

## AN OPERATOR FORMALISM FOR UNITARY MATRIX MODELS

K. N. ANAGNOSTOPOULOS\* and M. J. BOWICK\*  
*Institute for Theoretical Physics, University of California, Santa Barbara,  
CA 93106, USA*

N. ISHIBASHI  
*Department of Physics, University of California, Santa Barbara, CA 93106, USA*

Received 23 July 1991

We analyze the double scaling limit of unitary matrix models in terms of trigonometric orthogonal polynomials on the circle. In particular we find a compact formulation of the string equation at the  $k$ th multicritical point in terms of pseudodifferential operators and a corresponding action principle. We also relate this approach to the mKdV hierarchy which appears in the analysis in terms of conventional orthogonal polynomials on the circle.

### 1. Introduction

Random matrix models provide an elegant and powerful way to study the dynamics of random surfaces.<sup>1-4</sup> Random surfaces themselves appear in a wide variety of physical problems.<sup>5</sup> They correspond to statistical mechanical models in which the background geometry is allowed to fluctuate. The fluctuations of the geometry itself is characteristic of theories of gravity and thus one is really studying matter coupled to two-dimensional gravity. The simplest models (one-matrix models) are defined by a partition function which is a finite-dimensional ordinary integral over an  $N \times N$ -matrix  $M$

$$Z = \int DM \exp \left\{ -\frac{N}{\lambda} \text{Tr} V(M) \right\} . \quad (1)$$

Different models correspond to different classes of matrices  $M$  and different universality classes of potentials  $V(M)$ . The best understood case is when  $M$  is Hermitian. The integral (1) may then be expanded in a double power series in  $1/N^2$  and the coupling constant  $\lambda$ , and generates a set of Feynman diagrams which are dual to a discrete triangulation of a random manifold. Given powers of  $N$  correspond to surfaces of fixed genus. The partition function  $Z$  may be evaluated in the large- $N$  (planar) limit,<sup>6</sup> corresponding to spherical topology. In string theory,

\*Permanent address: Physics Department, Syracuse University, Syracuse, NY 13244-1130, USA.

and perhaps in 2D-gravity, one is interested in summing the complete topological expansion. This may be done via the “double scaling” continuum limit in which  $\lambda$  is tuned to a critical value  $\lambda_c$  and  $N$  tends to infinity with the scaling variable  $z = (\lambda_c - \lambda)N^{2k/(2k+1)}$  fixed.<sup>7-9</sup> The order of multicriticality is then  $k$ .<sup>10</sup> The case  $k = 2$  corresponds to pure 2D-gravity.  $N$  is then related to Newton’s constant  $G_0$  ( $N = e^{1/4G_0}$ ) and  $\lambda$  to the cosmological constant  $\mu$  ( $\lambda = e^{-\mu}$ ). To reach the  $k$ th multicritical point requires a potential of order at least  $2k$  for even potentials. In the double scaling limit the specific heat  $f_k$  (second derivative of  $Z$  with respect to  $z$ ) is determined by a nonlinear differential equation of order  $2k - 2$ . At criticality, however, the potential  $V_k$  for even-order multicritical points is unbounded from below and  $Z_k$  is not well-defined. This problem does not exist for odd-order multicritical points.<sup>11,12</sup>

Another case of great interest is that of unitary matrices  $U$

$$Z = \int DU \exp \left\{ -\frac{N}{\lambda} \text{Tr} V(U) \right\} . \quad (2)$$

This may be considered as a model of pure two-dimensional QCD.<sup>13,14</sup> It has the virtue of being well-defined at all multicritical points since the integration domain is compact. Ultimately one would like to formulate four-dimensional QCD as a matrix model corresponding to sums over world-sheets of string-like chromo-electric flux tubes. It is hoped that the double scaling limit of (2) will yield some new insights into this problem.<sup>15</sup>

In this paper we give a differential operator formulation of the continuum limit of (2) and the associated string equation mimicking as closely as possible the analysis of the Hermitian model. The organization of the paper is the following. Section 2 introduces some technical machinery, particularly orthogonal polynomials, appropriate for analyzing unitary matrix models. In Sec. 3 the continuum limit is carefully defined. In Sec. 4 we analyze the string equation for the  $k$ th multicritical point. In Sec. 5 an action principle is given for the string equation of Sec. 4 and the relation to the mKdV hierarchy is derived. Finally we conclude and list open problems.

