Geometry Transition in Covariant LQG

Charalampos Theofilis

Supervisors: Marios Christodoulou, IQOQI
Kostas Anagnostopoulos, NTUA

Structure

- Introduction: Why we need QG and why it's difficult to have it. The spirit of LQG
- Classical GR
- Kinematics of LQG
- Dynamics of LQG
- Estimation of the transition amplitude in a tree-order truncation
  Marios Christodoulou, Fabio D'Ambrosio, and Charalampos Theofilis, "Geometry Transition in Spinfoams," (2023) arXiv:2302.12622 [gr-qc]

Why we need Quantum Gravity

Since every interaction we are aware of is of quantum nature we expect the same to be true for Gravity. Except for that, the singularities that arise in GR imply that it is only an effective description of the physical reality and not a fundamental theory.

Why it's difficult to quantize Gravity

Two primary reasons

- The standard Dirac procedure leads to Wheeler-DeWitt equation which is ill defined.
- The standard QFT-inspired split $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ leads to a non-renormalizable theory.

The spirit of LQG

LQG is a background-independendet, non-perturbative theory of Quantum Gravity from the Relativist's perspective spacetime $\leftrightarrow$ gravitational field. The quantum object considered is spacetime itself. There are two versions of the theory; the canonical and the covariant. Here we present the covariant.

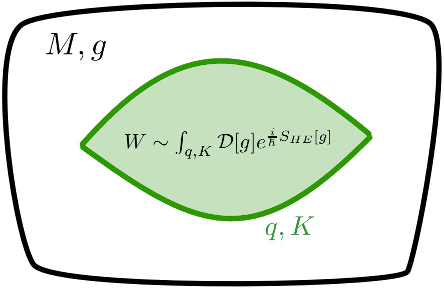
The triplet of CLQG

$$(\mathcal{H}, \mathcal{A}, \mathcal{W})$$

- $\mathcal{H}$: Hilbert space
- $\mathcal{A}$: Set of operators
- $\mathcal{W}$: rule for Dynamics. Here the path integral

We are going to construct everything step by step.

The essence of CLQG in one picture



Action

- $S[g]_{EH} = \int d^4x\sqrt{-g}R[g]$
- $e_I = e_I^\alpha \partial_\alpha$
- $e_I^\alpha e_\alpha^J = \delta_J^I$
  $e_I^\beta e_\alpha^I = \delta_\alpha^\beta$
- $e_{\alpha I} \equiv g_{\alpha\beta} e_I^\beta$
- $g = g_{\alpha\beta} dx^\alpha dx^\beta = e_{\alpha I} e_\beta^I dx^\alpha dx^\beta = e_{\alpha I} e_{\beta J} \eta^{IJ} dx^\alpha dx^\beta = e_I e_J \eta^{IJ} = e_I e^I$
- When $e^I \to \Lambda_K^I e^K$, $g \to g$. Local $SO(1,3)$ symmetry.

Action

- $S[e]_{EH} = \int d^4x |\det e| R[e]$
- $e^0 \wedge e^1 \wedge e^2 \wedge e^3 = \det(e) d^4x$
- $S_{EH}[e] = \int e^0 \wedge e^1 \wedge e^2 \wedge e^3 R(e)$
- $R(e) = \frac{1}{4} \epsilon_{IJCD} \epsilon^{CDKL} R^{IJ}{}_{KL}(e)$

Action

- $S[e]_{EH} = \int d^4x|\det e|R[e]$
- $e^0 \wedge e^1 \wedge e^2 \wedge e^3 = \det(e)d^4x$
- $S_{EH}[e] = \int e^0 \wedge e^1 \wedge e^2 \wedge e^3 R(e)$
- $R(e) = \frac{1}{4}\epsilon_{IJCD}\epsilon^{CDKL}R^{IJ}{}_{KL}(e)$
- $S_{EH}[e] = \frac{1}{2}\int \star(e \wedge e)_{IJ} \wedge F^{IJ}(e)$
- $F^{KL} := R^{KL}{}_{AB}e^A \wedge e^B$
- $\star(e \wedge e)_{KL} = \frac{1}{2}\epsilon_{IJKL}e^I \wedge e^J$

Action

- $S_{EH}[e] = \int \star(e \wedge e) \wedge F[e]$

Action

- $S_{EH}[e] = \int \star(e \wedge e) \wedge F[e]$
- Palatini formulation of GR: $S_P[g, \Gamma] = \int d^4x \sqrt{-g} R[g, \Gamma]$
- We do the same here: $S_P[e, \omega] = \int \star(e \wedge e) \wedge F[\omega]$

Action

- $S_{EH}[e] = \int \star(e \wedge e) \wedge F[e]$
- Palatini formulation of GR: $S_P[g,\Gamma] = \int d^4x\sqrt{-g}R[g,\Gamma]$
- We do the same here: $S_P[e,\omega] = \int \star(e \wedge e) \wedge F[\omega]$
- We add the extra term $\frac{1}{\gamma}\int e \wedge e \wedge F[\omega]$ where γ is called the Barbero-Immirzi parameter.

Action

- $S_{EH}[e] = \int \star(e \wedge e) \wedge F[e]$
- Palatini formulation of GR: $S_P[g,\Gamma] = \int d^4x\sqrt{-g}R[g,\Gamma]$
- We do the same here: $S_P[e,\omega] = \int \star(e \wedge e) \wedge F[\omega]$
- We add the extra term $\frac{1}{\gamma}\int e \wedge e \wedge F[\omega]$ where γ is called the Barbero-Immirzi parameter. This corresponds to the canonical transformation from ADM variables to the Ashtekar variables and doesn't affect the classical EOM.

Action

- $S_{EH}[e] = \int \star(e \wedge e) \wedge F[e]$
- Palatini formulation of GR: $S_P[g, \Gamma] = \int d^4x \sqrt{-g} R[g, \Gamma]$
- We do the same here: $S_P[e, \omega] = \int \star(e \wedge e) \wedge F[\omega]$
- We add the extra term $\frac{1}{\gamma} \int e \wedge e \wedge F[\omega]$ where γ is called the Barbero-Immirzi parameter. This corresponds to the canonical transformation from ADM variables to the Ashtekar variables and doesn't affect the classical EOM.
- $\frac{1}{\gamma} \int e \wedge e \wedge F[\omega] = \frac{1}{\gamma} \int d^4x \sqrt{-g} \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \to 0$ when on shell

Action

- $S_{EH}[e] = \int \star(e \wedge e) \wedge F[e]$
- Palatini formulation of GR: $S_P[g,\Gamma] = \int d^4x\sqrt{-g}R[g,\Gamma]$
- We do the same here: $S_P[e,\omega] = \int \star(e \wedge e) \wedge F[\omega]$
- We add the extra term $\frac{1}{\gamma}\int e \wedge e \wedge F[\omega]$ where γ is called the Barbero-Immirzi parameter. This corresponds to the canonical transformation from ADM variables to the Ashtekar variables and doesn't affect the classical EOM.
- $\frac{1}{\gamma}\int e \wedge e \wedge F[\omega] = \frac{1}{\gamma}\int d^4x\sqrt{-g}\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \rightarrow 0$ when on shell
- (Analogy with QCD $S_{QCD} = \int F \wedge \star F + \theta_{QCD}\int F \wedge F$)

Action

- Holst action $S_H[e,\omega] = \int \left(\star(e \wedge e) + \frac{1}{\gamma} e \wedge e \right) \wedge F[\omega]$

Action

- Holst action $S_H[e,\omega] = \int \left(\star(e \wedge e) + \frac{1}{\gamma} e \wedge e \right) \wedge F[\omega]$
- $S_H[e,\omega] = \int B[e] \wedge F[\omega]$
- $B[e] := \star(e \wedge e) + \frac{1}{\gamma} e \wedge e$. Simplicity constraint
- This type of theories are called "BF" theories and are well-studied. GR is special because the bivector B is simple.

Action

- Holst action $S_H[e,\omega] = \int \left(\star(e \wedge e) + \frac{1}{\gamma} e \wedge e \right) \wedge F[\omega]$
- $S_H[e,\omega] = \int B[e] \wedge F[\omega]$
- $B[e] := \star(e \wedge e) + \frac{1}{\gamma} e \wedge e$. Simplicity constraint
- This type of theories are called "BF" theories and are well-studied. GR is special because the bivector B is simple.
- $F = d\omega + \omega \wedge \omega$, the usual field strength of gauge theories. ω is an $so(1,3)$ or equivalently $\underline{sl(2,\mathbb{C})}$ valued form.

Action

- Holst action $S_H[e,\omega] = \int \left(\star(e\wedge e) + \frac{1}{\gamma} e\wedge e\right)\wedge F[\omega]$
- $S_H[e,\omega] = \int B[e]\wedge F[\omega]$
- $B[e] := \star(e\wedge e) + \frac{1}{\gamma} e\wedge e$. Simplicity constraint
- This type of theories are called "BF" theories and are well-studied. GR is special because the bivector B is simple.
- $F = d\omega + \omega\wedge\omega$, the usual field strength of gauge theories. ω is an $so(1,3)$ or equivalently $\underline{sl(2,\mathbb{C})}$ valued form.
- On a $t = const$ boundary, B is the derivative of the action with respect to $\partial\omega/\partial t$, since the quadratic part of the action is $\sim B\wedge d\omega$. Thus B is the momentum canonical to the connection, thus related to the Lorentz transformations.

