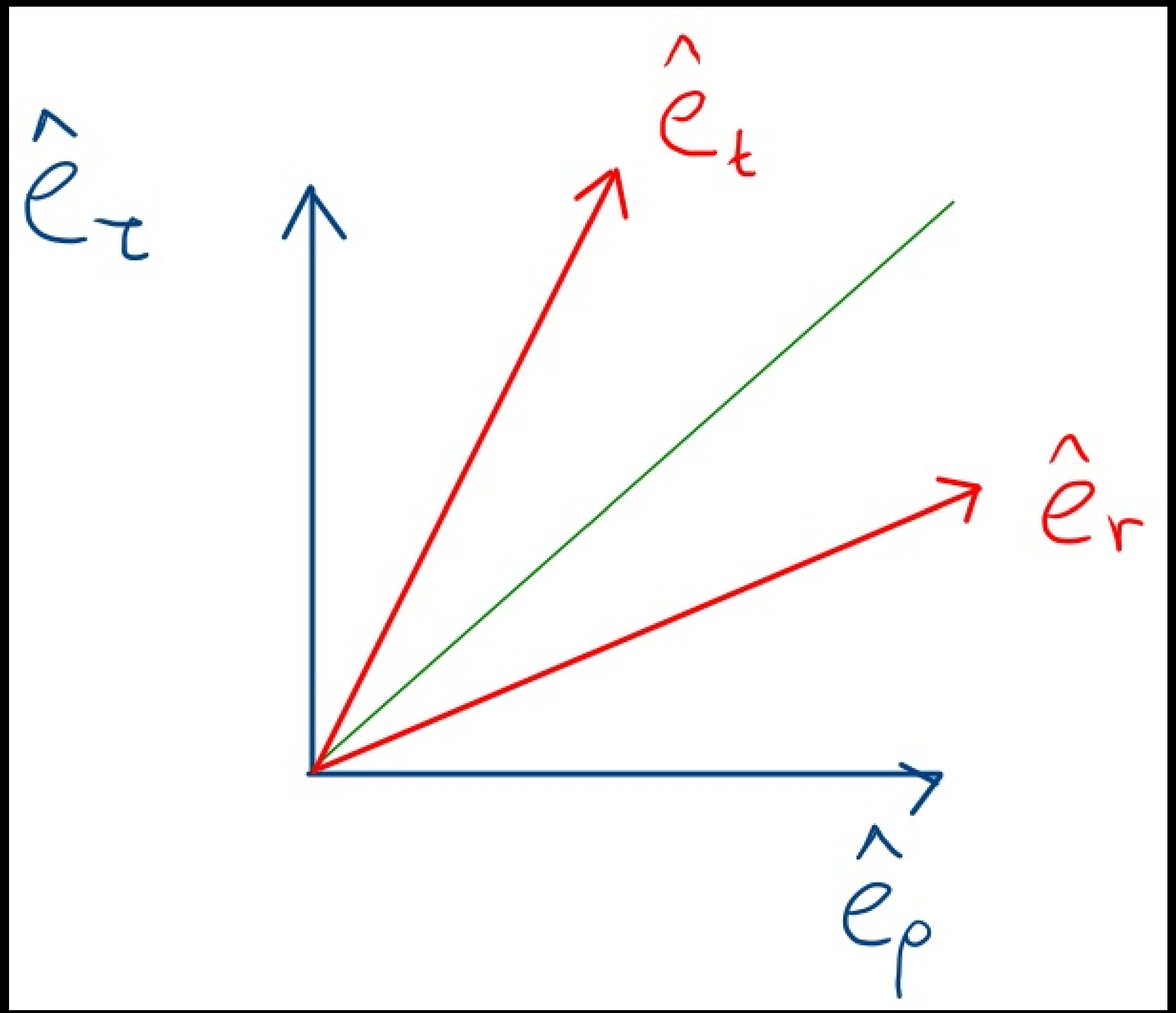
inverse matrix



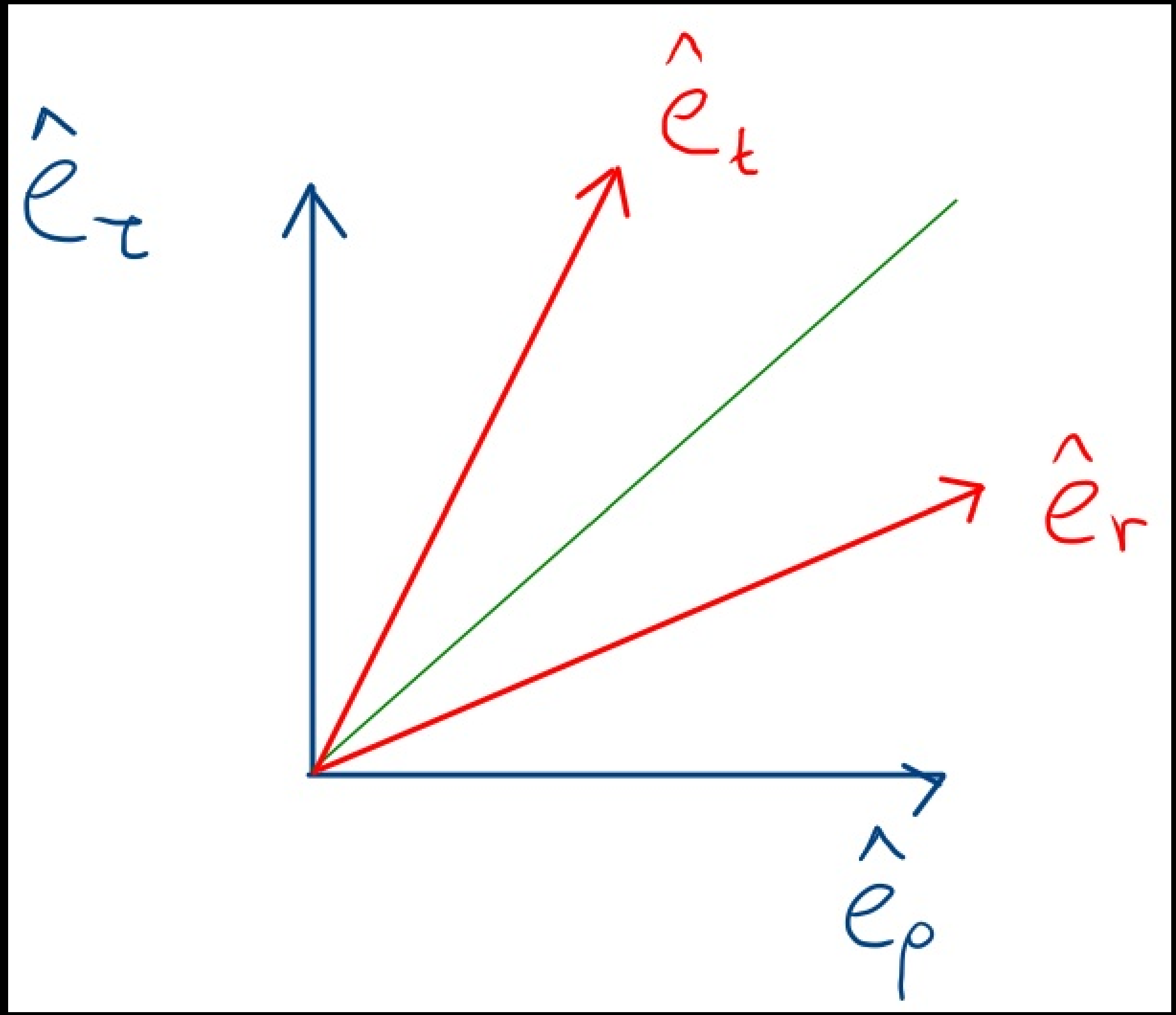
The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:

$$\begin{pmatrix} \hat{e}_\tau \\ \hat{e}_\rho \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} \hat{e}_t \\ \hat{e}_r \end{pmatrix}$$

$$\begin{pmatrix} V^{\hat{\tau}} \\ V^{\hat{\rho}} \end{pmatrix} = \begin{pmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} V^{\hat{t}} \\ V^{\hat{r}} \end{pmatrix}$$



The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:

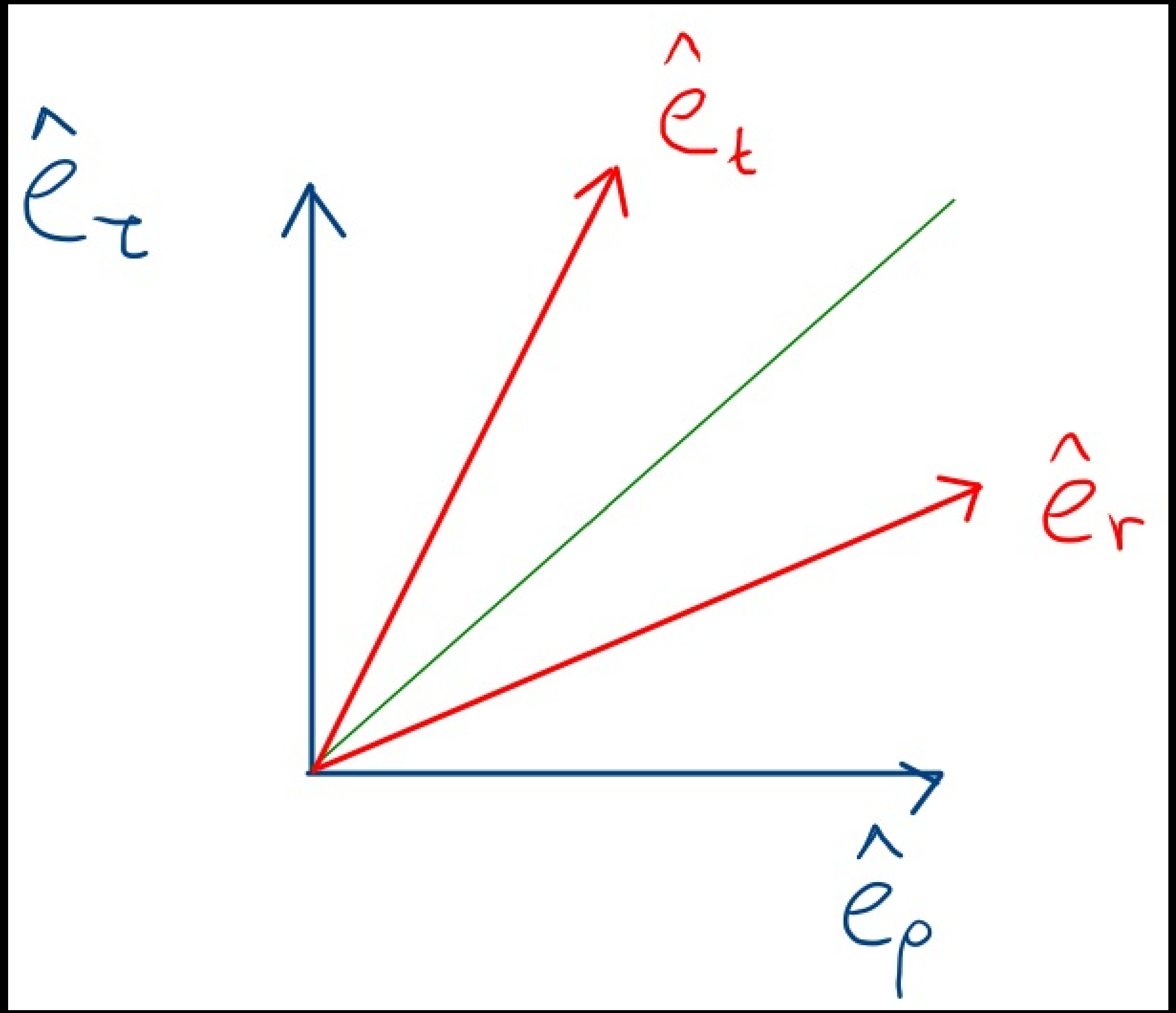


$$\begin{pmatrix} \hat{e}_\tau \\ \hat{e}_\rho \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} \hat{e}_t \\ \hat{e}_r \end{pmatrix}$$

$$\begin{pmatrix} V^{\hat{\tau}} \\ V^{\hat{\rho}} \end{pmatrix} = \begin{pmatrix} \cosh(-\beta) & \sinh(-\beta) \\ \sinh(-\beta) & \cosh(-\beta) \end{pmatrix} \begin{pmatrix} V^{\hat{t}} \\ V^{\hat{r}} \end{pmatrix}$$

$\hookrightarrow$ opposite velocity $\rightarrow$ inverse boost

The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:

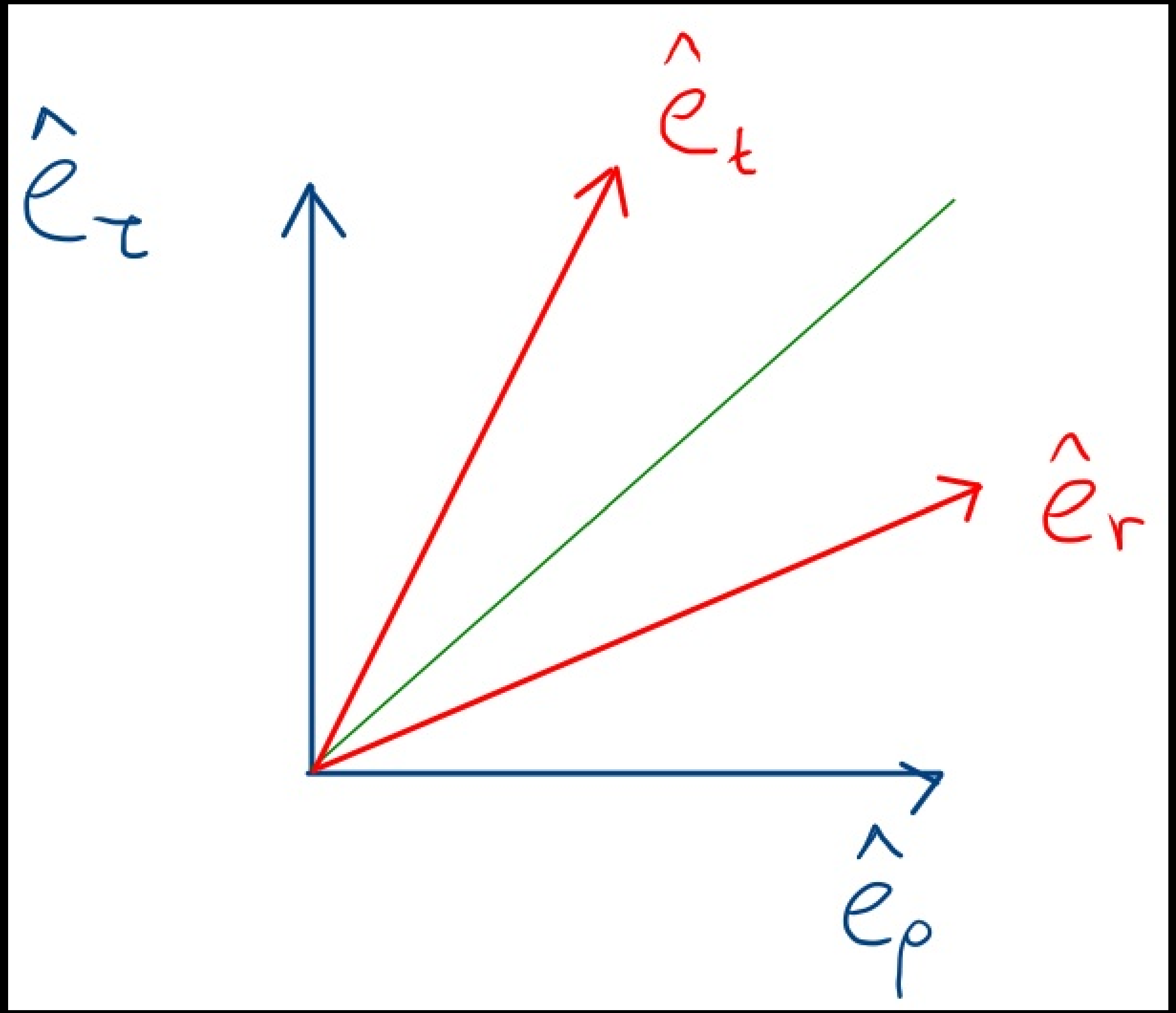


$$\begin{pmatrix} \hat{e}_\tau \\ \hat{e}_\rho \end{pmatrix} = \Lambda(\beta) \begin{pmatrix} \hat{e}_t \\ \hat{e}_r \end{pmatrix}$$

$$\begin{pmatrix} V^{\hat{\tau}} \\ V^{\hat{\rho}} \end{pmatrix} = \Lambda(-\beta) \begin{pmatrix} V^{\hat{t}} \\ V^{\hat{r}} \end{pmatrix}$$

$$\Lambda^{-1}(\beta) = \Lambda(-\beta)$$

The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:



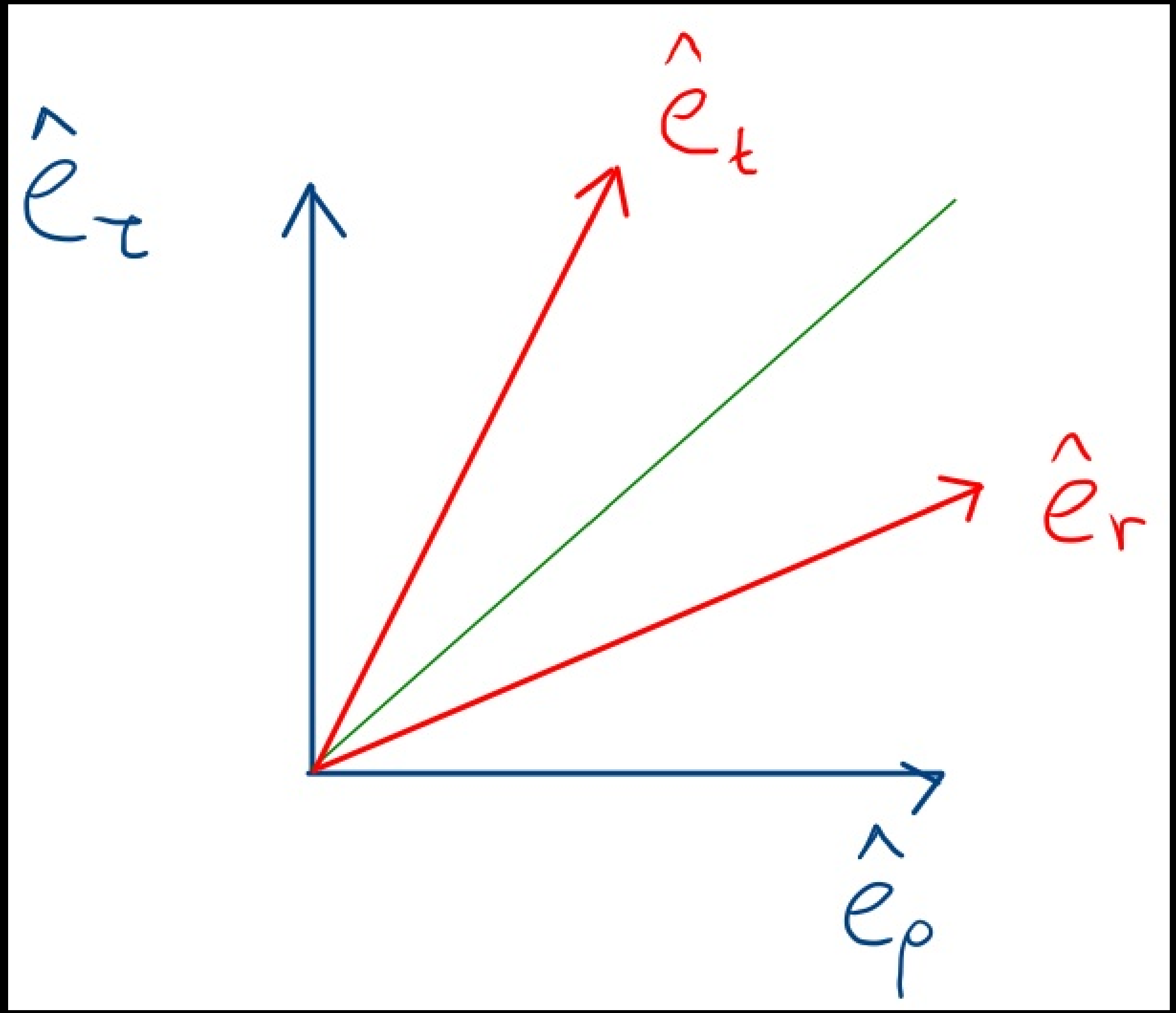
$$\begin{pmatrix} \omega_{\hat{\tau}} \\ \omega_{\hat{\rho}} \end{pmatrix} = \Lambda(\beta) \begin{pmatrix} \omega_{\hat{t}} \\ \omega_{\hat{r}} \end{pmatrix}$$

components of one form ω

$$\begin{pmatrix} V^{\hat{\tau}} \\ V^{\hat{\rho}} \end{pmatrix} = \Lambda(-\beta) \begin{pmatrix} V^{\hat{t}} \\ V^{\hat{r}} \end{pmatrix}$$

$$\Lambda^{-1}(\beta) = \Lambda(-\beta)$$

The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:



$$\begin{pmatrix} \omega_{\hat{\tau}} \\ \omega_{\hat{\rho}} \end{pmatrix} = \Lambda(\beta) \begin{pmatrix} \omega_{\hat{t}} \\ \omega_{\hat{r}} \end{pmatrix}$$

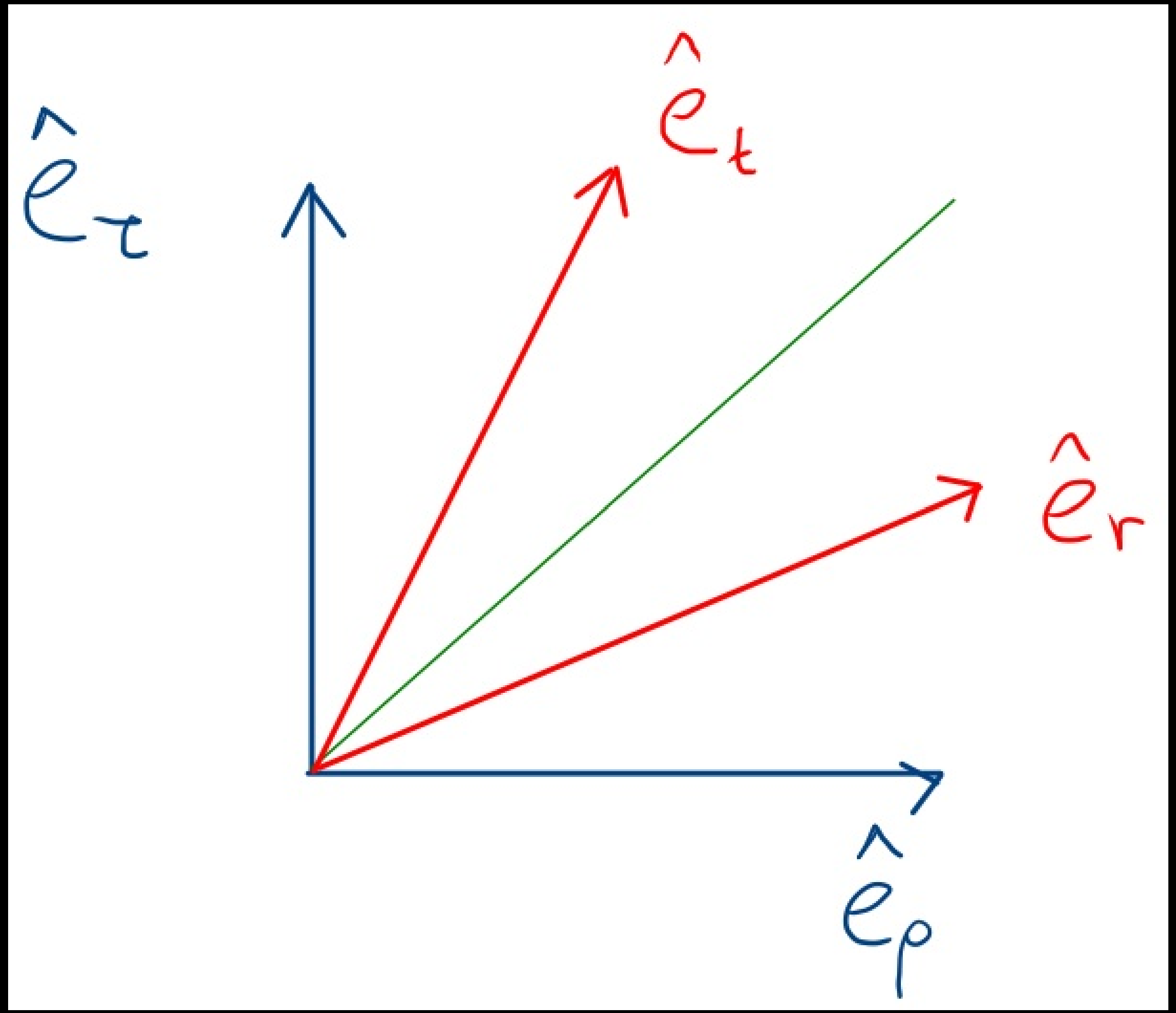
$$\omega_{\hat{\mu}'} = \Lambda_{\hat{\mu}'}{}^{\hat{\mu}} \, \omega_{\hat{\mu}}$$

$$\Lambda_{\hat{\tau}}{}^{\hat{t}} = \Lambda_{\hat{\rho}}{}^{\hat{r}} = \cosh\beta$$

$$\Lambda_{\hat{\rho}}{}^{\hat{t}} = \Lambda_{\hat{\tau}}{}^{\hat{r}} = \sinh\beta$$

$$\left.\begin{array}{l} \Lambda_{\hat{\theta}}{}^{\hat{\theta}} = \Lambda_{\hat{\phi}}{}^{\hat{\phi}} = 1 \\ \text{and the rest are zero} \end{array}\right\} \text{not transformed}$$

The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:

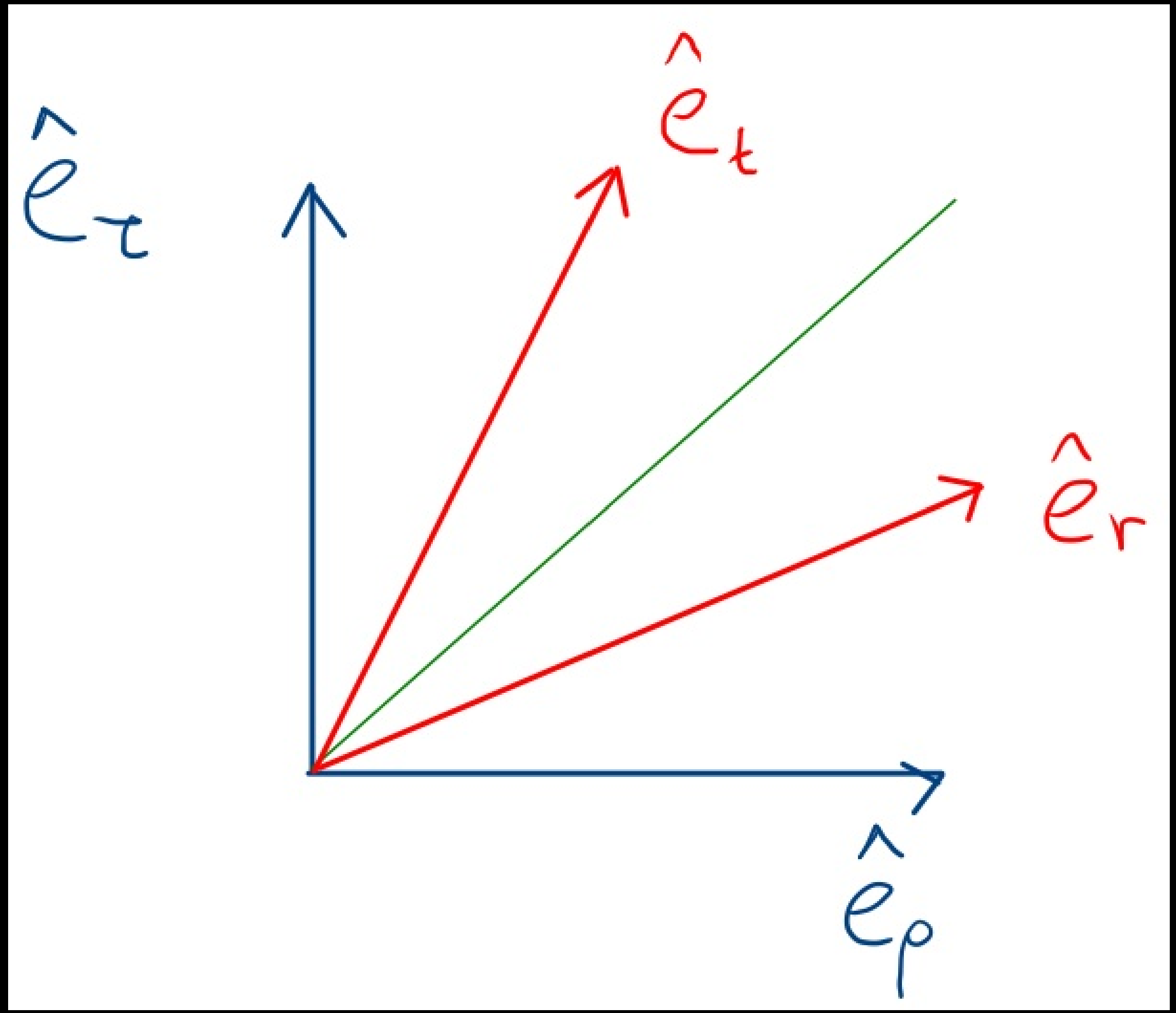


$$\begin{pmatrix} \omega_{\hat{\tau}} \\ \omega_{\hat{\rho}} \end{pmatrix} = \Lambda(\beta) \begin{pmatrix} \omega_{\hat{t}} \\ \omega_{\hat{r}} \end{pmatrix}$$

$$\omega_{\hat{\mu}'} = \Lambda_{\hat{\mu}'}{}^{\hat{\mu}} \, \omega_{\hat{\mu}}$$