## 2. Unitary Matrix Models

We will consider symmetric unitary matrix models of the form<sup>15-17</sup>

$$Z_N^U = \int DU \exp \left\{ -\frac{N}{\lambda} \text{Tr} V(U + U^\dagger) \right\} , \quad (3)$$

where

$$V(U) = \sum_{k \geq 0} g_k U^k , \quad (4)$$

and  $DU$  is the Haar measure for the unitary group. It is easy to show that  $Z_N^U$  reduces to

$$Z_N^U = \int \prod_j \frac{dz_j}{2\pi iz_j} |\Delta(z)|^2 \exp \left\{ -\frac{N}{\lambda} \sum_i V(z_i + z_i^*) \right\} , \quad (5)$$

where  $\Delta(z)$  is the Vandermonde determinant

$$\Delta \equiv \prod_{k < j} (z_k - z_j) \tag{6}$$

and  $z_i$ , the eigenvalues of  $U$ , live on the unit circle. The inner product is defined as a contour integral over the unit circle

$$\begin{aligned} \langle A(z), B(z) \rangle &= \oint \frac{dz}{2\pi iz} \bar{A}(z) B(z) \exp \left\{ -\frac{N}{\lambda} V(z + z^*) \right\} \\ &\equiv \int d\mu \bar{A}(z) B(z) . \end{aligned} \tag{7}$$

Introducing orthogonal polynomials with respect to this inner product

$$\langle P_n(z), P_m(z) \rangle = h_n \delta_{n,m} \tag{8}$$

one can show that the  $P_n(z)$  obey a recursion relation

$$z P_n(z) = P_{n+1}(z) - S_n z^n P_n(1/z) \tag{9}$$

with

$$S_n^2 = 1 - \frac{h_{n+1}}{h_n} . \tag{10}$$

The partition function is as usual given by

$$Z_N^U = \prod_{i=0}^{N-1} h_i = \prod_{i=0}^{N-1} (1 - S_i^2)^{N-i} \tag{11}$$

and is thus determined by the recursion coefficients  $S_n$  of the multiplication operator  $z$ . The dependence of  $S_n$  on the coefficients  $g_k$  of the potential  $V$  is easily shown to be described by the integral flows of the modified Volterra hierarchy,<sup>18</sup> the simplest flow being

$$\dot{S}_n = \frac{\partial S_n}{\partial g_1} = -(1 - S_n^2)(S_{n+1} - S_{n-1}) . \tag{12}$$

In the continuum limit the modified Volterra hierarchy becomes the modified KdV (mKdV) hierarchy. We will return to this later.

By taking appropriate linear combinations of the orthogonal polynomials  $\{P_n(z), P_n^*(z)\}$  which preserve the measure factor  $|\Delta(z)|^2$ , it is possible to find an alternative trigonometric basis of orthogonal polynomials<sup>19</sup> of the form

$$\begin{aligned} c_n^\pm &= z^n + \alpha_{n,n-1} z^{n-1} + \dots \pm \alpha_{n,n-1} z^{-n+1} \pm z^{-n} \\ &= \pm c_n^\pm(z^{-1}) , \end{aligned} \tag{13}$$

where  $n$  is an integer for  $U(2N + 1)$  and a half integer for  $U(2N)$ . The attractive feature of these polynomials is that they satisfy a three term recursion relation analogous to that of the Hermitian matrix model

$$z_+ c_n^\pm(z) = c_{n+1}^\pm(z) - r_n^\pm c_n^\pm(z) + R_n^\pm c_{n-1}^\pm(z) , \tag{14}$$

$$z_- c_n^\pm(z) = c_{n+1}^\mp(z) - q_n^\pm c_n^\mp(z) - Q_n^\pm c_{n-1}^\mp(z), \tag{15}$$

where  $z_\pm = z \pm 1/z$ . Let us denote the norms of  $c_n^\pm(z)$  by  $e^{\phi_n^\pm}$

$$\langle c_n^\pm, c_m^\pm \rangle = e^{\phi_n^\pm} \delta_{n,m}^{\pm,\pm}. \tag{16}$$

The integrable flows analogous to the modified Volterra hierarchy are now those of the Toda chain on the half line<sup>20</sup>

$$\frac{\partial^2 \phi_n^\pm}{\partial g_1^2} = e^{\phi_{n+1}^\pm - \phi_n^\pm} - e^{\phi_n^\pm - \phi_{n-1}^\pm}. \tag{17}$$

The norms  $e^{\phi_n^\pm}$  are related to the norms  $h_n$  of the  $P_n(z)$  polynomials by

$$e^{\phi_n^\pm} = 2(1 \mp S_{2n-1})h_{2n-1} \tag{18}$$

and

$$e^{\phi_0^\pm} = h_0. \tag{19}$$

Then one finds that

$$R_n^\pm = e^{(\phi_n^\pm - \phi_{n-1}^\pm)} = (1 \mp S_{2n-1})(1 - S_{2n-2}^2)(1 \pm S_{2n-3}), \tag{20}$$

$$r_n^\pm = \frac{\partial \phi_n^\pm}{\partial g_1} = \pm S_{2n}(1 \pm S_{2n-1}) \mp S_{2n-2}(1 \mp S_{2n-1}) \tag{21}$$

and

$$Q_n^\pm = e^{(\phi_n^\pm - \phi_{n-1}^\mp)} = (1 \mp S_{2n-1})(1 - S_{2n-2}^2)(1 \mp S_{2n-3}). \tag{22}$$