Linear Simplicity Constraints

Pick a spacelike surface Σ that bounds spacetime. n_I is the vector normal to the surface. We can decompose B into its electric and magnetic part

- Electric part: $K^I = n_J B^{IJ}$
- Magnetic part: $L^I = n_J (\star B)^{IJ}$

Linear Simplicity Constraints

Pick a spacelike surface Σ that bounds spacetime. n_I is the vector normal to the surface. We can decompose B into its electric and magnetic part

- Electric part: $K^I = n_J B^{IJ}$
- Magnetic part: $L^I = n_J (\star B)^{IJ}$
- $n_I K^I = n_I n_J B^{IJ} = 0$
- $n_I L^I = n_I n_J (\star B)^{IJ} = 0$
- $K^I \rightarrow K^i$
- $L^I \rightarrow L^i$

Linear Simplicity Constraints

Choose locally $n_I = (1,0,0,0)$ (time gauge). Then
$K^i = B^{i0}, \quad L^i = \frac{1}{2}\epsilon^i{}_{jk}B^{jk}.$

Linear Simplicity Constraints

Choose locally $n_I = (1,0,0,0)$ (time gauge). Then $K^i = B^{i0}, \quad L^i = \frac{1}{2}\epsilon^i{}_{jk}B^{jk}$.It is very easy to show that

- $\vec{K} = \gamma\vec{L}$. This is called the Linear Simplicity Constraint.

Linear Simplicity Constraints

Choose locally $n_I = (1,0,0,0)$ (time gauge). Then
$K^i = B^{i0}, \quad L^i = \frac{1}{2}\epsilon^i{}_{jk}B^{jk}$.It is very easy to show that

- $\vec{K} = \gamma\vec{L}$. This is called the Linear Simplicity Constraint.
- Physical meaning of $\vec{K}$ and $\vec{L}$: B is the Generator of Lorentz transformations. In the time gauge K^i is a boost in the i-direction and L^i is the generator of the rotation around the i-axis.

$SU(2)$, $SL(2,\mathbb{C})$ and the Y_Γ map

$SL(2,\mathbb{C})$ is the double cover of $SO(1,3)$. It has six generators and two Casimirs $C_1 = |\vec{K}|^2 - |\vec{L}|^2$ and $C_2 = \vec{K}\cdot\vec{L}$. The unitary irreps are labelled by $p \in \mathbb{R}$ and $k \in \mathbb{Z}/2$. The Hilbert space $V^{(p,k)}$ is infinite dimensional and can be decomposed as $V^{(p,k)} = \bigoplus_{j=k}^{+\infty} \mathcal{H}^j$, where $\mathcal{H}^j$ is the usual $2j+1$ dim irrep space that carries the usual j spin representation of $SU(2)$.

- states in $V^{(p,k)}$: $|p,k;j,m\rangle$

$SU(2)$, $SL(2,\mathbb{C})$ and the Y_Γ map

$SL(2,\mathbb{C})$ is the double cover of $SO(1,3)$. It has six generators and two Casimirs $C_1 = |\vec{K}|^2 - |\vec{L}|^2$ and $C_2 = \vec{K} \cdot \vec{L}$. The unitary irreps are labelled by $p \in \mathbb{R}$ and $k \in \mathbb{Z}/2$. The Hilbert space $V^{(p,k)}$ is infinite dimensional and can be decomposed as $V^{(p,k)} = \bigoplus_{j=k}^{+\infty} \mathcal{H}^j$, where $\mathcal{H}^j$ is the usual $2j+1$ dim irrep space that carries the usual j spin representation of $SU(2)$.

- states in $V^{(p,k)}$: $|p,k;j,m\rangle$
- Choose $p = \gamma k$ and $k = j$ thus these special states have the form $|\gamma j, j; j, m\rangle$

$SU(2)$, $SL(2,\mathbb{C})$ and the Y_Γ map

$SL(2,\mathbb{C})$ is the double cover of $SO(1,3)$. It has six generators and two Casimirs $C_1 = |\vec{K}|^2 - |\vec{L}|^2$ and $C_2 = \vec{K}\cdot\vec{L}$. The unitary irreps are labelled by $p \in \mathbb{R}$ and $k \in \mathbb{Z}/2$. The Hilbert space $V^{(p,k)}$ is infinite dimensional and can be decomposed as $V^{(p,k)} = \bigoplus_{j=k}^{+\infty} \mathcal{H}^j$, where $\mathcal{H}^j$ is the usual $2j+1$ dim irrep space that carries the usual j spin representation of $SU(2)$.

- states in $V^{(p,k)}$: $|p,k;j,m\rangle$
- Choose $p = \gamma k$ and $k = j$ thus these special states have the form $|\gamma j, j; j, m\rangle$
- $\left\langle K^i - \gamma L^i \right\rangle \approx 0$ in the large j limit. Central idea of the EPRL model that we will use.

$SU(2)$, $SL(2,\mathbb{C})$ and the Y_Γ map

Both $|j,m\rangle$ and $|\gamma j, j; j, m\rangle$ totally described in terms of j and m. Thus $\mathcal{H}^j$ and $V^{(p=\gamma j, k=j)}$ are isomorphic.

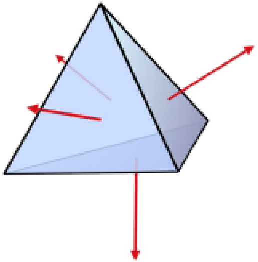
$$
\begin{aligned}
Y_\gamma : \ \mathcal{H}^j &\longrightarrow \ V^{(p=\gamma j, k=j)} \\
|j,m\rangle &\longmapsto |\gamma j, j; j, m\rangle
\end{aligned}
$$

Summary

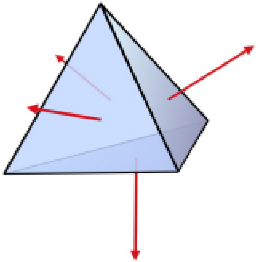
GR can be formulated as a BF theory with $SL(2,\mathbb{C})$ symmetry in the bulk, $SU(2)$ symmetry on the boundary together with the linear simplicity constraint $\vec{K} = \gamma \vec{L}$ on the boundary.

Elementary Geometry

The most economical and efficient way to describe a tetrahedron is in terms of four vectors $\vec{L}_0, \vec{L}_1, \vec{L}_2, \vec{L}_3$ normal to the faces that satisfy the closure condition $\vec{C} := \vec{L}_0 + \vec{L}_1 + \vec{L}_2 + \vec{L}_3 = 0$. Degrees of freedom: $4 \times 3 - 3 - 3 = 6$, the same number as the number of edges.



Elementrary Geometry



- $|\vec{L}_f| =$ area of the triangle f
- $V^2 = \frac{2}{9}(\vec{L}_1 \times \vec{L}_2) \cdot \vec{L}_3$

Quantum Spacetime

In GR spacetime $\leftrightarrow$ gravitational field. Thus, quantum gravitational field $\leftrightarrow$ quantum spacetime. We focus on space. We study a quantum of space which we take to be a tetrahedron and we promote every L_f^i into an operator that satisfy some algebra

- $[L_f^i, L_{f'}^j] =$

In GR spacetime $\leftrightarrow$ gravitational field. Thus, quantum gravitational field $\leftrightarrow$ quantum spacetime. We focus on space. We study a quantum of space which we take to be a tetrahedron and we promote every L_f^i into an operator that satisfy some algebra

- $[L_f^i, L_{f'}^j] = l_0^2 i\epsilon^{ij}{}_k L_f^k \delta_{ff'}$
- $l_0^2 = 8\pi\gamma\hbar G$, $L_{pl} = \sqrt{\hbar G}$

Quantum Spacetime

In GR spacetime $\leftrightarrow$ gravitational field. Thus, quantum gravitational field $\leftrightarrow$ quantum spacetime. We focus on space. We study a quantum of space which we take to be a tetrahedron and we promote every L^i_f into an operator that satisfy some algebra

- $[L^i_f, L^j_{f'}] = l_0^2 i\epsilon^{ij}{}_k L^k_f \delta_{ff'}$
- $l_0^2 = 8\pi\gamma\hbar G$, $L_{pl} = \sqrt{\hbar G}$
- $A = l_0^2\sqrt{j(j+1)}, \quad j = 0, 1/2, 1, 3/2, ...$ The areas are quantized!

Quantum Spacetime

A candidate for the Hilbert space is $\mathcal{H} = \mathcal{H}_{j_0} \otimes \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}$.

Quantum Spacetime

A candidate for the Hilbert space is $\mathcal{H} = \mathcal{H}_{j_0} \otimes \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}$. But we also have to impose the closure condition $\vec{C}\Psi = 0 \quad \Psi \in \mathcal{H}$.

Quantum Spacetime

A candidate for the Hilbert space is $\mathcal{H} = \mathcal{H}_{j_0} \otimes \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}$. But we also have to impose the closure condition $\vec{C}\Psi = 0 \quad \Psi \in \mathcal{H}$. Thus, the Hilbert space is $\mathcal{K} := \text{Inv}_{SU(2)}[\mathcal{H}_{j_0} \otimes \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}]$

Quantum Spacetime

A candidate for the Hilbert space is $\mathcal{H} = \mathcal{H}_{j_0} \otimes \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}$. But we also have to impose the closure condition $\vec{C}\Psi = 0 \quad \Psi \in \mathcal{H}$. Thus, the Hilbert space is $\mathcal{K} := \mathrm{Inv}_{SU(2)}[\mathcal{H}_{j_0} \otimes \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}]$

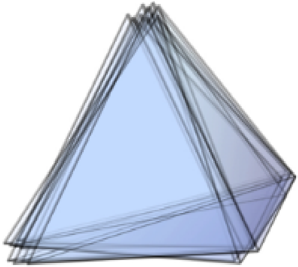
- $\dim(\mathcal{K}) = \min(j_0 + j_1, j_2 + j_3) - \max(|j_0 - j_1|, |j_2 - j_3|) + 1$
- $\dim(\mathcal{K}) = \min(j_0 + j_1, j_2 + j_3) - \max(|j_0 - j_1|, |j_2 - j_3|) + 1 > 0$
- The volume operator is well-defined in $\mathcal{K}$ and has discrete eigenvalues.

