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Nuclear Physics B (Proc. Suppl.) 88 (2000) 241–244

NUCLEAR PHYSICS B
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Making the gravitational path integral more Lorentzian *or* Life beyond Liouville gravity

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In two space-time dimensions, there is a theory of Lorentzian quantum gravity which can be defined by a rigorous, non-perturbative path integral and is inequivalent to the well-known theory of (Euclidean) quantum Liouville gravity. It has a number of appealing features: i) its quantum geometry is non-fractal, ii) it remains consistent when coupled to matter, even beyond the c=1 barrier, iii) it is closer to canonical quantization approaches than previous path-integral formulations, and iv) its construction generalizes to higher dimensions.

1. MOTIVATION

The ultimate aim of the work described below is to learn more about four-dimensional quantum gravity by relating non-perturbative canonical and covariant approaches, which so far have not been successful separately.

By 'covariant' we do not mean semi-classical gravitational path integrals, but genuine "sums over all metrics", which usually involve a discretization of space-time. A prototype of this ansatz is quantum Regge calculus. With the help of numerical simulations, one tries to find a non-trivial fixed point and an associated continuum theory of quantum gravity. A great deal of numerical expertise has been gathered in the approach of dynamical triangulations, a recent variant of the Regge method. Unfortunately, all investigations so far have concentrated on path integrals for unphysical space-time metrics of *Euclidean* signature. Unlike for some fixed background metrics, there is no prescription of how to "Wick-rotate" a general Euclidean metric to Lorentzian signature.

On the other hand, a lot of progress has been made in the last ten years in an analytic formulation of *canonical* quantum gravity based on a reformulation in terms of gauge-theoretic variables, called "loop quantum gravity". Although *a priori* based in the continuum, the quantum theory has a number of discrete features reminiscent of a generally covariant version of a lattice gauge field theory. However, in this approach some basic obstacles remain in defining a satisfactory quantum Hamiltonian evolution, and efficient numerical methods have not yet been developed.

It is tempting to try to combine the positive aspects of both approaches, but one soon realizes that in order to relate the two, a number of technical and conceptual difficulties have to be overcome. To narrow this gap, we want to define a *Lorentzian* path integral where individual regularized space-time geometries in the sum are required to be causal, reflected in a local "light-cone structure" and the absence of closed time-like curves. It should be appreciated that it is relatively easy to write down Feynman sums of amplitudes

$$\sum_{\text{causal geometries } \{I\}} e^{iS^{\text{Einstein}}(I)}, \qquad (1)$$

but that it is very hard to construct concrete models with a suitable regularization, such that the sum can be performed and leads to a non-trivial continuum theory.

PII S0920-5632(00)00776-3

2. AN IDEAL TESTING GROUND: d=2

The difficulties associated with defining the sum (1) can be overcome, at least in dimension $d=2$. There exists already a rigorous discretized path integral for *Euclidean* geometries, obtained by the method of dynamical triangulations, where the path-integral sum is performed over all possible triangulations T (i.e. gluings of equilateral triangles). The 2d gravity action for fixed space-time topology reduces to the cosmological-constant term

$$S=\lambda\int d^2x\,\sqrt{|\det g|}, \qquad (2)$$

for both Euclidean and Lorentzian metrics $g_{\mu\nu}$. After the discretization, this term becomes proportional to $\lambda N(T)$, with $N(T)$ counting the number of triangles contained in T. The Euclidean state sum is given by

$$Z^{\rm eu}(\lambda)=\sum_N e^{-\lambda N}Z^{\rm eu}(N)=\sum_N e^{-\lambda N}\sum_{T^{(N)}}1. \qquad (3)$$