Using the relation  $[z_+, z_-] = 0$  one can show that

$$\begin{aligned} q_n^\pm &= \frac{(Q_{n+1}^\pm - Q_n^\pm) + (R_{n+1}^\mp - R_n^\pm)}{r_n^\pm - r_n^\mp} \\ &= \mp(1 \mp S_{2n-1})(S_{2n} + S_{2n-2}). \end{aligned} \tag{23}$$

Next, we compute the action of the operator  $z\partial_z \equiv z(\partial/\partial z)$  on the  $c_n^\pm$  basis. One finds that

$$z\partial_z c_n^\pm = nc_n^\mp + \frac{N}{\lambda} \sum_{r=1}^k (\gamma_z^\pm)_{n,n-r} c_{n-r}^\mp, \tag{24}$$

where

$$(\gamma_z^\pm)_{n,n-r} = e^{-\phi_{n-r}^\mp} \int d\mu(c_{n-r}^\mp)^* (z\partial_z V(z_+)) c_n^\pm \tag{25}$$

and  $k$  is the highest power of  $z_+$  in the potential. For  $k = 1$ , for example, the above relation becomes

$$z\partial_z c_n^\pm = nc_n^\mp - \frac{N}{\lambda} Q_n^\pm c_{n-1}^\mp. \tag{26}$$

The operator  $z\partial_z$  acting on  $c_n^\pm$  is not Hermitian and is not appropriate for taking the continuum limit. We need to compute instead the action of  $z\partial_z$  on a basis of functions  $\pi_n^\pm$  orthonormal with respect to the “flat” measure  $dz/(2\pi iz)$ . Therefore we define

$$\pi_n^\pm(z) = e^{-\phi_n^\pm/2} e^{-\frac{N}{2\lambda}V(z_+)} c_n^\pm(z) \tag{27}$$

and find that

$$\langle \pi_n^\pm(z), \pi_m^\pm(z) \rangle = \oint \frac{dz}{2\pi iz} (\pi_n^\pm(z))^* (\pi_m^\pm(z)) = \delta_{n,m}^{\pm,\pm}. \tag{28}$$

The recursion relations (14) and (15) become

$$\begin{aligned} z_+ \pi_n^\pm(z) &= \sqrt{R_{n+1}^\pm} \pi_{n+1}^\pm(z) - r_n^\pm \pi_n^\pm(z) + \sqrt{R_n^\pm} \pi_{n-1}^\pm(z), \\ z_- \pi_n^\pm(z) &= \sqrt{Q_{n+1}^\mp} \pi_{n+1}^\mp(z) - q_n^\pm \sqrt{\frac{Q_n^\mp}{R_n^\pm}} \pi_n^\mp(z) - \sqrt{Q_n^\pm} \pi_{n-1}^\mp(z). \end{aligned} \tag{29}$$

The action of the operator  $z\partial_z$  on the  $\pi_n^\pm(z)$  basis is found to be

$$\begin{aligned} z\partial_z \pi_n^\pm(z) &= -\frac{N}{2\lambda} \sum_{r=1}^k (v_z^\pm)_{n,n+r} \pi_{n+r}^\mp(z) + \left\{ n \sqrt{\frac{Q_n^\mp}{R_n^\pm}} - \frac{N}{2\lambda} (v_z^\pm)_{n,n} \right\} \pi_n^\mp(z) \\ &\quad + \frac{N}{2\lambda} \sum_{r=1}^k (v_z^\pm)_{n,n-r} \pi_{n-r}^\mp(z), \end{aligned} \tag{30}$$

where

$$(v_z^\pm)_{n,n-r} = \oint \frac{dz}{2\pi iz} (\pi_{n-r}^\mp(z))^* (z\partial_z V(z_+)) \pi_n^\pm(z). \tag{31}$$

The  $k = 1$  case now becomes

$$\begin{aligned} z\partial_z \pi_n^\pm(z) &= -\frac{N}{2\lambda} \sqrt{Q_{n+1}^\mp} \pi_{n+1}^\mp(z) + \left( n + \frac{N}{2\lambda} q_n^\pm \right) \sqrt{\frac{Q_n^\mp}{R_n^\pm}} \pi_n^\mp(z) \\ &\quad - \frac{N}{2\lambda} \sqrt{Q_n^\pm} \pi_{n-1}^\mp(z). \end{aligned} \tag{32}$$

It is easy to check that the above operator is Hermitian. The string equation is now derived from the relation<sup>a</sup>  $[z\partial_z, z_\pm] = -z_\mp$ .<sup>19,21</sup> We are now ready to calculate the continuum limit of the operators  $z\partial_z$  and  $z_\pm$  near the critical region.