Quantum Spacetime

A state in $\mathcal{K}$ has the form $|j_0, j_1, j_2, j_3, v\rangle$.

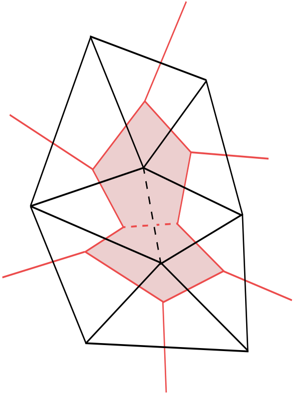
Quantum Spacetime

A state in $\mathcal{K}$ has the form $|j_0, j_1, j_2, j_3, v\rangle$.Five numbers instead of six! The tetrahedron is fuzzy!

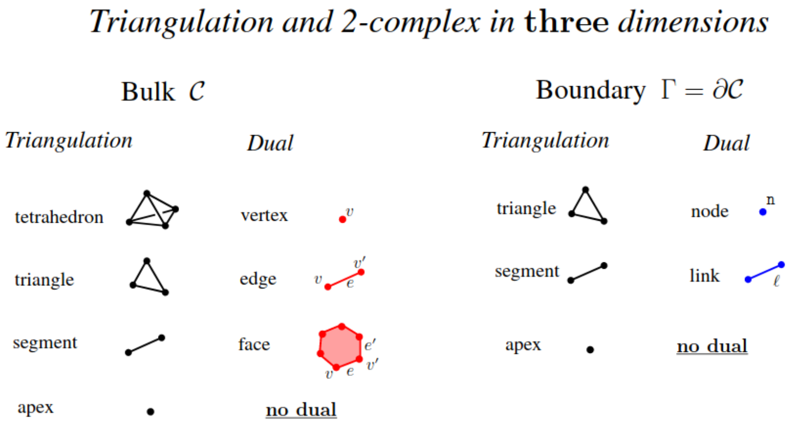


Triangulation and Dual Triangulation

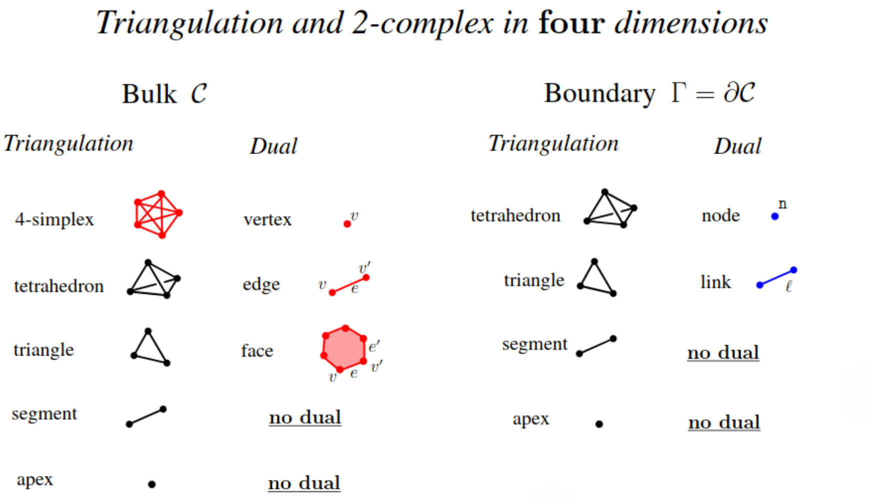
A triangulation in two dimensions. Each edge of the dual graph, shown in red, is common to two faces. As an example, the segment in dotted black is dual to the edge in dotted red, which is common to the two faces in pale red.



Triangulation and Dual Triangulation



Triangulation and Dual Triangulation



How to measure curvature

- In 2D: Pick a frame and take a tour around a “hinge” by following the edges. If the frame returns rotated you have detected curvature.

How to measure curvature

- In 2D: Pick a frame and take a tour around a "hinge" by following the edges. If the frame returns rotated you have detected curvature.
- In 3D: Pick a frame and take a tour around a segment by following the edges. If the frame returns rotated you have detected curvature.

How to measure curvature

- In 2D: Pick a frame and take a tour around a “hinge” by following the edges. If the frame returns rotated you have detected curvature.
- In 3D: Pick a frame and take a tour around a segment by following the edges. If the frame returns rotated you have detected curvature.
- In 4D: Pick a frame and take a tour around a triangle by following the edges. If the frame returns rotated you have detected curvature.

How to measure curvature

Inside the 4D bulk the frame rotation is the outcome of the individual rotations that take place every time we jump from one 4-simplex to the other by following the edges. Thus, we assign to each edge an group element $g \in SL(2, \mathbb{C})$.

How to measure curvature

Inside the 4D bulk the frame rotation is the outcome of the individual rotations that take place every time we jump from one 4-simplex to the other by following the edges. Thus, we assign to each edge an group element $g \in SL(2, \mathbb{C})$.By the same reasoning the assign to each link of the boundary dual graph an element $h \in SU(2)$. The boundary dual graph has now the structure of a spin network.

Hilbert Space

Hilbert space is defined on the boundary. The only variables we have are the $SU(2)$ elements of total number L. Thus, a candidate is $\tilde{\mathcal{H}}_\Gamma = L_2[SU(2)^L]$

Hilbert Space

Hilbert space is defined on the boundary. The only variables we have are the $SU(2)$ elements of total number L. Thus, a candidate is $\tilde{\mathcal{H}}_\Gamma = L_2[SU(2)^L]$. But we have the freedom to choose a rotated frame and do the same thing. In total we have N frames where N is the total number of the nodes of the spin network. Thus, the Hilbert space that corresponds to a triangulation Γ is $\mathcal{H}_\Gamma = L_2[SU(2)^L/SU(2)^N]_\Gamma$.

Hilbert Space

Hilbert space is defined on the boundary. The only variables we have are the $SU(2)$ elements of total number L. Thus, a candidate is $\tilde{\mathcal{H}}_\Gamma = L_2[SU(2)^L]$. But we have the freedom to choose a rotated frame and do the same thing. In total we have N frames where N is the total number of the nodes of the spin network. Thus, the Hilbert space that corresponds to a triangulation Γ is $\mathcal{H}_\Gamma = L_2[SU(2)^L/SU(2)^N]_\Gamma$.

- $\vec{C}_n\Psi = 0$ for $\Psi \in \tilde{\mathcal{H}}_\Gamma$, where $\vec{C}_n = \vec{L}_{l_1} + \vec{L}_{l_2} + \vec{L}_{l_3} + \vec{L}_{l_4}$ is the generator of the total $SU(2)$ transformation of the n node.

Operators

In standard QM he have the operators
$\hat{x}\Psi(x) = x\Psi(x), \quad \hat{p}\Psi(x) = -i\hbar\frac{d\Psi(x)}{dx}$. Here:

- $\hat{h}_l\Psi(h_l) = h_l\Psi(h_l)$

Operators

In standard QM he have the operators
$\hat{x}\Psi(x) = x\Psi(x), \quad \hat{p}\Psi(x) = -i\hbar\frac{d\Psi(x)}{dx}$. Here:

- $\hat{h}_l\Psi(h_l) = h_l\Psi(h_l)$
- What about the derivative?

Operators

There exist natural derivatives in $SU(2)$ which correspond to the left-invariant vector fields $(J^i\Psi)(h) = -i\frac{d}{dt}\Psi(he^{t\tau^i})|_{t=0}$, where $\tau^i = -\frac{\sigma^i}{2}$.
To be dimensionally correct we use the operators

- $\vec{L}_l = 8\pi\gamma\hbar G\vec{J}_l$.

Operators

There exist natural derivatives in $SU(2)$ which correspond to the left-invariant vector fields $(J^i\Psi)(h) = -i\frac{d}{dt}\Psi(he^{t\tau^i})|_{t=0}$, where $\tau^i = -\frac{\sigma^i}{2}$.
To be dimensionally correct we use the operators

- $\vec{L}_l = 8\pi\gamma\hbar G\vec{J}_l$.Well-defined in $\tilde{\mathcal{H}}_\Gamma$ but non in $\mathcal{H}_\Gamma$.

Operators

There exist natural derivatives in $SU(2)$ which correspond to the left-invariant vector fields $(J^i\Psi)(h) = -i\frac{d}{dt}\Psi(he^{t\tau^i})|_{t=0}$, where $\tau^i = -\frac{\sigma^i}{2}$. To be dimensionally correct we use the operators

- $\vec{L}_l = 8\pi\gamma\hbar G\vec{J}_l$.Well-defined in $\tilde{\mathcal{H}}_\Gamma$ but non in $\mathcal{H}_\Gamma$.
- Easy! We consider $G_{ll'} = \vec{L}_l \cdot \vec{L}'_l$

Operators

There exist natural derivatives in $SU(2)$ which correspond to the left-invariant vector fields $(J^i\Psi)(h) = -i\frac{d}{dt}\Psi(he^{t\tau^i})|_{t=0}$, where $\tau^i = -\frac{\sigma^i}{2}$. To be dimensionally correct we use the operators

- $\vec{L_l} = 8\pi\gamma\hbar G\vec{J_l}$.Well-defined in $\tilde{\mathcal{H}}_\Gamma$ but non in $\mathcal{H}_\Gamma$.
- Easy! We consider $G_{ll'} = \vec{L_l}\cdot\vec{L'_l}$
- When $l = l'$ the norm $A_l = \sqrt{G_l}$ is of course the area of the triangle punctured by the link l with spectrum $A_l = 8\pi\gamma\hbar G\sqrt{j_l(j_l+1)}$

Operators

There exist natural derivatives in $SU(2)$ which correspond to the left-invariant vector fields $(J^i\Psi)(h) = -i\frac{d}{dt}\Psi(he^{t\tau^i})|_{t=0}$, where $\tau^i = -\frac{\sigma^i}{2}$. To be dimensionally correct we use the operators

- $\vec{L}_l = 8\pi\gamma\hbar G\vec{J}_l$.Well-defined in $\tilde{\mathcal{H}}_\Gamma$ but non in $\mathcal{H}_\Gamma$.
- Easy! We consider $G_{ll'} = \vec{L}_l \cdot \vec{L}'_l$
- When $l = l'$ the norm $A_l = \sqrt{G_l}$ is of course the area of the triangle punctured by the link l with spectrum $A_l = 8\pi\gamma\hbar G\sqrt{j_l(j_l+1)}$
- $V_n^2 = \frac{2}{9}(\vec{L}_{l_1} \times \vec{L}_{l_2}) \cdot \vec{L}_{l_3}$
- State: $|\Gamma, j_l, v_n\rangle$

The path integral of a BF theory

BF path integral: $Z = \int \mathcal{D}B\mathcal{D}\omega e^{\frac{i}{\hbar}\int B\wedge F}$.

