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A study of the complex action problem in a simple model for dynamical compactification in superstring theory using the factorization method.

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The IIB matrix model proposes a mechanism for dynamically generating four dimensional space–time in string theory by spontaneous breaking of the ten dimensional rotational symmetry SO(10). Calculations using the Gaussian expansion method (GEM) lend support to this conjecture. We study a simple SO(4) invariant matrix model using Monte Carlo simulations and we confirm that its rotational symmetry breaks down, showing that lower dimensional configurations dominate the path integral. The model has a strong complex action problem and the calculations were made possible by the use of the factorization method on the density of states $\rho_n(x)$ of properly normalized eigenvalues $\tilde{\lambda}_n$ of the space–time moment of inertia tensor. We study scaling properties of the factorized terms of $\rho_n(x)$ and we find them in agreement with simple scaling arguments. These can be used in the finite size scaling extrapolation and in the study of the region of configuration space obscured by the large fluctuations of the phase. The computed values of $\tilde{\lambda}_n$ are in reasonable agreement with GEM calculations and a numerical method for comparing the free energy of the corresponding ansatze is proposed and tested.

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1. Introduction

Matrix models have been studied intensively in the past few years in the context of non-perturbative formulations of string theory and in the study of gauge/gravity duality. By dimensionally reducing $D = 10$ dimensional $\mathrm{U}(N)$ supersymmetric (SUSY) Yang–Mills theories to zero dimensions, one obtains the IIB Matrix Model [1] (IKKT model) which has been proposed as a non-perturbative definition of IIB superstring theory. In this model space–time is represented by the distribution of eigenvalues of the bosonic matrices, a feature that raises the possibility of *dynamical* compactification of the extra dimensions by Spontaneous Symmetry Breaking (SSB) of the $\mathrm{SO}(D)$ rotational symmetry of the model. Such a scenario is plausible as calculations using the Gaussian Expansion Method (GEM) indicate [2].

Monte Carlo simulations of matrix models [3, 4] could play an important role in understanding string theories in a similar fashion that lattice QCD has contributed to the understanding of the non–perturbative regime of quantum field theories. Unfortunately such simulations are plagued by the complex action problem which arises when one simulates the system after integrating out the fermionic degrees of freedom. This problem is particularly important in the lattice studies of finite density QCD [5]. The factorization method has been proposed in [6] as a general method to reduce the complex action problem and eliminate the overlap problem, see also [7]. The basic idea is to control an *appropriately chosen* variable in order to sample regions of the configuration space which are hard to sample using reweighting and whose contribution is crucial in the computation of the physical observables. The study of the scaling properties of the related density of states allow for useful extrapolations to the physical results.

We present preliminary results from calculations performed on a related zero–dimensional matrix model proposed in [8] which realizes the scenario of dynamical compactification of space–time dimensions [8, 9]. The model has a very strong complex action problem and bears strong similarities to the IIB matrix model, which makes it a useful playground for testing ideas to apply on the IIB matrix model and more generally on other interesting physical systems with a complex action problem. We are able to show that SSB occurs consistently with the predictions in [8, 9]. The scaling properties of the density of states are studied in detail and are found to agree with simple scaling arguments. This is possible only by sampling heavily suppressed regions by using the factorization method and it is crucial in the extrapolations used in order to compute the expectation values of the SSB order parameters.

2. The Model

Consider the partition function [8]

$$Z = \int dA\, d\psi\, d\bar{\psi}\, \mathrm{e}^{-(S_{\mathrm{b}}+S_{\mathrm{f}})} \qquad \text{where} \qquad S_{\mathrm{b}} = \frac{1}{2} N \,\mathrm{tr}(A_\mu)^2\,, \qquad S_{\mathrm{f}} = -\bar{\psi}^f_\alpha (\Gamma_\mu)_{\alpha\beta} A_\mu \psi^f_\beta\,. \quad (2.1)$$