With the help of ingenious combinatorial methods the counting of all triangulations $T^{(N)}$ of volume N in the sum on the right can be done explicitly. Moreover, there is good evidence that the method is diffeomorphism-invariant, since it reproduces the results of continuum Liouville gravity in the continuum limit. How can this framework be adapted to the Lorentzian situation? We have substituted the fundamental equilateral building blocks (with squared edge lengths $a^2=1$) by triangles with two time-like edges with $a^2=-1$ and one space-like edge with $a^2=1$ [1]. To obtain allowed histories, these must be glued causally: consecutive spatial slices (consisting entirely of space-like edges) of variable length l are connected by sets of time-like edges. For simplicity, these slices are compactified to circles S^1. A typical triangulated 2d Lorentzian geometry of t time-steps (t pointing up) is depicted in Fig.1. Note that the local geometric degrees of freedom (apart from the edge lengths) are encoded in the variable coordination numbers of edges meeting at vertices, giving a direct measure of curvature. It turns out that also in this discrete Lorentzian model, the combinatorics can be solved explicitly and yields the Lorentzian analogue $Z^{\rm lor}(\lambda)$ of (3). The partition function exhibits critical behaviour as $\lambda\to\lambda_{\rm crit}$, where a continuum limit can be taken. After appropriate renormalization, one obtains a *new* quantum gravity theory inequivalent to Liouville gravity. It is rather surprising that there is a second universality class of models describing fluctuating two-geometries!

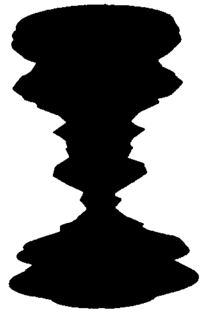

Figure 1. A typical triangulated 2d Lorentzian space-time (of topology $[0,1]\times S^1$).

The central dynamical quantity of the theory is the continuum propagator $G_\Lambda(L_1,L_2;T)$. It describes the transition from an initial spatial geometry of length L_1 to a final one of length L_2 in proper time T and takes the form [1]

$$G_\Lambda(L_1,L_2;T)=e^{-\coth(\sqrt{i\Lambda}T)\sqrt{i\Lambda}(L_1+L_2)}\times\frac{\sqrt{i\Lambda L_1L_2}}{\sinh(\sqrt{i\Lambda}T)}\,I_1\Big(\frac{2\sqrt{i\Lambda L_1L_2}}{\sinh(\sqrt{i\Lambda}T)}\Big), \qquad (4)$$

where I_1 denotes the modified Bessel function.

In order to illustrate our claim that the Lorentzian quantum gravity theory differs from Liouville gravity, let us look at the behaviour of a simple observable. A good example is the so-called Hausdorff dimension d_H, which contains information about the bulk properties of the quantum geometry in the ground state of the theory. It is measured by looking at the volume $V\sim r^{d_H}$ of geodesic balls (discs in dimension 2) of radius r. Liouville gravity has a fractal Hausdorff dimension $d_H=4$. This may be surprising at first, but has to do with the fact that the dominant contributions to the path integral are highly branched geometries, with many "baby universes". By contrast, in the Lorentzian theory we have $d_H = 2$, which is the "canonical" dimension expected from naïve semi-classical considerations. The difference arises because there are no baby universes in Lorentzian gravity. At a point where a baby universe branches off, the Lorentzian metric structure must inevitably go bad, thereby violating causality. This also implies that in Lorentzian gravity the topology of the spatial slices cannot change. Note that this is exactly the situation described by canonical approaches to gravity.

3. COUPLING MATTER TO LORENTZIAN GRAVITY

The discussion of the previous section suggests that the geometry of Lorentzian quantum gravity is "better" behaved than its Euclidean counterpart. This is also illustrated by Fig.1 (taken from a Monte Carlo simulation of pure Lorentzian gravity). In spite of strong fluctuations $\langle \Delta l \rangle$ of the length of spatial slices, the geometry is still effectively two-dimensional. The geometry of the Lorentzian model therefore lies somewhere in between the wildly fluctuating and fractal quantum geometry of the Liouville model and that of a fixed classical two-dimensional space-time.

It is an interesting question how matter will behave under coupling to the Lorentzian model. To investigate this issue, we have considered a model of Ising spins with nearest-neighbour interaction. Coupling this to *Euclidean* dynamical triangulations yields an exactly soluble model of Euclidean gravity plus matter. Its matter behaviour is governed by the critical exponents

$$\alpha = -1, \quad \beta = 0.5, \quad \gamma = 2, \tag{5}$$

characterizing the singularity structure of the specific heat, the spontaneous magnetization, and the magnetic susceptibility as functions of the bare Ising coupling constant β_I. This should be contrasted with the Onsager values of these exponents found on fixed, flat lattices, which are given by

$$\alpha = 0, \quad \beta = 0.125, \quad \gamma = 1.75. \tag{6}$$

The partition function for Lorentzian gravity coupled to Ising spins $\sigma_i = \pm 1$ is the sum

$$Z(\lambda, \beta_I) = \sum_N e^{-\lambda N} \sum_{T^{(N)}} Z_{T^{(N)}}(\beta_I), \tag{7}$$

where the partition function $Z_T(\beta_I)$ of the Ising model on the Lorentzian triangulation T is

$$Z_T(\beta_I) = \sum_{\{\sigma_i\}} e^{\beta_I/2 \sum_{\langle ij \rangle} \sigma_i \sigma_j}. \tag{8}$$