The path integral of a BF theory

BF path integral: $Z = \int \mathcal{D}B\mathcal{D}\omega e^{\frac{i}{\hbar}\int B\wedge F}$.$B$ is a two-form, thus integrated on triangles or **faces** of the dual graph. ω is an one-form and is integrated on edges of the dual graph. From it we extract a group element U_e via $U_e = \mathcal{P}e^{\int \omega_e}$. Hence, we have

- $Z = \int \mathcal{D}B_f \int_G dU_e e^{\frac{i}{\hbar}\sum_f B_f \prod_{e\in f} U_e}$

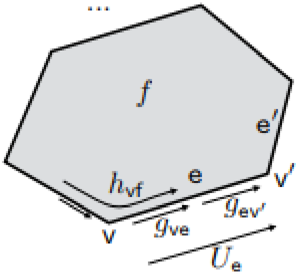
The path integral of a BF theory

BF path integral: $Z = \int \mathcal{D}B\mathcal{D}\omega e^{\frac{i}{\hbar}\int B\wedge F}$.$B$ is a two-form, thus integrated on triangles or **faces** of the dual graph. ω is an one-form and is integrated on edges of the dual graph. From it we extract a group element U_e via $U_e = \mathcal{P}e^{\int \omega_e}$. Hence, we have

- $Z = \int \mathcal{D}B_f \int_G dU_e e^{\frac{i}{\hbar}\sum_f B_f \prod_{e\in f} U_e}$
- From $\int dp e^{ipx} \sim \delta(x)$ we can write $Z = \int_G dU_e \prod_f \delta(\prod_{e\in f} U_e)$

The path integral of a BF theory

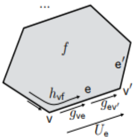
$Z = \int_G dU_e \prod_f \delta(\prod_{e \in f} U_e)$



By definition ${g_{ve}}^{-1} = g_{ev}$
$Z = \int_G dg_{ev} \prod_f \delta(g_{ve} g_{ev'} g_{v'e'} g_{e'v''} \dots)$

The path integral of a BF theory

$Z = \int_G dg_{ev} \prod_f \delta(g_{ve} g_{ev'} g_{v'e'} g_{e'v''} \dots)$
We focus on a face f and we trade the group element that terminates in vertex and the group element that emanates from the same vertex with one group element h_{vf}. We do it for every vertex of the face and for every face in the bulk.



$$Z = \int_{G'} dh_{vf} \int_G dg_{ev} \prod_f \delta(h_{vf} h_{v'f} \dots) \prod_v \prod_{f \in v} \delta(g_{e'v} g_{ve} h_{vf})$$

The path integral of a BF theory

$$Z = \int_{G'} dh_{\mathrm{v}f} \int_G dg_{e\mathrm{v}} \prod_f \delta(h_{\mathrm{v}f} h_{\mathrm{v}'f} \dots) \prod_{\mathrm{v}} \prod_{f \in \mathrm{v}} \delta(g_{e'\mathrm{v}} g_{\mathrm{v}e} h_{\mathrm{v}f}).$$

The path integral of a BF theory

$Z = \int_{G'} dh_{\mathrm{v}f} \int_G dg_{e\mathrm{v}} \prod_f \delta(h_{\mathrm{v}f} h_{\mathrm{v}'f} \ldots) \prod_{\mathrm{v}} \prod_{f \in \mathrm{v}} \delta(g_{e'\mathrm{v}} g_{\mathrm{v}e} h_{\mathrm{v}f})$.By rearranging some terms we can write it as

- $Z = \int_{G'} dh_{\mathrm{v}f} \prod_f \delta(h_{\mathrm{v}f} h_{\mathrm{v}'f} \ldots) \prod_{\mathrm{v}} A_{\mathrm{v}}(h_{\mathrm{v}f})$
- $A_{\mathrm{v}}(h_{\mathrm{v}f}) = \int_G dg_{e\mathrm{v}} \prod_{f \in \mathrm{v}} \delta(g_{e'\mathrm{v}} g_{\mathrm{v}e} h_{\mathrm{v}f})$ is called the vertex amplitude.

The path integral of a BF theory

$Z = \int_{G'} dh_{\mathrm{v}f} \int_G dg_{e\mathrm{v}} \prod_f \delta(h_{\mathrm{v}f} h_{\mathrm{v}'f} \ldots) \prod_{\mathrm{v}} \prod_{f \in \mathrm{v}} \delta(g_{e'\mathrm{v}} g_{\mathrm{v}e} h_{\mathrm{v}f})$.By rearranging some terms we can write it as

- $Z = \int_{G'} dh_{\mathrm{v}f} \prod_f \delta(h_{\mathrm{v}f} h_{\mathrm{v}'f} \ldots) \prod_{\mathrm{v}} A_{\mathrm{v}}(h_{\mathrm{v}f})$
- $A_{\mathrm{v}}(h_{\mathrm{v}f}) = \int_G dg_{e\mathrm{v}} \prod_{f \in \mathrm{v}} \delta(g_{e'\mathrm{v}} g_{\mathrm{v}e} h_{\mathrm{v}f})$ is called the vertex amplitude.
- The delta function on a group can be expanded as $\delta(U) = \sum_{j_f=0}^{+\infty} (2j_f + 1) \mathrm{Tr}_{j_f}[U]$ (similar to $\delta(\theta) = \sum_n e^{in\theta}$).

The path integral of a BF theory

$Z = \int_{G'} dh_{\mathrm{v}f} \int_G dg_{e\mathrm{v}} \prod_f \delta(h_{\mathrm{v}f} h_{\mathrm{v}'f} \dots) \prod_{\mathrm{v}} \prod_{f \in \mathrm{v}} \delta(g_{e'\mathrm{v}} g_{\mathrm{v}e} h_{\mathrm{v}f})$.By rearranging some terms we can write it as

- $Z = \int_{G'} dh_{\mathrm{v}f} \prod_f \delta(h_{\mathrm{v}f} h_{\mathrm{v}'f} \dots) \prod_{\mathrm{v}} A_{\mathrm{v}}(h_{\mathrm{v}f})$
- $A_{\mathrm{v}}(h_{\mathrm{v}f}) = \int_G dg_{e\mathrm{v}} \prod_{f \in \mathrm{v}} \delta(g_{e'\mathrm{v}} g_{\mathrm{v}e} h_{\mathrm{v}f})$ is called the vertex amplitude.
- The delta function on a group can be expanded as $\delta(U) = \sum_{j_f=0}^{+\infty} (2j_f + 1) \mathrm{Tr}_{j_f}[U]$ (similar to $\delta(\theta) = \sum_n e^{in\theta}$).
- $A_{\mathrm{v}}(h_{\mathrm{v}f}) = \sum_{\{j_f\}} \int_G dg_{e\mathrm{v}} \prod_f (2j_f + 1) \mathrm{Tr}_{j_f}[g_{e'\mathrm{v}} g_{\mathrm{v}e} h_{\mathrm{v}f}]$

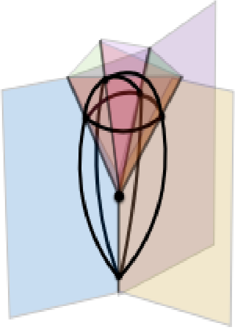
The path integral of a special BF theory, GR

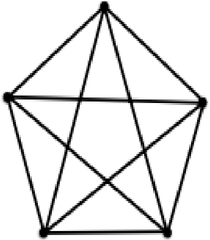
$A_{\mathrm{v}}(h_{\mathrm{v}f}) = \sum_{\{j_f\}} \int_G dg_{\mathrm{ev}} \prod_f (2j_f + 1) \mathrm{Tr}_{j_f}[g_{e'\mathrm{v}} g_{\mathrm{v}e} h_{\mathrm{v}f}]$

- A vertex is dual to a 4-simplex thus, vertex amplitude $\leftrightarrow$ 4-simplex amplitude

The path integral of a special BF theory, GR

- A 4-simplex is bounded by five tetraedra $\leftrightarrow$ five nodes around the vertex, one on every edge.
- Between the 5 nodes there are 10 links that correspond to the 10 h_{vf}.





The path integral of a special BF theory, GR

Now, we need to remember that we are not quantizing a general BF theory but GR. GR is characterised by $SL(2,\mathbb{C})$ symmetry in the bulk, $SU(2)$ on the boundary plus $\vec{K} = \gamma \vec{L}$ on the boundary.

- h_{vf} are assigned to the **links** of the boundary of the 4-simplex thus are $SU(2)$ elements.