A_μ ($\mu = 1,\ldots,D$, D even) are $N\times N$ hermitian matrices, and $\bar{\psi}^f_\alpha$ and ψ^f_α are N-dimensional row and column vectors.The actions S_{b} and S_{f} have an $\mathrm{SU}(N)$ symmetry. The spinor index $\alpha = 1,\ldots,p$, where p represents the number of components of a D-dimensional Weyl spinor, $p = 2^{D/2-1}$, and the flavor index $f = 1,\ldots,N_{\mathrm{f}}$, where N_{f} represents the number of flavors. The $p\times p$ matrices Γ_μ are

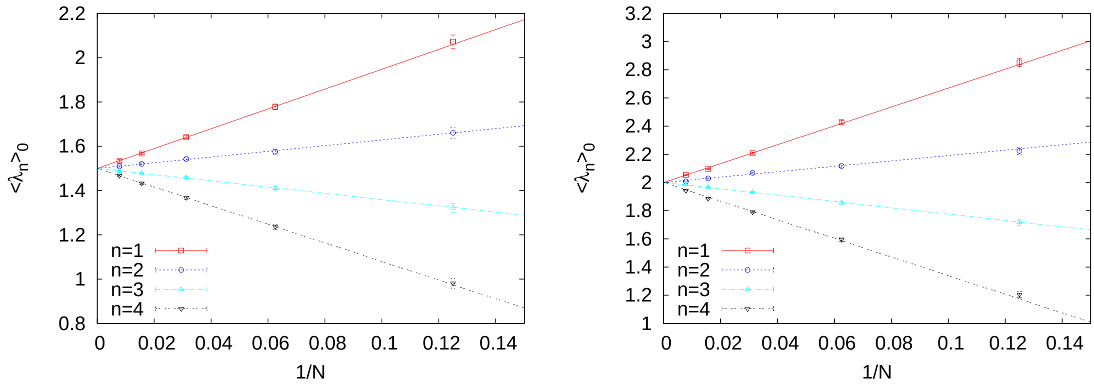

Figure 1: The VEV $\langle \lambda_n \rangle_0$ ($n = 1, 2, 3$ and 4) in the *phase quenched model* Z_0 are plotted for $r = 1$ (left) and $r = 2$ (right) against $\dfrac{1}{N}$. The data for each r can be nicely fitted to straight lines meeting at the same point $(1 + r/2)$ at $N = \infty$, which demonstrates the absence of SSB.

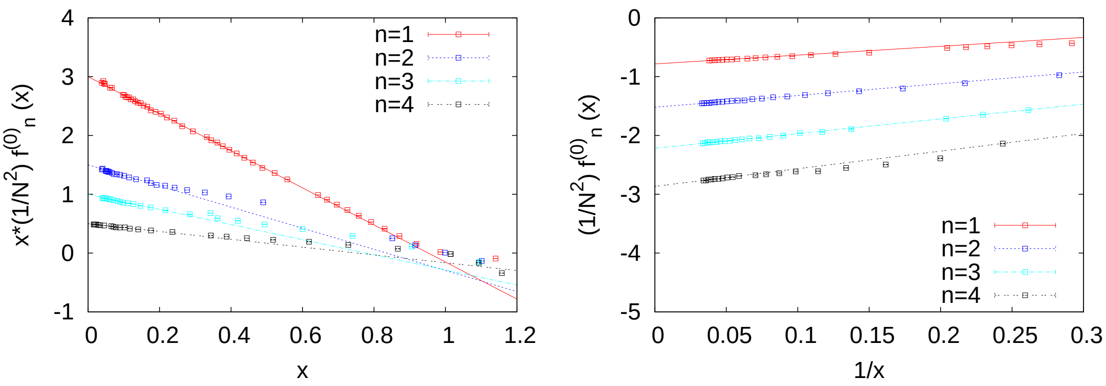

Figure 2: The small-x (left) and large-x (right) behavior of $\dfrac{1}{N^2} f_n^{(0)}(x)$ for $r = 1$. The straight lines are fits to the theoretical behavior (3.1).