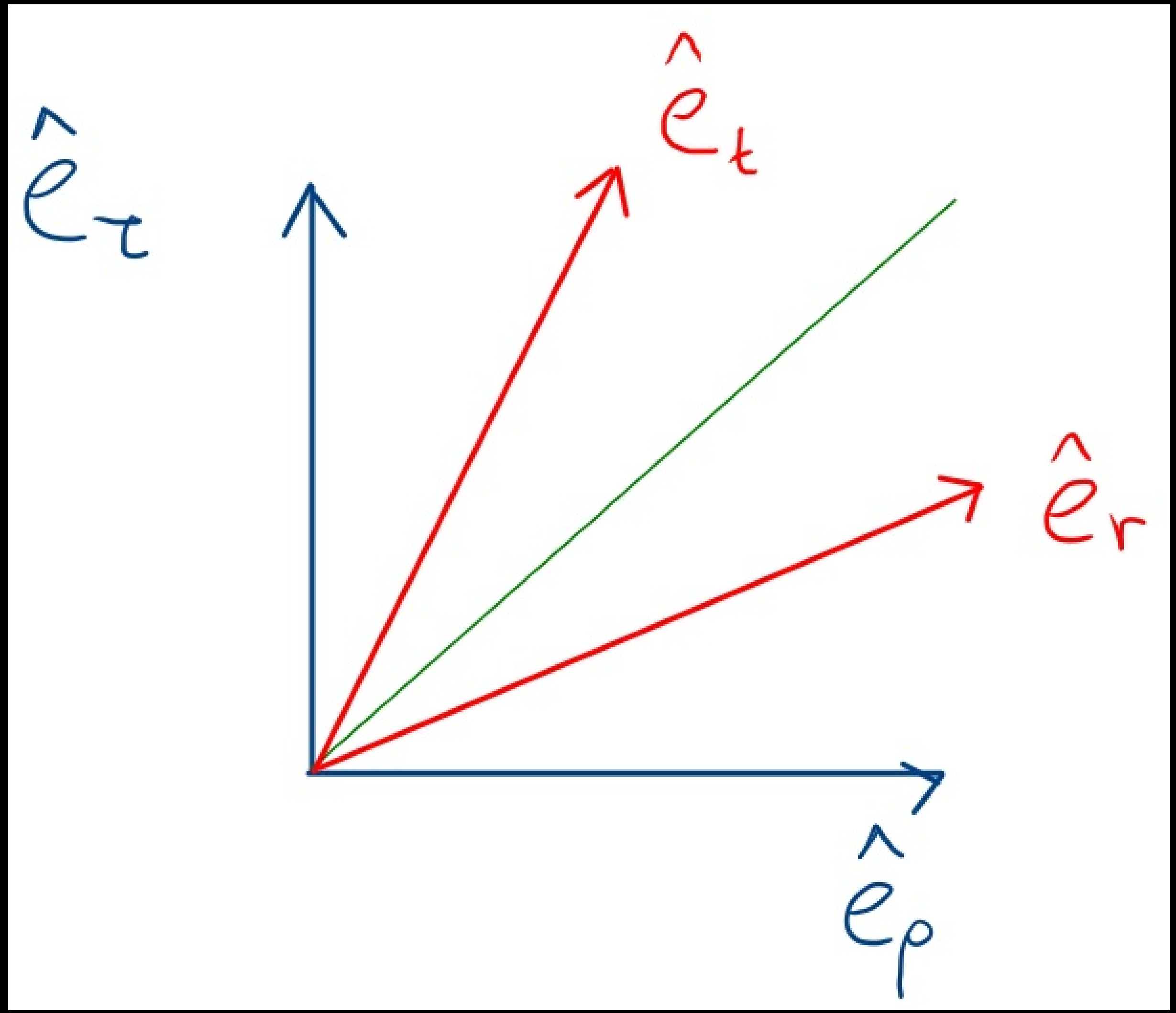
$$R_{\hat{\mu}'\hat{\nu}'\hat{\lambda}'\hat{\sigma}'} = \Lambda_{\hat{\mu}'}{}^{\hat{\mu}} \Lambda_{\hat{\nu}'}{}^{\hat{\nu}} \Lambda_{\hat{\lambda}'}{}^{\hat{\lambda}} \Lambda_{\hat{\sigma}'}{}^{\hat{\sigma}} R_{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\sigma}}$$

So: • R $\hat{\theta}$ $\hat{\phi}$ $\hat{\vartheta}$ $\hat{\varphi}$ unchanged



So: • $R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}}$ unchanged

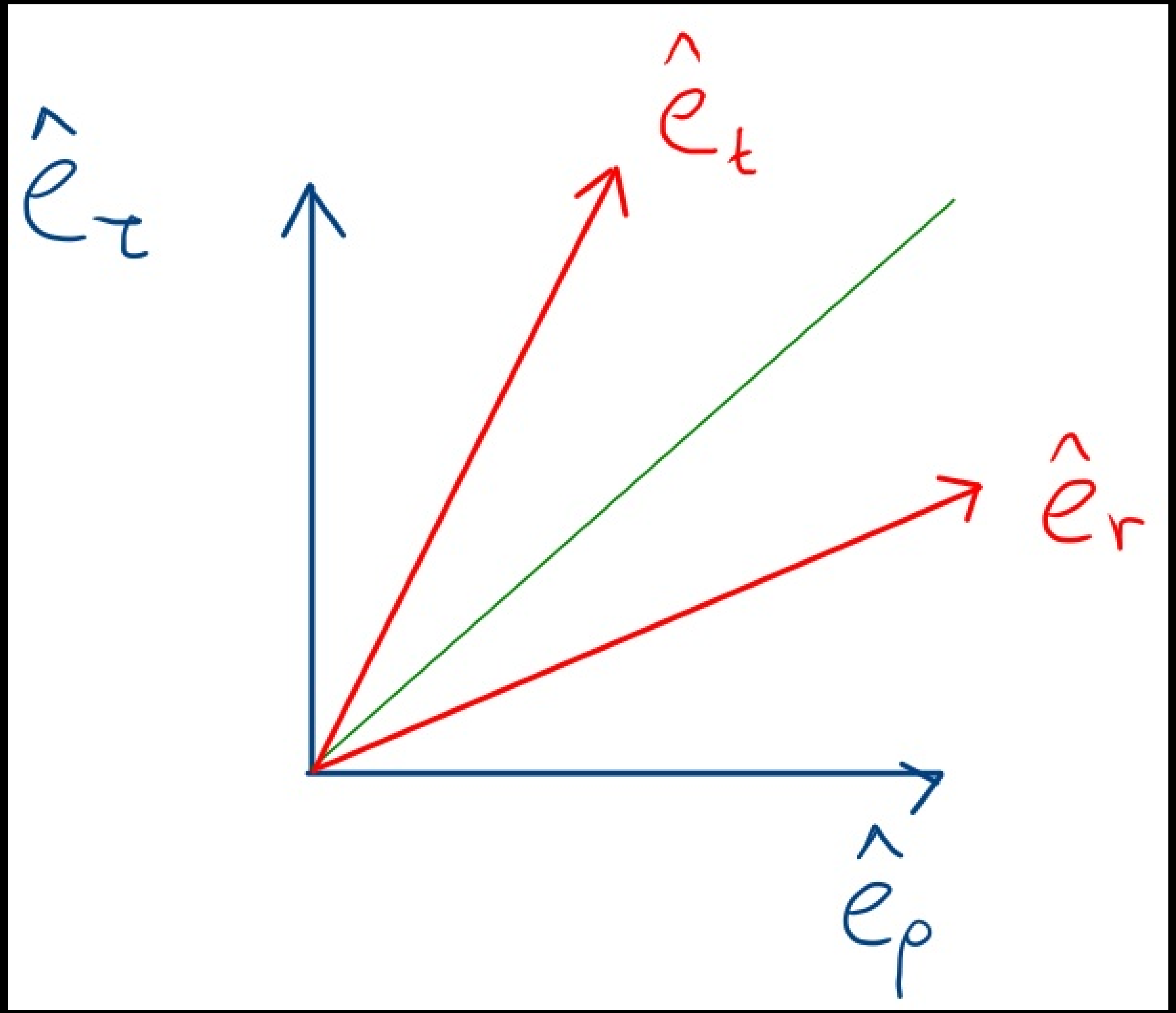
• $R_{\hat{\tau}\hat{\theta}\hat{\tau}\hat{\theta}} = \Lambda_{\hat{\tau}}{}^{\hat{\mu}} \Lambda_{\hat{\tau}}{}^{\hat{\nu}} R_{\hat{\mu}\hat{\theta}\hat{\nu}\hat{\theta}}$



So: • $R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}}$ unchanged

• $R_{\hat{\tau}\hat{\theta}\hat{\tau}\hat{\theta}} = \Lambda_{\hat{\tau}}{}^{\hat{\mu}} \Lambda_{\hat{\tau}}{}^{\hat{\nu}} R_{\hat{\mu}\hat{\theta}\hat{\nu}\hat{\theta}}$

$= \Lambda_{\hat{\tau}}{}^{\hat{t}} \Lambda_{\hat{\tau}}{}^{\hat{t}} R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} + \Lambda_{\hat{\tau}}{}^{\hat{r}} \Lambda_{\hat{\tau}}{}^{\hat{r}} R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}}$



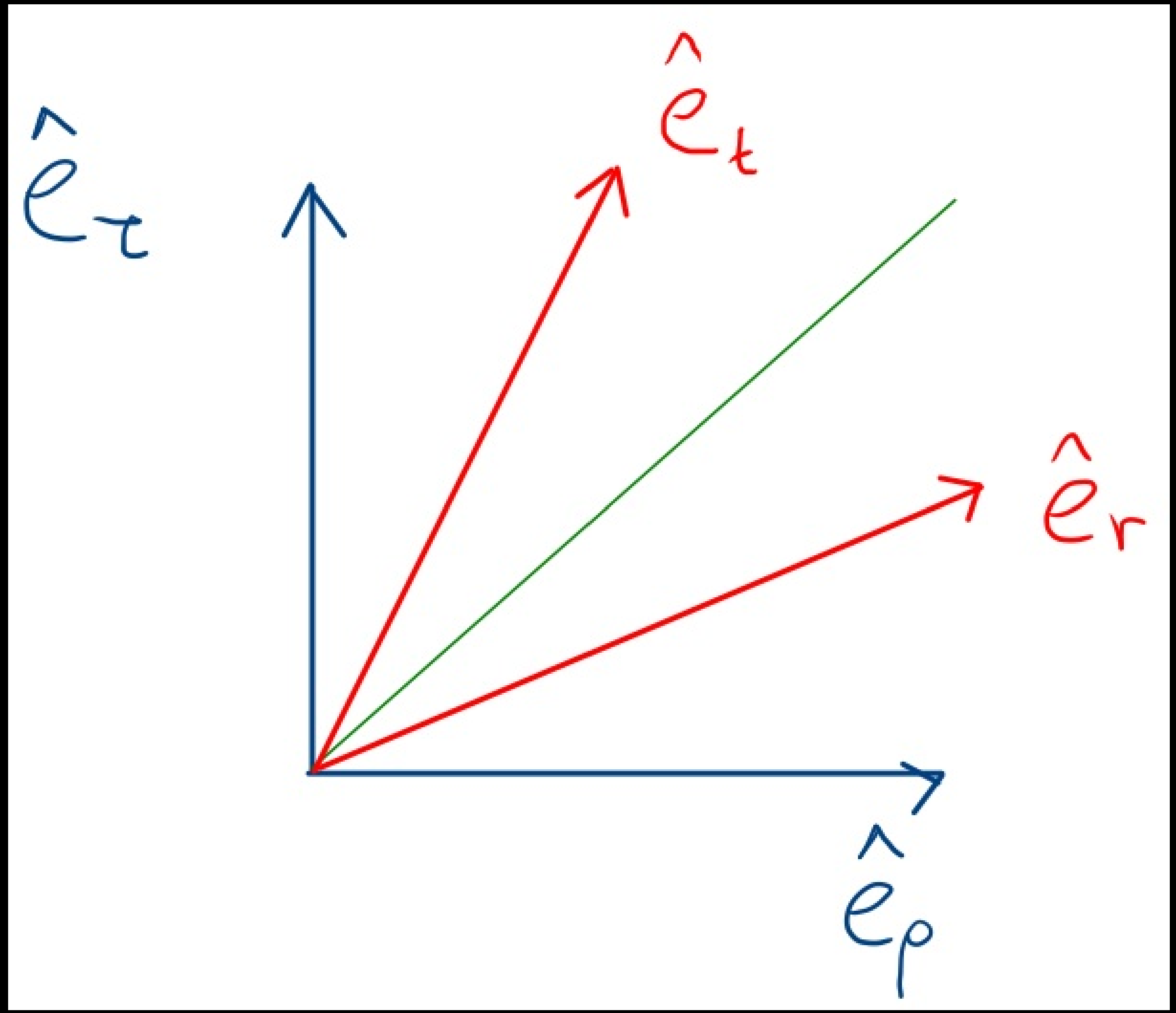
So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged

• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = \Lambda_{\hat\tau}{}^{\hat\mu}\,\Lambda_{\hat\tau}{}^{\hat\nu}\, R_{\hat\mu\hat\theta\hat\nu\hat\theta}$

$$= \Lambda_{\hat\tau}{}^{\hat t}\,\Lambda_{\hat\tau}{}^{\hat t}\, R_{\hat t\hat\theta\hat t\hat\theta} + \Lambda_{\hat\tau}{}^{\hat r}\,\Lambda_{\hat\tau}{}^{\hat r}\, R_{\hat r\hat\theta\hat r\hat\theta}$$

$$= \cosh^2\beta \;\frac{M}{r^3} \;+\; \sinh^2\beta \left(-\frac{M}{r^3}\right)$$

notice the effect
of opposite values



So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged

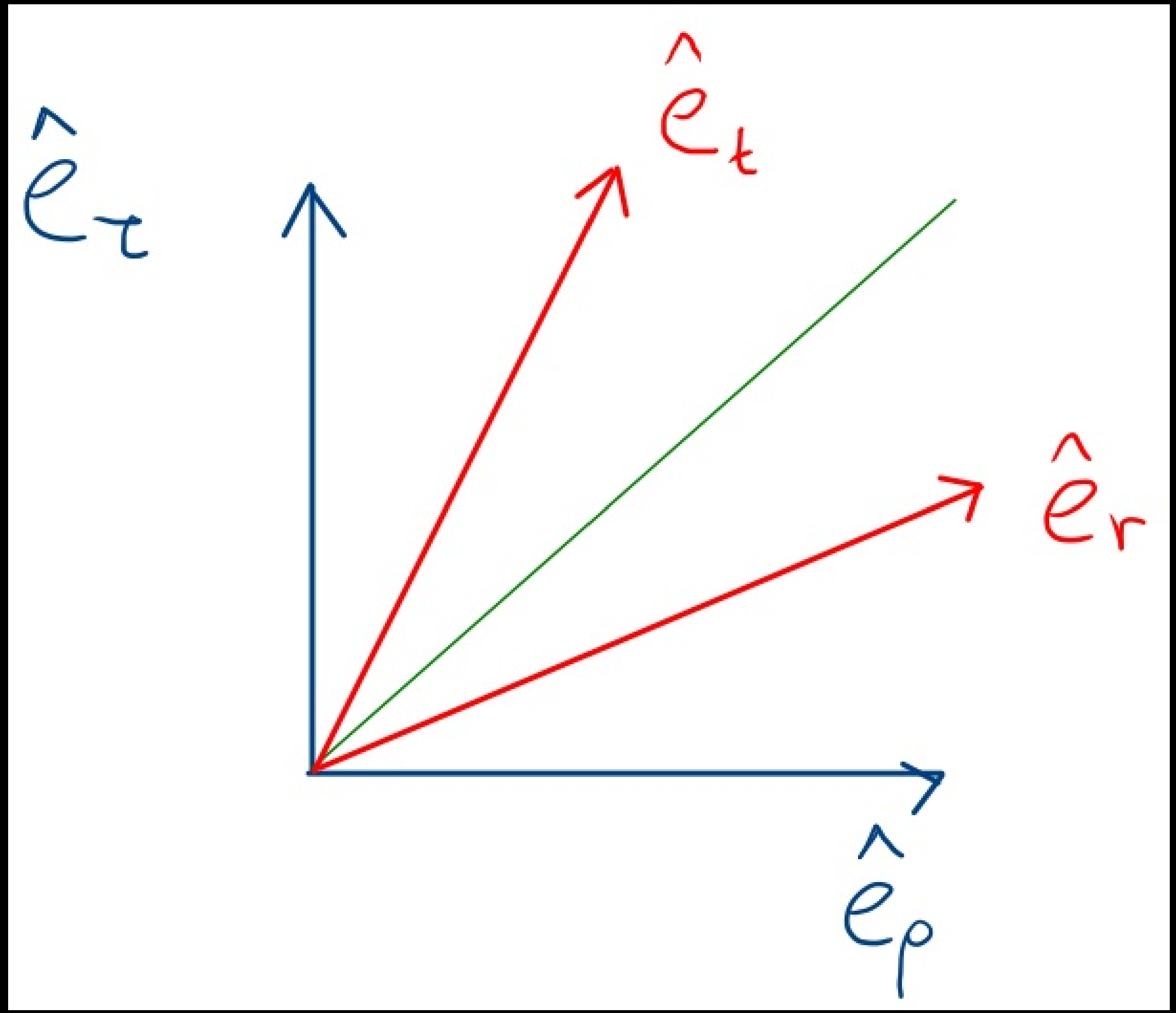
• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = \Lambda_{\hat\tau}{}^{\hat\mu}\Lambda_{\hat\tau}{}^{\hat\nu} R_{\hat\mu\hat\theta\hat\nu\hat\theta}$

$$= \Lambda_{\hat\tau}{}^{\hat t}\Lambda_{\hat\tau}{}^{\hat t} R_{\hat t\hat\theta\hat t\hat\theta} + \Lambda_{\hat\tau}{}^{\hat r}\Lambda_{\hat\tau}{}^{\hat r} R_{\hat r\hat\theta\hat r\hat\theta}$$

$$= \cosh^2\beta \, \frac{M}{r^3} + \sinh^2\beta \left(-\frac{M}{r^3}\right)$$

$$= \left(\cosh^2\beta - \sinh^2\beta\right)\frac{M}{r^3} =$$

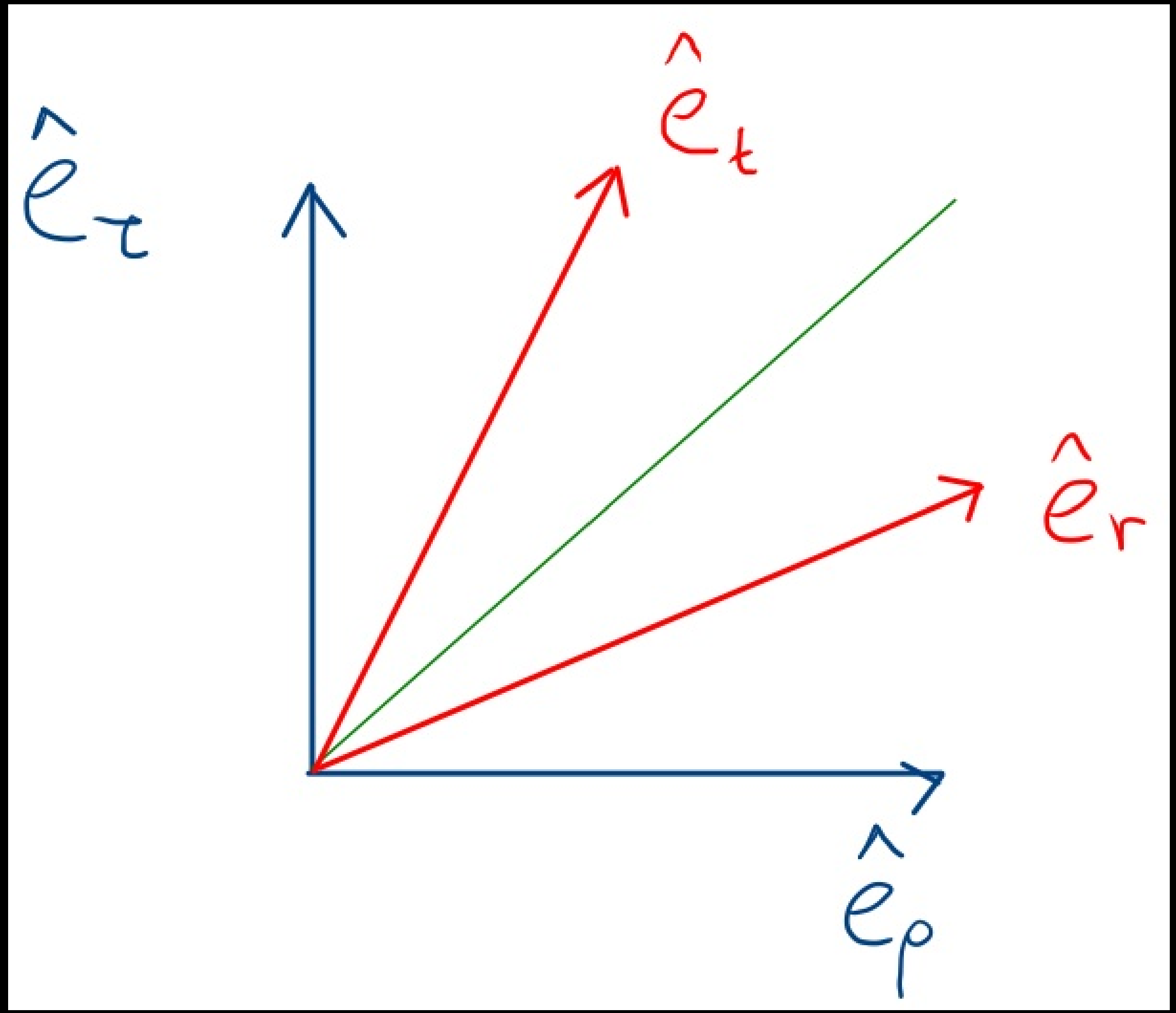
$$= \frac{M}{r^3} = R_{\hat t\hat\theta\hat t\hat\theta}$$



So: • $R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}}$ unchanged

• $R_{\hat{\tau}\hat{\theta}\hat{\tau}\hat{\theta}} = R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}}$

• $R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = \Lambda_{\hat{\rho}}{}^{\hat{t}}\Lambda_{\hat{\rho}}{}^{\hat{t}} R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} + \Lambda_{\hat{\rho}}{}^{\hat{r}}\Lambda_{\hat{\rho}}{}^{\hat{r}} R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}}$

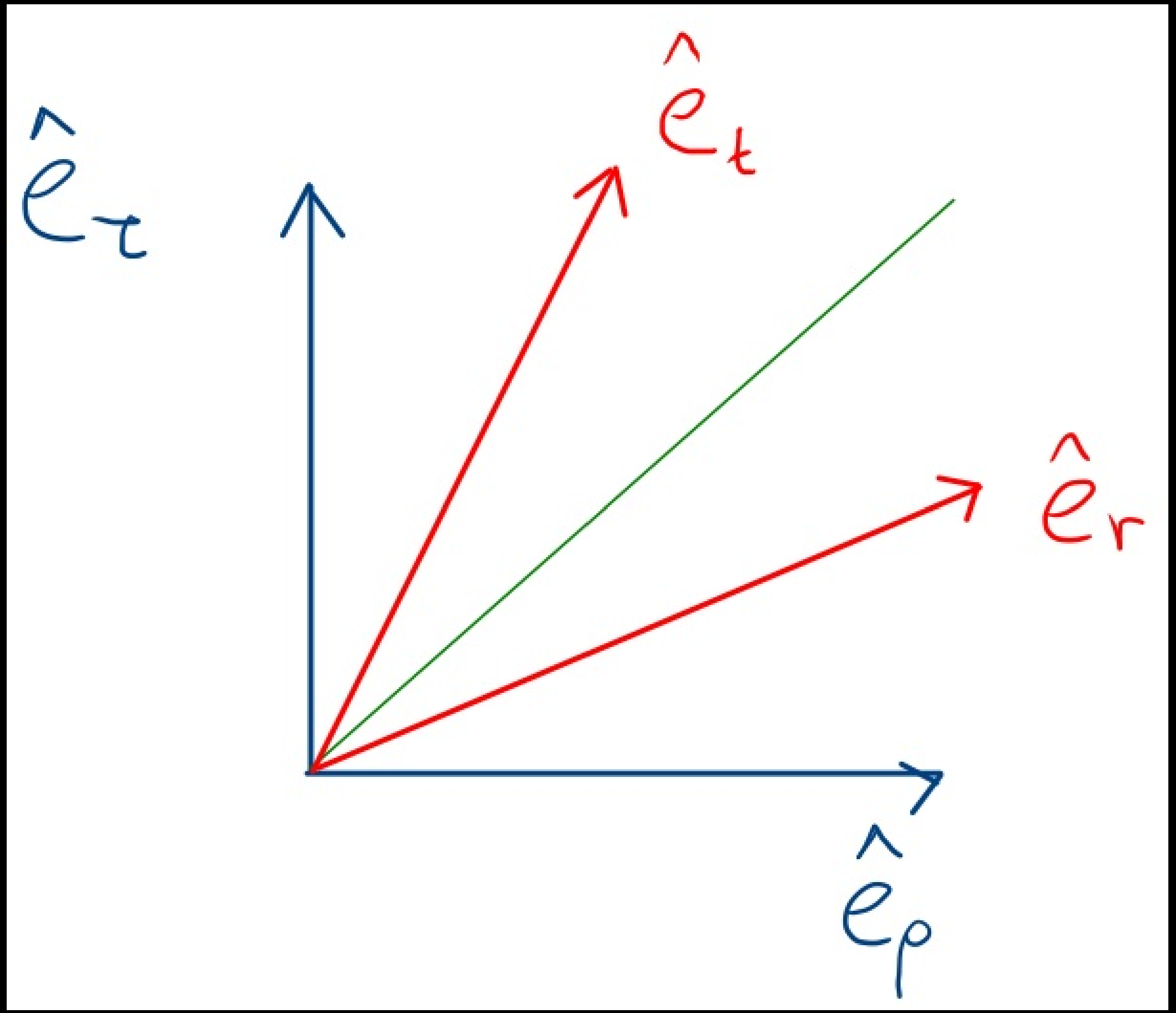


So: • $R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}}$ unchanged

• $R_{\hat{\tau}\hat{\theta}\hat{\tau}\hat{\theta}} = R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}}$

• $R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = \Lambda_{\hat{\rho}}{}^{\hat{t}}\Lambda_{\hat{\rho}}{}^{\hat{t}} R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} + \Lambda_{\hat{\rho}}{}^{\hat{r}}\Lambda_{\hat{\rho}}{}^{\hat{r}} R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}}$

$$= \sinh^2\beta \quad \frac{M}{r^3} + \cosh^2\beta \left(-\frac{M}{r^3}\right)$$