The path integral of a special BF theory, GR

Now, we need to remember that we are not quantizing a general BF theory but GR. GR is characterised by $SL(2,\mathbb{C})$ symmetry in the bulk, $SU(2)$ on the boundary plus $\vec{K} = \gamma \vec{L}$ on the boundary.

- h_{vf} are assigned to the **links** of the boundary of the 4-simplex thus are $SU(2)$ elements.
- g_{ev} are assigned to the **edges** of the bulk dual graph thus are $SL(2,\mathbb{C})$ elements.

The path integral of a special BF theory, GR

Now, we need to remember that we are not quantizing a general BF theory but GR. GR is characterised by $SL(2,\mathbb{C})$ symmetry in the bulk, $SU(2)$ on the boundary plus $\vec{K} = \gamma \vec{L}$ on the boundary.

- $h_{\mathrm{v}f}$ are assigned to the **links** of the boundary of the 4-simplex thus are $SU(2)$ elements.
- $g_{e\mathrm{v}}$ are assigned to the **edges** of the bulk dual graph thus are $SL(2,\mathbb{C})$ elements.
- $Z = \int_{SU(2)} dh_{\mathrm{v}f} \prod_f \delta(h_{\mathrm{v}f} h_{\mathrm{v}'f} \ldots) A_{\mathrm{v}}(h_{\mathrm{v}f})$
- $A_{\mathrm{v}}(h_{\mathrm{v}f}) = \sum_{\{j_f\}} \int_{SL(2,\mathbb{C})} dg_{e\mathrm{v}} \prod_f (2j_f + 1) \mathrm{Tr}_{j_f}[g_{e'\mathrm{v}} g_{\mathrm{v}e} h_{\mathrm{v}f}]$

The transition amplitude

- $A_{\mathrm{v}}(h_{\mathrm{v}f}) = \sum_{\{j_f\}} \int_{SL(2,\mathbb{C})} dg_{e\mathrm{v}} \prod_f (2j_f + 1) \mathrm{Tr}_{j_f}[g_{e'\mathrm{v}} g_{\mathrm{v}e} h_{\mathrm{v}f}]$
- The trace in the vertex amplitude seems to involve both $SU(2)$ and $SL(2,\mathbb{C})$ elements. Odd, but easy to fix

The transition amplitude

- $A_{\mathrm{v}}(h_{\mathrm{v}f}) = \sum_{\{j_f\}} \int_{SL(2,\mathbb{C})} dg_{e\mathrm{v}} \prod_f (2j_f + 1) \mathrm{Tr}_{j_f}[g_{e'\mathrm{v}} g_{\mathrm{v}e} h_{\mathrm{v}f}]$
- The trace in the vertex amplitude seems to involve both $SU(2)$ and $SL(2,\mathbb{C})$ elements. Odd, but easy to fix
- $A_{\mathrm{v}}(h_{\mathrm{v}f}) = \sum_{j_f} \int_{SL(2,\mathbb{C})} dg_{e\mathrm{v}} \prod_f (2j_f + 1) \mathrm{Tr}_{j_f}[Y_\gamma^\dagger g_{e'\mathrm{v}} g_{\mathrm{v}e} Y_\gamma h_{\mathrm{v}f}]$
- $\mathrm{Tr}_j[Y_\gamma^\dagger g Y_\gamma h] = \sum_m \langle j,m| Y_\gamma^\dagger g Y_\gamma h |j,m\rangle =$
  $\sum_m \sum_n \langle j,m| Y_\gamma^\dagger g Y_\gamma |j,n\rangle \langle j,n| h |j,m\rangle = \sum_{m,n} D^{(\gamma j,j)}_{jm,jn}(g) D^{(j)}_{nm}(h)$

The transition amplitude

- $W_{\mathcal{C}}(h_\ell) = \mathcal{N} \int_{SU(2)} dh_{\mathrm{v}f} \prod_f \delta(h_{\mathrm{v}f} h_{\mathrm{v}'f} \ldots) \prod_{\mathrm{v}} A_{\mathrm{v}}(h_{\mathrm{v}f})$: function of the variables on the boundary.

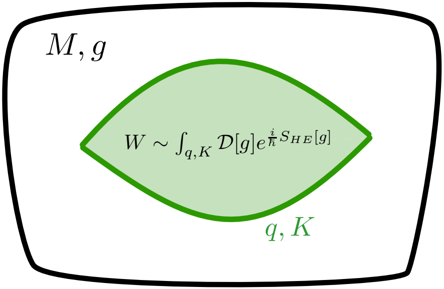
The transition amplitude

- $W_{\mathcal{C}}(h_\ell) = \mathcal{N} \int_{SU(2)} dh_{\mathrm{v}f} \prod_f \delta(h_{\mathrm{v}f} h_{\mathrm{v}'f} \ldots) \prod_{\mathrm{v}} A_{\mathrm{v}}(h_{\mathrm{v}f})$: function of the variables on the boundary.
- The final transition amplitude is abstractly defined as the limit in the most possible refined truncation

The transition amplitude

- $W_{\mathcal{C}}(h_\ell) = \mathcal{N} \int_{SU(2)} dh_{\mathrm{v}f} \prod_f \delta(h_{\mathrm{v}f} h_{\mathrm{v}'f} \ldots) \prod_{\mathrm{v}} A_{\mathrm{v}}(h_{\mathrm{v}f})$: function of the variables on the boundary.
- The final transition amplitude is abstractly defined as the limit in the most possible refined truncation
- UV finite.
- There can be IR divergences but the version of the theory with cosmological constant is proved to be IR finite.

Semiclassical states



In the boundary we have a semiclassical (also known as coherent) state of geometry. Semiclassical states are quantum states that resemble classical states as much as possible.

Semiclassical states

In standard QM a semiclassical state (Gaussian wavepacket) has the form $\Psi^t_{x_0,p_0}(x) \propto \int dp e^{-(p-p_0)^2 t + ipx_0} \psi(p,x)$, where $\psi(p,x) = e^{-ipx}$. It is peaked in momentum p_0 and position x_0.

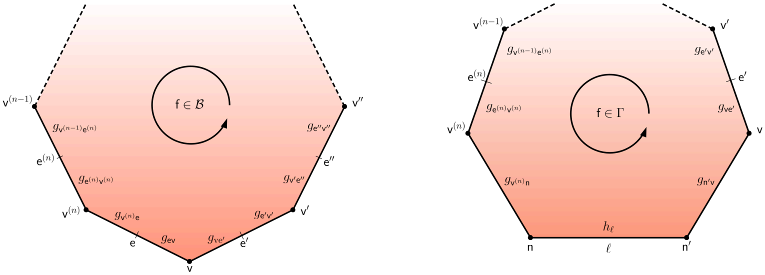
Semiclassical states

In standard QM a semiclassical state (Gaussian wavepacket) has the form $\Psi^t_{x_0,p_0}(x) \propto \int dp e^{-(p-p_0)^2 t + ipx_0} \psi(p,x)$, where $\psi(p,x) = e^{-ipx}$. It is peaked in momentum p_0 and position x_0. In LQG a semiclassical state has the form $\Psi^t_{\Gamma,H_\ell}(h_\ell) \propto \sum_{\{j_\ell\}} \prod_\ell d_{j_\ell} \, \mathrm{e}^{-(j_\ell-\omega_\ell)^2 t + i\gamma\zeta_\ell j_\ell} \; \psi_{\Gamma,j_\ell,\vec{n}_{s(\ell)},\vec{n}_{t(\ell)}}(h_\ell)$, where $\psi_{\Gamma,j_\ell,\vec{n}_{s(\ell)},\vec{n}_{t(\ell)}}(h_\ell) = \sum_{m_s,m_t} D^{j_\ell}_{j_\ell m_t}(n^\dagger_{t(\ell)}) \, D^{j_\ell}_{m_t m_s}(h_\ell) \, D^{j_\ell}_{m_s j_\ell}(n_{s(\ell)})$

- $t = \left(\frac{l_p^2}{A}\right)^n$ with $n \in [0,2]$ controls the spread of the Gaussians. Since the area A is macroscopic $t \ll 1$.
- $\omega_\ell := \frac{\eta_\ell - t}{2t} \approx \frac{\eta_\ell}{2t}$ where $\eta_\ell \in \mathbb{R}^+$ is related to the area dual to the link ℓ and is taken $\gg 1$.
- $\zeta_\ell \in [0, 4\pi)$ is the distributional extrinsic curvature.
- $n_{t_{(\ell)}}$, $n_{s_{(\ell)}}$ nodes of the source and the tagret of the link ℓ.

The transition amplitude again

An equivalent and useful form of the transition amplitude is $W_{\mathcal{C}}(h_\ell) = \mathcal{N} \int_{SL(2,\mathbb{C})} \left(\prod_{\mathsf{v}} \mathrm{d}g_{\mathsf{ve}}\right) \left(\prod_{\mathsf{f}\in\mathcal{B}} A_{\mathsf{f}}\right) \left(\prod_{\mathsf{f}\in\Gamma} A_{\mathsf{f}}(h_\ell)\right)$ where A_f are the internal (bulk) faces and $A_{\mathsf{f}}(h_\ell)$ are boundary faces.