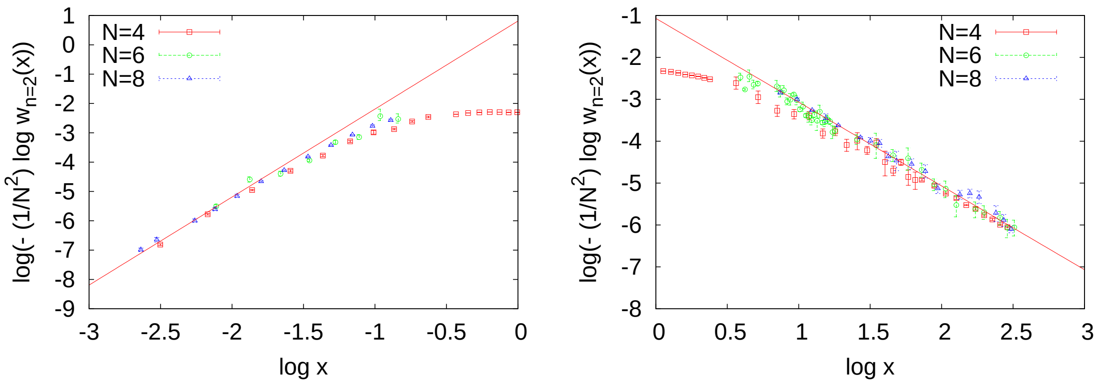

Figure 3: The asymptotic behavior (3.2) for $r = 1$ and $n = 2$.The straight lines are fits to the predicted power-law behavior using $N = 8$ data. The same power law is obeyed also by smaller N data, and a clear trend towards large-N scaling is observed.

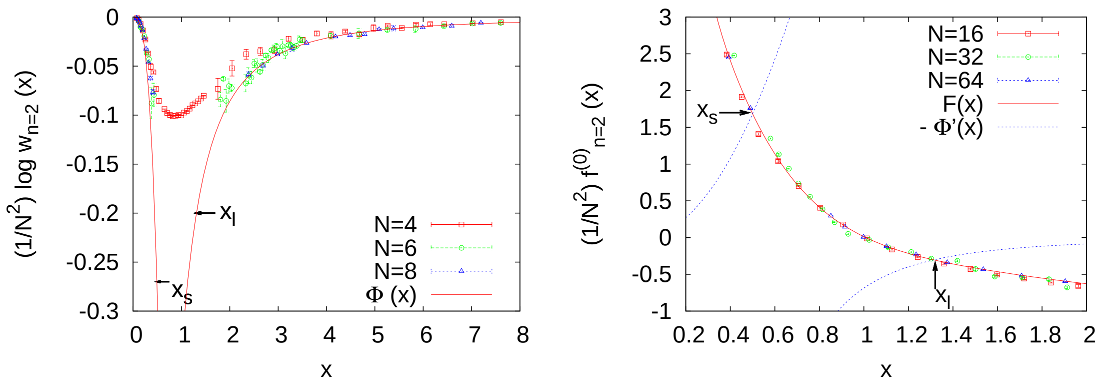

Figure 4: (Left) The function $\frac{1}{N^2}\log w_n(x)$ is plotted together with the scaling function $\Phi_n(x)$ extracted from the asymptotic behaviors (3.2) for $r = 1$ and $n = 2$. (Right) The solution to (2.5) is obtained by finding the intersections of $\frac{1}{N^2} f_n^{(0)}(x)$ and $\frac{d}{dx}\Phi_n(x)$ for $r = 1$ and $n = 2$. The position of the intersections is indicated by the arrows with the symbols x_s and x_l for the regions $x < 1$ and $x > 1$, respectively.