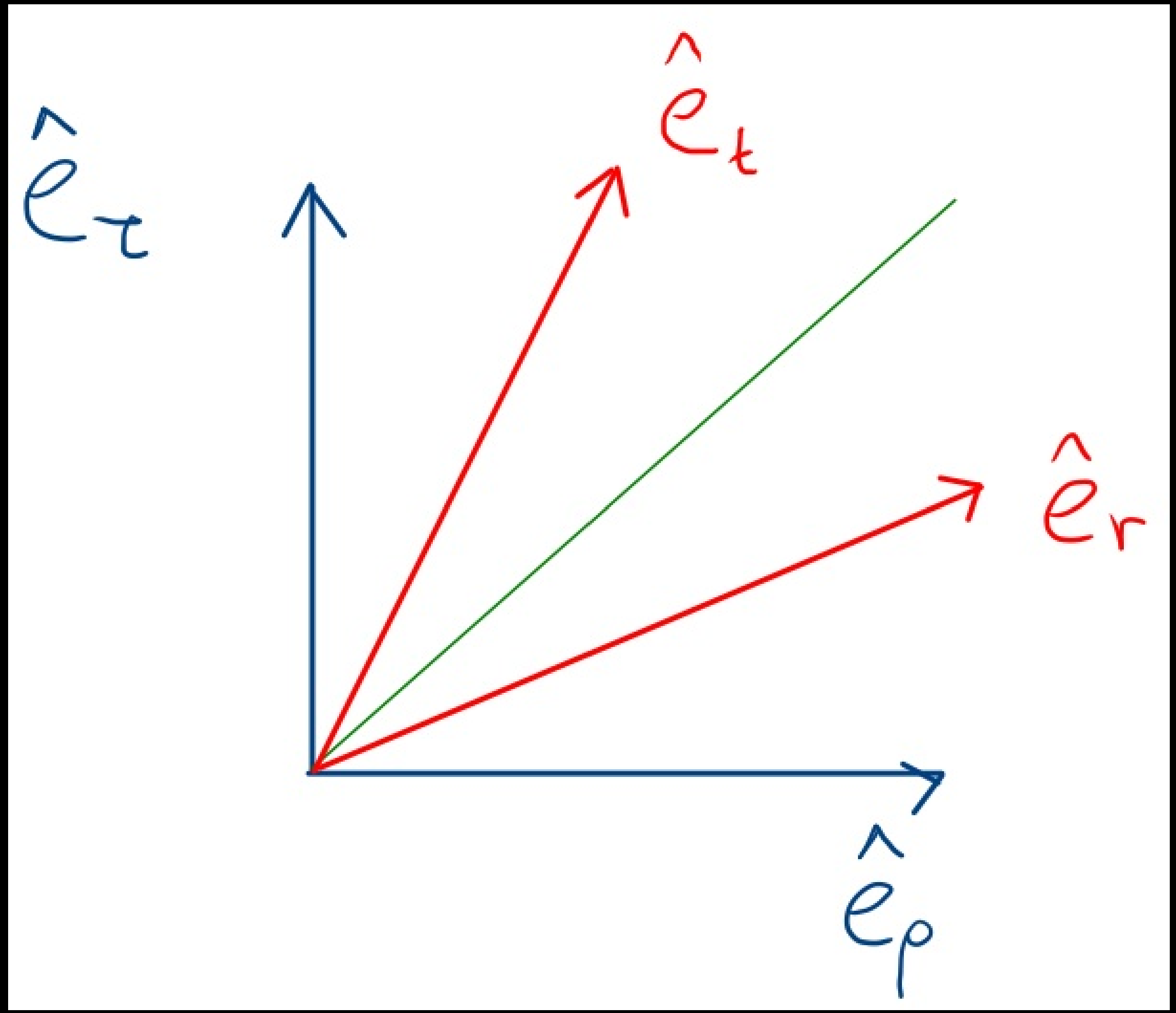
So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged

• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = R_{\hat t\hat\theta\hat t\hat\theta}$

• $R_{\hat\rho\hat\theta\hat\rho\hat\theta} = \Lambda_{\hat\rho}{}^{\hat t}\Lambda_{\hat\rho}{}^{\hat t} R_{\hat t\hat\theta\hat t\hat\theta} + \Lambda_{\hat\rho}{}^{\hat r}\Lambda_{\hat\rho}{}^{\hat r} R_{\hat r\hat\theta\hat r\hat\theta}$

$$= \sinh^2\beta \; \frac{M}{r^3} + \cosh^2\beta \left(-\frac{M}{r^3}\right)$$

$$= -\frac{M}{r^3} = R_{\hat r\hat\theta\hat r\hat\theta}$$

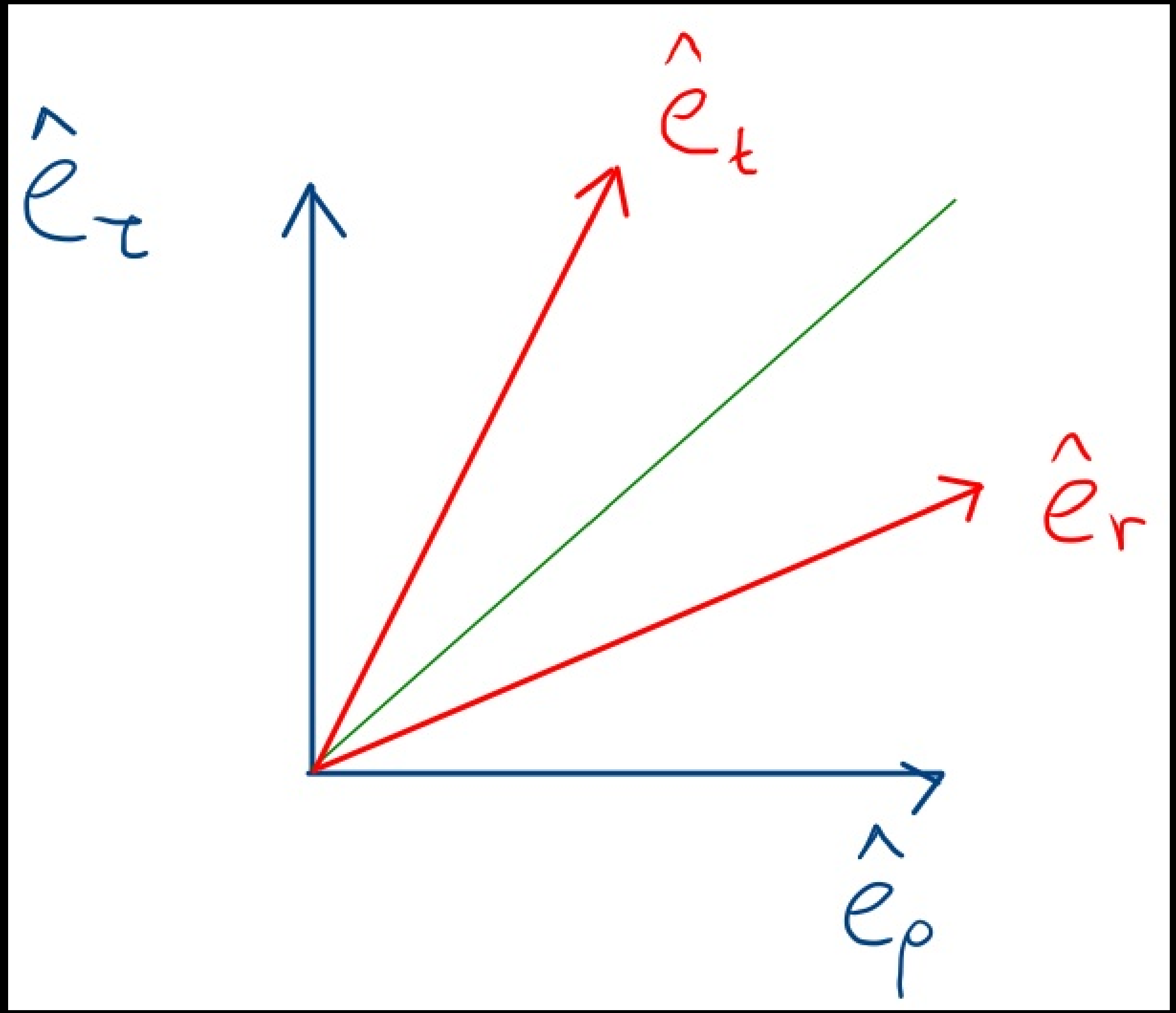


So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged

• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = R_{\hat t\hat\theta\hat t\hat\theta}$

• $R_{\hat\rho\hat\theta\hat\rho\hat\theta} = R_{\hat r\hat\theta\hat r\hat\theta}$

• $R_{\hat\rho\hat\phi\hat\rho\hat\phi} = R_{\hat r\hat\phi\hat r\hat\phi}$ (same as before!)



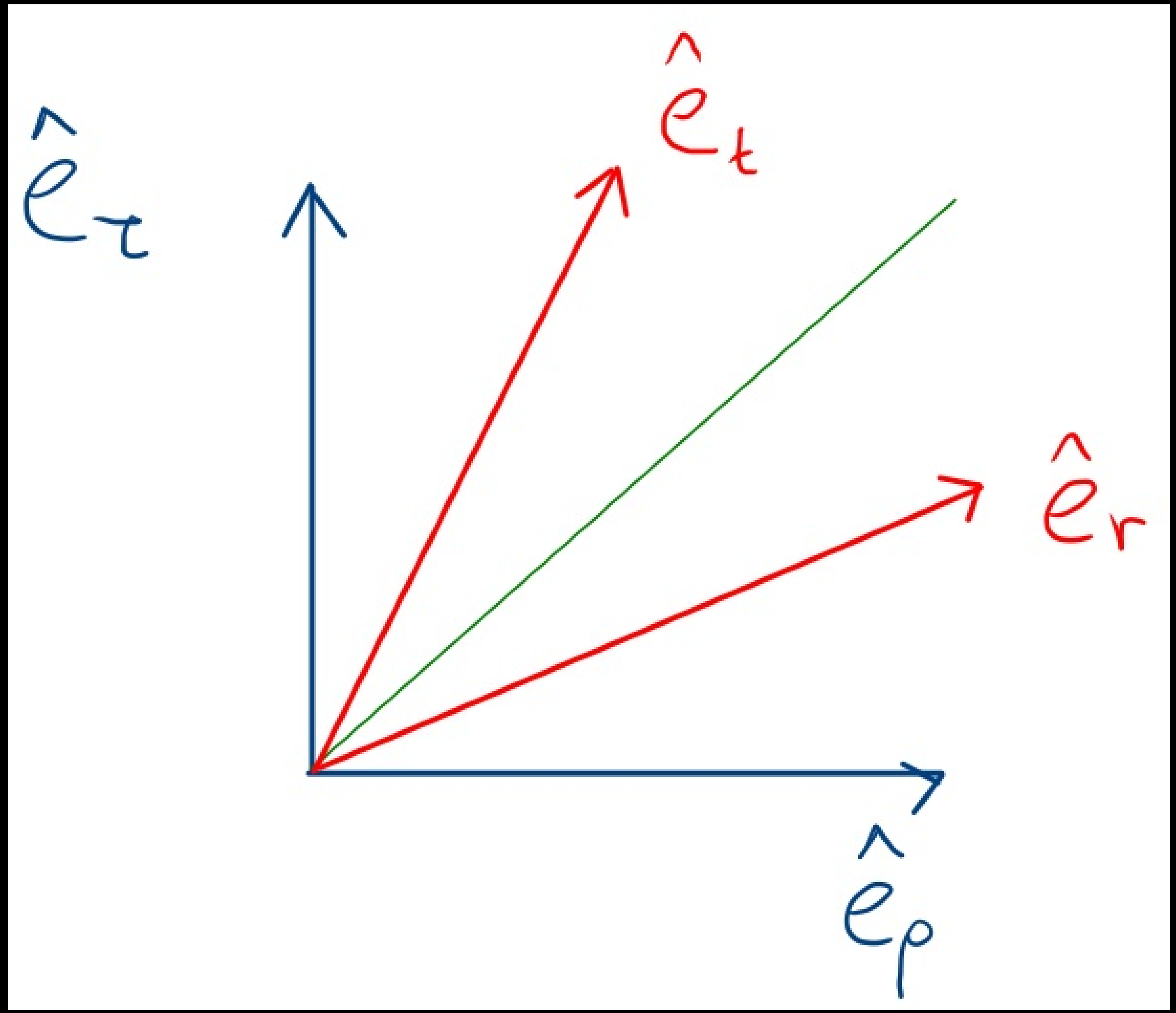
So: • $R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}}$ unchanged

• $R_{\hat{\tau}\hat{\theta}\hat{\tau}\hat{\theta}} = R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}}$

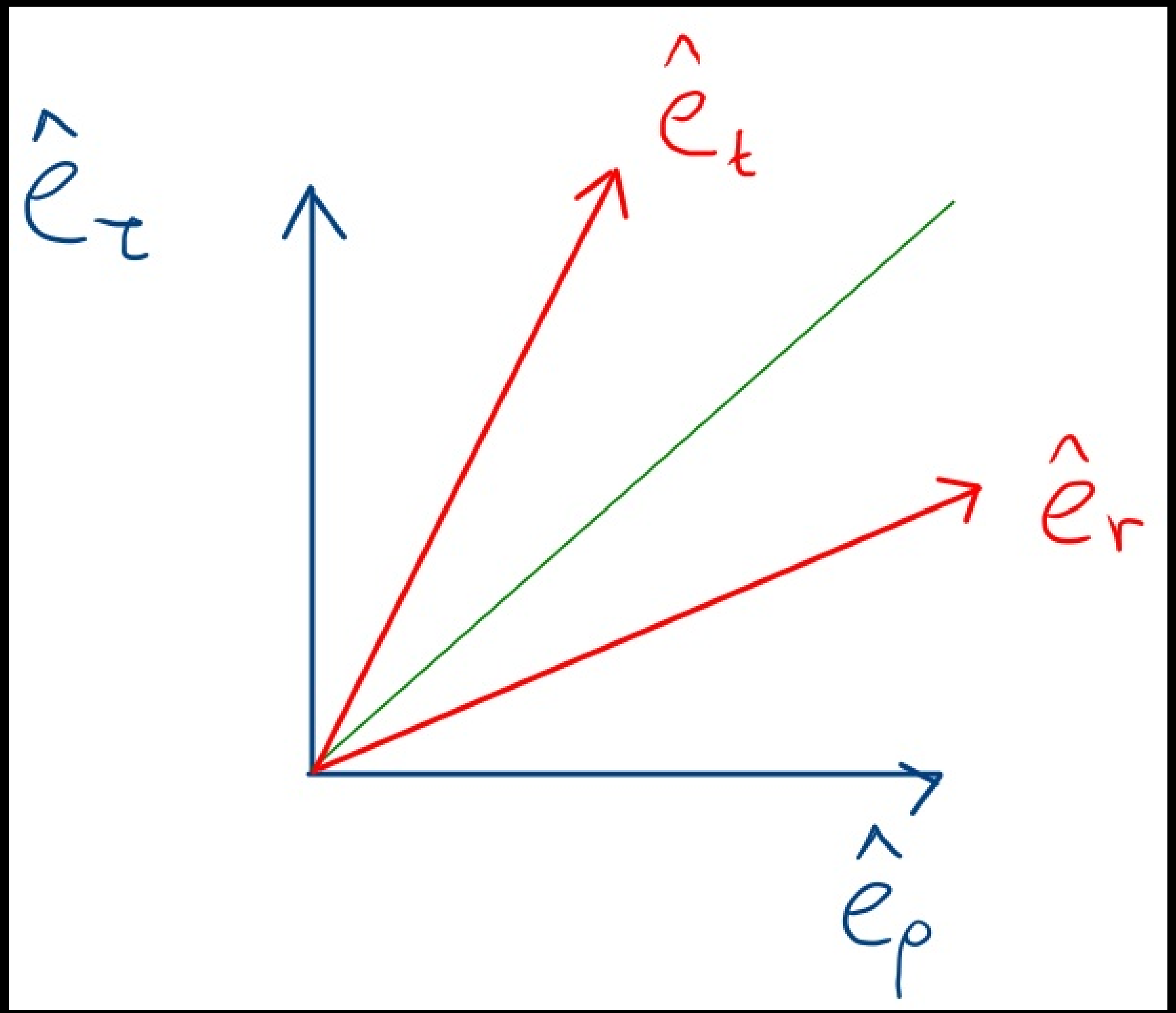
• $R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}}$

• $R_{\hat{\rho}\hat{\phi}\hat{\rho}\hat{\phi}} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}}$

• $R_{\hat{\tau}\hat{\rho}\hat{\tau}\hat{\rho}} = \Lambda_{\hat{\tau}}{}^{\hat{\mu}} \Lambda_{\hat{\rho}}{}^{\hat{\lambda}} \Lambda_{\hat{\tau}}{}^{\hat{\sigma}} \Lambda_{\hat{\rho}}{}^{\hat{\nu}} R_{\hat{\mu}\hat{\lambda}\hat{\sigma}\hat{\nu}}$



So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged



• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = R_{\hat t\hat\theta\hat t\hat\theta}$

• $R_{\hat\rho\hat\theta\hat\rho\hat\theta} = R_{\hat r\hat\theta\hat r\hat\theta}$

• $R_{\hat\rho\hat\phi\hat\rho\hat\phi} = R_{\hat r\hat\phi\hat r\hat\phi}$

• $R_{\hat\tau\hat\rho\hat\tau\hat\rho} = \Lambda^{\hat\mu}{}_{\hat\tau}\Lambda^{\hat\lambda}{}_{\hat\rho}\Lambda^{\hat\sigma}{}_{\hat\tau}\Lambda^{\hat\nu}{}_{\hat\rho}\, R_{\hat\mu\hat\lambda\hat\sigma\hat\nu}$

$$\left.\begin{aligned}
&= + \Lambda^{\hat t}{}_{\hat\tau}\Lambda^{\hat r}{}_{\hat\rho}\Lambda^{\hat t}{}_{\hat\tau}\Lambda^{\hat r}{}_{\hat\rho}\, R_{\hat t\hat r\hat t\hat r} \\
&\quad + \Lambda^{\hat t}{}_{\hat\tau}\Lambda^{\hat r}{}_{\hat\rho}\Lambda^{\hat r}{}_{\hat\tau}\Lambda^{\hat t}{}_{\hat\rho}\, R_{\hat t\hat r\hat r\hat t} \\
&\quad + \Lambda^{\hat r}{}_{\hat\tau}\Lambda^{\hat t}{}_{\hat\rho}\Lambda^{\hat t}{}_{\hat\tau}\Lambda^{\hat r}{}_{\hat\rho}\, R_{\hat r\hat t\hat t\hat r} \\
&\quad + \Lambda^{\hat r}{}_{\hat\tau}\Lambda^{\hat t}{}_{\hat\rho}\Lambda^{\hat r}{}_{\hat\tau}\Lambda^{\hat t}{}_{\hat\rho}\, R_{\hat r\hat t\hat r\hat t}
\end{aligned}\right\}\ \text{only nonzero terms (exercise!)}$$

So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged

• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = R_{\hat t\hat\theta\hat t\hat\theta}$

• $R_{\hat\rho\hat\theta\hat\rho\hat\theta} = R_{\hat r\hat\theta\hat r\hat\theta}$

• $R_{\hat\rho\hat\phi\hat\rho\hat\phi} = R_{\hat r\hat\phi\hat r\hat\phi}$

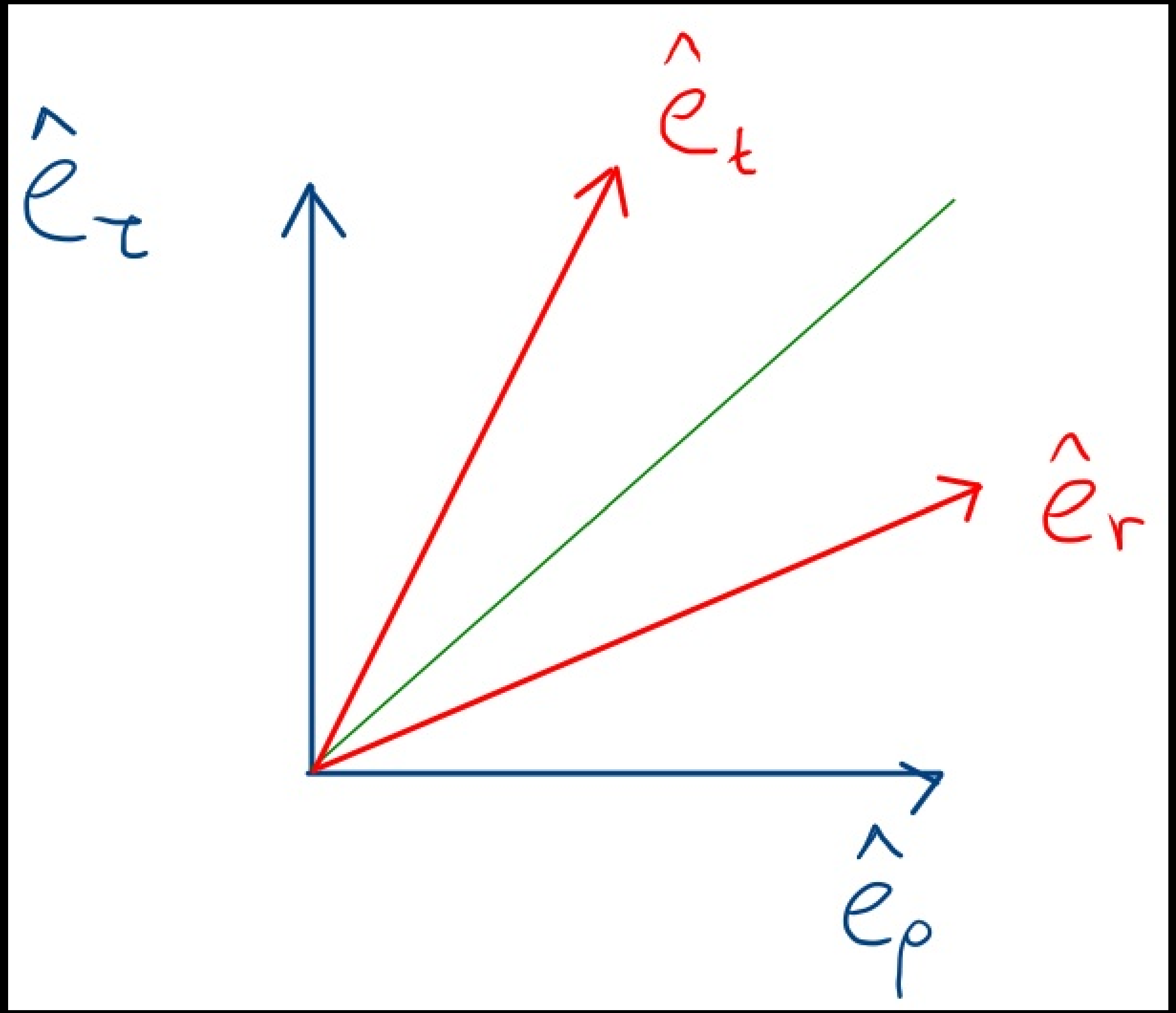
• $R_{\hat\tau\hat\rho\hat\tau\hat\rho} = \Lambda^{\hat\mu}{}_{\hat\tau}\Lambda^{\hat\lambda}{}_{\hat\rho}\Lambda^{\hat\sigma}{}_{\hat\tau}\Lambda^{\hat\nu}{}_{\hat\rho} R_{\hat\mu\hat\lambda\hat\sigma\hat\nu}$

$$= +\Lambda^{\hat t}{}_{\hat\tau}\Lambda^{\hat r}{}_{\hat\rho}\Lambda^{\hat t}{}_{\hat\tau}\Lambda^{\hat r}{}_{\hat\rho} R_{\hat t\hat r\hat t\hat r}$$

$$-\Lambda^{\hat t}{}_{\hat\tau}\Lambda^{\hat r}{}_{\hat\rho}\Lambda^{\hat r}{}_{\hat\tau}\Lambda^{\hat t}{}_{\hat\rho} R_{\hat t\hat r\hat t\hat r}$$

$$-\Lambda^{\hat r}{}_{\hat\tau}\Lambda^{\hat t}{}_{\hat\rho}\Lambda^{\hat t}{}_{\hat\tau}\Lambda^{\hat r}{}_{\hat\rho} R_{\hat t\hat r\hat t\hat r}$$

$$+\Lambda^{\hat r}{}_{\hat\tau}\Lambda^{\hat t}{}_{\hat\rho}\Lambda^{\hat r}{}_{\hat\tau}\Lambda^{\hat t}{}_{\hat\rho} R_{\hat t\hat r\hat t\hat r}$$



So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged

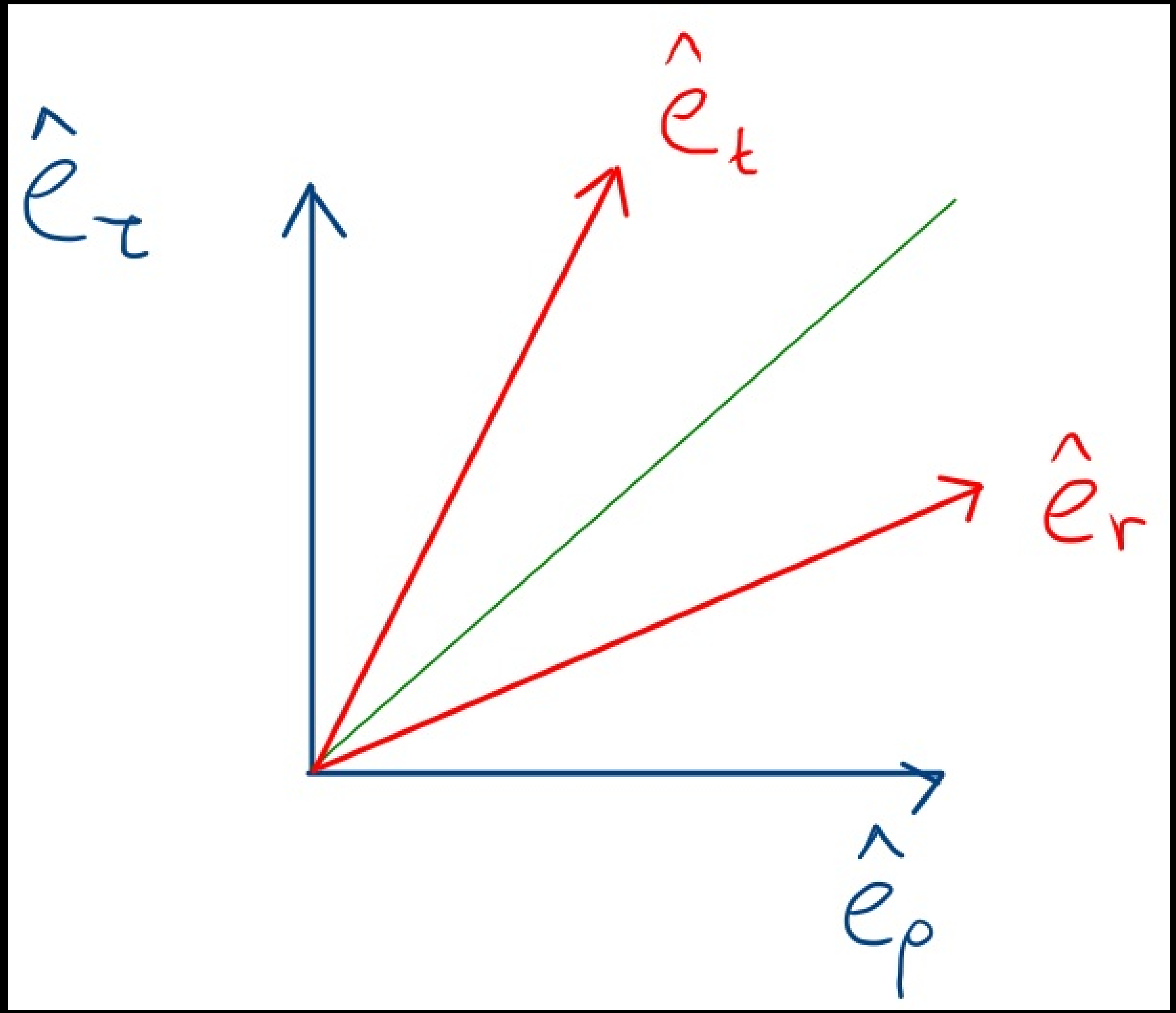
• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = R_{\hat t\hat\theta\hat t\hat\theta}$

• $R_{\hat\rho\hat\theta\hat\rho\hat\theta} = R_{\hat r\hat\theta\hat r\hat\theta}$