### 3. The Continuum Limit

In this section we wish to study the continuum limit of the operators  $z_\pm$  and  $z\partial_z$  as defined in (29) and (30). At the discrete level, the above mentioned operators

<sup>a</sup>We use the convention here that  $\mathcal{O}\pi_n = \mathcal{O}_{n,m}\pi_m$ , for  $\mathcal{O}$  any of the operators  $z\partial_z$  or  $z_\pm$ . The skew-Hermitian character of  $z_-$  then leads to the minus sign on the right-hand side of the string equation.

act on an infinite-dimensional inner product space of complex functions on the unit circle, spanned by the functions  $\pi_n^\pm$  defined in (27). Taking the continuum limit means letting  $N \rightarrow \infty$ . But  $N$  appears only as the limit of the product (11). In the continuum limit, therefore, only the indices  $n$  in a small neighborhood of  $N$  will contribute to the singular part of  $Z_N^U$ . For the  $k$ th multicritical point the relevant index space is described by the scaling variable<sup>16,17,22,23</sup>

$$t = \left(1 - \frac{n}{N}\right) N^{2k/(2k+1)}. \tag{33}$$

The double scaling limit ansatz of Refs. 16 and 17 entails taking  $\lambda \rightarrow \lambda_c$  according to the scaling relation

$$z = \left(1 - \frac{\lambda}{\lambda_c}\right) N^{2k/(2k+1)}, \tag{34}$$

and scaling the recursion coefficients  $S_n$  of (10) as

$$S_{2n} \rightarrow f(t, z) N^{-1/(2k+1)} \tag{35}$$

where  $f^2(0, z)$  is the specific heat of the unitary matrix model. Then the elements of the space spanned by the functions  $\pi_n^\pm$  and all quantities defined in the previous section become functions of  $t$  and  $z$ . The operators, (29) and (30), have non-zero matrix elements  $(z_\pm)_{m,n}$  and  $(z\partial_z)_{m,n}$ , only for  $|m - n| \leq 1$  and  $|m - n| \leq 2k$  respectively. Therefore in the continuum limit they become finite order differential operators.<sup>24</sup> Using the scaling of equations (33)–(35), the Taylor expansions

$$\begin{aligned} S_{2n-m} &\rightarrow N^{1/(2k+1)} f\left(t + \frac{m}{2N} N^{2k/(2k+1)}, z\right) \\ &= N^{-1/(2k+1)} f\left(t + \frac{m}{2} N^{-1/(2k+1)}, z\right) \\ &= N^{-1/(2k+1)} f(t, z) + \frac{m}{2} N^{-2/(2k+1)} f'(t, z) + \dots \\ &\quad + \left(\frac{m}{2}\right)^r \frac{1}{r!} N^{-(r+1)/(2k+1)} f^{(r)}(t, z) + \dots, \end{aligned} \tag{36}$$

and

$$\begin{aligned} \pi_{n-m}^\pm(z) &\rightarrow \pi_n^\pm(z) + m N^{-1/(2k+1)} (\pi_n^\pm(z))' + \dots \\ &\quad + \frac{m^r}{r!} N^{-(r+1)/(2k+1)} (\pi_n^\pm(z))^{(r)} + \dots, \end{aligned} \tag{37}$$

and Eqs. (18)–(23), we find that

$$\begin{aligned} Q_n^\pm(t, z) &= 1 \mp 2N^{-1/(2k+1)} f(t, z) \mp 2N^{-2/(2k+1)} f'(t, z) + \mathcal{O}(N^{-3/(2k+1)}), \\ R_n^\pm(t, z) &= 1 + N^{-2/(2k+1)} (\pm f'(t, z) - 2f^2(t, z)) + \mathcal{O}(N^{-3/(2k+1)}), \\ r_n^\pm(t, z) &= N^{-2/(2k+1)} (\mp f'(t, z) + 2f^2) + \mathcal{O}(N^{-3/(2k+1)}), \\ q_n^\pm(t, z) &= \mp 2N^{-1/(2k+1)} f(t, z) + N^{-2/(2k+1)} (\mp f'(t, z) \\ &\quad + 2f^2(t, z)) + \mathcal{O}(N^{-3/(2k+1)}). \end{aligned} \tag{38}$$

Substituting into Eq. (29) and keeping terms of order  $N^{-2/(2k+1)}$  and  $N^{-1/(2k+1)}$  respectively we obtain

$$z_+ \rightarrow 2 + N^{-2/(2k+1)}Q_+, \quad z_- \rightarrow -N^{-1/(2k+1)}Q_-, \quad (39)$$

where  $Q_{\pm}$  are given by

$$Q_+ = \begin{pmatrix} \partial_t^2 - v' - v^2 & 0 \\ 0 & \partial_t^2 + v' - v^2 \end{pmatrix},$$

$$Q_- = 2 \begin{pmatrix} 0 & \partial_t + v \\ \partial_t - v & 0 \end{pmatrix}. \quad (40)$$