(a) Bulk face

(b) Boundary face

Bulk face amplitude

$A_{\mathsf{f}} := \sum_{j_{\mathsf{f}}} d_{j_{\mathsf{f}}} \,\mathrm{Tr}_{j_{\mathsf{f}}} \left[\prod_{\mathsf{v}\in\mathsf{f}} Y^{\dagger} g_{\mathsf{ve}}^{-1} g_{\mathsf{ve}'} Y\right] :=$
$\sum_{j_{\mathsf{f}}} d_{j_{\mathsf{f}}} \,\mathrm{Tr}_{j_{\mathsf{f}}} \left[Y^{\dagger} g_{\mathsf{ev}} g_{\mathsf{ve}'} Y Y^{\dagger} g_{\mathsf{e}'\mathsf{v}'} g_{\mathsf{v}'\mathsf{e}''} Y \dots Y^{\dagger} g_{\mathsf{e}^{(n)}\mathsf{v}^{(n)}} g_{\mathsf{v}^{(n)}\mathsf{e}} Y\right]$ for $\mathsf{f} \in \mathcal{B}$
where $\mathrm{Tr}_{j_{\mathsf{f}}} \left[\prod_{\mathsf{v}\in\mathsf{f}} Y^{\dagger} g_{\mathsf{ve}}^{-1} g_{\mathsf{ve}'} Y\right] =$
$\sum_{\{me\}} D^{(\gamma j_{\mathsf{f}}, j_{\mathsf{f}})}_{j_{\mathsf{f}} m_{\mathsf{e}} j_{\mathsf{f}} m_{\mathsf{e}'}} (g_{\mathsf{ev}} g_{\mathsf{ve}'}) D^{(\gamma j_{\mathsf{f}}, j_{\mathsf{f}})}_{j_{\mathsf{f}} m_{\mathsf{e}'} j_{\mathsf{f}} m_{\mathsf{e}''}} (g_{\mathsf{e}'\mathsf{v}'} g_{\mathsf{v}'\mathsf{e}''}) \dots D^{(\gamma j_{\mathsf{f}}, j_{\mathsf{f}})}_{j_{\mathsf{f}} m_{\mathsf{e}^{(n)}} j_{\mathsf{f}} m_{\mathsf{e}}} (g_{\mathsf{e}^{(n)}\mathsf{v}^{(n)}} g_{\mathsf{v}^{(n)}\mathsf{e}})$

Bulk face amplitude

To calculate this we are going to work in the principal series representation

Bulk face amplitude

To calculate this we are going to work in the principal series representation. A representation space of the j-irrep of $SU(2)$, $\mathcal{V}^j$, is spanned by the homogeneous complex polynomials of degree $2j$.

$P^j_m(\mathbf{z}) = \left[\frac{(2j)!}{(j+m)!(j-m)!}\right]^{\frac{1}{2}} z_0^{j+m} z_1^{j-m}, \quad m \in \{-j, ..., j\}, \quad \mathbf{z} = (z_0, z_1)^\intercal \in \mathbb{C}^2$. By acting on P with the Y map we obtain the principal series representation of $SL(2, \mathbb{C})$

$\phi_m^{(\gamma j, j)}(\mathbf{z}) := Y \triangleright P^j_m(\mathbf{z}) = \sqrt{\frac{d_j}{\pi}} \langle \mathbf{z}|\mathbf{z}\rangle^{i\gamma j - j - 1} P^j_m(\mathbf{z})$.

Bulk face amplitude

To calculate this we are going to work in the principal series representation. A representation space of the j-irrep of $SU(2)$, $\mathcal{V}^j$, is spanned by the homogeneous complex polynomials of degree $2j$.

$P^j_m(\mathbf{z}) = \left[\frac{(2j)!}{(j+m)!(j-m)!}\right]^{\frac{1}{2}} z_0^{j+m} z_1^{j-m}, \quad m \in \{-j, ..., j\}, \quad \mathbf{z} = (z_0, z_1)^\intercal \in \mathbb{C}^2$. By acting on P with the Y map we obtain the principal series representation of $SL(2, \mathbb{C})$

$\phi^{(\gamma j, j)}_m(\mathbf{z}) := Y \triangleright P^j_m(\mathbf{z}) = \sqrt{\frac{d_j}{\pi}} \langle \mathbf{z}|\mathbf{z}\rangle^{i\gamma j - j - 1} P^j_m(\mathbf{z})$.Then,

$D^{(\gamma j, j)}_{j\,m\,j\,m'}(g) \equiv \langle jm|\, Y^\dagger g Y \,|jm'\rangle = \int_{\mathbb{C}\mathrm{P}^1} \mathrm{d}\Omega\, \overline{\phi^{(\gamma j, j)}_m(\mathbf{z})}\, \phi^{(\gamma j, j)}_{m'}(g^\intercal \mathbf{z})$ where $\mathrm{d}\Omega = \frac{i}{2}(z^0 \mathrm{d}z^1 - z^1 \mathrm{d}z^0) \wedge (\overline{z}^0 \mathrm{d}\overline{z}^1 - \overline{z}^1 \mathrm{d}\overline{z}^0)$ is a homogeneous and $SL(2, \mathbb{C})$ invariant measure on $\mathbb{C}^2 \backslash \{0\} \simeq \mathbb{C}\mathrm{P}^1$

Bulk face amplitude

The face amplitude takes the form

$A_{\mathsf{f}} = \sum_{j_{\mathsf{f}}} d_{j_{\mathsf{f}}} \prod_{\mathsf{e}\in\mathsf{f}} \frac{d_{j_{\mathsf{f}}}}{\pi} \int_{\mathbb{C}\mathsf{P}^1} \mathrm{d}\tilde{\Omega}_{\mathsf{vef}}\; \mathsf{e}^{j_{\mathsf{f}} S_{\mathsf{f}}[g_{\mathsf{ve}}, \mathbf{z}_{\mathsf{vf}}]} \quad \forall \mathsf{f} \in \mathcal{B}$ where

$S_{\mathsf{f}}[g_{\mathsf{ve}}, \mathbf{z}_{\mathsf{vf}}] := \log \frac{\langle \mathbf{Z}_{\mathsf{v'e'f}} | \mathbf{Z}_{\mathsf{ve'f}} \rangle^2}{\langle \mathbf{Z}_{\mathsf{vef}} | \mathbf{Z}_{\mathsf{vef}} \rangle \langle \mathbf{Z}_{\mathsf{ve'f}} | \mathbf{Z}_{\mathsf{ve'f}} \rangle} + i\gamma \log \frac{\langle \mathbf{Z}_{\mathsf{ve'f}} | \mathbf{Z}_{\mathsf{ve'f}} \rangle}{\langle \mathbf{Z}_{\mathsf{vef}} | \mathbf{Z}_{\mathsf{vef}} \rangle},$

$\mathbf{Z}_{\mathsf{vef}} := g^{\dagger}_{\mathsf{ve}}\, \mathbf{z}_{\mathsf{vf}} \quad , \quad \mathbf{Z}_{\mathsf{ve'f}} := g^{\dagger}_{\mathsf{ve'}}\, \mathbf{z}_{\mathsf{vf}}$

Boundary face amplitude

$A_{\mathsf{f}}(h_\ell) :=$
$\sum_{j_{\mathsf{f}}} d_{j_{\mathsf{f}}} \, \mathrm{Tr}_{j_{\mathsf{f}}} \left[Y^\dagger g_{\mathsf{vn}'}^{-1} g_{\mathsf{ve}'} Y \left(\prod_{\mathsf{v} \in \mathsf{f}} Y^\dagger g_{\mathsf{ve}'}^{-1} \, g_{\mathsf{ve}} Y \right) Y^\dagger g_{\mathsf{v}^{(n)} \mathsf{e}^{(n)}}^{-1} g_{\mathsf{v}^{(n)} \mathsf{n}} Y h_\ell^{-1} \right]$ for
$\mathsf{f} \in \Gamma$

Boundary face amplitude

$A_{\mathsf{f}}(h_\ell) :=$
$\sum_{j_{\mathsf{f}}} d_{j_{\mathsf{f}}} \mathrm{Tr}_{j_{\mathsf{f}}} \left[Y^\dagger g_{\mathsf{vn}'}^{-1} g_{\mathsf{ve}'} Y \left(\prod_{\mathsf{v}\in\mathsf{f}} Y^\dagger g_{\mathsf{ve}'}^{-1} g_{\mathsf{ve}} Y \right) Y^\dagger g_{\mathsf{v}(n)_{\mathsf{e}(n)}}^{-1} g_{\mathsf{v}(n)_{\mathsf{n}}} Y h_\ell^{-1} \right]$ for $\mathsf{f} \in \Gamma$. By the same technique of the principal series representation we obtain.

$\mathrm{Tr}_{j_{\mathsf{f}}} \left[Y^\dagger g_{\mathsf{vn}'}^{-1} g_{\mathsf{ve}'} Y \left(\prod_{\mathsf{v}\in\mathsf{f}} Y^\dagger g_{\mathsf{ve}'}^{-1} g_{\mathsf{ve}} Y \right) Y^\dagger g_{\mathsf{v}(n)_{\mathsf{e}(n)}}^{-1} g_{\mathsf{v}(n)_{\mathsf{n}}} Y h_\ell^{-1} \right] =$
$\left(\prod_{\mathsf{e}\in\mathsf{f}} \frac{d_{j_{\mathsf{f}}}}{\pi} \int_{\mathbb{C}\mathrm{P}^1} \mathrm{d}\tilde{\Omega}_{\mathsf{vef}} \right) \left(\frac{d_{j_{\mathsf{f}}}^3}{\pi^3} \int_{(\mathbb{C}\mathrm{P}^1)^3} \mathrm{d}\tilde{\Omega}_{\mathsf{n}\ell\mathsf{n}'} \right) e^{j_{\mathsf{f}} S_{\mathsf{f}}[g_{\mathsf{ve}}, \mathbf{z}_{\mathsf{vf}}] + j_{\mathsf{f}} B_\ell[g_{\mathsf{vn}}, h_\ell, \mathbf{z}_\ell]}$, where