• $R_{\hat\rho\hat\phi\hat\rho\hat\phi} = R_{\hat r\hat\phi\hat r\hat\phi}$

• $R_{\hat\tau\hat\rho\hat\tau\hat\rho} = \Lambda^{\hat\mu}{}_{\hat\tau}\Lambda^{\hat\lambda}{}_{\hat\rho}\Lambda^{\hat\sigma}{}_{\hat\tau}\Lambda^{\hat\nu}{}_{\hat\rho} R_{\hat\mu\hat\lambda\hat\sigma\hat\nu}$

$$
\begin{aligned}
&= + \cosh^4\beta \, R_{\hat t\hat r\hat t\hat r} \\
&\quad - \cosh^2\beta \sinh^2\beta \, R_{\hat t\hat r\hat t\hat r} \\
&\quad - \cosh^2\beta \sinh^2\beta \, R_{\hat t\hat r\hat t\hat r} \\
&\quad + \sinh^4\beta \, R_{\hat t\hat r\hat t\hat r}
\end{aligned}
$$

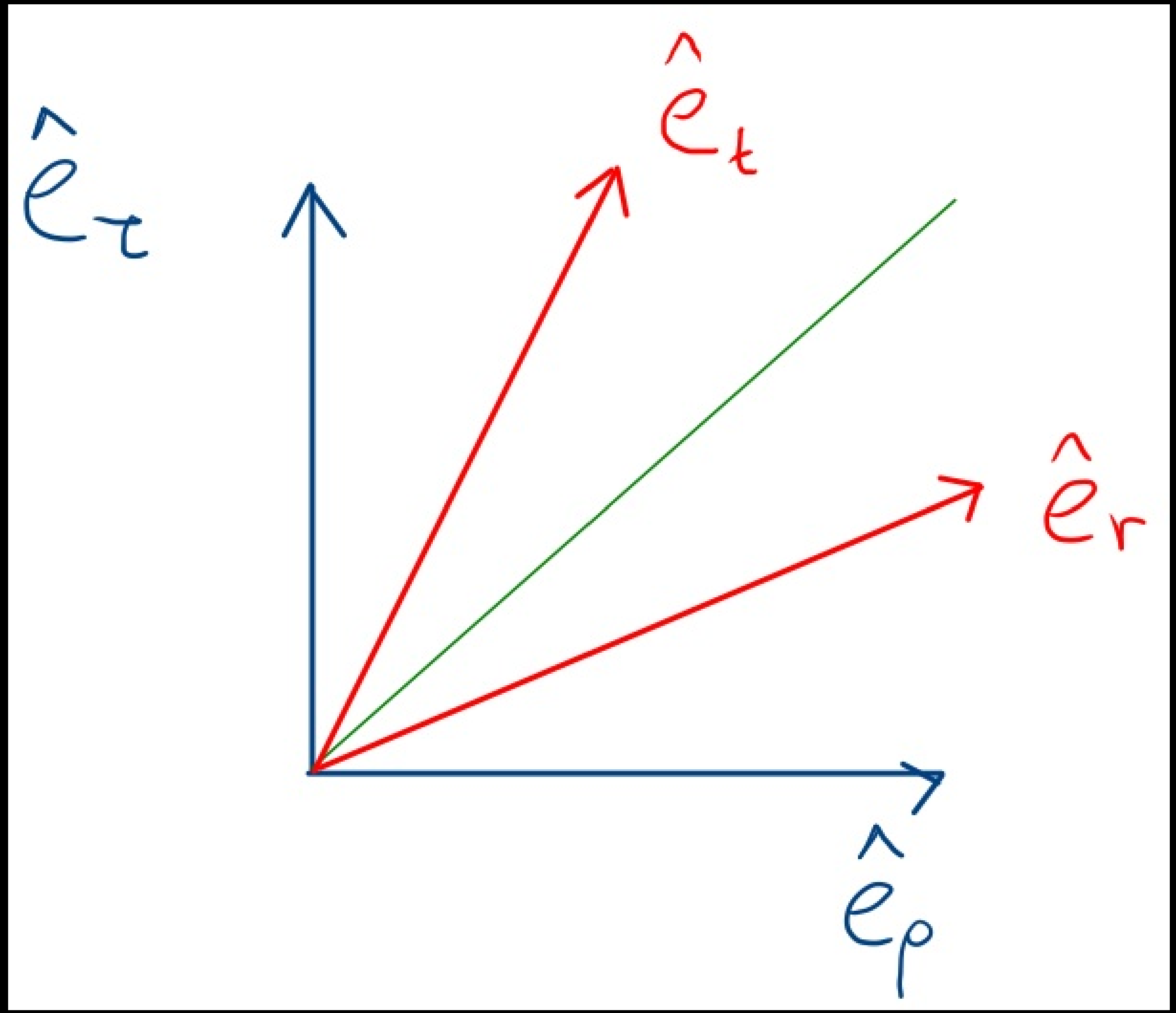


So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged

• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = R_{\hat t\hat\theta\hat t\hat\theta}$

• $R_{\hat\rho\hat\theta\hat\rho\hat\theta} = R_{\hat r\hat\theta\hat r\hat\theta}$

• $R_{\hat\rho\hat\phi\hat\rho\hat\phi} = R_{\hat r\hat\phi\hat r\hat\phi}$



• $R_{\hat\tau\hat\rho\hat\tau\hat\rho} = \Lambda_{\hat\tau}{}^{\hat\mu}\Lambda_{\hat\rho}{}^{\hat\lambda}\Lambda_{\hat\tau}{}^{\hat\sigma}\Lambda_{\hat\rho}{}^{\hat\nu} R_{\hat\mu\hat\lambda\hat\sigma\hat\nu}$

$$= + \cosh^4\beta \quad R_{\hat t\hat r\hat t\hat r}$$
$$- \cosh^2\beta \sinh^2\beta \quad R_{\hat t\hat r\hat t\hat r}$$
$$- \cosh^2\beta \sinh^2\beta \quad R_{\hat t\hat r\hat t\hat r} \qquad = 1 \cdot R_{\hat t\hat r\hat t\hat r}$$
$$+ \sinh^4\beta \quad R_{\hat t\hat r\hat t\hat r}$$

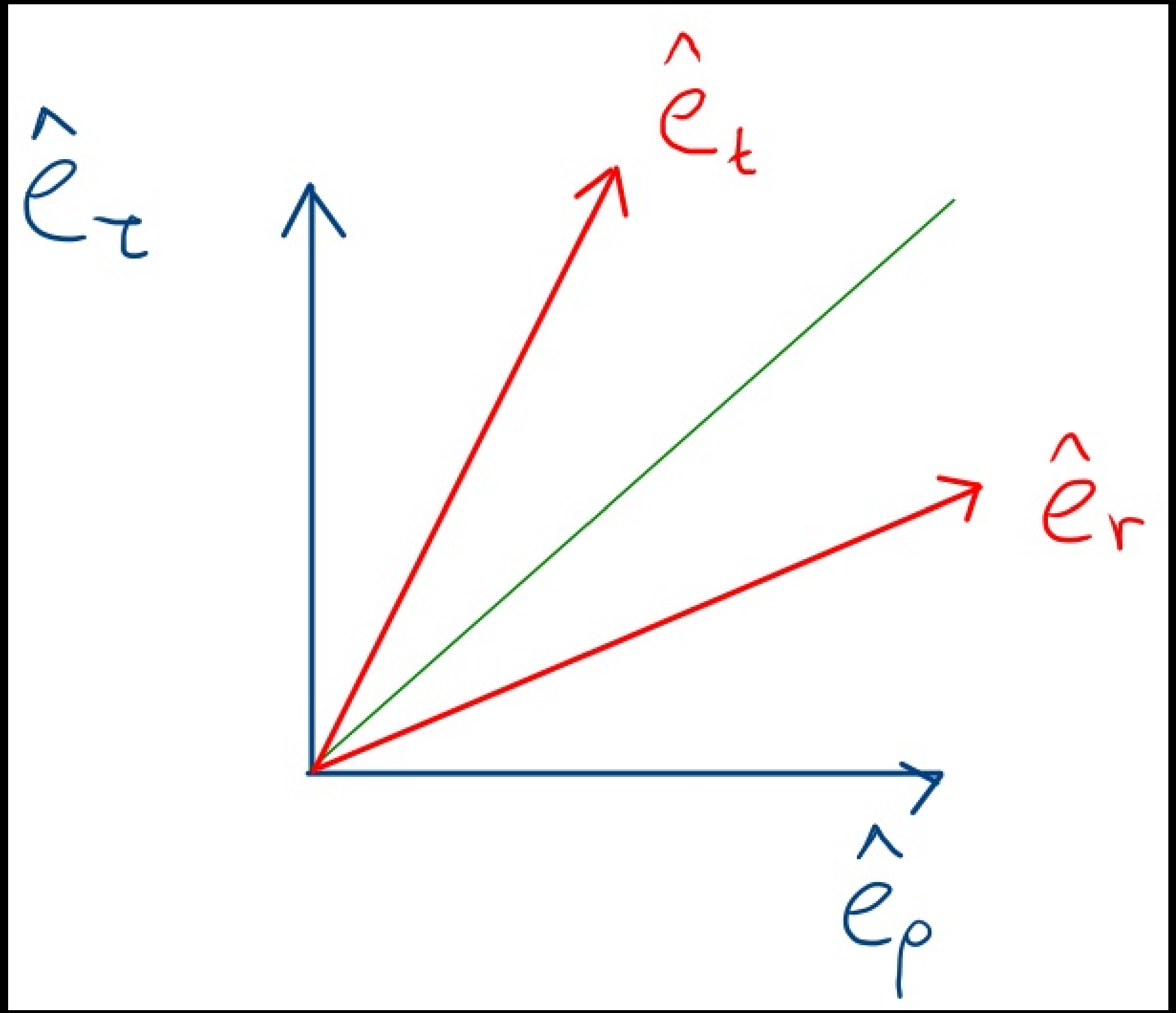
So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged

• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = R_{\hat t\hat\theta\hat t\hat\theta}$

• $R_{\hat\rho\hat\theta\hat\rho\hat\theta} = R_{\hat r\hat\theta\hat r\hat\theta}$

• $R_{\hat\rho\hat\phi\hat\rho\hat\phi} = R_{\hat r\hat\phi\hat r\hat\phi}$

• $R_{\hat\tau\hat\rho\hat\tau\hat\rho} = R_{\hat t\hat r\hat t\hat r}$



• Components of R are invariant under this Lorentz boost!

• The freely falling observer measures the same R-components!

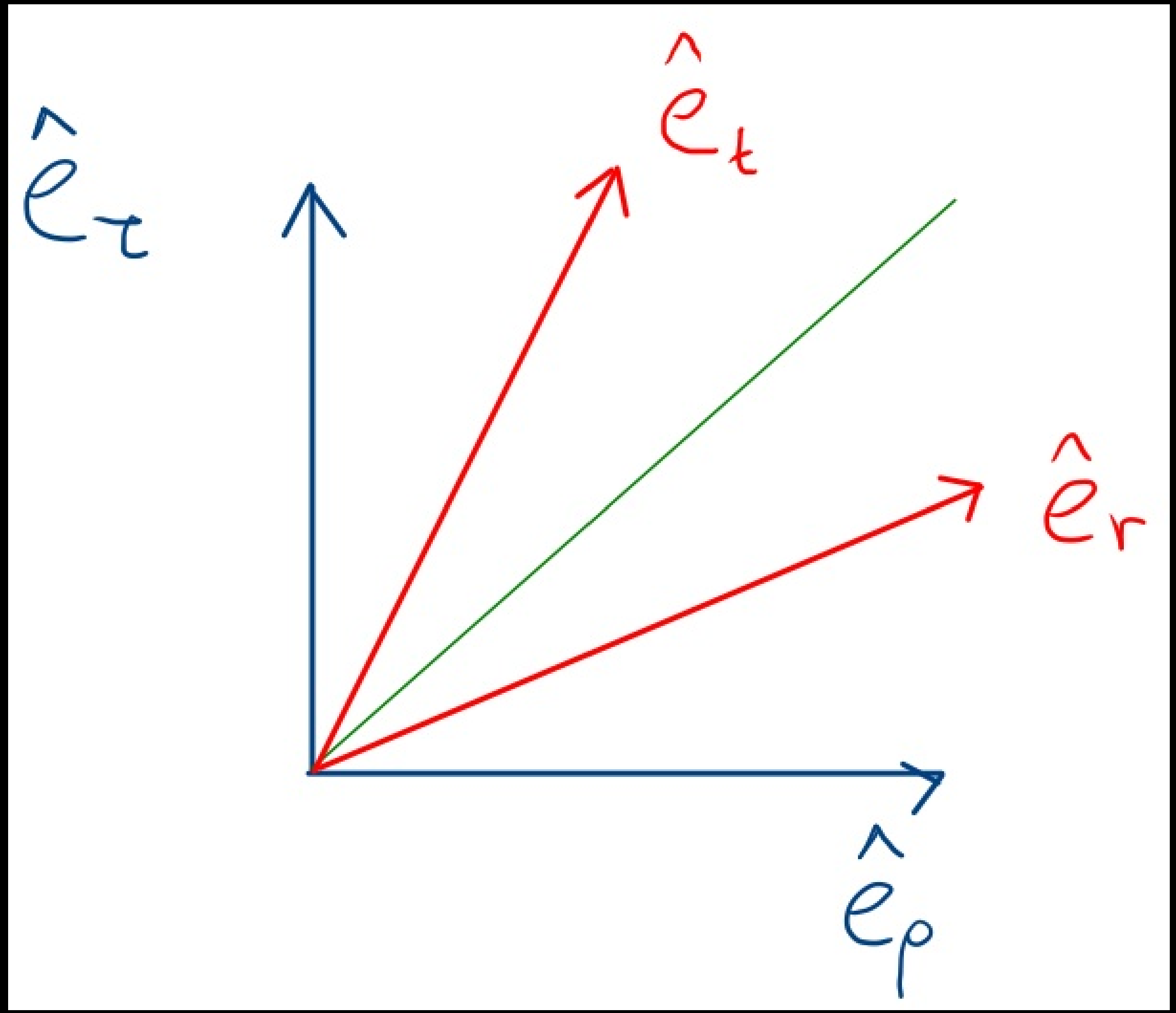
So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged

• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = R_{\hat t\hat\theta\hat t\hat\theta}$

• $R_{\hat\rho\hat\theta\hat\rho\hat\theta} = R_{\hat r\hat\theta\hat r\hat\theta}$

• $R_{\hat\rho\hat\phi\hat\rho\hat\phi} = R_{\hat r\hat\phi\hat r\hat\phi}$

• $R_{\hat\tau\hat\rho\hat\tau\hat\rho} = R_{\hat t\hat r\hat t\hat r}$



• special for the Schwarzschild metric:

- structure of non zero $R_{\hat\mu\hat\nu\hat\lambda\hat\sigma}$ components

- values of components, e.g. we used $R_{\hat t\hat\theta\hat t\hat\theta} = -R_{\hat r\hat\theta\hat r\hat\theta}$, etc use results from

$\{e_{\hat{t}}, e_{\hat{r}}, e_{\hat{\theta}}, e_{\hat{\phi}}\}$

calculation

$$\left\{ \begin{aligned} -R_{\hat{\rho}\hat{\tau}\hat{\rho}\hat{\tau}} &= R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3} \\ R_{\hat{\theta}\hat{\tau}\hat{\theta}\hat{\tau}} &= R_{\hat{\phi}\hat{\tau}\hat{\phi}\hat{\tau}} = \frac{M}{r^3} \\ R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} &= R_{\hat{\rho}\hat{\phi}\hat{\rho}\hat{\phi}} = -\frac{M}{r^3} \end{aligned} \right.$$

$$\frac{D^2\xi^{\hat{\tau}}}{d\tau^2} = -\underbrace{R^{\hat{\tau}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}}_{\text{no nonzero } R \text{ with 3 } \hat{t}\text{s}}\xi^{\hat{\sigma}} = 0$$

$$-R_{\hat{\rho}\hat{\tau}\hat{\rho}\hat{\tau}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{\tau}\hat{\theta}\hat{\tau}} = R_{\hat{\phi}\hat{\tau}\hat{\phi}\hat{\tau}} = \frac{M}{r^3}$$

$$R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = R_{\hat{\rho}\hat{\phi}\hat{\rho}\hat{\phi}} = -\frac{M}{r^3}$$

- Relative accelerations given by geodesic deviation equation:

$$\frac{D^2\xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}}u^{\hat{\nu}}u^{\hat{\lambda}}\xi^{\hat{\sigma}} = -R^{\hat{\mu}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\xi^{\hat{\sigma}}$$

antisymmetric change

$$\frac{D^2 \xi^{\hat{\tau}}}{d\tau^2} = - R^{\hat{\tau}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}} \xi^{\hat{\sigma}} = 0$$

$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} = - R^{\hat{\rho}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}} \xi^{\hat{\sigma}} = - R_{\hat{\rho}\hat{\tau}\hat{\rho}\hat{\tau}} \xi^{\hat{r}}$$

lower index with $\eta_{\hat{\mu}\hat{\nu}}$

$$-R_{\hat{\rho}\hat{\tau}\hat{\rho}\hat{\tau}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{\tau}\hat{\theta}\hat{\tau}} = R_{\hat{\phi}\hat{\tau}\hat{\phi}\hat{\tau}} = \frac{M}{r^3}$$

$$R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = R_{\hat{\rho}\hat{\phi}\hat{\rho}\hat{\phi}} = -\frac{M}{r^3}$$

- Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}} u^{\hat{\nu}} u^{\hat{\lambda}} \xi^{\hat{\sigma}} = - R^{\hat{\mu}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}} \xi^{\hat{\sigma}}$$

$$\frac{D^2 \xi^{\hat{\tau}}}{d\tau^2} = - R^{\hat{\tau}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}} \xi^{\hat{\sigma}} = 0$$

$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} = - R^{\hat{\rho}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}} \xi^{\hat{\sigma}} = - R_{\hat{\rho}\hat{\tau}\hat{\rho}\hat{\tau}} \xi^{\hat{r}}$$

$$= -\left(-\frac{2M}{r^3}\right) \xi^{\hat{\rho}} = + \frac{2M}{r^3} \xi^{\hat{\rho}}$$

$$-R_{\hat{\rho}\hat{\tau}\hat{\rho}\hat{\tau}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{\tau}\hat{\theta}\hat{\tau}} = R_{\hat{\phi}\hat{\tau}\hat{\phi}\hat{\tau}} = \frac{M}{r^3}$$

$$R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = R_{\hat{\rho}\hat{\phi}\hat{\rho}\hat{\phi}} = -\frac{M}{r^3}$$

- Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}} u^{\hat{\nu}} u^{\hat{\lambda}} \xi^{\hat{\sigma}} = -R^{\hat{\mu}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}} \xi^{\hat{\sigma}}$$

$$\frac{D^2 \xi^{\hat{\tau}}}{d\tau^2} = -R^{\hat{\tau}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\, \xi^{\hat{\sigma}} = 0$$

$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} = \frac{2M}{r^3}\, \xi^{\hat{\rho}}$$

↳ relative acceleration of head w.r.t. waist

$$-R_{\hat{\rho}\hat{\tau}\hat{\rho}\hat{\tau}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{\tau}\hat{\theta}\hat{\tau}} = R_{\hat{\phi}\hat{\tau}\hat{\phi}\hat{\tau}} = \frac{M}{r^3}$$

$$R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = R_{\hat{\rho}\hat{\phi}\hat{\rho}\hat{\phi}} = -\frac{M}{r^3}$$

- Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}}\, u^{\hat{\nu}} u^{\hat{\lambda}} \xi^{\hat{\sigma}} = -R^{\hat{\mu}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\, \xi^{\hat{\sigma}}$$

$$\frac{D^2\xi^{\hat\tau}}{d\tau^2} = -R^{\hat\tau}{}_{\hat\tau\hat\sigma\hat\tau}\,\xi^{\hat\sigma} = 0$$

$$\frac{D^2\xi^{\hat\rho}}{d\tau^2} = \frac{2M}{r^3}\,\xi^{\hat\rho}$$

↳ relative acceleration of head w.r.t. waist

→ there is no $\left(1-\frac{2M}{r}\right)$

$$-R_{\hat\rho\hat\tau\hat\rho\hat\tau} = R_{\hat\theta\hat\phi\hat\theta\hat\phi} = \frac{2M}{r^3}$$

$$R_{\hat\theta\hat\tau\hat\theta\hat\tau} = R_{\hat\phi\hat\tau\hat\phi\hat\tau} = \frac{M}{r^3}$$

$$R_{\hat\rho\hat\theta\hat\rho\hat\theta} = R_{\hat\rho\hat\phi\hat\rho\hat\phi} = -\frac{M}{r^3}$$

- Relative accelerations given by geodesic deviation equation:

$$\frac{D^2\xi^{\hat\mu}}{d\tau^2} = R^{\hat\mu}{}_{\hat\nu\hat\lambda\hat\sigma}\,u^{\hat\nu}u^{\hat\lambda}\xi^{\hat\sigma} = -R^{\hat\mu}{}_{\hat\tau\hat\sigma\hat\tau}\,\xi^{\hat\sigma}$$

$$\frac{D^2 \xi^{\hat{\tau}}}{d\tau^2} = -R^{\hat{\tau}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\, \xi^{\hat{\sigma}} = 0$$

$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} = \frac{2M}{r^3}\, \xi^{\hat{\rho}}$$

↳ relative acceleration of head w.r.t. waist

→ there is no $\left(1-\frac{2M}{r}\right)$

→ blows up as $r \to 0$

$$-R_{\hat{\rho}\hat{\tau}\hat{\rho}\hat{\tau}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{\tau}\hat{\theta}\hat{\tau}} = R_{\hat{\phi}\hat{\tau}\hat{\phi}\hat{\tau}} = \frac{M}{r^3}$$

$$R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = R_{\hat{\rho}\hat{\phi}\hat{\rho}\hat{\phi}} = -\frac{M}{r^3}$$

– Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}}\, u^{\hat{\nu}} u^{\hat{\lambda}} \xi^{\hat{\sigma}} = -R^{\hat{\mu}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\, \xi^{\hat{\sigma}}$$

$$\frac{D^2 \xi^{\hat\tau}}{d\tau^2} = -R^{\hat\tau}{}_{\hat\tau\hat\sigma\hat\tau}\,\xi^{\hat\sigma} = 0$$

$$\frac{D^2 \xi^{\hat\rho}}{d\tau^2} = \frac{2M}{r^3}\,\xi^{\hat\rho}$$

↳ relative acceleration of head w.r.t. waist

→ there is no $\left(1 - \frac{2M}{r}\right)$

→ blows up as $r \to 0$

→ head moves away from feet

$$-R_{\hat\rho\hat\tau\hat\rho\hat\tau} = R_{\hat\theta\hat\phi\hat\theta\hat\phi} = \frac{2M}{r^3}$$

$$R_{\hat\theta\hat\tau\hat\theta\hat\tau} = R_{\hat\phi\hat\tau\hat\phi\hat\tau} = \frac{M}{r^3}$$

$$R_{\hat\rho\hat\theta\hat\rho\hat\theta} = R_{\hat\rho\hat\phi\hat\rho\hat\phi} = -\frac{M}{r^3}$$

– Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat\mu}}{d\tau^2} = R^{\hat\mu}{}_{\hat\nu\hat\lambda\hat\sigma}\,u^{\hat\nu} u^{\hat\lambda}\,\xi^{\hat\sigma} = -R^{\hat\mu}{}_{\hat\tau\hat\sigma\hat\tau}\,\xi^{\hat\sigma}$$

$$\frac{D^2 \xi^{\hat{\tau}}}{d\tau^2} = -R^{\hat{\tau}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\,\xi^{\hat{\sigma}} = 0$$

$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} = \frac{2M}{r^3}\,\xi^{\hat{\rho}}$$

$$\frac{D^2 \xi^{\hat{\theta}}}{d\tau^2} = -R^{\hat{\theta}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\,\xi^{\hat{\sigma}} = -R_{\hat{\theta}\hat{\tau}\hat{\theta}\hat{\tau}}\,\xi^{\hat{\theta}}$$

$$= -\frac{M}{r^3}\,\xi^{\hat{\theta}}$$

$$-R_{\hat{\rho}\hat{\tau}\hat{\rho}\hat{\tau}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{\tau}\hat{\theta}\hat{\tau}} = R_{\hat{\phi}\hat{\tau}\hat{\phi}\hat{\tau}} = \frac{M}{r^3}$$

$$R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = R_{\hat{\rho}\hat{\phi}\hat{\rho}\hat{\phi}} = -\frac{M}{r^3}$$

- Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}}\,u^{\hat{\nu}}u^{\hat{\lambda}}\,\xi^{\hat{\sigma}} = -R^{\hat{\mu}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\,\xi^{\hat{\sigma}}$$

$$\frac{D^2 \xi^{\hat{\tau}}}{d\tau^2} = -R^{\hat{\tau}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\,\xi^{\hat{\sigma}} = 0$$

$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} = \frac{2M}{r^3}\,\xi^{\hat{\rho}}$$

$$\frac{D^2 \xi^{\hat{\theta}}}{d\tau^2} = -\frac{M}{r^3}\,\xi^{\hat{\theta}}$$

$$-R_{\hat{\rho}\hat{\tau}\hat{\rho}\hat{\tau}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{\tau}\hat{\theta}\hat{\tau}} = R_{\hat{\phi}\hat{\tau}\hat{\phi}\hat{\tau}} = \frac{M}{r^3}$$

$$R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = R_{\hat{\rho}\hat{\phi}\hat{\rho}\hat{\phi}} = -\frac{M}{r^3}$$

- Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}}\,u^{\hat{\nu}}u^{\hat{\lambda}}\xi^{\hat{\sigma}} = -R^{\hat{\mu}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\,\xi^{\hat{\sigma}}$$

$$\frac{D^2\xi^{\hat{\tau}}}{d\tau^2} = -R^{\hat{\tau}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\,\xi^{\hat{\sigma}} = 0$$

$$\frac{D^2\xi^{\hat{\rho}}}{d\tau^2} = \frac{2M}{r^3}\,\xi^{\hat{\rho}}$$

$$\frac{D^2\xi^{\hat{\theta}}}{d\tau^2} = -\frac{M}{r^3}\,\xi^{\hat{\theta}}$$

→ nothing wrong at $r=2M$

→ left + right hand approach squeezed as $r \to 0$

$$-R_{\hat{\rho}\hat{\tau}\hat{\rho}\hat{\tau}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{\tau}\hat{\theta}\hat{\tau}} = R_{\hat{\phi}\hat{\tau}\hat{\phi}\hat{\tau}} = \frac{M}{r^3}$$

$$R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = R_{\hat{\rho}\hat{\phi}\hat{\rho}\hat{\phi}} = -\frac{M}{r^3}$$

– Relative accelerations given by geodesic deviation equation:

$$\frac{D^2\xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}}\,u^{\hat{\nu}}u^{\hat{\lambda}}\xi^{\hat{\sigma}} = -R^{\hat{\mu}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\,\xi^{\hat{\sigma}}$$

$$\frac{D^2\xi^{\hat{\tau}}}{d\tau^2} = -R^{\hat{\tau}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\,\xi^{\hat{\sigma}} = 0$$

$$\frac{D^2\xi^{\hat{\rho}}}{d\tau^2} = \frac{2M}{r^3}\,\xi^{\hat{\rho}}$$

$$\frac{D^2\xi^{\hat{\theta}}}{d\tau^2} = -\frac{M}{r^3}\,\xi^{\hat{\theta}}$$

$$\frac{D^2\xi^{\hat{\phi}}}{d\tau^2} = -\frac{M}{r^3}\,\xi^{\hat{\phi}}$$

$$-R_{\hat{\rho}\hat{\tau}\hat{\rho}\hat{\tau}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{\tau}\hat{\theta}\hat{\tau}} = R_{\hat{\phi}\hat{\tau}\hat{\phi}\hat{\tau}} = \frac{M}{r^3}$$

$$R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = R_{\hat{\rho}\hat{\phi}\hat{\rho}\hat{\phi}} = -\frac{M}{r^3}$$

– Relative accelerations given by geodesic deviation equation:

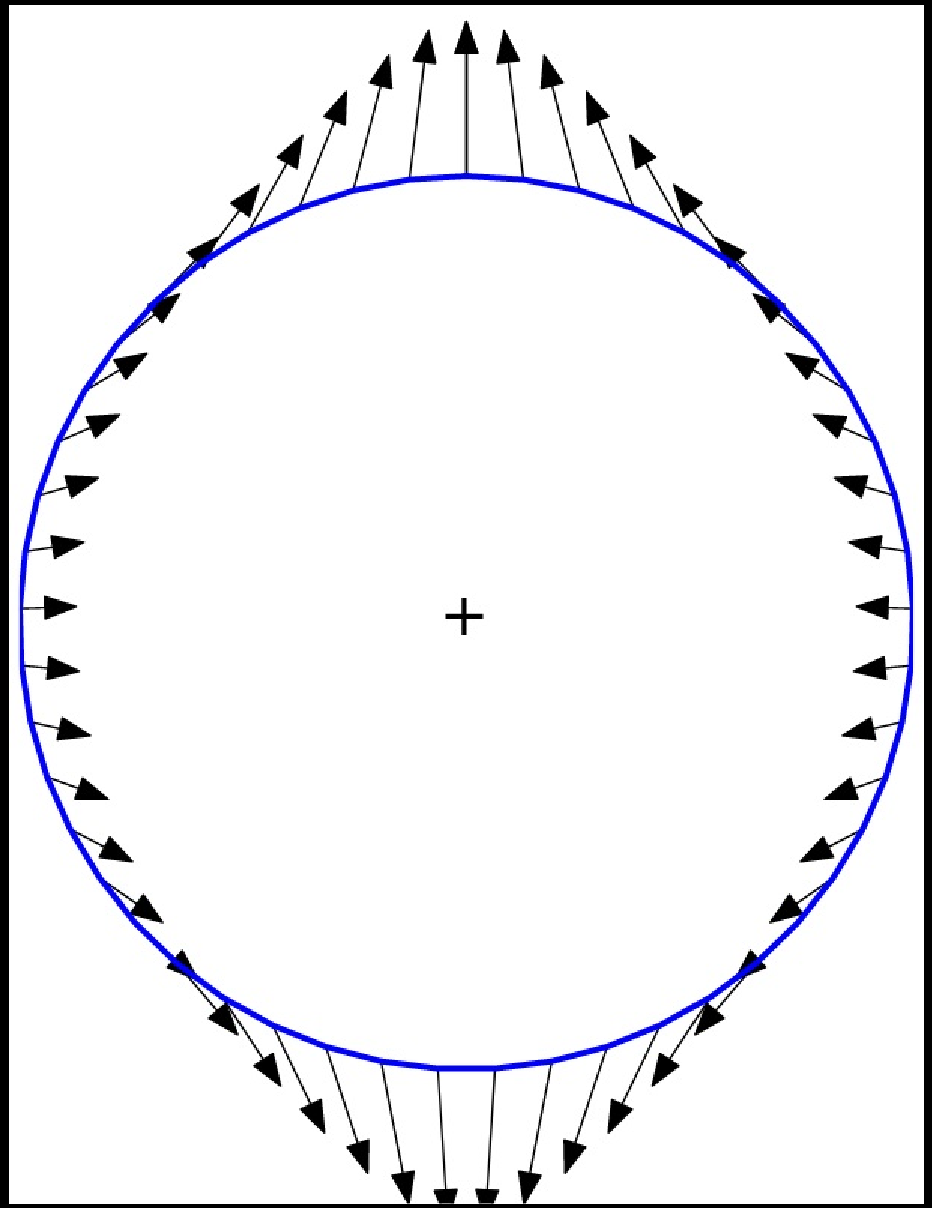
$$\frac{D^2\xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}}\,u^{\hat{\nu}}u^{\hat{\lambda}}\,\xi^{\hat{\sigma}} = -R^{\hat{\mu}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\,\xi^{\hat{\sigma}}$$

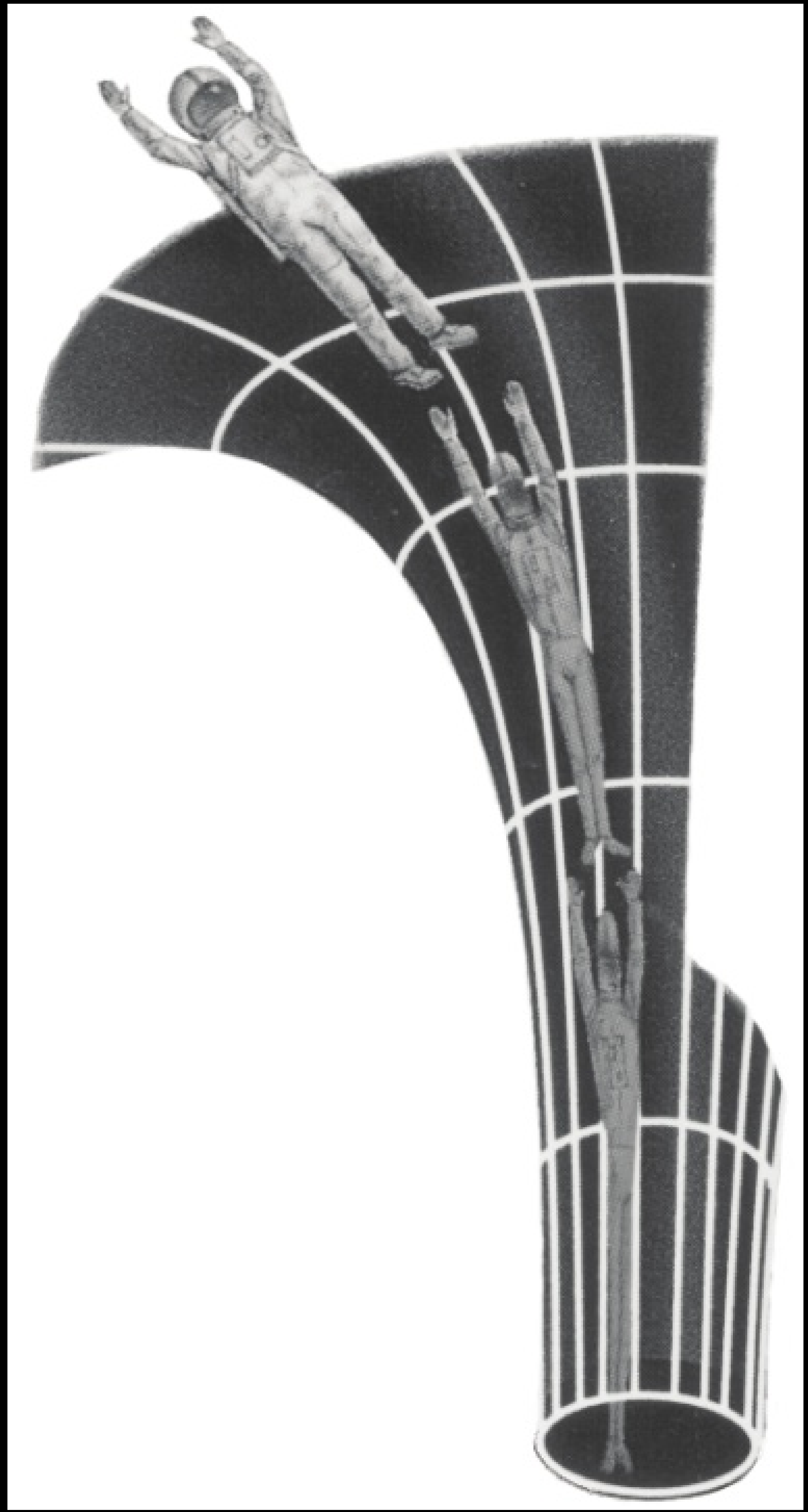
$$\frac{D^2 \xi^{\hat{\tau}}}{d\tau^2} = - R^{\hat{\tau}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\, \xi^{\hat{\sigma}} = 0$$

$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} = \frac{2M}{r^3}\, \xi^{\hat{\rho}}$$

$$\frac{D^2 \xi^{\hat{\theta}}}{d\tau^2} = -\frac{M}{r^3}\, \xi^{\hat{\theta}}$$

$$\frac{D^2 \xi^{\hat{\phi}}}{d\tau^2} = -\frac{M}{r^3}\, \xi^{\hat{\phi}}$$





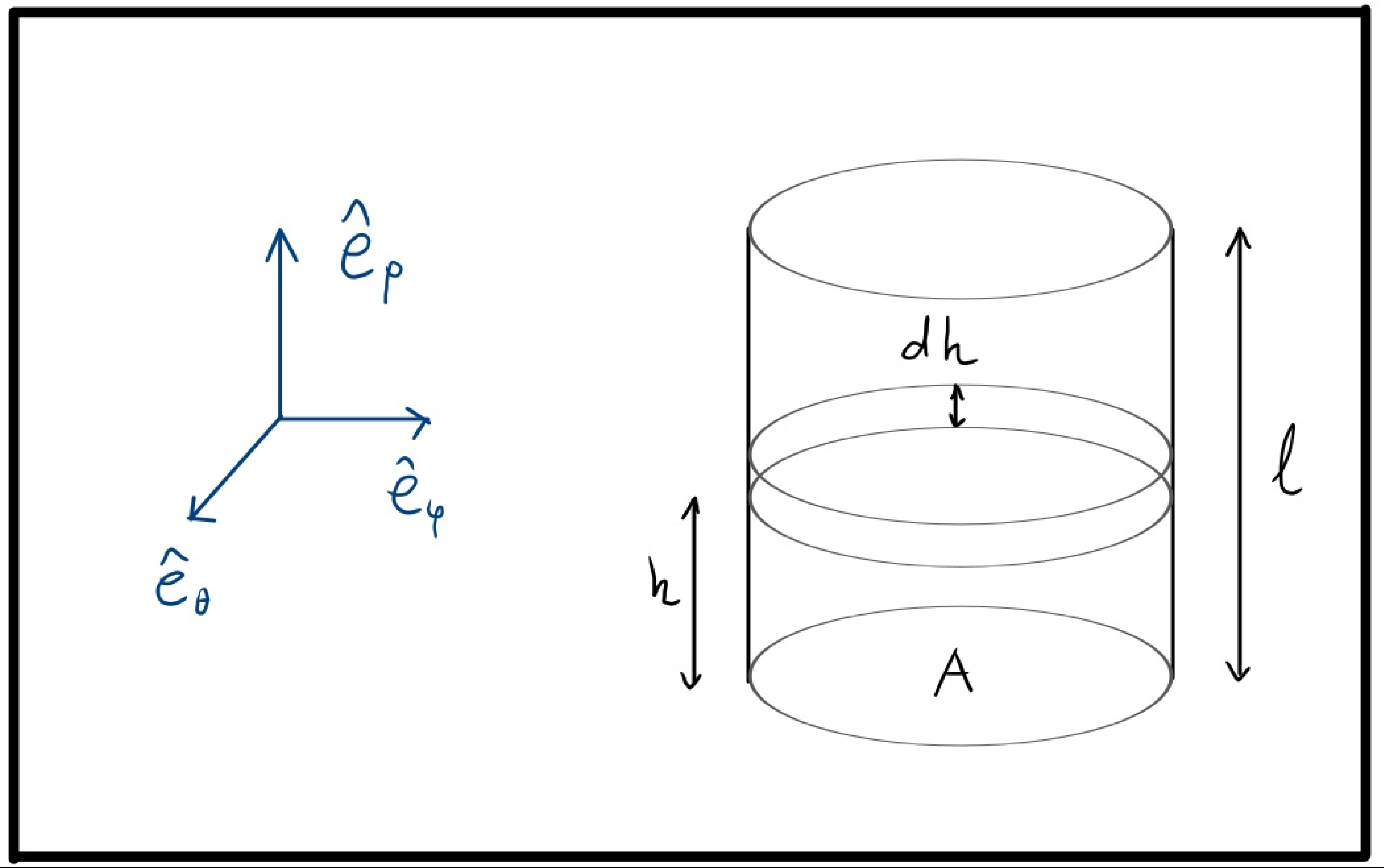
spaghettification ...

$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} = \frac{2M}{r^3} \xi^{\hat{\rho}}$$

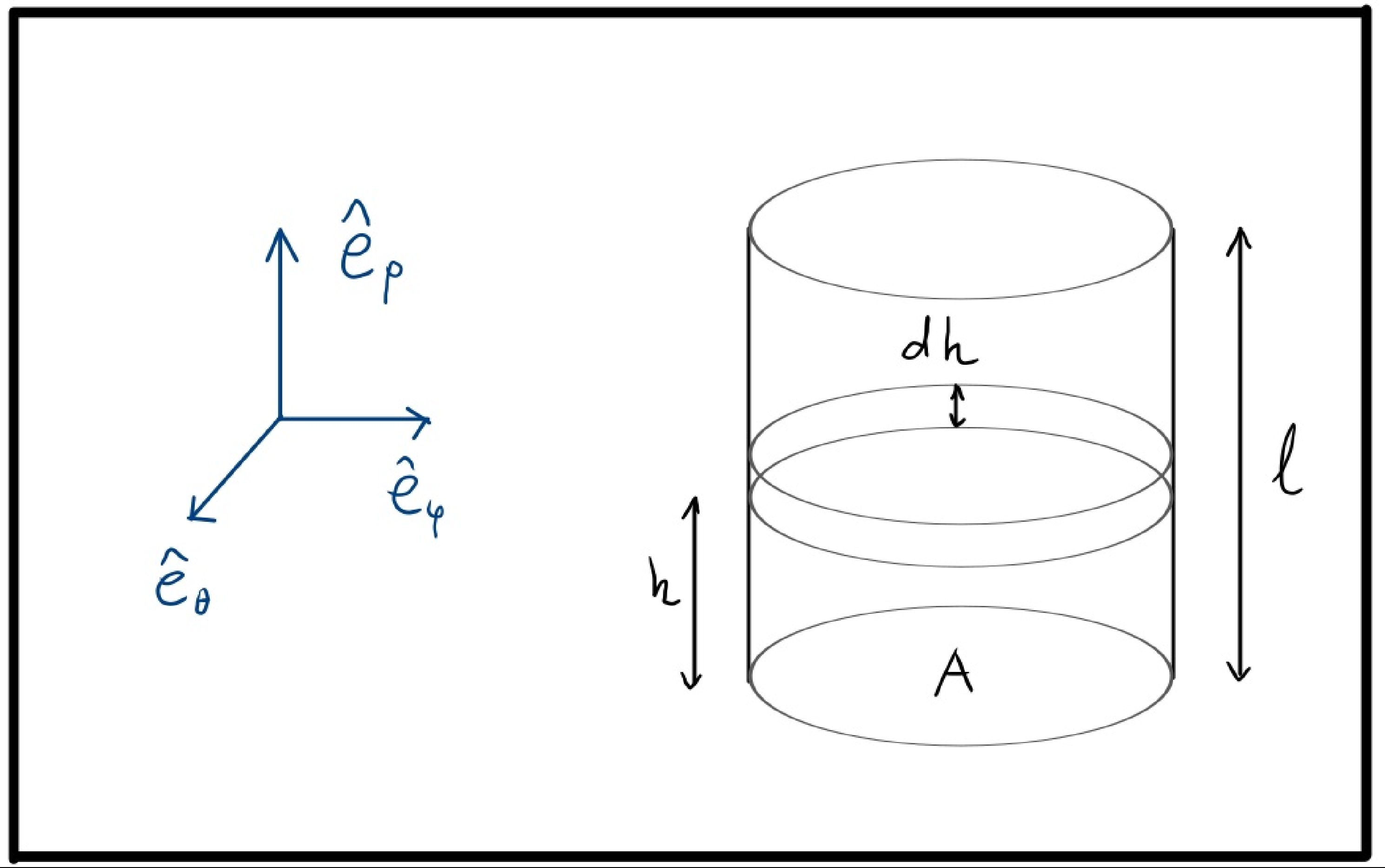
Relative acceleration of disks at distance h:

$$a = \frac{2M}{r^3} \cdot h$$

$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} \qquad \qquad \xi^{\hat{\rho}}$$



$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} = \frac{2M}{r^3} \xi^{\hat{\rho}}$$



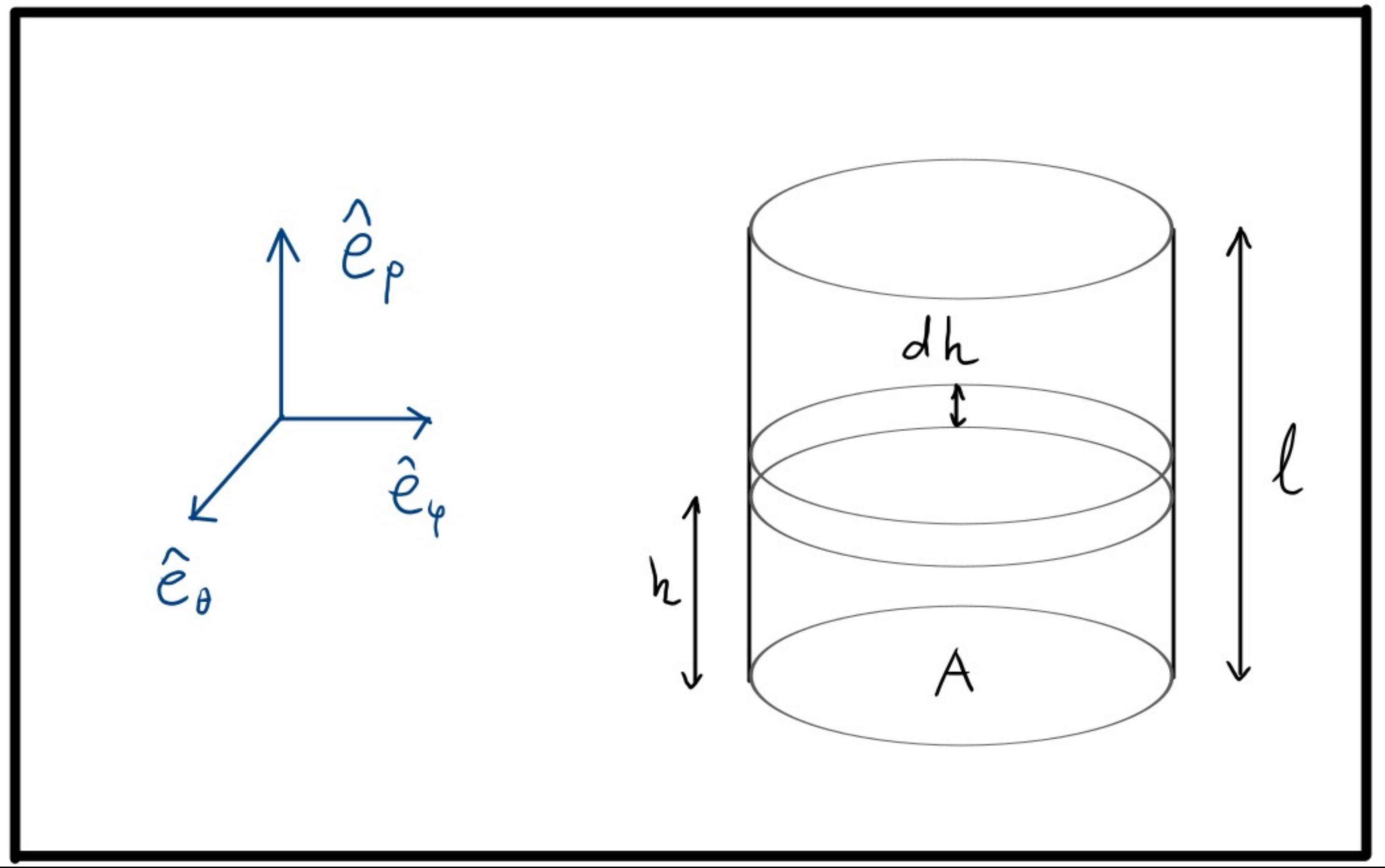
Relative acceleration of disks at distance h:

$$a = \frac{2M}{r^3} \cdot h$$

Astronaut of height l:

$$\left(\begin{array}{c} \text{Pressure on} \\ \text{waist} \end{array} \right) = \frac{1}{A} \int_0^{l/2} dF = \frac{1}{A} \int_0^{l/2} a \, dm$$

$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} = \frac{2M}{r^3} \xi^{\hat{\rho}}$$



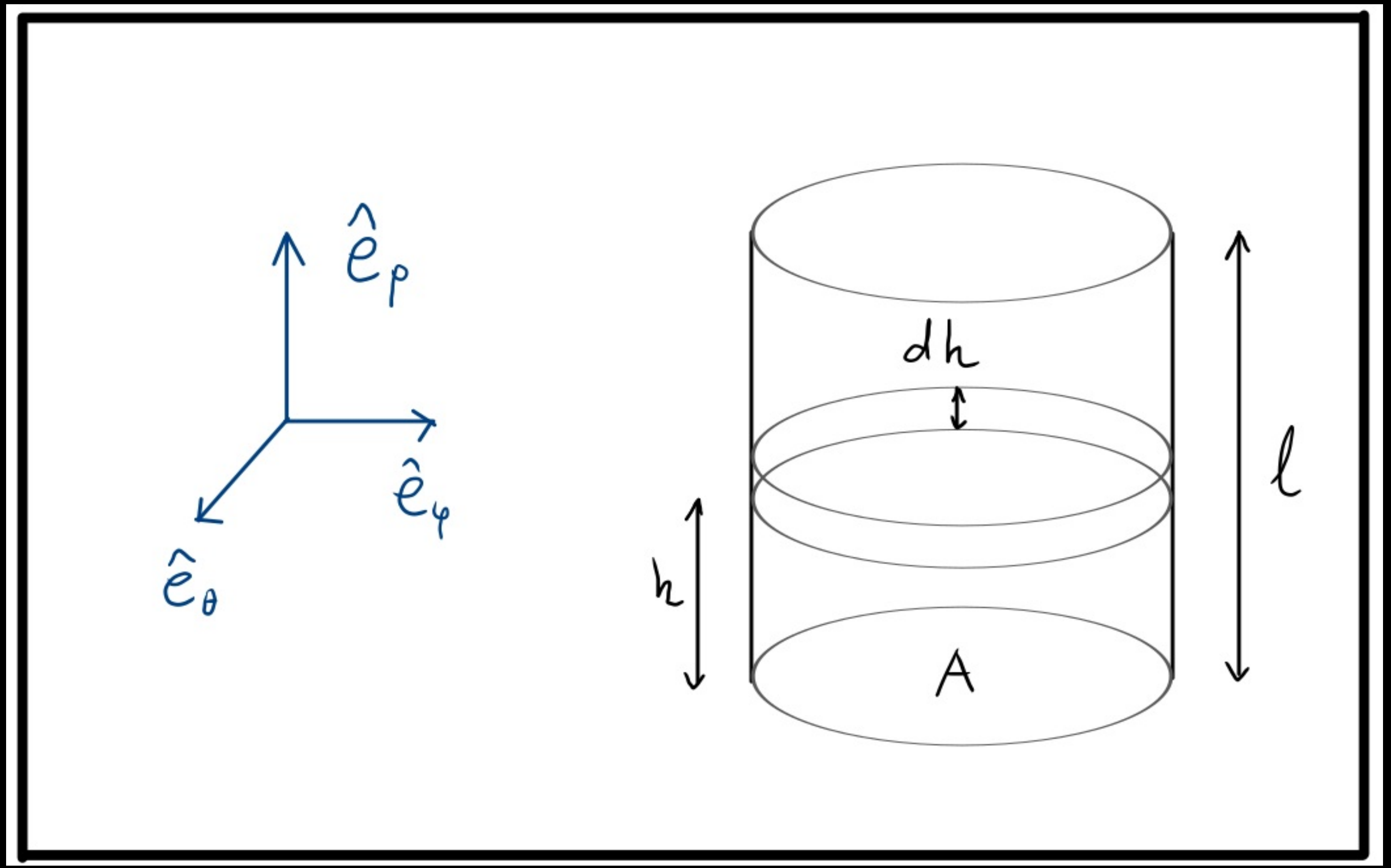
Relative acceleration of disks at distance h:

$$a = \frac{2M}{r^3} \cdot h$$

Astronaut of height ℓ:

$$\left(\begin{array}{c}\text{Pressure on}\\ \text{waist}\end{array}\right) = \frac{1}{A}\int_0^{\ell/2} dF = \frac{1}{A}\int_0^{\ell/2} a\, dm = \frac{1}{A}\int_0^{\ell/2} \underbrace{\left(\frac{2M}{r^3}h\right)}_{a} \underbrace{\left(\frac{m}{\ell A}\right)}_{\rho} \underbrace{(A\, dh)}_{dV}$$

$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} = \frac{2M}{r^3} \xi^{\hat{\rho}}$$



Relative acceleration of disks at distance h:

$$a = \frac{2M}{r^3} \cdot h$$

Astronaut of height ℓ:

$$\begin{pmatrix}\text{Pressure on} \\ \text{waist}\end{pmatrix} = \frac{1}{A}\int_0^{\ell/2} dF = \frac{1}{A}\int_0^{\ell/2} a\, dm = \frac{1}{A}\int_0^{\ell/2} \underbrace{\left(\frac{2M}{r^3} h\right)}_{a} \underbrace{\left(\frac{m}{\ell A}\right)}_{\rho} \underbrace{(A\, dh)}_{dV}$$

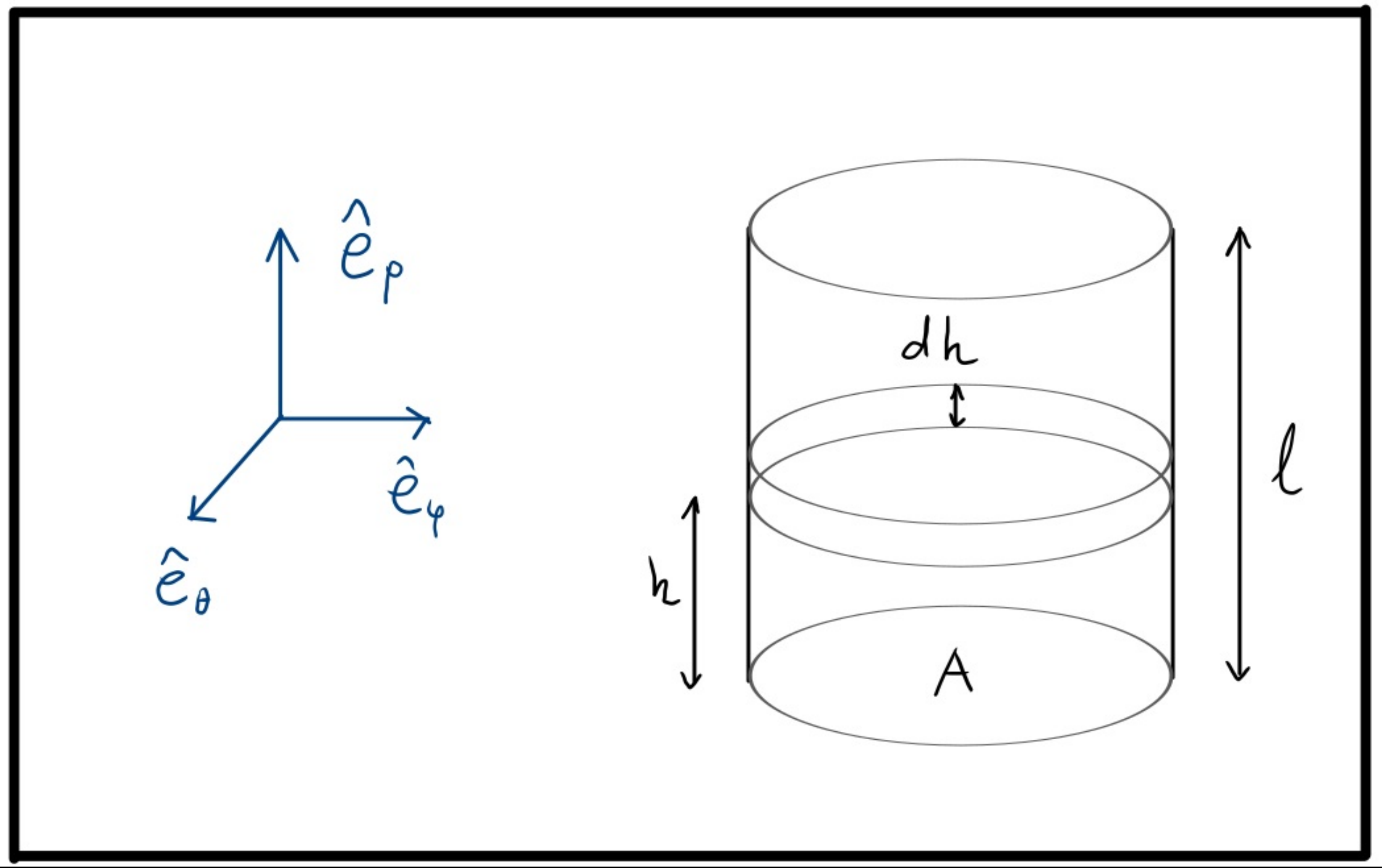
$$= \frac{1}{4}\frac{mM\ell}{Ar^3}$$

$m = 75\,\text{kg} \quad A = (0.2\,\text{m})^2 \quad \ell = 1.8\,\text{m}$

$\Rightarrow$

$$(\text{Pressure}) \approx 1.1 \times 10^9 \, \frac{(M/M_\odot)}{(r/1\,\text{km})^3} \,\text{Atm}$$