In the above formula  $v = -2f$ ,  $\partial_t \equiv \partial/\partial t$  and  $z_{\pm}$  act on the column vector  $\begin{pmatrix} \pi_+^t \\ \pi_-^t \end{pmatrix}$ . In the continuum limit the operator  $z\partial_z$  becomes

$$z\partial_z \rightarrow \frac{1}{a_k}N^{1/(2k+1)}\mathcal{P}_k. \quad (41)$$

The matrix operator  $\mathcal{P}_k$  has the form

$$\mathcal{P}_k = \begin{pmatrix} 0 & \mathbf{P}_k \\ \mathbf{P}_k^\dagger & 0 \end{pmatrix}, \quad (42)$$

with

$$\mathbf{P}_k = \partial_t^{2k} + p_{k,2k-1}\partial_t^{2k-1} + \dots - a_k(t+z). \quad (43)$$

The coefficient  $a_k$  may be calculated from the action of  $z\partial_z$  given in Eq. (30) and the  $k$ -multicritical potentials found in Ref. 17. The result is

$$a_k^{-1} = 2(2k+1) \sum_{l=1}^k (-1)^l l^{2k} \frac{B(k+1, k+1)}{\Gamma(k-l+1)\Gamma(k+l+1)}. \quad (44)$$

The computation of  $\mathbf{P}_k$  is straightforward, but becomes quite tedious for high values of  $k$ . For  $k = 1$ , for example,  $a_1 = -2$  and the explicit form of  $z\partial_z$  is

$$z\partial_z \rightarrow -\frac{1}{2}N^{1/3}\mathcal{P}_1, \quad (45)$$

where  $\mathcal{P}_1$  is given by

$$\mathcal{P}_1 = \begin{pmatrix} 0 & \mathbf{P}_1 \\ \mathbf{P}_1^\dagger & 0 \end{pmatrix} \quad (46)$$

with

$$\mathbf{P}_1 = \partial_t^2 + v\partial_t + \frac{1}{2}(v' - v^2) + 2(t+z). \quad (47)$$

The calculation is done by substituting Eqs. (36)–(38) into Eq. (32). The value of  $\lambda_c$  is found by using the string equation<sup>16</sup>

$$\lambda \frac{2n+1}{N} S_{2n}^2 = S_{2n}(S_{2n+1} + S_{2n-1})(1 - S_{2n}^2). \quad (48)$$

By letting  $n \rightarrow N$  and  $S_{2n} \rightarrow S$  (spherical limit), we obtain  $2\lambda S^2 = 2S^2(1 - S^2)$  or  $\lambda = 1 - S^2$ . As the critical solutions for the  $k$ th multicritical point are given by  $\lambda = \lambda_c(1 - S^{2k})$ , we deduce that  $\lambda_c = 1$ . In the above computation we have used the minimal  $k = 1$  potential  $V(z_+) = z_+$ .

The string equation is computed from  $[z\partial_z, z_\pm] = -z_\mp$ . As expected, we find that  $v$  obeys

$$\frac{1}{2}\partial_x^2 v(t, z) - v(t, z)^3 = -4v(t, z)(t + z) . \tag{49}$$

Therefore  $v$  is a function of  $x = t + z$  and is a solution of

$$\frac{1}{2}v''(x) - v(x)^3 = -4v(x)x , \tag{50}$$

which is the Painlevé II equation.

As already noted, the computation of  $z\partial_z$  and of the string equation following the steps described above is quite tedious for general  $k$ . In the next section we describe a more elegant way of computing them that will give the operator formalism for the unitary matrix models and its relation to the mKdV hierarchy.

#### 4. The Operator Formalism and the String Equation

In this section we present the form of the operator  $\mathcal{P}_k$  of Eqs. (45) and (46) and of the string equation (50) for general  $k$ . We find that  $\mathcal{P}_k$  is given as the positive part of a pseudodifferential operator as in the case of the Hermitian one-matrix model<sup>24</sup> and that the string equation is closely related to the mKdV hierarchy as in Ref. 17. The string equation  $[z\partial_z, z_\pm] = -z_\mp$  in terms of the operators  $\mathcal{P}_k, \mathcal{Q}_\pm$  is given by

$$\begin{aligned} [z\partial_z, z_+] &= -z_- \Rightarrow [\mathcal{P}_k, \mathcal{Q}_+] = a_k \mathcal{Q}_- \\ \Rightarrow \mathbf{P}_k(D - v)(D + v) - (D + v)(D - v)\mathbf{P}_k &= 2a_k(D + v) \end{aligned} \tag{51}$$

and

$$\begin{aligned} [z\partial_z, z_-] &= -z_+ \Rightarrow [\mathcal{P}_k, \mathcal{Q}_-] = 2a_k \\ \Rightarrow \mathbf{P}_k(D - v) - (D + v)\mathbf{P}_k^\dagger &= a_k \\ \mathbf{P}_k^\dagger(D + v) - (D - v)\mathbf{P}_k &= a_k , \end{aligned} \tag{52}$$

where  $D = \partial/\partial x$ .