$B_\ell[g_{\mathsf{vn}}, h_\ell, \mathbf{z}_\ell] :=$
$\log \frac{\langle \mathbf{Z}_{\mathsf{vn}'\mathsf{f}} | \mathbf{z}_\ell \rangle^2}{\langle \mathbf{Z}_{\mathsf{vn}'\mathsf{f}} | \mathbf{Z}_{\mathsf{vn}'\mathsf{f}} \rangle \langle \mathbf{z}_\ell | \mathbf{z}_\ell \rangle} + \log \frac{\left\langle h_\ell^{\mathsf{T}} \mathbf{z}_\ell | \mathbf{Z}_{\mathsf{v}(n)_{\mathsf{nf}}} \right\rangle^2}{\langle \mathbf{z}_\ell | \mathbf{z}_\ell \rangle \left\langle \mathbf{Z}_{\mathsf{v}(n)_{\mathsf{nf}}} | \mathbf{Z}_{\mathsf{v}(n)_{\mathsf{nf}}} \right\rangle} + i\gamma \log \frac{\left\langle \mathbf{Z}_{\mathsf{v}(n)_{\mathsf{nf}}} | \mathbf{Z}_{\mathsf{v}(n)_{\mathsf{nf}}} \right\rangle}{\langle \mathbf{Z}_{\mathsf{vn}'\mathsf{f}} | \mathbf{Z}_{\mathsf{vn}'\mathsf{f}} \rangle}$

The full amplitude

$$W_{\mathcal{C}}(h_\ell) = \mathcal{N} \sum_{\{j_\mathsf{f}\}} \int_{SL(2,\mathbb{C})} \left(\prod_{\mathsf{v}} \mathrm{d}g_{\mathsf{ve}} \right) \left(\prod_{\mathsf{f}\in\mathcal{C}} d_{j_\mathsf{f}} \prod_{\mathsf{e}\in\mathsf{f}} \frac{d_{j_\mathsf{f}}}{\pi} \int_{\mathbb{C}\mathsf{P}^1} \mathrm{d}\tilde{\Omega}_{\mathsf{vef}} \right) \times \\ \times \left(\prod_{\ell\in\Gamma} \frac{d^3_{j_\mathsf{f}}}{\pi^3} \int_{(\mathbb{C}\mathsf{P}^1)^3} \mathrm{d}\tilde{\Omega}_{\mathsf{n}\ell\mathsf{n}'} \right) \mathrm{e}^{\sum_{\mathsf{f}\in\mathcal{C}} j_\mathsf{f} S_\mathsf{f} + \sum_{\ell\in\Gamma} j_\mathsf{f} B_\ell} \quad (1)$$

The homomorphic amplitude

We contract the full amplitude with the coherent states to impose the semiclassicality of the geometry

$$W_{\mathcal{C}}^{t_\ell}(H_\ell) := \left\langle W_{\mathcal{C}} \middle| \Psi^{t_\ell}_{\Gamma,H_\ell} \right\rangle := \int_{SU(2)^L} \left(\prod_{\ell\in\Gamma} \mathrm{d}h_\ell \right) W_{\mathcal{C}}(h_\ell)\, \Psi^{t_\ell}_{\Gamma,H_\ell}(h_\ell)$$

Approximations

- We are going to consider tree-level two-complexes $\mathcal{T}$: there are no faces which lie completely in the bulk. $W^t_{\mathcal{T}}(H_\ell) = \mathcal{N} \sum_{\{j_\ell\} \in D^k_\omega} \mu_j \, e^{-t \sum_\ell (j_\ell - \omega_\ell)^2} \, e^{i\gamma \sum_\ell \zeta_\ell j_\ell} \int_{D_{g,\mathbf{z}}} \mathrm{d}\mu_{g,\Omega} \, e^{\sum_\ell j_\ell F_\ell(g, \mathbf{z}; \mathbf{n}_{\ell(\mathrm{n})})}$
- $F_\ell[g_{\mathrm{ve}}, \mathbf{z}_{\mathrm{n}\ell}; \mathbf{n}_{\mathrm{n}(\ell)}] := S_\ell[g_{\mathrm{ve}}, \mathbf{z}_{\mathrm{n}\ell}] + \log \frac{\langle \overline{\mathbf{n}}_{s(\ell)} | \mathbf{z}_{\mathrm{n}\ell} \rangle^2 \langle \mathbf{z}_{\mathrm{n}'\ell} | \overline{\mathbf{n}}_{t(\ell)} \rangle^2}{\langle \mathbf{z}_{\mathrm{n}\ell} | \mathbf{z}_{\mathrm{n}\ell} \rangle^2 \langle \mathbf{z}_{\mathrm{n}'\ell} | \mathbf{z}_{\mathrm{n}'\ell} \rangle^2} + \log \frac{\langle \mathbf{Z}_{\mathrm{vn}'\ell} | \mathbf{z}_{\mathrm{n}'\ell} \rangle^2 \langle \mathbf{z}_{\mathrm{n}\ell} | \mathbf{Z}_{\mathrm{v}(n)_{\mathrm{n}\ell}} \rangle^2}{\langle \mathbf{Z}_{\mathrm{vn}'\ell} | \mathbf{Z}_{\mathrm{vn}'\ell} \rangle \langle \mathbf{Z}_{\mathrm{v}(n)_{\mathrm{n}\ell}} | \mathbf{Z}_{\mathrm{v}(n)_{\mathrm{n}\ell}} \rangle} + i\gamma \log \frac{\langle \mathbf{Z}_{\mathrm{v}(n)_{\mathrm{n}\ell}} | \mathbf{Z}_{\mathrm{v}(n)_{\mathrm{n}\ell}} \rangle}{\langle \mathbf{Z}_{\mathrm{vn}'\ell} | \mathbf{Z}_{\mathrm{vn}'\ell} \rangle}$
- $\mu_j := \left(\prod_{\mathrm{f}\in\Gamma} \prod_{\mathrm{e}\in\mathrm{f}} d_{j_\ell}\right) \left(\prod_{\ell\in\Gamma} d^4_{j_\ell}\right)$
- $\int_{D_{g,\mathbf{z}}} \mathrm{d}\mu_{g,\Omega} := \int_{SL(2,\mathbb{C})} \left(\prod_{\mathrm{v}} \mathrm{d}g_{\mathrm{ve}}\right) \left(\prod_{\mathrm{f}\in\Gamma} \prod_{\mathrm{e}\in\mathrm{f}} \int_{\mathbb{C}\mathrm{P}^1} \mathrm{d}\tilde{\Omega}_{\mathrm{vef}}\right) \left(\prod_{\ell\in\Gamma} \int_{(\mathbb{C}\mathrm{P}^1)^4} \mathrm{d}\tilde{\Omega}_{s\ell t}\right)$
- D^k_ω: an appropriate domain that satisfies the triangular inequalities between the spins
- $j_\ell = \lambda a_\ell + s_\ell \quad$ with $\quad \omega_\ell \equiv \lambda a_\ell$

Tree-level holomorphic amplitude

$W^t_\Gamma(H_\ell) = \mathcal{N} \int_{D_{g,\mathbf{z}}} \mu_j \mathrm{d}\mu_{g,\Omega} \mathcal{U}(g, \mathbf{z}; t, H_\ell)\, \mathrm{e}^{\lambda \Sigma(a_\ell, g, \mathbf{z}; \mathbf{n}_{\ell(\mathrm{n})})}$ where

$\mathcal{U}(g, \mathbf{z}; t, H_\ell) := \prod_\ell \left(\sum_{s_\ell \in D^k_\omega} \mathrm{e}^{-s^2_\ell t + (i\gamma\zeta_\ell + F_\ell(g, \mathbf{z}; \mathbf{n}_{\ell(\mathrm{n})}))s_\ell} \right)$

$\Sigma(a_\ell, g, \mathbf{z}; \mathbf{n}_{\ell(\mathrm{n})}) := \sum_\ell (a_\ell F_\ell(g, \mathbf{z}; \mathbf{n}_{\ell(\mathrm{n})}) + i\gamma\zeta_\ell a_\ell)$

Tree-level holomorphic amplitude

$W^t_\Gamma(H_\ell) = \mathcal{N} \int_{D_{g,\mathbf{z}}} \mu_j \mathrm{d}\mu_{g,\Omega} \mathcal{U}(g,\mathbf{z};t,H_\ell)\, \mathrm{e}^{\lambda\Sigma(a_\ell,g,\mathbf{z};\mathbf{n}_{\ell(\mathrm{n})})}$ where

$\mathcal{U}(g,\mathbf{z};t,H_\ell) := \prod_\ell \left(\sum_{s_\ell \in D^k_\omega} \mathrm{e}^{-s_\ell^2 t + (i\gamma\zeta_\ell + F_\ell(g,\mathbf{z};\mathbf{n}_{\ell(\mathrm{n})}))s_\ell}\right)$

$\Sigma(a_\ell,g,\mathbf{z};\mathbf{n}_{\ell(\mathrm{n})}) := \sum_\ell (a_\ell F_\ell(g,\mathbf{z};\mathbf{n}_{\ell(\mathrm{n})}) + i\gamma\zeta_\ell a_\ell)$.