SO(D) gamma matrices after the Weyl projection. Thus the actions (2.1) have an SO(D) symmetry, where the bosonic variables A_μ transform as vectors and the fermionic variables transform as Weyl spinors. Integrating out the fermions, we obtain $Z = \int dA\, \mathrm{e}^{-S_b} Z_f[A]$, where $Z_f[A] = (\det \mathscr{D})^{N_f}$ and $\mathscr{D} = \Gamma_\mu A_\mu$ is a $pN \times pN$ matrix. The fermion determinant $\det \mathscr{D}$ for a single flavor is complex in general. Under parity transformation $A_D \to -A_D$, $A_i \to A_i$ $(i \neq D)$, the fermion determinant transforms as $\det \mathscr{D} \to (\det \mathscr{D})^*$. This implies that $\det \mathscr{D}$ is real for configurations with $A_D = 0$ and that the phase of the determinant becomes stationary for configurations with $A_D = A_{D-1} = 0$. We take the large-N limit with $r = N_f/N$ fixed, which corresponds to the Veneziano limit. Whether the SSB of SO(D) occurs in that limit is the issue we would like to address. For that purpose, we consider the "moment of inertia tensor" $T_{\mu\nu} = \frac{1}{N}\mathrm{tr}(A_\mu A_\nu)$ and its real positive eigenvalues λ_n $(n = 1, \ldots, D)$ ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_D$. The vacuum expectation values (VEV) of these eigenvalues $\langle \lambda_n \rangle$ play the role of the order parameters. If they turn out to be unequal in the large-N limit, it signals the SSB of SO(D). Consider the $D = 4$ case where we have $p = 2$ and $\Gamma_i = \sigma_i$, $i = 1, 2, 3$ and $\Gamma_4 = i\sigma_4$. The "phase–quenched model" is defined by

$$Z_0 = \int dA\, e^{-S_0[A]}\,, \qquad S_0[A] = S_b[A] - N_f \log|\det \mathscr{D}[A]|\,. \tag{2.2}$$

It is easy to show that the absence of SSB implies

$$\langle \lambda_n \rangle_0 = 1 + \frac{r}{2} \qquad \text{for all} \quad n = 1, 2, 3, 4\,, \tag{2.3}$$

where the VEVs $\langle \cdot \rangle_0$ are taken with respect to (2.2), which is confirmed at infinitesimal r [8], and also at $r = 1$ and $r = 2$ numerically in this work. In the full model, GEM calculation up to 9-th order [9] indicate that the true vacuum is only SO(2) invariant and the $\langle \lambda_n \rangle$ are not all equal.

In order to simulate (2.1) we rewrite it as $Z = \int dA\, \mathrm{e}^{-S_0[A]}\, \mathrm{e}^{i\Gamma[A]}$. This system is very hard to simulate using simple reweighting due to the complex action and overlap problem which make the simulations of large systems exponentially hard. In this work we use the factorization method proposed in ref. [6] where one computes the density of states of a *properly chosen* observable

	$r=1$				$r=2$			
n	x_s	x_l	$x_{\mathrm{SO}(2)}$	$x_{\mathrm{SO}(3)}$	x_s	x_l	$x_{\mathrm{SO}(2)}$	$x_{\mathrm{SO}(3)}$
1		2.12	1.4	1.2		1.94	1.7	1.2
2	0.49	**1.29**	**1.4**	1.2	0.48	**1.36**	**1.7**	1.2
3	**0.67**	*1.13*	**0.7**	*1.2*	**0.53**	*1.16*	**0.5**	*1.2*
4	*0.75*		0.5	*0.5*	*0.51*		0.1	*0.3*

Table 1: The solutions $(x_\mathrm{s}, x_\mathrm{l})$ to eq. (2.4) that correspond to the (local) maxima of $\rho_n(x)$ are shown. We also add the corresponding VEV $\langle\tilde{\lambda}_n\rangle$ obtained by the Gaussian expansion method in [9]: $x_{\mathrm{SO}(2)} \equiv \langle\tilde{\lambda}_n\rangle$ obtained using the SO(2) ansatz and $x_{\mathrm{SO}(3)} \equiv \langle\tilde{\lambda}_n\rangle$ obtained using the SO(3) ansatz. Bold/italic numbers for same r and n are to be compared according to the text.