We have investigated this model by means of a high-T (that is, small inverse temperature β_I) expansion and by Monte Carlo simulations [2]. An exact solution has not yet been constructed. Note that eq. (7) describes the Euclidean sector of Lorentzian gravity plus matter, i.e. with real weights and therefore Euclidean values for the coupling constants. This is the form suitable for numerical simulations. What we have found is that both methods agree with good precision in their estimates of the critical matter exponents, which turn out to be the Onsager exponents. The Hausdorff dimension of the geometry is unaltered, $d_H = 2$, and the typical Monte-Carlo-generated geometries look qualitatively similar to the ones in pure gravity. There *are* effects of the gravity-matter coupling at the discretized level, for example, on the distribution of coordination numbers, but we have not investigated whether this is reflected in a change of universal properties of the geometry that would survive in the continuum limit.

4. COUPLING MORE MATTER

The previous picture is changed drastically when several Ising models instead of one are coupled to Lorentzian gravity. For the case of n Ising models, the partition function (7) is replaced by

$$Z(\lambda, \beta_I) = \sum_N e^{-\lambda N} \sum_{T^{(N)}} Z^n_{T^{(N)}}(\beta_I). \tag{9}$$

At the critical point, this model describes a conformal field theory with central charge $c = n/2$ coupled to gravity. Our motivation for coupling more matter is the fact that *Euclidean* 2d gravity becomes inconsistent for $n > 2$, that is, beyond the so-called $c = 1$ barrier. In the presence of Ising spins it is energetically favourable to have short boundaries between regions of opposite spins. In a theory of fluctuating geometry the effect of the spins is to try and "squeeze off" parts of the space-time manifold. In Euclidean gravity, where the geometry is very branched to start with, this mechanism seems to be so effective that for $n > 2$ the theory seizes to make sense.

In order to get a clear picture of what goes on "well beyond the $c=1$ barrier" in Lorentzian gravity, we have investigated its properties at $n=8$ by numerical simulations [3]. One observes a very strong interaction of gravity and matter, to the extent that the geometry is now in a different phase from before: time and space directions acquire an anomalous relative scaling and the Hausdorff dimension is changed to $d_H = 3$! This is illustrated by Fig.2. The effect of the matter on the geometry is reflected in the presence of the long, stalk-like part of the space-time, which is effectively one-dimensional. All interesting physics (that survives the limit as $N \to \infty$) happens in the extended bulk phase. However, in spite of these drastic changes in the geometrical properties, we have found that the critical matter exponents retain their Onsager values!

5. CONCLUSIONS

There are a number of lessons to be learned from this two-dimensional model of quantum gravity. The choice of Lorentzian over Euclidean, which in our case consisted in the imposition of a causality condition on individual path-integral histories, made a big difference. In two dimensions, it led us to the discovery of a new universality class of quantum gravity models, besides that of Liouville gravity. In Lorentzian gravity, the quantum geometry is much smoother, and better behaved in the sense that one can cross the infamous $c=1$ barrier without any problems. Conversely, the coupled model with eight Ising models illustrated that the matter behaviour is rather robust: the geometry can undergo drastic changes without the critical matter behaviour being affected. From this we also learn that Onsager exponents by no means imply that the underlying space-time is flat.

Figure 2. Lorentzian gravity coupled to 8 Ising models (N=18816, t=168).

The difference between the Euclidean and Lorentzian theories can be traced entirely to the presence of branchings or baby universes [1,3]. Since this is a purely kinematical effect which has to do with an *a priori* restriction on the sum-over-geometries, it will be present in higher dimensions as well. To date, the problem with dynamically triangulated path integrals for Euclidean geometries in $d > 2$ has been the dominance of highly degenerate geometries, including a proliferation of baby universes. Our hope is that also in these cases a causality requirement will lead to an effective "smoothing out" of the quantum geometry. An investigation of the case $d=3$ is under way.

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