It is convenient to write the above equations in terms of

$$\tilde{\mathcal{P}} = \mathcal{P} + a_k \mathcal{X} , \tag{53}$$

where

$$\tilde{\mathcal{P}} = \begin{pmatrix} 0 & \tilde{\mathbf{P}}_k \\ \tilde{\mathbf{P}}_k^\dagger & 0 \end{pmatrix} \quad \mathcal{X} = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} . \tag{54}$$

Then Eqs. (51) and (52) become

$$\tilde{\mathbf{P}}_k(D - v)(D + v) - (D + v)(D - v)\tilde{\mathbf{P}}_k = 2a_k(vx)' \tag{55}$$



and

$$\begin{aligned} \tilde{P}_k(D - v) - (D + v)\tilde{P}_k^\dagger &= -2a_k v x , \\ \tilde{P}_k^\dagger(D + v) - (D - v)\tilde{P}_k &= 2a_k v x . \end{aligned} \tag{56}$$

Eliminating  $\tilde{P}_k$  ( $\tilde{P}_k^\dagger$ ) yields Eq. (55) and its Hermitian conjugate respectively. The left-hand side of Eqs. (56) are differential operators of order  $2k$ . We get, therefore, a total of  $4k + 2$  equations, which is an overdetermined system of differential equations for the  $2k + 1$  functions  $p_{k,i}$  and  $v$ . By checking the first few values of  $k$  we find that, remarkably, only  $2k + 1$  of them are independent. We conjecture that this is true for all  $k$ , although we have no general proof. If this is the case, Eq. (56) uniquely determines the operator  $\tilde{P}_k$  and the string equation.

It is instructive to examine the  $k = 1$  case in this formalism. First note that in this case Eqs. (47) and (53) give

$$\begin{aligned} \tilde{P}_1 &= D^2 + vD + \frac{1}{2}(v' - v^2) \\ &= \{(D + v)A_1\}_+ , \end{aligned} \tag{57}$$

where  $A_1 = [(D - v)(D + v)]^{1/2}$ , and as usual  $\{\dots\}_+$  denotes the differential part of the pseudodifferential operator in the brackets. An obvious generalization of Eq. (57) for the  $k$ th multicritical point is

$$\tilde{P}_k = \{(D + v)A_k\}_+ , \tag{58}$$

where

$$\begin{aligned} A_k &= [(D - v)(D + v)]^{k-1/2} \\ &= D^{2k-1} + g_{k,2k-2}D^{2k-2} + \dots + g_{k,0} + f_{k,1}D^{-1} + f_{k,2}D^{-2} + \dots . \end{aligned} \tag{59}$$

$\tilde{P}_k$  is then a differential operator of order  $2k$  as in Eq. (43). Equation (59) then determines the coefficients  $p_{k,i}$  and Eq. (56) gives two copies of the string equation for the function  $v$ . The latter is found to be

$$(\text{Res } A_k)' + 2(\text{Res } A_k)v = 2a_k v x , \tag{60}$$

where  $\text{Res } A_k = f_{k,1}$ . Note that because  $\text{Res } A_1 = 1/2(v' - v^2)$ , Eq. (60) trivially gives Eq. (50)

For the derivation of Eq. (60), we observe that the trivial equations

$$(D + v)A_k(D - v) - (D + v)A_k(D - v) = 0$$

and

$$\mathcal{O} = \mathcal{O}_+ + \mathcal{O}_-$$

for any pseudodifferential operator  $\mathcal{O}$  give

$$\tilde{P}_k(D - v) - (D + v)\tilde{P}_k^\dagger = -\{(D + v)A_k\}_-(D - v) + (D + v)\{A_k(D - v)\}_- . \quad (61)$$

Since the only overlap of the pseudodifferential operators on each side of Eq. (61) is the constant part this establishes that the left-hand side of Eq. (61) is a purely multiplicative operator in the ring of pseudodifferential operators. Computing the right-hand side of Eq. (61) and equating it to that of Eq. (56) we obtain the string equation (60).