by studying a set of systems where the observable is constrained to a given fixed value. The choice of the eigenvalues λ_n is promising since restricting their values to be large or small favors configurations with relatively small fluctuations of the phase Γ. In this respect it is convenient to define $\tilde{\lambda}_n = \frac{\lambda_n}{\langle\lambda_n\rangle_0}$ and the density of states $\rho_n(x) = \left\langle \delta(x-\tilde{\lambda}_n)\right\rangle$ and $\rho_n^{(0)}(x) = \left\langle \delta(x-\tilde{\lambda}_n)\right\rangle_0$. Then it is easy to show that $\rho_n(x) = \frac{1}{C}\rho_n^{(0)}(x)\,w_n(x)$, where $C = \langle e^{i\Gamma}\rangle_0 = \langle\cos\Gamma\rangle_0$. It follows that $\langle\tilde{\lambda}_n\rangle = \int_0^\infty dx\, x\, \rho_n(x)$ and the deviation of its value from one is a measure of the effect of the phase. The function $w_n(x)$ is defined by $w_n(x) = \langle e^{i\Gamma}\rangle_{n,x} = \langle\cos\Gamma\rangle_{n,x}$, where $\langle\,\cdot\,\rangle_{n,x}$ denotes a VEV with respect to the partition function $Z_{n,x} = \int dA\, \mathrm{e}^{-S_0}\,\delta(x-\tilde{\lambda}_n)$. It turns out that $w_n(x) > 0$, which simplifies our analysis significantly. Using the saddle point approximation, the problem of determining $\langle\tilde{\lambda}_n\rangle$ can be reduced to that of minimizing the "free energy" $\mathscr{F}_n(x) = -\log\rho_n(x)$ by solving the saddle point equation

$$\frac{d}{dx}\log\rho_n(x) = f_n^{(0)}(x) + \frac{d}{dx}\log w_n(x) = 0\,, \tag{2.4}$$

where $f_n^{(0)}(x) = \frac{d}{dx}\log\rho_n^{(0)}(x)$. It is important that the errors due to statistics and finite N do not propagate exponentially to $\langle\tilde{\lambda}_n\rangle$ as a direct computation would imply.

The implementation of the above system is obtained by studying $Z_{n,V} = \int dA\, e^{-\{S_0+V(\lambda_n)\}}$, where $V(z) = \frac{1}{2}\gamma(z-\xi)^2$ and γ and ξ are real parameters. The parameter γ controls the position and width of the peak of $\tilde{\lambda}_n$ and it is chosen large enough so that the results become independent of its value. In our simulations we used γ in the range 10^3–10^7. Using the fact that $\rho_{n,V}(x) = \left\langle \delta(x-\tilde{\lambda}_n)\right\rangle_{n,V} \propto \rho_n^{(0)}(x)\exp\left\{-V\left(x\langle\lambda_n\rangle_0\right)\right\}$, where $\langle\,\cdot\,\rangle_{n,V}$ is a VEV with respect to $Z_{n,V}$, the position of the peak of the distribution function $\rho_{n,V}(x)$ is given by the solution of

$$f_n^{(0)}(x) - \langle\lambda_n\rangle_0\, V'\left(x\langle\lambda_n\rangle_0\right) = 0\,. \tag{2.5}$$

If we denote the solution by x_p, we use the estimators $x_p = \langle\tilde{\lambda}_n\rangle_{n,V}$, $w_n(x_p) = \langle\cos\Gamma\rangle_{n,V}$ and $f_n^{(0)}(x_p) = \langle\lambda_n\rangle_0\, V'\left(\langle\lambda_n\rangle_{n,V}\right) = \gamma\langle\lambda_n\rangle_0\left(\langle\lambda_n\rangle_{n,V} - \xi\right)$.

3. Results

First we study the phase quenched model Z_0. We simulate the system for $r = 1,2$ and compute the eigenvalues $\langle\lambda_n\rangle_0$. We find that $\langle\lambda_n\rangle_0(N) = 1 + r/2 + \mathscr{O}(1/N)$ as can be seen from fig. 1. We conclude that no SSB occurs in the phase quenched model and verify eq. (2.3). We calculate $f_n^{(0)}(x)$ from eq. (2.5) by simulating $Z_{n,V}$. Using simple scaling arguments we find that its asymptotic behavior at $x \ll 1$ and $x \gg 1$ is

$$\frac{1}{N^2} f_n^{(0)}(x) \simeq \begin{cases} \left(\frac{1}{2}(5-n) + r\delta_{n1}\right)\frac{1}{x} + a_n & x \ll 1\ , \\ -\frac{1}{2}n\langle\lambda_n\rangle_0 + \left(\frac{n}{2}+r\right)\frac{1}{x} & x \gg 1\ . \end{cases} \tag{3.1}$$