Misner § 32.6



$$\begin{pmatrix}\text{Pressure on}\\ \text{waist}\end{pmatrix} = \frac{1}{A}\int_0^{\ell/2} dF = \frac{1}{A}\int_0^{\ell/2} a\,dm = \frac{1}{A}\int_0^{\ell/2} \underbrace{\left(\frac{2M}{r^3}h\right)}_{a}\underbrace{\left(\frac{m}{\ell A}\right)}_{\rho}\underbrace{(A\,dh)}_{dV}$$

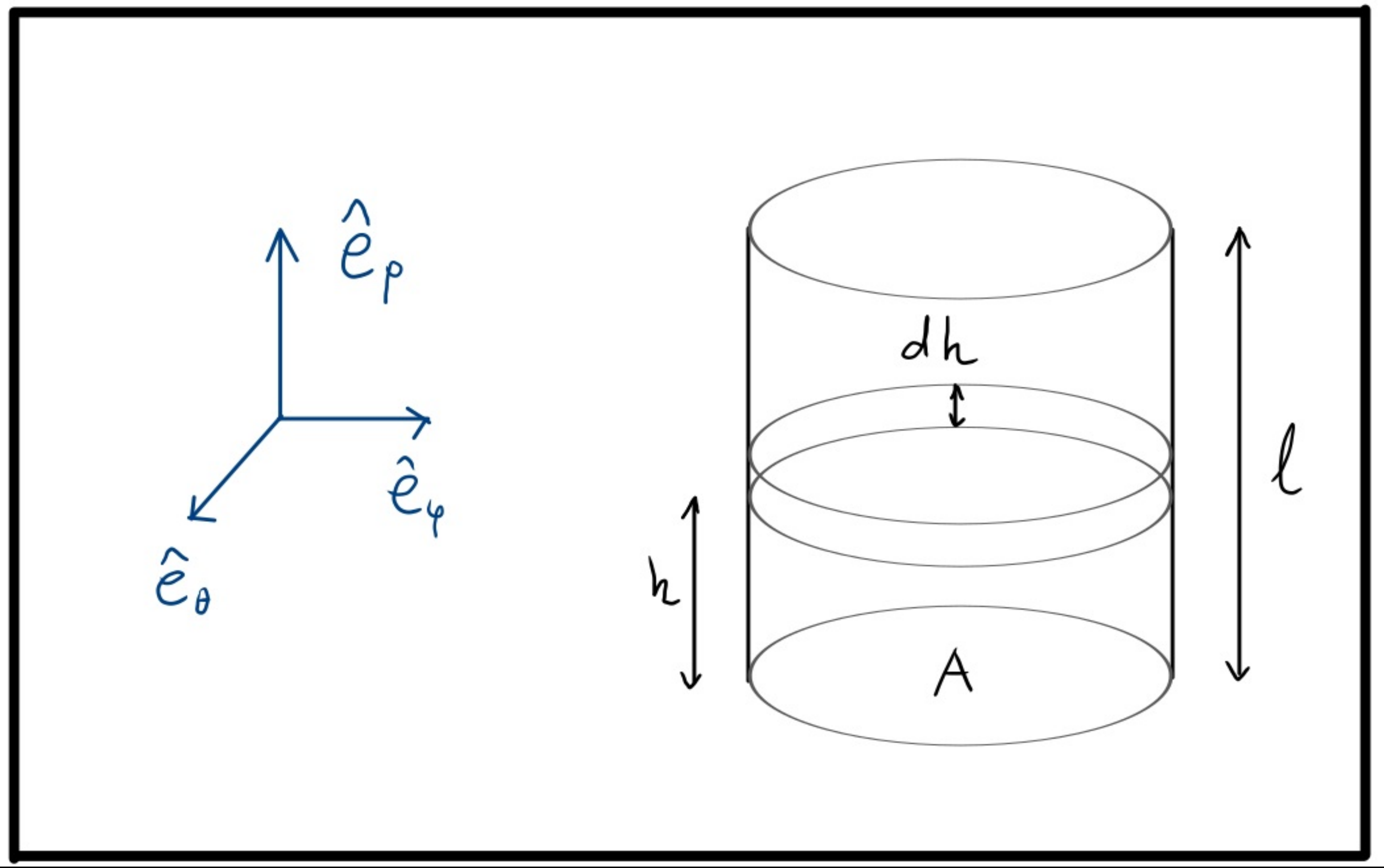
$$= \frac{1}{4}\,\frac{mM\ell}{Ar^3}$$

$m = 75\,\text{kg} \quad A = (0.2\,\text{m})^2 \quad \ell = 1.8\,\text{m}$

$\Rightarrow$

$$(\text{pressure}) \approx 1.1 \times 10^9 \, \frac{(M/M_\odot)}{(r/1\,\text{km})^3} \; \text{Atm}$$

Misner § 32.6



For a stellar black hole

$M \approx M_\odot \approx 1.5\,\text{km} \qquad r_s \approx 3.0\,\text{km}$

so at $r = r_s$ $\quad (\text{pressure}) \approx 10^9 \frac{1}{3^3}\,\text{Atm} \approx 10^7\,\text{Atm}$

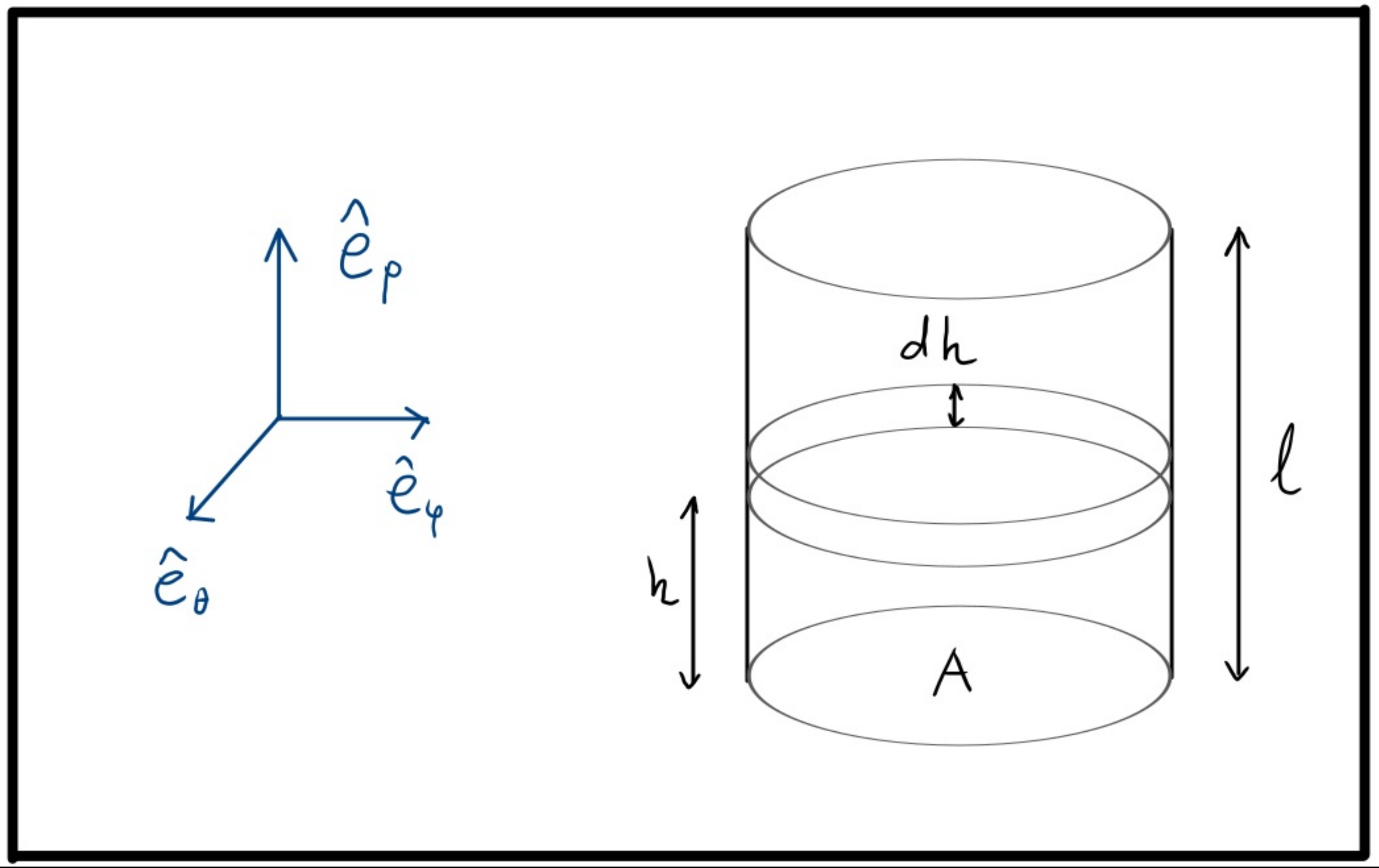
Human body may withstand $\approx 10^2\,\text{Atm}$

$m = 75\,kg \quad A = (0.2\,m)^2 \quad l = 1.8\,m$

$\Rightarrow$

$$(\text{pressure}) \approx 1.1 \times 10^9 \, \frac{(M/M_\odot)}{(r/1\,km)^3} \; \text{Atm}$$

Misner § 32.6



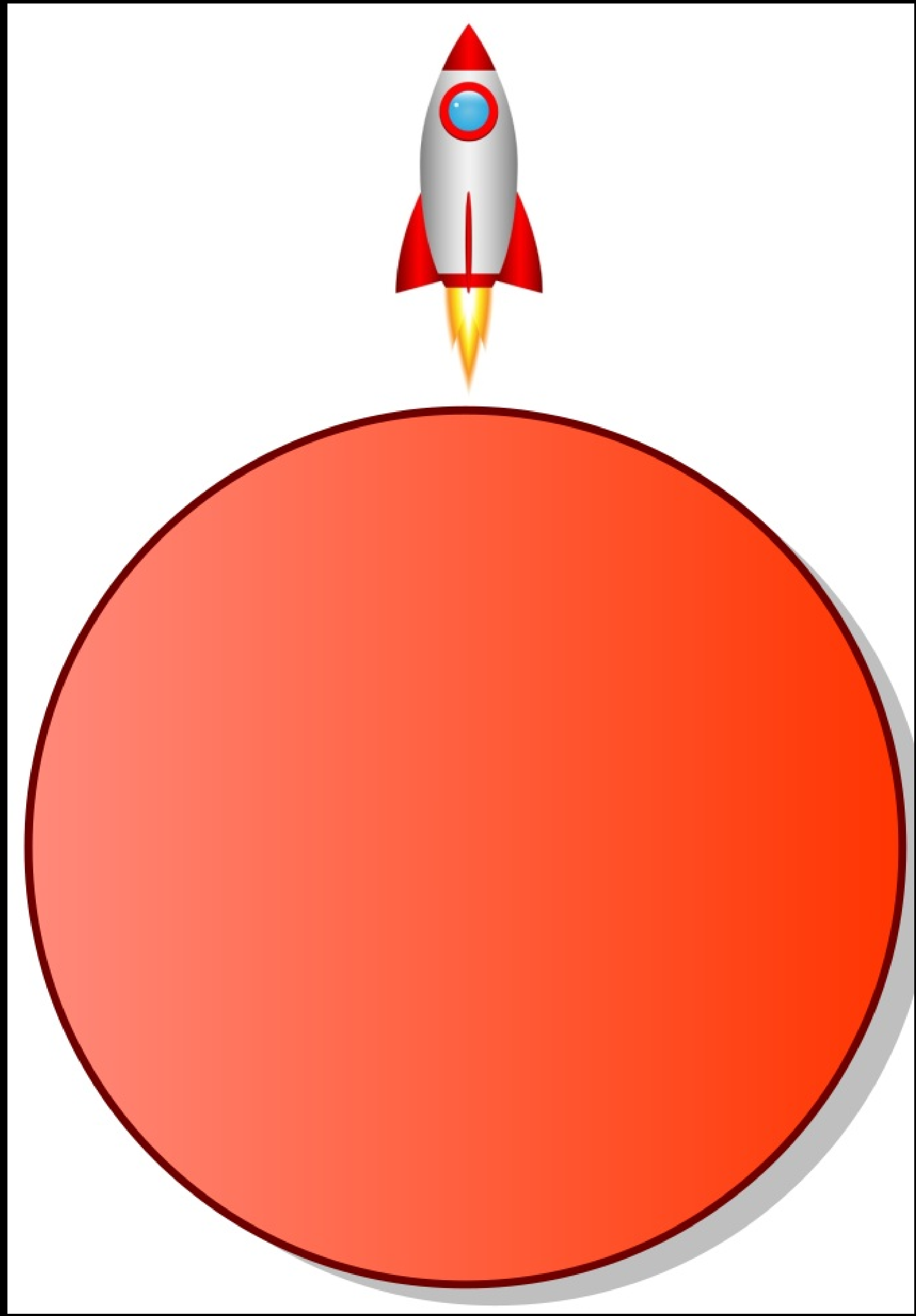
For supermassive black hole

$M \approx 10^9 M_\odot \approx 10^9\,km \qquad r_s \approx 10^9\,km$

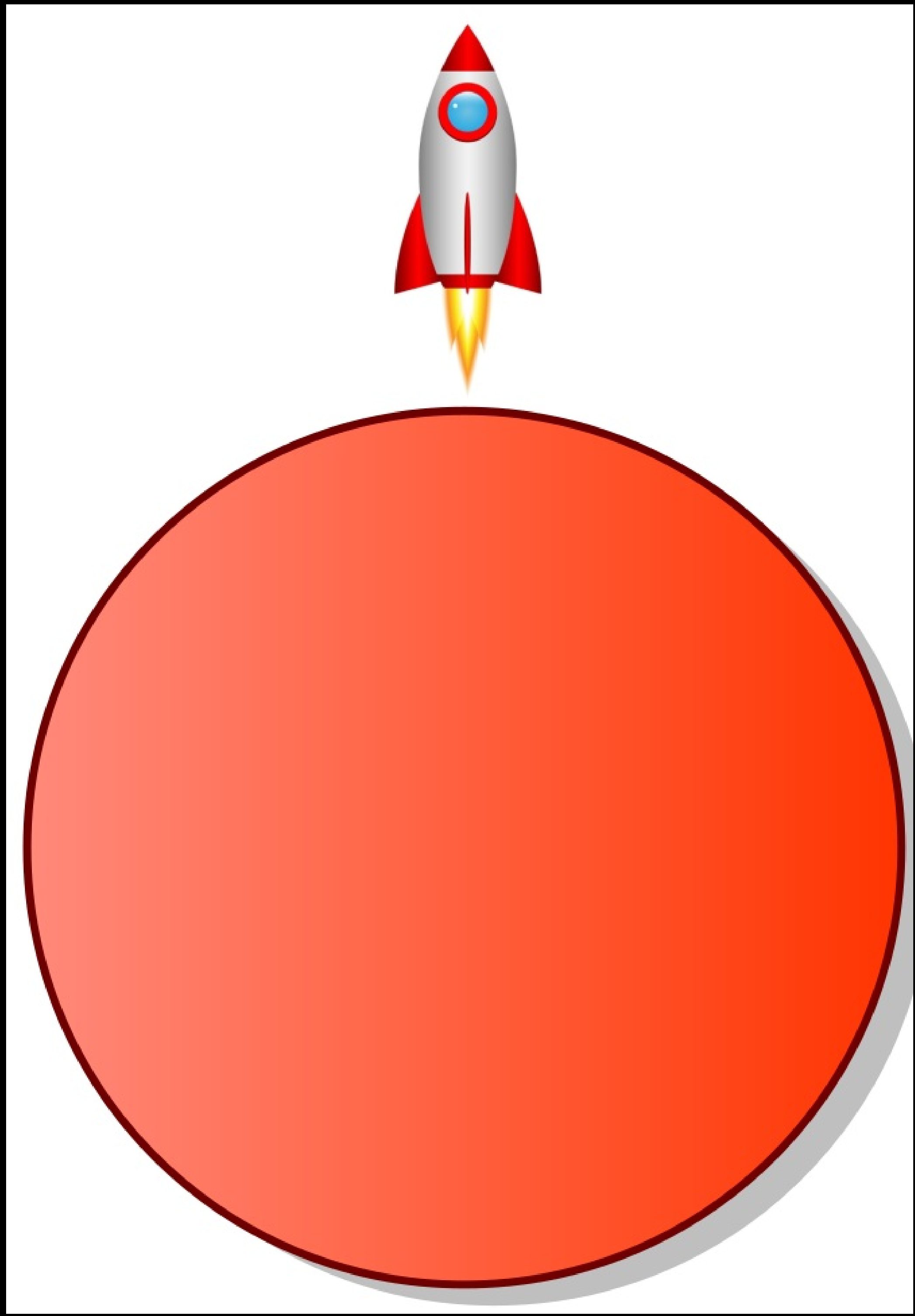
so at $r = r_s$ $\quad (\text{pressure}) \approx 10^9 \dfrac{10^9}{(10^9)^3} \text{Atm} \approx 10^{18-27}\,\text{Atm} \lesssim 10^{-9}\,\text{Atm}$

Human body may withstand $\approx 10^2\,\text{Atm}$

- Freely falling observer sees nothing special crossing the horizon

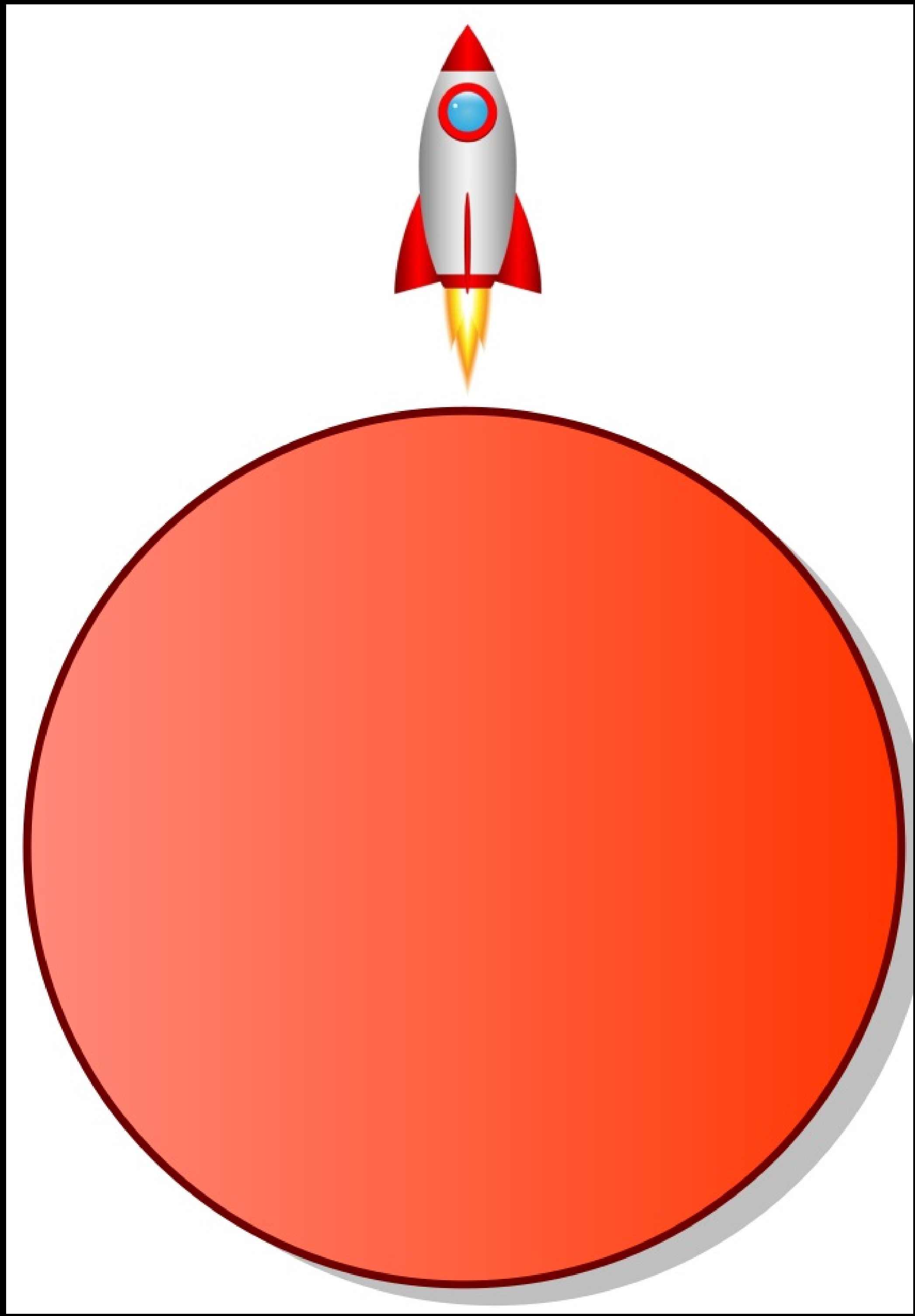


- Freely falling observer sees nothing special crossing the horizon
- But an (accelerated) observer struggles infinitely hard to remain stationary infinitesimally close to the horizon



- Freely falling observer sees nothing special crossing the horizon
- But an (accelerated) observer struggles infinitely hard to remain stationary infinitesimally close to the horizon
- Indeed, 4-force per unit mass needed is:

$$f^\mu = \frac{Du^\mu}{d\tau} = u^\nu \nabla_\nu u^\mu = \frac{du^\mu}{d\tau} + \Gamma^\mu{}_{\nu\rho} u^\nu u^\rho$$

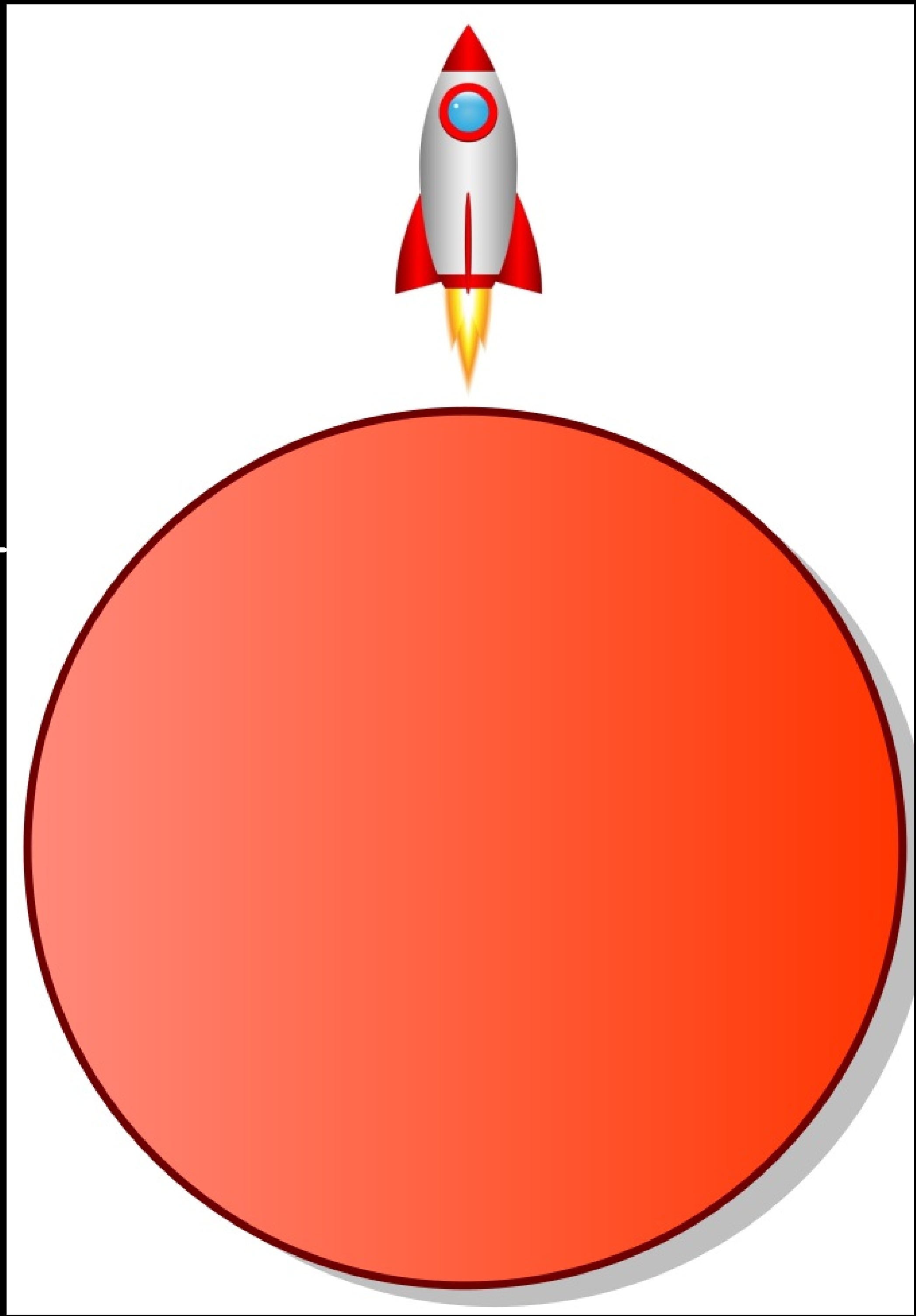


$$\Gamma^0{}_{10} = \frac{M}{r(r-2M)} \qquad \Gamma^1{}_{00} = \frac{M(r-2M)}{r^3} \qquad \Gamma^1{}_{11} = \frac{M}{2Mr - r^2}$$

$$\Gamma^1{}_{22} = 2M - r \qquad \Gamma^1{}_{33} = (2M-r)\sin^2\theta \qquad \Gamma^2{}_{21} = \frac{1}{r}$$

$$\Gamma^2{}_{33} = -\cos\theta\sin\theta \qquad \Gamma^3{}_{31} = \frac{1}{r} \qquad \Gamma^3{}_{32} = \cot\theta$$

$$f^\mu = \frac{Du^\mu}{d\tau} = u^\nu \nabla_\nu u^\mu = \frac{du^\mu}{d\tau} + \Gamma^\mu{}_{\nu\rho} u^\nu u^\rho$$



$$\Gamma^0{}_{10} = \frac{M}{r(r-2M)} \qquad \Gamma^1{}_{00} = \frac{M(r-2M)}{r^3} \qquad \Gamma^1{}_{11} = \frac{M}{2Mr-r^2}$$