Stationary phase theorem:

$W^t_T(H_\ell) =$

$N \sum_c \mu_j \lambda^{M^c_{\mathcal{C}}} \mathcal{H}_c(a_\ell,\mathbf{n}_{\ell(\mathrm{n})}) \mathcal{U}(g_c,z_c;t,H_l)\, \mathrm{e}^{\lambda\Sigma(a_\ell,g,\mathbf{z};\mathbf{n}_{\ell(\mathrm{n})})} \left(1+\mathcal{O}(\lambda^{-1})\right)$

- c: the critical points. Each critical point comes with a 2^N degeneracy, corresponding to the different configurations for the orientation $s(\mathrm{v})$ where $s(\mathrm{v})$ takes the values ± 1 on each vertex of $\mathcal{C}$
- $\mathcal{H}_c$: the Hessian of Σ which we are going to ignore

Evaluation at critical points

Evaluation at critical points

- $F_\ell(g, \mathbf{z}; \mathbf{n}_{\ell(\mathsf{n})}) = -i\gamma\,\phi_\ell(s_{c(\mathsf{v})}, a_\ell, \mathbf{n}_{\ell(\mathsf{n})})$, where $\phi_\ell(s_{c(\mathsf{v})}, a_\ell, \mathbf{n}_{\ell(\mathsf{n})})$ is the Palatini deficit angle which also depends on $s(\mathsf{v})$ and reduces to the usual Regge deficit angle when $s(\mathsf{v})$ is uniform, i.e. it is either $+1$ or -1 for all vertices of $\mathcal{C}$
- $\mathcal{U}(g_c, \mathbf{z}_c; t, H_\ell) = \prod_\ell \left(\sum_{s_\ell \in D^k_\omega} \mathrm{e}^{-s_\ell^2 t + i\gamma(\zeta_\ell - \phi_\ell(g, \mathbf{z}; \mathbf{n}_{\ell(\mathsf{n})}))s_\ell} \right)$

Evaluation at critical points

- $F_\ell(g,\mathbf{z};\mathbf{n}_{\ell(\mathrm{n})}) = -i\gamma\,\phi_\ell(s_{c(\mathrm{v})},a_\ell,\mathbf{n}_{\ell(\mathrm{n})})$, where $\phi_\ell(s_{c(\mathrm{v})},a_\ell,\mathbf{n}_{\ell(\mathrm{n})})$ is the Palatini deficit angle which also depends on $s(\mathrm{v})$ and reduces to the usual Regge deficit angle when $s(\mathrm{v})$ is uniform, i.e. it is either $+1$ or -1 for all vertices of $\mathcal{C}$
- $\mathcal{U}(g_c,\mathbf{z}_c;t,H_\ell) = \prod_\ell \left(\sum_{s_\ell\in D^k_\omega} \mathrm{e}^{-s_\ell^2 t + i\gamma(\zeta_\ell - \phi_\ell(g,\mathbf{z};\mathbf{n}_{\ell(\mathrm{n})}))s_\ell}\right)$
- The sum is dominated by the exponential damping factor $\exp\left(-s_\ell^2 t\right)$. It can reasonably be expected that due to this exponential damping the sum converges very fast and that it is therefore a good approximation to remove the cut-off k and sum s_ℓ from $-\infty$ to ∞ for all $\ell\in\Gamma$
- $\sum_{s_\ell=-\infty}^{\infty} \mathrm{e}^{-s_\ell^2 t + i\gamma(\zeta_\ell-\phi_\ell)s_\ell}$

Evaluation at critical points

- $\sum_{s_\ell=-\infty}^{\infty} \mathrm{e}^{-s_\ell^2 t + i\gamma(\zeta_\ell - \phi_\ell)s_\ell} =$
  $2\sqrt{\frac{\pi}{t}}\, \mathrm{e}^{-\frac{\gamma^2}{4t}(\zeta_\ell - \phi_\ell)^2}\, \vartheta_3\left(-\frac{i\pi\gamma(\zeta_\ell - \phi_\ell)}{t}, \mathrm{e}^{-\frac{4\pi^2}{t}}\right)$
- $\vartheta_3(u, q) := 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos(2nu)$:third Jacobi theta function

Evaluation at critical points

- $\sum_{s_\ell=-\infty}^{\infty} \mathrm{e}^{-s_\ell^2 t + i\gamma(\zeta_\ell - \phi_\ell)s_\ell} =$
  $2\sqrt{\frac{\pi}{t}}\,\mathrm{e}^{-\frac{\gamma^2}{4t}(\zeta_\ell-\phi_\ell)^2}\,\vartheta_3\left(-\frac{i\pi\gamma(\zeta_\ell-\phi_\ell)}{t}, \mathrm{e}^{-\frac{4\pi^2}{t}}\right)$
- $\vartheta_3(u,q) := 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nu)$:third Jacobi theta function
- $\mathcal{U}(g_c, \mathbf{z}_c; t, H_\ell) \approx \prod_\ell 2\sqrt{\frac{\pi}{t}}\,\mathrm{e}^{-\frac{\gamma^2}{4t}(\zeta_\ell-\phi_\ell)^2}\,\vartheta_3\left(-\frac{i\pi\gamma(\zeta_\ell-\phi_\ell)}{t}, \mathrm{e}^{-\frac{4\pi^2}{t}}\right)$

Final estimation of the transition amplitude

- $\mathcal{U}(g_c, \mathbf{z}_c; t, H_\ell) \approx \prod_\ell 2\sqrt{\frac{\pi}{t}}\, \mathrm{e}^{-\frac{\gamma^2}{4t}(\zeta_\ell - \phi_\ell)^2}\, \vartheta_3\left(-\frac{i\pi\gamma(\zeta_\ell - \phi_\ell)}{t}, \mathrm{e}^{-\frac{4\pi^2}{t}}\right)$

Final estimation of the transition amplitude

- $\mathcal{U}(g_c, \mathbf{z}_c; t, H_\ell) \approx \prod_\ell 2\sqrt{\frac{\pi}{t}}\, \mathrm{e}^{-\frac{\gamma^2}{4t}(\zeta_\ell - \phi_\ell)^2}\, \vartheta_3 \left(-\frac{i\pi\gamma(\zeta_\ell - \phi_\ell)}{t}, \mathrm{e}^{-\frac{4\pi^2}{t}} \right)$
- If $\gamma \leq \frac{1}{2}$ (as it seems to be from the LQG derived BH entropy formula) then $\theta_3 \approx 1$
- $\mathcal{U}(g_c, \mathbf{z}_c; t, H_\ell) \approx \prod_\ell 2\sqrt{\frac{\pi}{t}}\, \mathrm{e}^{-\frac{\gamma^2}{4t}(\zeta_\ell - \phi_\ell)^2}$

Final estimation of the transition amplitude

- $\mathcal{U}(g_c, \mathbf{z}_c; t, H_\ell) \approx \prod_\ell 2\sqrt{\frac{\pi}{t}}\, \mathrm{e}^{-\frac{\gamma^2}{4t}(\zeta_\ell - \phi_\ell)^2}\, \vartheta_3\left(-\frac{i\pi\gamma(\zeta_\ell - \phi_\ell)}{t}, \mathrm{e}^{-\frac{4\pi^2}{t}}\right)$
- If $\gamma \leq \frac{1}{2}$ (as it seems to be from the LQG derived BH entropy formula) then $\theta_3 \approx 1$
- $\mathcal{U}(g_c, \mathbf{z}_c; t, H_\ell) \approx \prod_\ell 2\sqrt{\frac{\pi}{t}}\, \mathrm{e}^{-\frac{\gamma^2}{4t}(\zeta_\ell - \phi_\ell)^2}$
- By substituting everything to the transition amplitude to obtain the estimation $W_T^t(H_\ell) \approx$
  $\mathcal{N} \sum_{\{s(\mathrm{v})\}} \lambda^N \mu(a) \prod_\ell e^{\frac{-\gamma^2}{4t}(\zeta_\ell - \phi_\ell)^2 + i\gamma(\zeta_\ell - \phi_\ell)\omega_\ell} \left(1 + \mathcal{O}(\lambda^{-1})\right)$

Applications

- Black Hole to White Hole transition $p \sim e^{-\frac{m^2}{m_{pl}^2}}$
- Bouncing Cosmology?

Future work

- What happens when we include bulk faces?

Thank you!

Thank you!

The sign

- $S_{EH}[g] = \int d^4x \sqrt{-det(g)} R$
- $S_T[e] = \int \star(e \wedge e) \wedge F$

The sign

- $S_{EH}[g] = \int d^4x \sqrt{-det(g)} R$
- $S_T[e] = \int \star(e \wedge e) \wedge F$
- $S_{EH}[e] = \int d^4x |det(e)| R[e]$
- $S_T[e] = \int d^4x det(e) R[e]$

The classical limit

$$A_v \sim c e^{iS_{Regge}} + c' e^{-iS_{Regge}}$$

Why there is no critical parameter

In the path integral of harmonic oscillator if we consider $q = q(t)$ then $S_N(q_n) = \sum_{n=1}^{N} m\frac{(q_{n+1}-q_n)^2}{2} - \frac{\Omega^2}{2}q_n^2$. We then take the limit $N \to \infty$ **and** $\Omega \to 0$.

Why there is no critical parameter

In the path integral of harmonic oscillator if we consider $q = q(t)$ then $S_N(q_n) = \sum_{n=1}^{N} m\frac{(q_{n+1}-q_n)^2}{2} - \frac{\Omega^2}{2}q_n^2$. We then take the limit $N \to \infty$ **and** $\Omega \to 0$.If we consider $t = t(\tau)$ and $q = q(\tau)$ then $S_N = \sum_{n=1}^{N} \frac{m}{2}\frac{(q_{n+1}-q_n)^2}{(t_{n+1}-t_n)} - (t_{n+1} - t_n)\frac{1}{2}\omega^2 q_n^2$.

Why there is no critical parameter

In the path integral of harmonic oscillator if we consider $q = q(t)$ then $S_N(q_n) = \sum_{n=1}^{N} m\frac{(q_{n+1}-q_n)^2}{2} - \frac{\Omega^2}{2} q_n^2$. We then take the limit $N \to \infty$ **and** $\Omega \to 0$.If we consider $t = t(\tau)$ and $q = q(\tau)$ then $S_N = \sum_{n=1}^{N} \frac{m}{2}\frac{(q_{n+1}-q_n)^2}{(t_{n+1}-t_n)} - (t_{n+1} - t_n)\frac{1}{2}\omega^2 q_n^2$.We only have to take $N \to \infty$, there is no critical parameter!