Similar arguments lead to the respective asymptotic behavior of $w_n(x)$

$$\frac{1}{N^2}\ln w_n(x) \simeq \Phi_n(x) = \begin{cases} -c_n x^{5-n} & (x \ll 1, n = 2,3,4)\ , \\ -d_n x^{-(4-n)} & (x \gg 1, n = 1,2,3)\ . \end{cases} \tag{3.2}$$

By varying the constants a_n, c_n and d_n, we fit our data to eqs. (3.1) and (3.2). We verify the expected asymptotic behaviors and use the coefficients c_n and d_n in order to extrapolate $\Phi_n(x)$ to the region in x where we find the solution to the saddle point equation (2.5). In fig. 2 we show the scaling (3.1) for $r = 1$ and in fig. 3 the scaling (3.2) for $r = 1$ and $n = 2$. The solution to eq. (2.5) is determined from the intersection of the curves $\frac{1}{N^2} f_n^{(0)}(x)$ with $-\Phi_n'(x)$ for each n. The results for $r = 1$ and $n = 2$ are shown in fig. 4.

We use the notation x_s and x_l for the solutions in the $x < 1$ and $x > 1$ regions respectively which correspond to the local maxima of $\rho_n(x)$. For $n = 1$ we obtain only $x_l \equiv \langle\tilde{\lambda}_1\rangle$ and for $n = 4$ we obtain only $x_s \equiv \langle\tilde{\lambda}_4\rangle$. We tabulate the results in table 1 and we compare them with those obtained using GEM in [9]. We note that since the dominant configurations near x_l for $n = 2$ and x_s for $n = 3$ are typically two dimensional, these are to be compared with the SO(2) ansatz. Similarly, since the dominant configurations near x_l for $n = 3$ and x_s for $n = 4$ are typically three dimensional, these are to be compared with the SO(3) ansatz.

We find that $\langle\tilde{\lambda}_1\rangle > 1 > \langle\tilde{\lambda}_4\rangle$, a relation that is clearly going to survive the large–N limit. Therefore we conclude that SO(4) SSB manifests in the model. In order to determine the group that SO(4) breaks to, we need to calculate the dominant peak as $N \to \infty$. We consider the quantity $\Delta_n = \frac{1}{N^2}\left\{\log\rho_n(x_l) - \log\rho_n(x_s)\right\} = \Phi_n(x_l) - \Phi_n(x_s) + \Xi_n$, where $\Xi_n = \int_{x_s}^{x_l} dx \left\{\frac{1}{N^2} f_n^{(0)}(x)\right\}$. If $\Delta_n > 0$ the peak at x_l dominates, otherwise x_s. We find $\Delta_2 \approx 0.34$ for $r = 1$ and $\Delta_2 \approx 0.25$ for $r = 2$ and we conclude that SSB breaks at least down to SO(2). Unfortunately Δ_3 turns out to be very close to 0, so we are unable to determine if SO(4) breaks to SO(2) as GEM predicts or to SO(3).

4. Conclusions

We have tested a scenario for dynamical compactification of space–time by simulating a toy matrix model related to the IIB matrix model of string theory. We have shown SSB of SO(4) rotational symmetry consistent with GEM analysis and small r calculations[8, 9]. The phase quenched model has no SSB, confirming the expectation that the wild fluctuations of the phase of the fermionic partition function plays a crucial role in the mechanism of SSB. Large and small length scales are dynamically generated by these fluctuations which make the calculation of the dominant ones in the thermodynamic limit a challenging problem. Our results indicate how to proceed with the study of the IIB matrix model. Although the latter case is computationally more demanding, SUSY could make SSB easier to see.

Calculations were possible because of the use of the factorization method. By effectively sampling large and small x regions we are able to exploit the asymptotic behaviors of $\rho_n^{(0)}(x)$ and $w_n(x)$ in order to extrapolate the results to regions in x and system size which are inaccessible by direct simulations of the phase quenched model. It is possible that a remaining overlap problem makes the results of table 1 slightly differ from GEM results. By a generalization of the factorization method[10] this problem can be overcome and achieve also better quantitative agreement. Then Monte Carlo studies of many interesting systems hindered by the complex action problem are hopefully going to be made possible by using the factorization method.

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