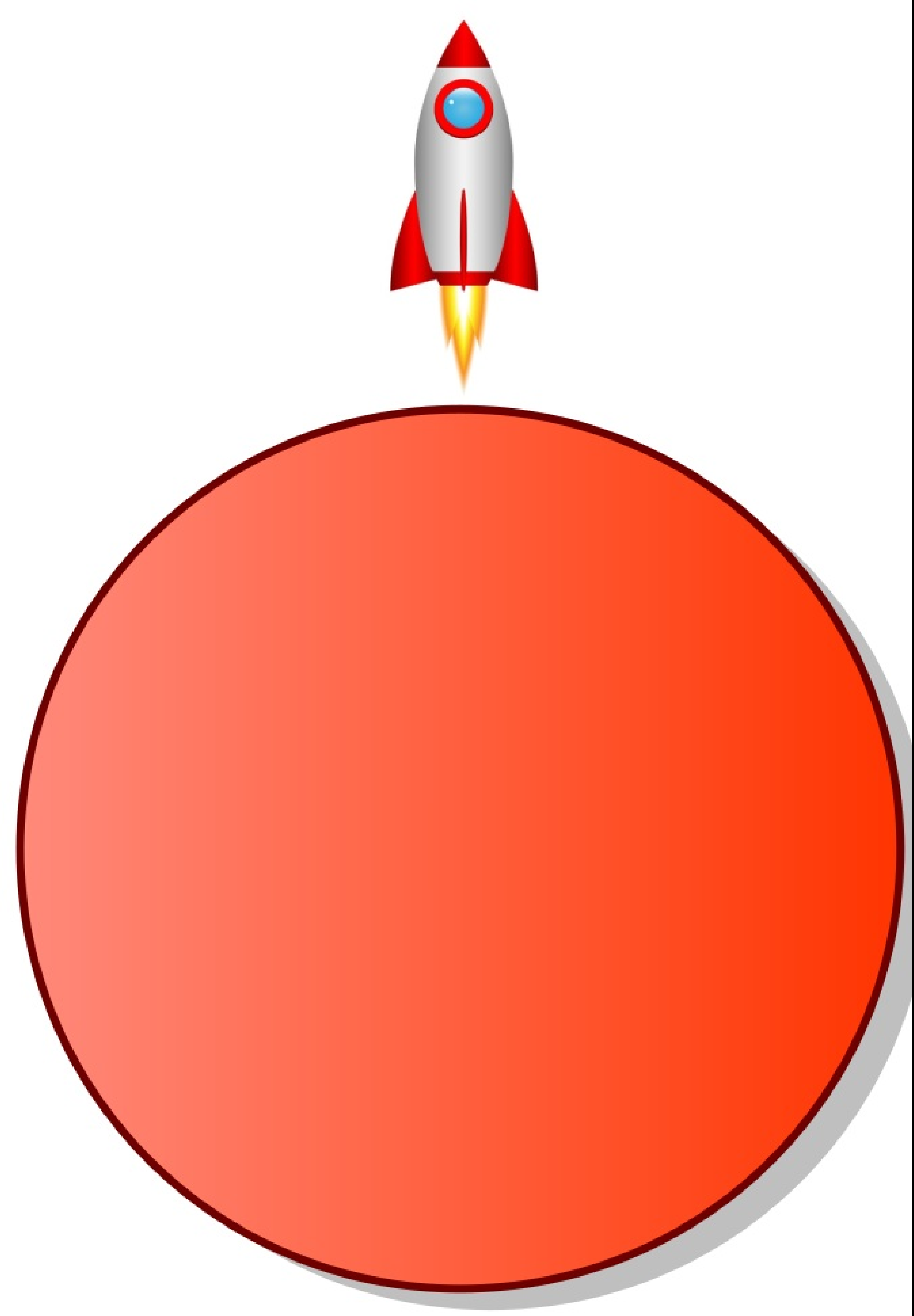
$$\Gamma^1{}_{22} = 2M - r \qquad \Gamma^1{}_{33} = (2M-r)\sin^2\theta \qquad \Gamma^2{}_{21} = \frac{1}{r}$$

$$\Gamma^2{}_{33} = -\cos\theta\sin\theta \qquad \Gamma^3{}_{31} = \frac{1}{r} \qquad \Gamma^3{}_{32} = \cot\theta$$

$$u^\mu = [u^0, 0, 0, 0]$$

$$u_\mu u^\mu = -1 \Rightarrow -\left(1-\frac{2M}{r}\right)(u^0)^2 = -1 \Rightarrow u^0 = \left(1-\frac{2M}{r}\right)^{-1/2}$$

$$f^\mu = \frac{Du^\mu}{d\tau} = u^\nu \nabla_\nu u^\mu = \frac{du^\mu}{d\tau} + \Gamma^\mu{}_{\nu\rho} u^\nu u^\rho$$



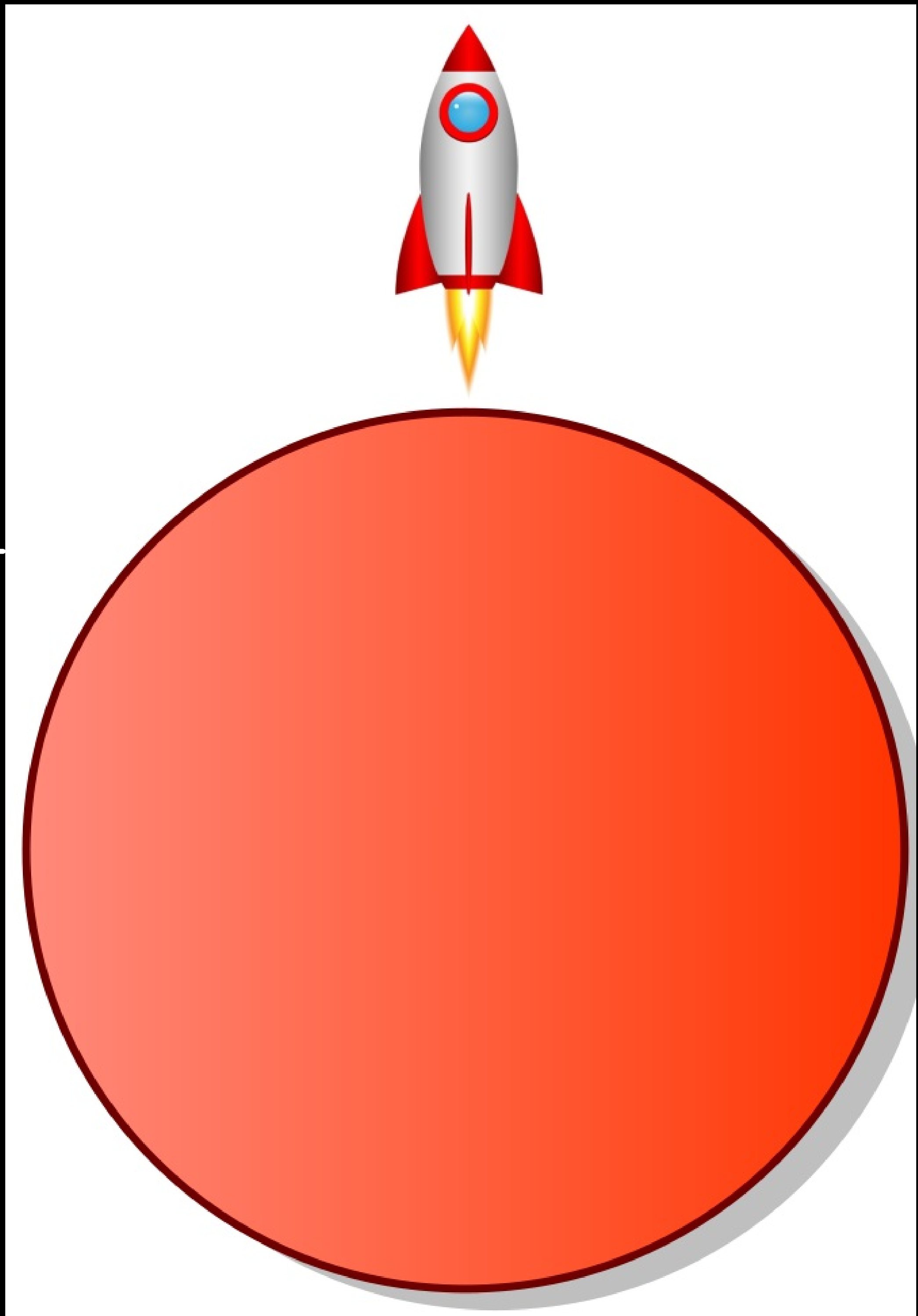
$$\Gamma^0{}_{10} = \frac{M}{r(r-2M)} \qquad \Gamma^1{}_{00} = \frac{M(r-2M)}{r^3} \qquad \Gamma^1{}_{11} = \frac{M}{2Mr - r^2}$$

$$\Gamma^1{}_{22} = 2M - r \qquad \Gamma^1{}_{33} = (2M-r)\sin^2\theta \qquad \Gamma^2{}_{21} = \frac{1}{r}$$

$$\Gamma^2{}_{33} = -\cos\theta\sin\theta \qquad \Gamma^3{}_{31} = \frac{1}{r} \qquad \Gamma^3{}_{32} = \cot\theta$$

$$u^\mu = \left[\left(1 - \frac{2M}{r}\right)^{-1/2}, 0, 0, 0\right]$$

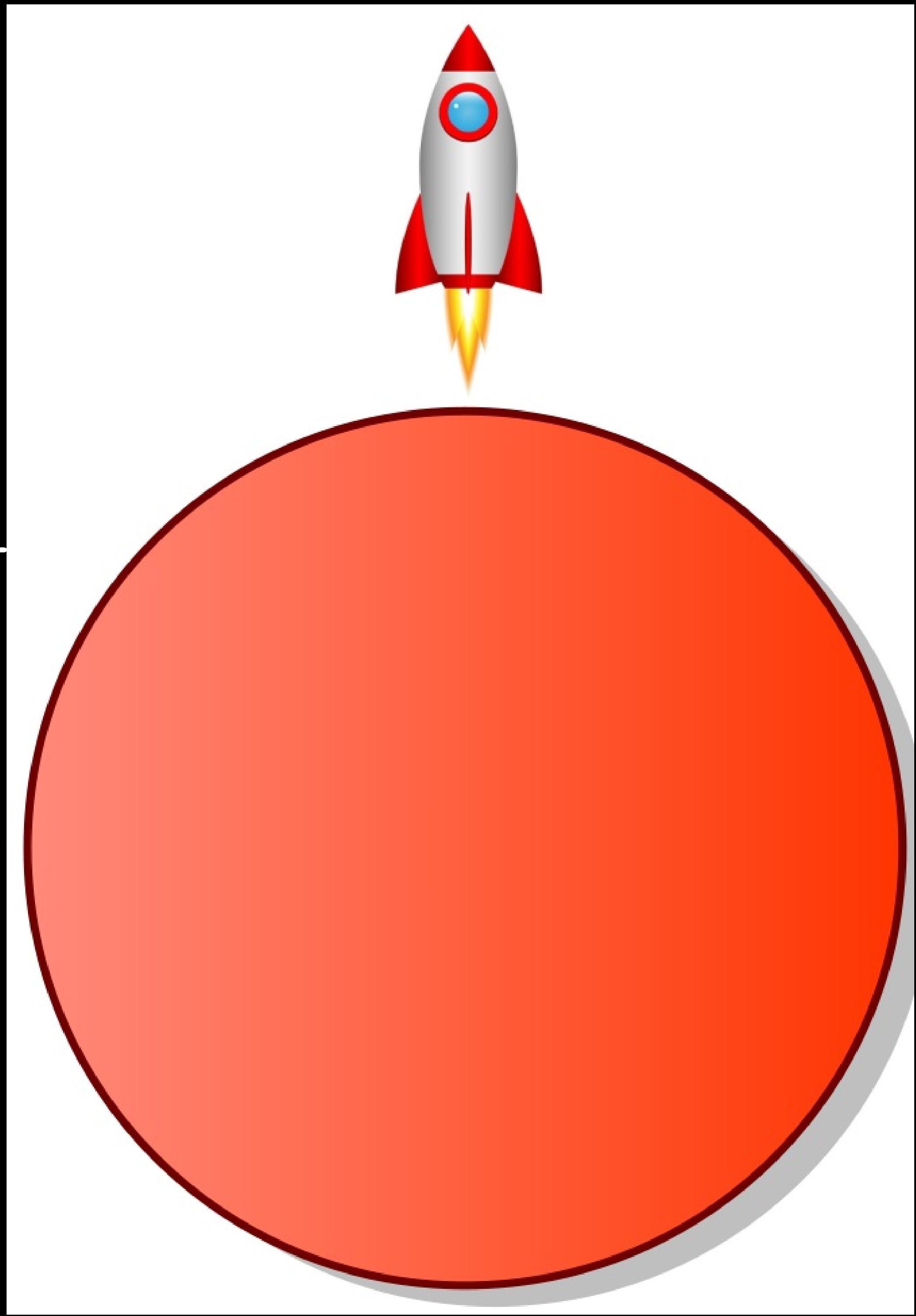
$$f^\mu = \frac{Du^\mu}{d\tau} = u^\nu \nabla_\nu u^\mu = \frac{du^\mu}{d\tau} + \Gamma^\mu{}_{\nu\rho} u^\nu u^\rho$$



$$\Gamma^0{}_{10} = \frac{M}{r(r-2M)} \qquad \Gamma^1{}_{00} = \frac{M(r-2M)}{r^3} \qquad \Gamma^1{}_{11} = \frac{M}{2Mr - r^2}$$

$$\Gamma^1{}_{22} = 2M - r \qquad \Gamma^1{}_{33} = (2M-r)\sin^2\theta \qquad \Gamma^2{}_{21} = \frac{1}{r}$$

$$\Gamma^2{}_{33} = -\cos\theta\sin\theta \qquad \Gamma^3{}_{31} = \frac{1}{r} \qquad \Gamma^3{}_{32} = \cot\theta$$



$$u^\mu = \left[\left(1 - \frac{2M}{R}\right)^{-1/2}, 0, 0, 0\right]$$

Since she stays at fixed $r = R$

$$\frac{du^\mu}{d\tau} = 0$$

$$f^\mu = \frac{Du^\mu}{d\tau} = u^\nu \nabla_\nu u^\mu = \frac{du^\mu}{d\tau} + \Gamma^\mu{}_{\nu\rho} u^\nu u^\rho$$

$\Gamma^0{}_{10} = \dfrac{M}{r(r-2M)}$ $\quad \Gamma^1{}_{00} = \dfrac{M(r-2M)}{r^3}$ $\quad \Gamma^1{}_{11} = \dfrac{M}{2Mr - r^2}$

$\Gamma^1{}_{22} = 2M - r$ $\quad \Gamma^1{}_{33} = (2M-r)\sin^2\theta$ $\quad \Gamma^2{}_{21} = \dfrac{1}{r}$

$\Gamma^2{}_{33} = -\cos\theta \sin\theta$ $\quad \Gamma^3{}_{31} = \dfrac{1}{r}$ $\quad \Gamma^3{}_{32} = \cot\theta$

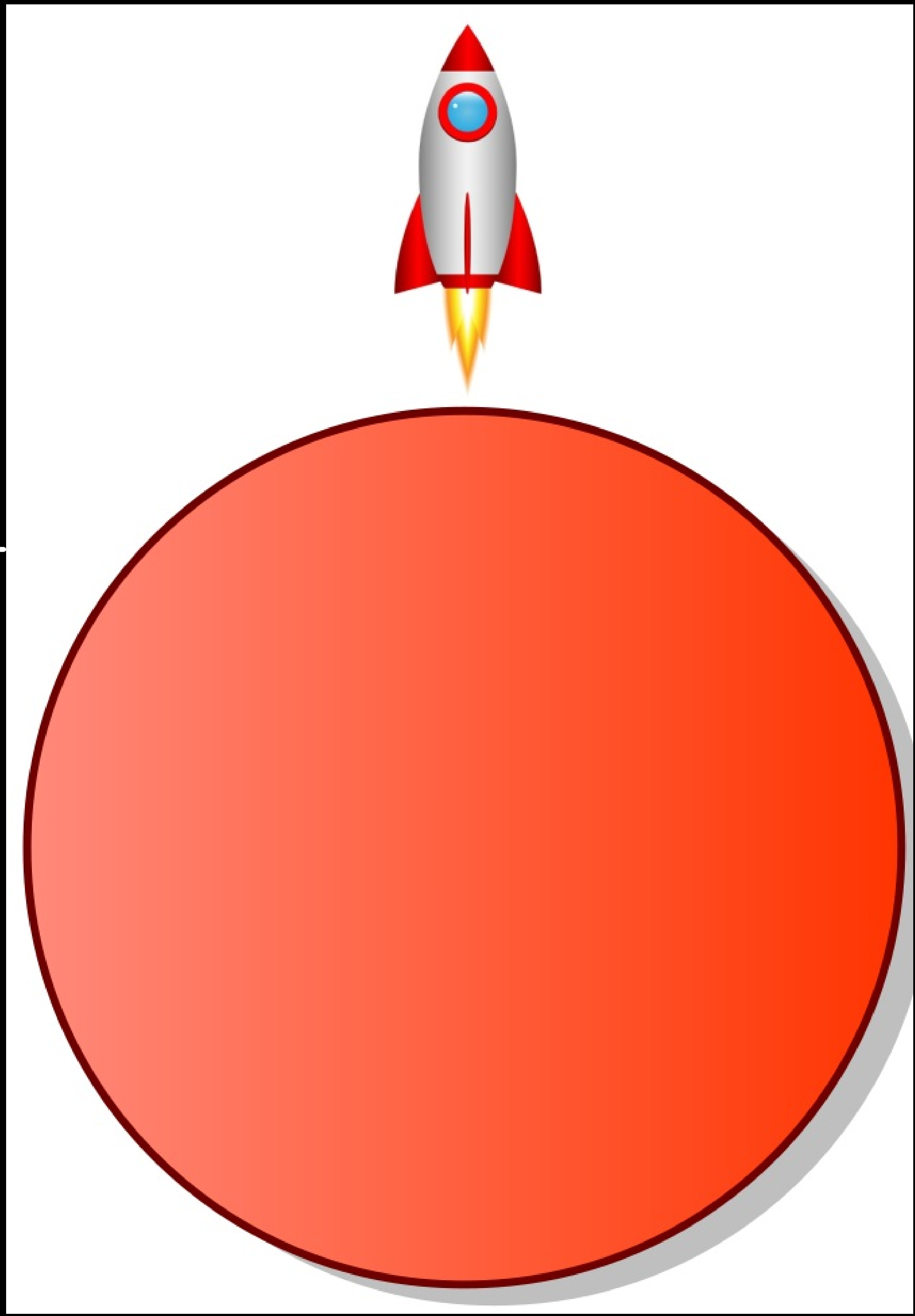
$$u^\mu = \left[\left(1 - \frac{2M}{R}\right)^{-1/2}, 0, 0, 0\right]$$

Since she stays at fixed $r = R$

$$\frac{du^\mu}{d\tau} = 0 \quad \Rightarrow$$

all other components $= 0$

$$\frac{Du^\mu}{d\tau} = 0 + \Gamma^\mu{}_{00} u^0 u^0$$



$$\Gamma^0{}_{10} = \frac{M}{r(r-2M)} \qquad \Gamma^1{}_{00} = \frac{M(r-2M)}{r^3} \qquad \Gamma^1{}_{11} = \frac{M}{2Mr - r^2}$$

$$\Gamma^1{}_{22} = 2M - r \qquad \Gamma^1{}_{33} = (2M-r)\sin^2\theta \qquad \Gamma^2{}_{21} = \frac{1}{r}$$

$$\Gamma^2{}_{33} = -\cos\theta\sin\theta \qquad \Gamma^3{}_{31} = \frac{1}{r} \qquad \Gamma^3{}_{32} = \cot\theta$$

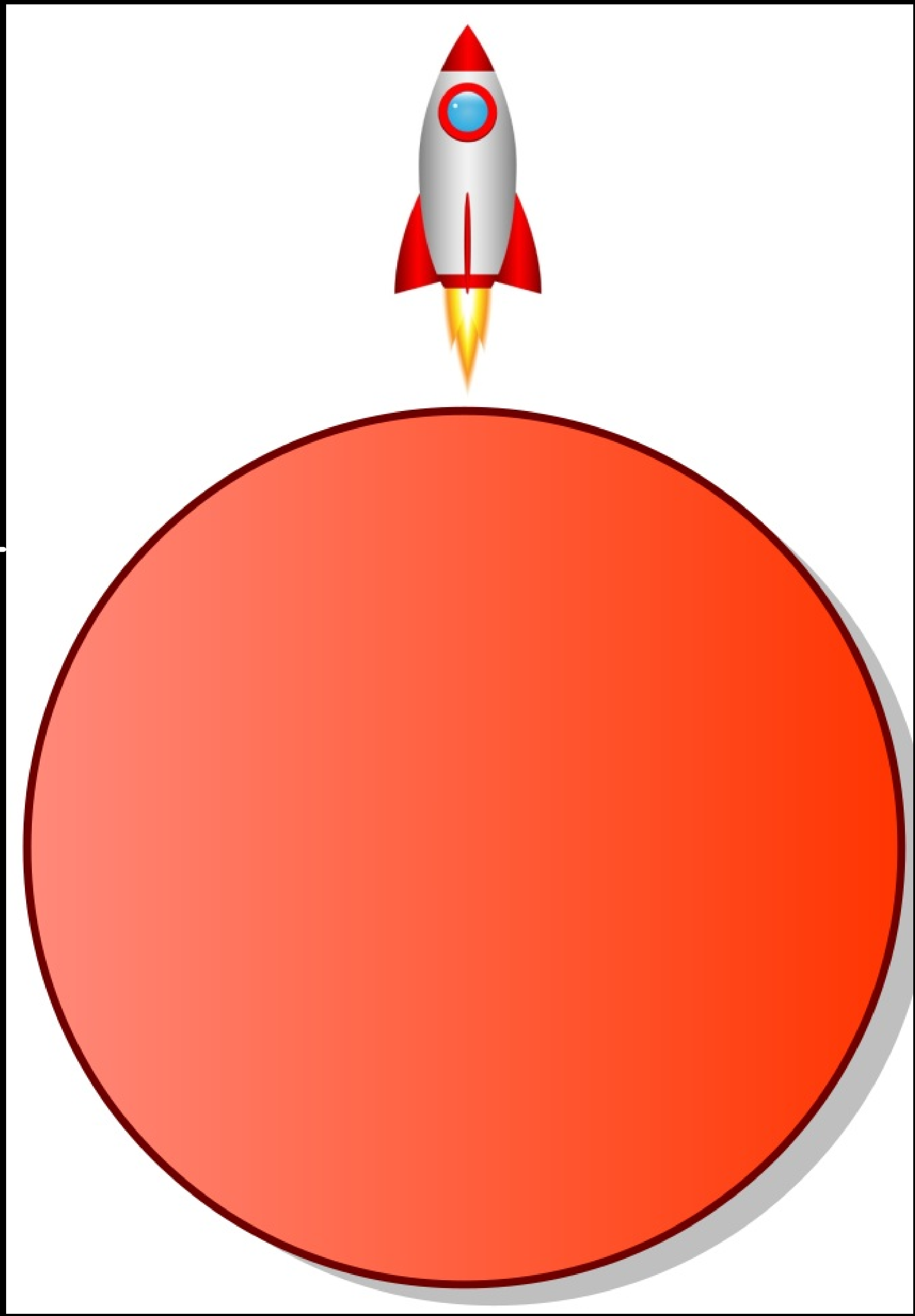
$$u^\mu = \left[\left(1 - \frac{2M}{R}\right)^{-1/2}, 0, 0, 0\right]$$

Since she stays at fixed $r = R$

$$\frac{du^\mu}{d\tau} = 0 \quad \Rightarrow$$

$$\frac{Du^\mu}{d\tau} = 0 + \Gamma^\mu{}_{00}\, u^0 u^0$$

Only $\Gamma^1{}_{00} = \frac{M}{r^2}\left(1 - \frac{2M}{r}\right) \neq 0$



$$\Gamma^0{}_{10} = \frac{M}{r(r-2M)} \qquad \Gamma^1{}_{00} = \frac{M(r-2M)}{r^3} \qquad \Gamma^1{}_{11} = \frac{M}{2Mr - r^2}$$

$$\Gamma^1{}_{22} = 2M - r \qquad \Gamma^1{}_{33} = (2M - r)\sin^2\theta \qquad \Gamma^2{}_{21} = \frac{1}{r}$$

$$\Gamma^2{}_{33} = -\cos\theta\sin\theta \qquad \Gamma^3{}_{31} = \frac{1}{r} \qquad \Gamma^3{}_{32} = \cot\theta$$

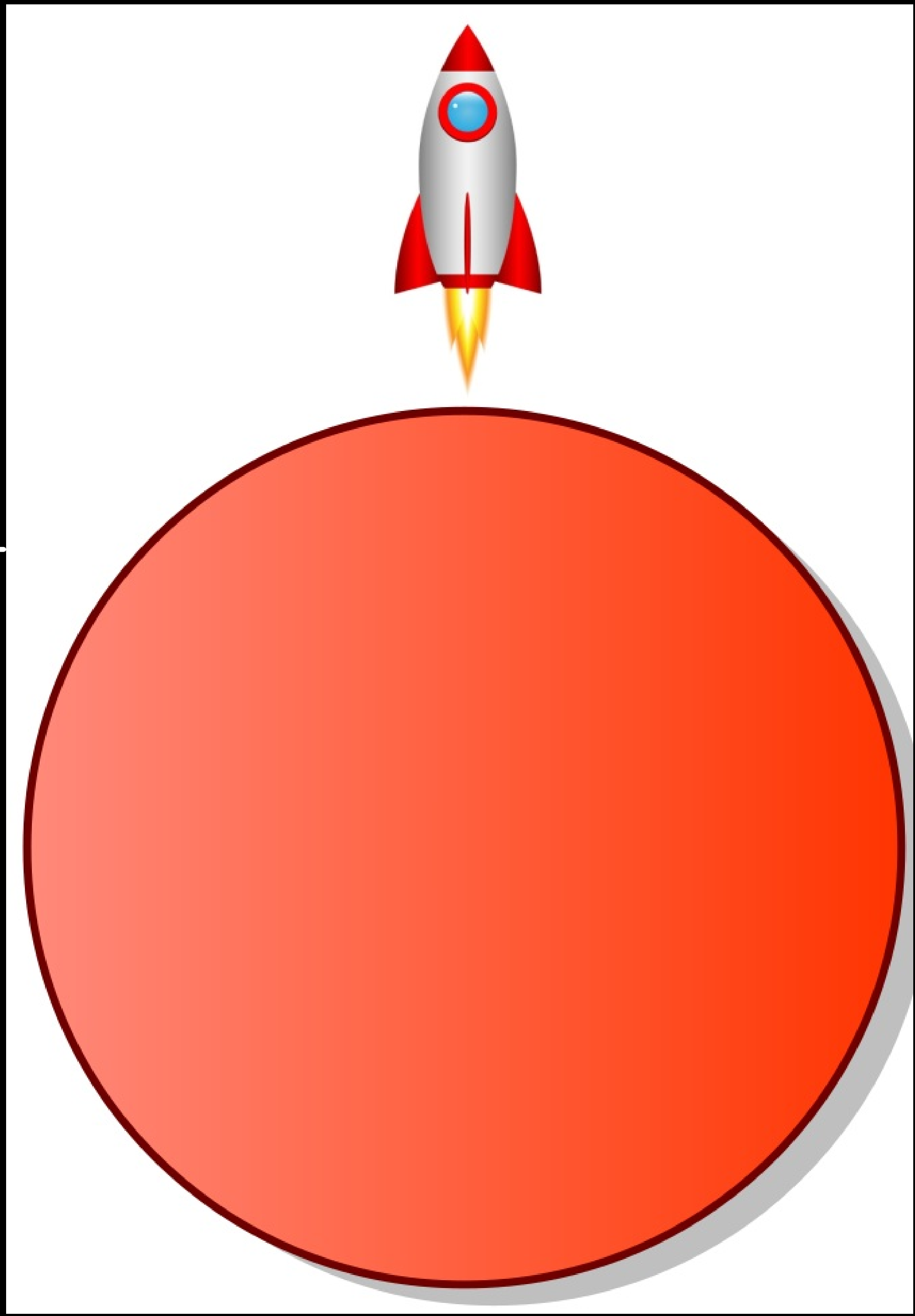
$$u^\mu = \left[\left(1 - \frac{2M}{R}\right)^{-1/2}, 0, 0, 0\right]$$

Since she stays at fixed $r = R$

$$\frac{du^\mu}{d\tau} = 0 \quad \Rightarrow$$

$$\frac{Du^1}{d\tau} = \Gamma^1{}_{00}\, u^0 u^0 = \frac{M}{r^2}\cancel{\left(1 - \frac{2M}{r}\right)}\left[\cancel{\left(1 - \frac{2M}{r}\right)}^{-\frac{1}{2}}\right]^2$$