### 5. The Relation to the mKdV Hierarchy and the Action Principle

In this section we discuss the relation between Eq. (60) and the mKdV hierarchy and we find an action principle from which Eq. (60) is derived.<sup>25</sup> A simple way to see that Eq. (60) is related to the mKdV hierarchy is the following. First observe that

$$(D - v)(D + v) = D^2 + (v' - v^2) \equiv D^2 - u , \quad (62)$$

where  $u$  is related to  $v$  by the Miura transformation<sup>26,19</sup>

$$u = v^2 - v' . \quad (63)$$

It is a standard result that

$$\begin{aligned} A_k &= (D^2 - u)^{k-1/2} = \sum_{i=-\infty}^k \{e_{2i-1}, D^{2i-1}\} \\ &= D^{2k-1} - \frac{2k-1}{4}\{u, D^{2k-3}\} + \dots + \{R_k[u], D^{-1}\} + \dots . \end{aligned} \quad (64)$$

Therefore

$$\text{Res } A_k = 2R_k[u] . \quad (65)$$

The Gelfand–Dikii potentials  $R_k[u]$  are defined through the recursion relation

$$DR_{k+1}[u] = M^{\text{KdV}} R_k[u], \quad R_0[u] = \frac{1}{2} , \quad (66)$$

where  $M^{\text{KdV}} = 1/4D^3 - 1/2(Du + uD)$ . The KdV flows are given by

$$u_{t_k} + M^{\text{KdV}} R_k[u] = 0 . \quad (67)$$

The mKdV flows are similarly generated by the potentials<sup>27</sup>

$$DR_{k+1}^{\text{mKdV}}[v] = M^{\text{mKdV}} R_k^{\text{mKdV}}[v], \quad R_1^{\text{mKdV}}[v] = v , \quad (68)$$

and are given by

$$v_{t_k} + M^{\text{mKdV}} R_k^{\text{mKdV}}[v] = 0 . \quad (69)$$

The operator  $M^{\text{mKdV}} = 1/2D^3 - 2v^2D - 2vv' + 2v'D^{-1}v' = 1/2D^3 - (v^2D + Dv^2) + 2v'D^{-1}v'$ . With the normalization chosen  $M^{\text{mKdV}}R_1^{\text{mKdV}}[v]$  is equal to the derivative of the left-hand side of Eq. (50).

Using the Miura transformation Eq. (63) we find that

$$R_{k+1}^{\text{mKdV}}[v] = 2^{k-2}(2vR_k^{\text{KdV}}[u] + DR_k^{\text{KdV}}[u]) . \tag{70}$$

Comparing with Eq. (60) and Eq. (65) we can write the string equation in the form<sup>17,23</sup>

$$R_{k+1}^{\text{mKdV}}[v] = 2^{k-2}a_k v x . \tag{71}$$

In order to see the relation of Eq. (71) with the one given in Ref. 17, one must use Eq. (68) and  $R_2^{\text{mKdV}}[v] = 1/2v'' - v^3$

$$\begin{aligned} 2^{k-2}a_k v x &= R_{k+1}^{\text{mKdV}}[v] \\ &= D^{-1}M^{\text{mKdV}}R_k^{\text{mKdV}}[v] \\ &= (D^{-1}M^{\text{mKdV}})^{k-1}R_2^{\text{mKdV}}[v] \\ &= D^{-1}(M^{\text{mKdV}}D^{-1})^{k-1}DR_2^{\text{mKdV}}[v] \\ &= D^{-1}\mathcal{D}_{\text{PS}}^{k-1}\left(\frac{1}{2}v''' - 3v^2v'\right) \end{aligned} \tag{72}$$

where  $\mathcal{D}_{\text{PS}} = (M^{\text{mKdV}}D^{-1}) = 1/2D^2 - 2v^2 - 2v'D^{-1}v$ . This is, up to rescalings, the form of the string equation for the  $k$ th multicritical point given in Ref. 17. From the third line of Eq. (72) we see that an alternative way of writing the string equation is

$$2^{k-2}a_k v x = \mathcal{D}^{k-1}R_2^{\text{mKdV}}[v] , \tag{73}$$

where  $\mathcal{D} = (D^{-1}M^{\text{mKdV}}) = 1/2D^2 - 2v^2 + 2vD^{-1}v'$ .

It is remarkable that we can write an action principle quite similar to the one of the Hermitian one-matrix model. Using the relation

$$\frac{\delta}{\delta u} \int dx R_{k+1}[u] = -\left(k + \frac{1}{2}\right) R_k[u] , \tag{74}$$

we find that by minimizing the action

$$I = \int dx \left\{ \text{Res}A_{k+1} + a_k \left(k + \frac{1}{2}\right) v^2 x \right\} , \tag{75}$$

we obtain the string equation (60). Indeed using (63), (65), (74) we get

$$\begin{aligned} \delta I &= -(2k+1) \int dx (R_k[u]\delta u - a_k v x \delta v) \\ &= -(2k+1) \int dx (R_k[u](2v\delta v - \delta v') - a_k v x \delta v) \\ &= -(2k+1) \int dx (2vR_k[u] + DR_k[u] - a_k v x)\delta v \\ &= 0 . \end{aligned} \tag{76}$$