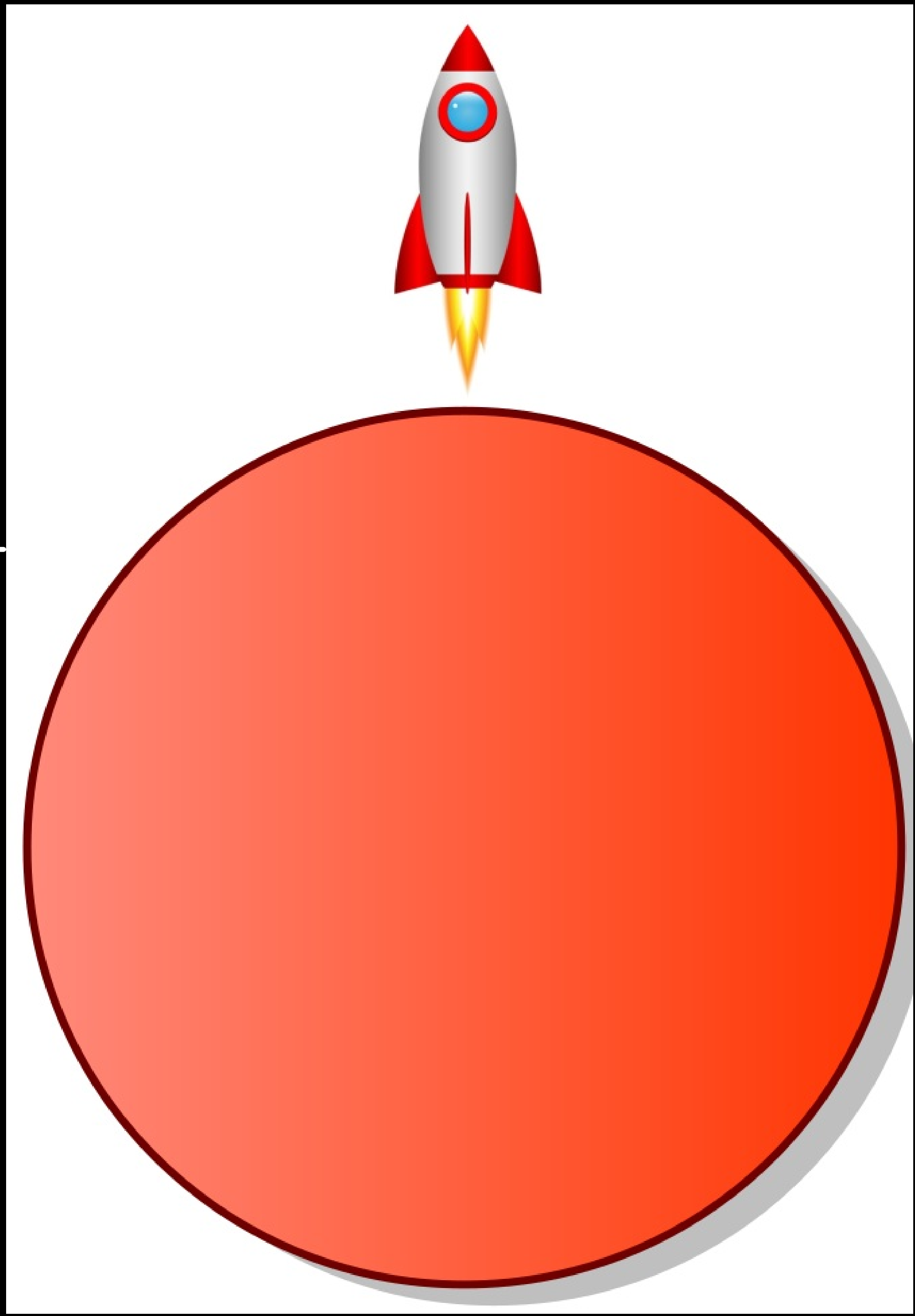
Only $\Gamma^1{}_{00} = \dfrac{M}{r^2}\left(1 - \dfrac{2M}{r}\right) \neq 0$



$$\Gamma^0{}_{10} = \frac{M}{r(r-2M)} \qquad \Gamma^1{}_{00} = \frac{M(r-2M)}{r^3} \qquad \Gamma^1{}_{11} = \frac{M}{2Mr-r^2}$$

$$\Gamma^1{}_{22} = 2M - r \qquad \Gamma^1{}_{33} = (2M-r)\sin^2\theta \qquad \Gamma^2{}_{21} = \frac{1}{r}$$

$$\Gamma^2{}_{33} = -\cos\theta\sin\theta \qquad \Gamma^3{}_{31} = \frac{1}{r} \qquad \Gamma^3{}_{32} = \cot\theta$$



$$u^\mu = \left[\left(1-\frac{2M}{R}\right)^{-1/2}, 0, 0, 0\right]$$

Since she stays at fixed $r = R$

$$\frac{du^\mu}{d\tau} = 0 \quad \Rightarrow$$

$$\frac{Du^1}{d\tau} = \qquad \Gamma^1{}_{00} u^0 u^0 = \frac{M}{r^2}\cancel{\left(1-\frac{2M}{r}\right)}\left[\cancel{\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}}\right]^2$$

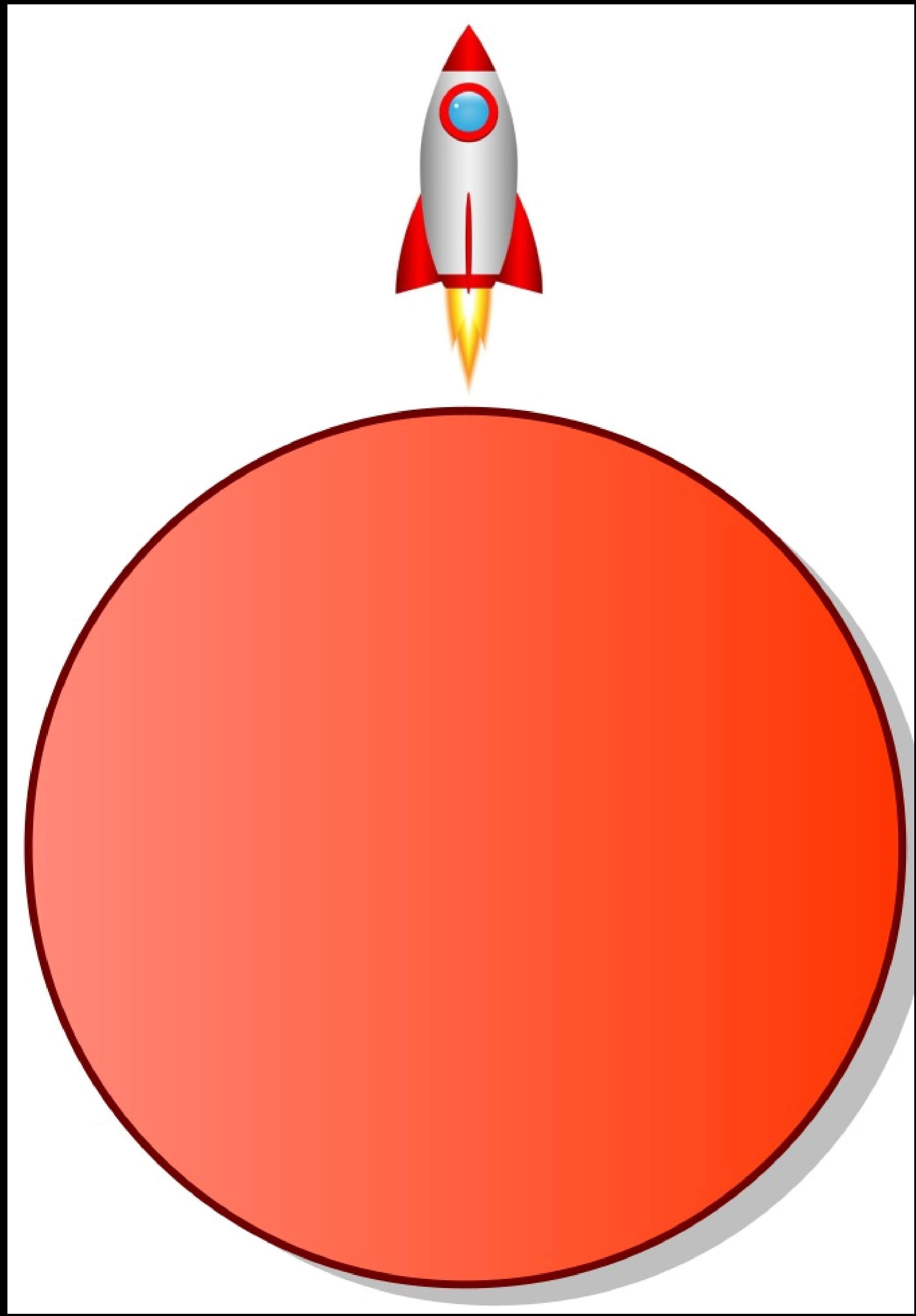
$$f^r = \frac{M}{r^2}$$

$$f^\mu = \left[0, \frac{M}{R^2}, 0, 0\right]$$

But the observer measures force in her local inertial frame $\{\hat{e}_0, \hat{e}_r, \hat{e}_\theta, \hat{e}_\phi\}$

$$\Rightarrow f^{\hat{r}} = |g_{rr}|^{1/2} f^r$$

$$= \left(1 - \frac{2M}{R}\right)^{-1/2} \frac{M}{R^2}$$



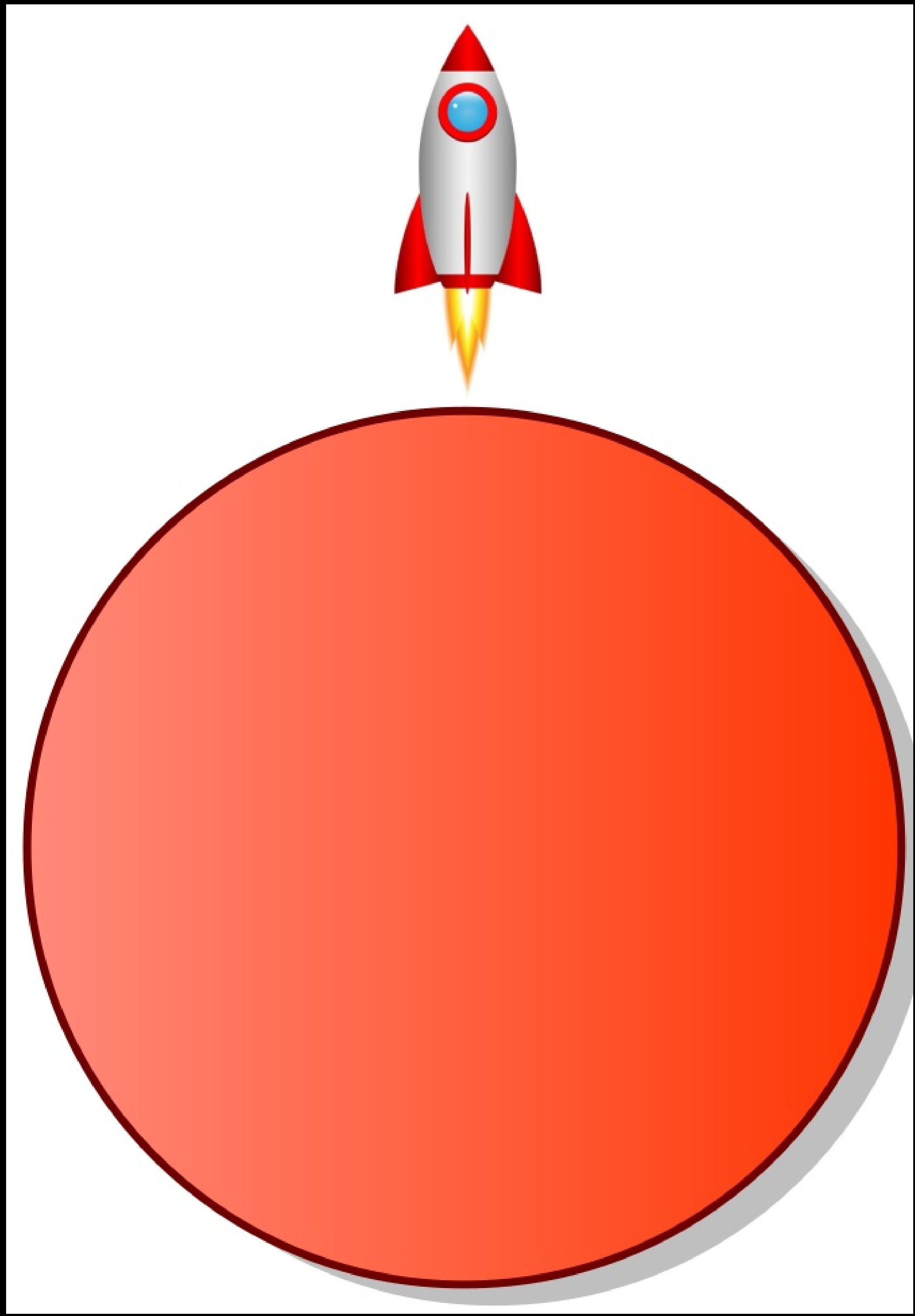
$$f^\mu = \left[0, \frac{M}{R^2}, 0, 0\right]$$

But the observer measures force in her local inertial frame $\{\hat{e}_0, \hat{e}_r, \hat{e}_\theta, \hat{e}_\phi\}$

$$\Rightarrow f^{\hat{r}} = |g_{rr}|^{1/2} f^r$$

$$= \left(1 - \frac{2M}{R}\right)^{-1/2} \frac{M}{R^2}$$

Blows up as $R \to 2M$!



$$f^\mu = \left[0, \frac{M}{R^2}, 0, 0\right]$$

- Although it takes an infinite t for an observer to fall radially to $r = 2M$, her proper time is finite. Even worse, after a finite τ, she will crush on $r = 0$ (her time ends! - think about it carefully ...)

- Although it takes an infinite t for an observer to fall radially to $r=2M$, her proper time is finite. Even worse, after a finite τ, she will crush on $r=0$ (her time ends! – think about it carefully ...)
- Consider an observer that starts from rest, relative to a stationary observer at $r=10M$. How much time will elapse on the observer's clock before hitting the singularity?

Hartle, problem 5, ch. 12

- $R = 10M$
- $\ell = 0$ (radial motion)
- stationary's observer 4-velocity:

$$\left.\begin{array}{l} U^\mu = [U^0, 0, 0, 0] \\ U^\mu U_\mu = -1 \end{array}\right\} \Rightarrow U^\mu = \left[\left(1 - \frac{2M}{R}\right)^{-1/2}, 0, 0, 0\right]$$

- Consider an observer that starts from rest, relative to a stationary observer at $r = 10\,M$. How much time will elapse on the observer's clock before hitting the singularity?

Hartle, problem 5, ch. 12

- $R = 10M$
- $\ell = 0$ (radial motion)
- stationary's observer 4-velocity: $U^\mu = \left[\frac{\sqrt{5}}{2}, 0, 0, 0\right]$

$$\left.\begin{aligned} U^\mu &= [U^0, 0, 0, 0] \\ U^\mu U_\mu &= -1 \end{aligned}\right\} \Rightarrow U^\mu = \left[\left(1 - \frac{2M}{10M}\right)^{-1/2}, 0, 0, 0\right] = \left[\left(1 - \frac{2}{10}\right)^{-1/2}, 0, 0, 0\right]$$

- $R = 10M$
- $\ell = 0$ (radial motion)
- stationary's observer 4-velocity: $U^\mu = \left[\frac{\sqrt{5}}{2}, 0, 0, 0\right]$

- $R = 10M$
- $\ell = 0$ (radial motion)
- stationary's observer 4-velocity: $U^\mu = \left[\frac{\sqrt{5}}{2}, 0, 0, 0 \right]$
- falling observer initially at rest w.r.t. U^μ, so has $u^\mu = U^\mu$

$$u^\mu = \left[\frac{\sqrt{5}}{2}, 0, 0, 0 \right]$$

- $R = 10M$
- $\ell = 0$ (radial motion)
- stationary's observer 4-velocity: $U^\mu = [\sqrt{5}, 0, 0, 0]$
- falling observer initially at rest w.r.t. U^μ, so has $u^\mu = U^\mu$

$$u^\mu = \left[\frac{\sqrt{5}}{2}, 0, 0, 0\right]$$

- $e = \left(1 - \frac{2M}{R}\right) \cdot u^0 = \left(1 - \frac{2M}{R}\right)\left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} = \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} = \frac{2}{\sqrt{5}}$

- $R = 10M$
- $\ell = 0$ (radial motion)
- stationary's observer 4-velocity: $U^\mu = [\sqrt{5}, 0, 0, 0]$
- falling observer initially at rest w.r.t. U^μ, so has $u^\mu = U^\mu$

$$u^\mu = \left[\frac{\sqrt{5}}{2}, 0, 0, 0\right]$$

- $e = \left(1 - \frac{2M}{R}\right) \cdot u^0 = \left(1 - \frac{2M}{R}\right)\left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} = \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} = \frac{2}{\sqrt{5}}$
- $\frac{e^2 - 1}{2} = \mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \underbrace{\frac{M}{r}}_{V_{eff}(r) \text{ for } \ell = 0}$

- $R = 10M$
- $\ell = 0$ (radial motion)
- stationary's observer 4-velocity: $U^\mu = [\sqrt{5}, 0, 0, 0]$
- falling observer initially at rest w.r.t. U^μ, so has $u^\mu = U^\mu$

$$u^\mu = \left[\frac{\sqrt{5}}{2}, 0, 0, 0\right]$$

- $e = \left(1 - \frac{2M}{R}\right) \cdot u^0 = \left(1 - \frac{2M}{R}\right)\left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} = \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} = \frac{2}{\sqrt{5}}$
- $\frac{e^2 - 1}{2} = \mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} \Rightarrow \left(\frac{dr}{d\tau}\right)^2 = e^2 - 1 + \frac{2M}{r} = \frac{2M}{r} - \frac{1}{5}$

- $R = 10M$
- $\ell = 0$ (radial motion)
- stationary's observer 4-velocity: $U^\mu = [\sqrt{5}, 0, 0, 0]$
- falling observer initially at rest w.r.t. U^μ, so has $u^\mu = U^\mu$

$$u^\mu = \left[\frac{\sqrt{5}}{2}, 0, 0, 0\right]$$

- $$e = \left(1 - \frac{2M}{R}\right) \cdot u^0 = \left(1 - \frac{2M}{R}\right)\left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} = \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} = \frac{2}{\sqrt{5}}$$
- $$\frac{e^2 - 1}{2} = \mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} \Rightarrow \left(\frac{dr}{d\tau}\right)^2 = e^2 - 1 + \frac{2M}{r} = \frac{2M}{r} - \frac{1}{5}$$

$$\Rightarrow \frac{dr}{d\tau} = -\left(\frac{2M}{r} - \frac{1}{5}\right)^{1/2}$$

$\hookrightarrow$ r decreases

- $R = 10M$
- $\ell = 0$ (radial motion)
- stationary's observer 4-velocity: $U^\mu = [\sqrt{5}, 0, 0, 0]$
- falling observer initially at rest w.r.t. U^μ, so has $u^\mu = U^\mu$

$$u^\mu = \left[\frac{\sqrt{5}}{2}, 0, 0, 0\right]$$

- $e = \left(1 - \frac{2M}{R}\right) \cdot u^0 = \left(1 - \frac{2M}{R}\right)\left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} = \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} = \frac{2}{\sqrt{5}}$
- $\frac{e^2 - 1}{2} = \mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} \Rightarrow \left(\frac{dr}{d\tau}\right)^2 = e^2 - 1 + \frac{2M}{r} = \frac{2M}{r} - \frac{1}{5}$

$$\Rightarrow \frac{dr}{d\tau} = -\left(\frac{2M}{r} - \frac{1}{5}\right)^{1/2} \Rightarrow \int_{10M}^{0}\left(\frac{2M}{r} - \frac{1}{5}\right)^{-1/2} dr = -\int_0^\tau d\tau$$

$$\Rightarrow \quad \frac{\tau}{M} = 5\sqrt{5}\,\pi$$

- $e = \left(1 - \frac{2M}{R}\right) \cdot u^0 = \left(1 - \frac{2M}{R}\right)\left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} = \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} = \frac{2}{\sqrt{5}}$

- $\frac{e^2 - 1}{2} = \mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} \Rightarrow \left(\frac{dr}{d\tau}\right)^2 = e^2 - 1 + \frac{2M}{r} = \frac{2M}{r} - \frac{1}{5}$

$$\Rightarrow \frac{dr}{d\tau} = -\left(\frac{2M}{r} - \frac{1}{5}\right)^{1/2} \Rightarrow \int_{10M}^{0} \left(\frac{2M}{r} - \frac{1}{5}\right)^{-1/2} dr = -\int_{0}^{\tau} d\tau$$

Eddington - Finkelstein Coordinates

$$(t, r, \theta, \phi) \to (v, r, \theta, \phi)$$

$$t = v - r - 2M \ln\left|\frac{r}{2M} - 1\right|$$

Eddington - Finkelstein Coordinates

$$(t, r, \theta, \phi) \to (v, r, \theta, \phi)$$

$$t = v - r - 2M \ln\left|\frac{r}{2M} - 1\right|$$

- regular at $r = 2M$, show explicitly that it is not true singularity

Eddington - Finkelstein Coordinates

$$(t, r, \theta, \phi) \to (v, r, \theta, \phi)$$

$$t = v - r - 2M \ln\left|\frac{r}{2M} - 1\right|$$

- regular at $r = 2M$, show explicitly that it is not true singularity

$$dt = dv - dr - 2M \frac{\frac{dr}{2M}}{\frac{r}{2M} - 1}$$

Eddington - Finkelstein Coordinates

$$(t, r, \theta, \phi) \to (v, r, \theta, \phi)$$

$$t = v - r - 2M \ln\left|\frac{r}{2M} - 1\right|$$

- regular at $r = 2M$, show explicitly that it is not true singularity

$$dt = dv - dr - 2M \frac{\frac{dr}{2M}}{\frac{r}{2M} - 1} = dv - dr\left[1 + \frac{1}{\frac{r}{2M} - 1}\right]$$

$$= dv - dr \frac{\cancel{\frac{r}{2M}}}{\cancel{\frac{r}{2M}}\left(1 - \frac{2M}{r}\right)} = dv - \left(1 - \frac{2M}{r}\right)^{-1} dr$$

Eddington - Finkelstein Coordinates

$$t = v - r - 2M \ln\left|\frac{r}{2M} - 1\right|$$

$$dt = dv - \left(1 - \frac{2M}{r}\right)^{-1} dr \quad \Rightarrow dt^2 = dv^2 + \left(1 - \frac{2M}{r}\right)^{-2} dr^2 - 2\left(1 - \frac{2M}{r}\right)^{-1} dv\,dr$$

Eddington - Finkelstein Coordinates

$$t = v - r - 2M \ln\left|\frac{r}{2M} - 1\right|$$

$$dt = dv - \left(1 - \frac{2M}{r}\right)^{-1} dr \qquad \Rightarrow dt^2 = dv^2 + \left(1 - \frac{2M}{r}\right)^{-2} dr^2 - 2\left(1 - \frac{2M}{r}\right)^{-1} dv\,dr$$

$$-\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 =$$

$$= -\left(1 - \frac{2M}{r}\right)\left(dv^2 + \left(1 - \frac{2M}{r}\right)^{-2} dr^2 - 2\left(1 - \frac{2M}{r}\right)^{-1} dv\,dr\right) + \left(1 - \frac{2M}{r}\right)^{-1} dr^2$$

Eddington - Finkelstein Coordinates

$$t = v - r - 2M \ln\left|\frac{r}{2M} - 1\right|$$

$$dt = dv - \left(1 - \frac{2M}{r}\right)^{-1} dr \quad \Rightarrow dt^2 = dv^2 + \left(1 - \frac{2M}{r}\right)^{-2} dr^2 - 2\left(1 - \frac{2M}{r}\right)^{-1} dv\,dr$$

$$-\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 =$$

$$= -\left(1 - \frac{2M}{r}\right)\left(dv^2 + \left(1 - \frac{2M}{r}\right)^{-2} dr^2 - 2\left(1 - \frac{2M}{r}\right)^{-1} dv\,dr\right) + \left(1 - \frac{2M}{r}\right)^{-1} dr^2$$

$$= -\left(1 - \frac{2M}{r}\right) dv^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + 2\,dv\,dr + \left(1 - \frac{2M}{r}\right)^{-1} dr^2$$

Eddington - Finkelstein Coordinates

$$t = v - r - 2M \ln\left|\frac{r}{2M} - 1\right|$$

$$dt = dv - \left(1-\frac{2M}{r}\right)^{-1} dr \quad \Rightarrow dt^2 = dv^2 + \left(1-\frac{2M}{r}\right)^{-2} dr^2 - 2\left(1-\frac{2M}{r}\right)^{-1} dv\, dr$$

$$-\left(1-\frac{2M}{r}\right) dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 =$$

$$= -\left(1-\frac{2M}{r}\right)\left(dv^2 + \left(1-\frac{2M}{r}\right)^{-2} dr^2 - 2\left(1-\frac{2M}{r}\right)^{-1} dv\, dr\right) + \left(1-\frac{2M}{r}\right)^{-1} dr^2$$

$$= -\left(1-\frac{2M}{r}\right) dv^2 - \cancel{\left(1-\frac{2M}{r}\right)^{-1} dr^2} + 2\, dv\, dr + \cancel{\left(1-\frac{2M}{r}\right)^{-1} dr^2}$$

$$= -\left(1-\frac{2M}{r}\right) dv^2 + 2\, dv\, dr$$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

$$= -\left(1-\frac{2M}{r}\right)dv^2 + 2\,dv\,dr + r^2 d\Omega^2$$

$$-\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 =$$

$$= -\left(1-\frac{2M}{r}\right)\left(dv^2 + \left(1-\frac{2M}{r}\right)^{-2}dr^2 - 2\left(1-\frac{2M}{r}\right)^{-1}dv\,dr\right) + \left(1-\frac{2M}{r}\right)^{-1}dr^2$$

$$= -\left(1-\frac{2M}{r}\right)dv^2 - \cancel{\left(1-\frac{2M}{r}\right)^{-1}dr^2} + 2\,dv\,dr + \cancel{\left(1-\frac{2M}{r}\right)^{-1}dr^2}$$

$$= -\left(1-\frac{2M}{r}\right)dv^2 + 2\,dv\,dr$$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$= -\left(1-\frac{2M}{r}\right)dv^2 + 2\,dv\,dr + r^2 d\Omega^2$$

- $r > 2M$ and $r < 2M$ are smoothly connected

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

$$= -\left(1-\frac{2M}{r}\right)dv^2 + 2\,dv\,dr + r^2 d\Omega^2$$

- $r > 2M$ and $r < 2M$ are smoothly connected
- $r \to \infty \Rightarrow ds^2$ flat , $t = v - r$

$$t = v - r - 2M \ln\left|\frac{r}{2M} - 1\right|$$

$$\left(r \to \infty \quad \frac{r}{2M} \gg \ln \frac{r}{2M} \right)$$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$= -\left(1-\frac{2M}{r}\right)dv^2 + 2\,dv\,dr + r^2 d\Omega^2$$

- $r > 2M$ and $r < 2M$ are smoothly connected
- $r \to \infty \Rightarrow ds^2$ flat, $t = v - r$
- $r \to 0$ singularity

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$
$$= -\left(1-\frac{2M}{r}\right)dv^2 + 2\,dv\,dr + r^2 d\Omega^2$$

- $r > 2M$ and $r < 2M$ are smoothly connected
- $r \to \infty \Rightarrow ds^2$ flat, $t = v - r$
- $r \to 0$ singularity

$g_{rv} = g_{vr} = 1$

Careful: $2\,dv\,dr$ is $(dv\,dr + dr\,dv)$

- $g_{\mu\nu}$ non-diagonal

$$(g_{\mu\nu}) = \begin{pmatrix} -\left(1-\frac{2M}{r}\right) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \qquad (g^{\mu\nu}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & \left(1-\frac{2M}{r}\right) & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix}$$

Radial null geodesics

$d\theta = d\varphi = 0$

Radial null geodesics

$d\theta = d\varphi = 0$

$$ds^2 = 0 \Rightarrow -\left(1 - \frac{2M}{r}\right) du^2 + 2\,du\,dr = 0$$

Radial null geodesics

$d\theta = d\varphi = 0$

$$ds^2 = 0 \Rightarrow -\left(1 - \frac{2M}{r}\right) du^2 + 2\,du\,dr = 0$$

$$\Rightarrow \left[-\left(1 - \frac{2M}{r}\right) du + 2\,dr\right] du = 0$$

Radial null geodesics

$d\theta = d\varphi = 0$

$$ds^2 = 0 \Rightarrow -\left(1 - \frac{2M}{r}\right) dv^2 + 2\,dv\,dr = 0$$

$$\Rightarrow \left[-\left(1 - \frac{2M}{r}\right) dv + 2dr\right] dv = 0$$

$$\Rightarrow \begin{cases} dv = 0 & (\alpha) \\ -\left(1 - \frac{2M}{r}\right) dv + 2dr = 0 & (\beta) \end{cases}$$

$$r = 2M \quad (dr = 0) \qquad (\gamma)$$

Radial null geodesics

$$d\theta = d\varphi = 0$$

$$ds^2 = 0 \Rightarrow -\left(1-\frac{2M}{r}\right)dv^2 + 2\,dv\,dr = 0$$

$$\Rightarrow \left[-\left(1-\frac{2M}{r}\right)dv + 2dr\right]dv = 0$$

$$\Rightarrow \begin{cases} dv = 0 & (\alpha) \\ -\left(1-\frac{2M}{r}\right)dv + 2dr = 0 & (\beta) \end{cases}$$

$$r = 2M \qquad (dr=0) \qquad (\gamma)$$

$$\Rightarrow \begin{cases} v = \text{const} & (\alpha) \\ v = 2\left(r + 2M \ln\left|\frac{r}{2M}-1\right|\right) + c & (\beta) \\ r = 2M & (\gamma) \end{cases}$$