### Conclusions

We have seen that the basis of trigonometric orthogonal polynomials on the circle allows an analysis of unitary matrix models which closely parallels that of the Hermitian models. There is a finite-term recursion relation for the multiplication operators  $z_{\pm}$  and the derivative operator  $z\partial_z$ , which leads in the continuum limit to an explicit representation in terms of pseudodifferential operators. The string equation has a simple formulation in terms of these operators and follows from an elegant action principle. The most pressing open problem is to find a world-sheet interpretation of unitary matrix models analogous to that of 2D-gravity coupled to  $(p, q)$  conformal matter in the case of the Hermitian models. In this respect some of the recent results of Minahan<sup>28</sup> are interesting.

### Acknowledgements

The research of K.A. and M.B. was supported by the Outstanding Junior Investigator Grant DOE DE-FG02-85ER40231, the Office of Sponsored Research of Syracuse University and NSF grant PHY 89-04035. The research of N.I. was supported by the Nishina Memorial Foundation and NSF grant PHY 86-14185. We would like to thank Mike Douglas, Vipul Periwal and Rob Myers for several enlightening conversations. K.A. and M.B. would like to thank the Institute for Theoretical Physics and its staff for providing the stimulating environment which made this work possible.

### References

1. F. David, *Nucl. Phys.* **B257**, 45 (1985).
2. V. Kazakov, *Phys. Lett.* **B150**, 28 (1985).
3. V. Kazakov, I. Kostov and A. Migdal, *Phys. Lett.* **B157**, 295 (1985).
4. J. Ambjørn, B. Durhuus and J. Fröhlich, *Nucl. Phys.* **B257**, 433 (1985).
5. For an excellent introduction, see *Statistical Mechanics of Membranes and Surfaces*, ed. D. Nelson, T. Piran and S. Weinberg (World Scientific, 1989).
6. E. Brézin, C. Itzykson, G. Parisi and J. B. Zuber, *Commun. Math. Phys.* **59**, 35 (1978).
7. E. Brézin and V. Kazakov, *Phys. Lett.* **B236**, 144 (1990).
8. M. Douglas and S. Shenker, *Nucl. Phys.* **B335**, 635 (1990).
9. D. Gross and A. Migdal, *Phys. Rev. Lett.* **64**, 127 (1990); *Nucl. Phys.* **B340**, 333 (1990).
10. V. Kazakov, *Mod. Phys. Lett.* **A4**, 2125 (1989).
11. E. Brézin, E. Marinari and G. Parisi, *Phys. Lett.* **B242**, 35 (1990).
12. M. Douglas, N. Seiberg and S. Shenker, *Phys. Lett.* **B244**, 381 (1990).
13. D. Gross and E. Witten, *Phys. Rev.* **D21**, 446 (1980).
14. S. R. Wadia, *Phys. Rev.* **D24**, 970 (1981).
15. H. Neuberger, *Nucl. Phys.* **B340**, 703 (1990); **B179**, 253 (1981).
16. V. Periwal and D. Shevitz, *Phys. Rev. Lett.* **64**, 1326 (1990).
17. V. Periwal and D. Shevitz, *Nucl. Phys.* **B344**, 731 (1990).
18. A. Gerasimov, A. Marshakov, A. Mironov, A. Morozov and A. Orlov, *Nucl. Phys.* **B357**, 565 (1991).
19. R.C. Myers and V. Periwal, *Phys. Rev. Lett.* **65**, 1088 (1990).
20. M. J. Bowick, A. Morozov and D. Shevitz, *Nucl. Phys.* **B354**, 496 (1991).
21. E. Martinec, *Commun. Math. Phys.* **138**, 437 (1991).
22. K. Demeterfi and C. I. Tan, *Mod. Phys. Lett.* **A5**, 1563 (1990).

23. Č. Crnković, M. Douglas and G. Moore, "Physical solutions for unitary matrix models", to appear in *Nucl. Phys. B* (YCTP-P6-90, RU-90-62).
24. M. R. Douglas, *Phys. Lett.* **B238**, 176 (1990).
25. P. Ginsparg, M. Goulian, M. R. Plesser and J. Zinn-Justin, *Nucl. Phys.* **B342**, 539 (1990).
26. R. Miura, *J. Math. Phys.* **9**, 1202 (1968).
27. F. Magri, *J. Math. Phys.* **19**, 1156 (1978).
28. J. Minahan, "Matrix models with boundary terms and the generalized Painlevé II equation", Univ. Virginia preprint PRINT-91-0105; "Schwinger-Dyson equations for unitary matrix models with boundaries," Univ. Virginia preprint UVA-HET-91-04.