Flavor and Fluctuations

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Workshop on the Standard Model and Beyond
Corfu 2023
August 31, 2023

Based on arXiv:2303.17875 and forthcoming.

Outline

Some history: No round date anniversaries, yet

- 1936: Anderson and Neddermayer discover the muon. Its decay implied the existence of the corresponding neutrino and signaled the existence of a new "flavor" of matter and of more than one "flavor" or "family".
- Rabi's question: "Who ordered that?" has remained open since then.
- As Feynman noted, when one asks a "why?" question (and Rabi's question is such a question) one must specify what's assumed known and what's not. Is the number of flavors an input and their properties the output, or can the number of flavors itself be an output? Within what framework?
- We shall present a framework, where the least number of flavors can be found to be an output, rather than an input. The new ingredient is an assumption about the description of the fluctuations–which, also, highlights the relevance of supersymmetry.

Noisy SUSY

Since its invention/discovery, SUSY has been considered an *optional* feature of natural phenomena;
is there any way in which it might be understood as an *inevitable* feature of natural phenomena?
More than forty years ago G. Parisi and N. Sourlas, in "Supersymmetric field theories and stochastic differential equations", *Nucl. Phys.* **B206** (1982) 321
made the case that supersymmetry is an inevitable property of a physical system in equilibrium with a bath of fluctuations.
A key role is played by a quantity introduced, in 1980, by H. Nicolai–within the context of supersymmetric theories, only, however–and known, since, as "the Nicolai map".
For lattice Yang-Mills theories a related idea, known as the "trivializing map", was introduced by M. Lüscher in 2009.

The story of a physical system and its fluctuations

The description of the properties of a physical system relies on *two* distinct, but equally important, groups: The dynamical degrees of freedom, that describe the "classical" dynamics and their *super*partners, that can resolve the fluctuations, with which they are in equilibrium.

These are (some of) their stories...

The story starts with the noise–that describes the bath of fluctuations.

And what's at stake is identifying the degrees of freedom, whose dynamics can describe such fluctuations.

The noise fields

They are defined by their partition function. For white noise, this means

$$Z = \int [\mathscr{D}\eta_I(x)]\, e^{-\int d^Dx\, \frac{1}{2}\frac{\eta_I(x)\eta_J(x)\delta^{IJ}}{\sigma^2}} \equiv 1$$

by definition of the measure.
This expression is equivalent to

$$\begin{aligned} \langle \eta_I(x) \rangle &= 0 \\ \langle \eta_I(x)\eta_J(x') \rangle &= \sigma^2 \delta_{IJ}\delta(x - x') \end{aligned}$$

and the other correlation functions are given by Wick's theorem.
We may choose units such that $\sigma = 1$.

The physical meaning of the noise

- $\sigma^2 = \hbar$: The bath describes quantum fluctuations.
- $\sigma^2 = k_\mathrm{B} T$: The bath describes thermal fluctuations.
- $\sigma^2 =$ strength of the disorder: (Annealed) Disordered systems.

A non-trivial issue concerns combining different baths.

From noise to dynamical fields

Now we must provide a map between the noise fields and putative dynamical fields.

The dynamics

For a single particle, whose dynamics is defined by its position, $\phi_I(t)$, where $I = 1, 2, \ldots D$ labels the dimensionality of the target space, its dynamics, in equilibrium with a bath of fluctuations, is given by the Lagevin equation

$$\eta_I(x) \equiv \frac{d}{dt}\phi_I + \frac{\partial W}{\partial \phi_I}$$

Here, in fact, t is the *Euclidian* time.
Following Parisi and Sourlas, we take this relation as the injunction to change variables in the partition function...

Changing variables

If we perform the change of variables in the partition function, we find

$$Z = 1 = \int [\mathscr{D}\phi_I] \left|\det \frac{\delta\eta_I}{\delta\phi_J}\right| e^{-S[\phi_I]}$$

and we notice that, absent anomalies, the value of the integral does not change. Therefore the absolute value of the determinant describes *all* of the fluctuations of the action of the scalars, $S[\phi_I]$, which, in the present case, is the Euclidian action for the particle.
Now we can write

$$\left|\det \frac{\delta\eta_I}{\delta\phi_J}\right| = e^{-\mathrm{i}\theta_{\det}} \det \frac{\delta\eta_I}{\delta\phi_J} =$$
$$e^{-\mathrm{i}\theta_{\det}} \int [\mathscr{D}\psi_I][\mathscr{D}\chi_I]\, e^{\int\, d^D x\, \psi_I \left\{ \frac{d}{dt}\delta_{IJ} + \frac{\partial^2 W}{\partial\phi_I \partial\phi_J} \right\} \chi_J}$$

From the point of view of the (super)partners

$$Z = 1 = \int [\mathscr{D}\phi_I][\mathscr{D}\psi_I][\mathscr{D}\chi_I]\, e^{-\mathrm{i}\theta_{\mathrm{det}}} \times e^{-S[\phi_I]+\int\, d^Dx\, \psi_I\left\{\delta_{IJ}\frac{d}{dt}+\frac{\partial^2 W}{\partial\phi_I\partial\phi_J}\right\}\chi_J}$$

This expression can be understood in two, equivalent, ways:

- The fluctuations of the scalars are described by the action of the anticommuting fields–along with the phase of the determinant!
- The fluctuations of the anticommuting fields, in interaction with the scalars, are described by the phase of the determinant, along with the action of the scalars.

From the point of view of the (super)partners

Said differently:

The anticommuting fields resolve the bath of fluctuations, with which the scalars are in equilibrium, as do the scalars for the anticommuting fields, when they are part of a supermultiplet.

It is in this way that the no-go theorem pertaining, in particular, to Bell's inequalities can be evaded; this was, in fact, noted by P. G. O. Freund, already, in 1981 in the paper "Fermionic hidden variables and EPR correlations", *Phys. Rev.* **D24** (1981) 1526.

Curiously, this idea wasn't followed up–nor was the relation to the work of Parisi and Sourlas, after it appeared, investigated further…

[Cf. also the talk by J. Moreno on entanglement at this Workshop]

We, also, remark that these expressions don't seem to be sensitive to the number of particles (more precisely the dimensionality of the target space)…

From particles to fields: The case $D = 2$

If we consider two-dimensional worldvolume, that carries a representation of the Lorentz group, things become quite interesting. We realize that, when we write the corresponding Langevin equation in a worldvolume of more than one dimension, viz.

$$\eta_I = (?)\partial_\mu \phi_J + \frac{\partial W}{\partial \phi_I}$$

that (?) must carry a spacetime index, to saturate the $\mu = 1, 2$ (we're always in Euclidian signature).
The question, also, is, whether I, J can take only the value 1, or *must* take greater values–appearance of flavors!

“Fiddling around”–but with a system!

(Cf. Feynman’s Les Houches lectures, 1976) How can we find what should fill in (?) ?

- When we compute $\eta_I \eta_J \delta^{IJ}$, we want rotation invariant kinetic terms;
- When we look at the Jacobian

$$\det \frac{\delta \eta_I}{\delta \phi_J} = (?)\partial_\mu + \frac{\partial^2 W}{\partial \phi_I \partial \phi_J}$$

  it should have the correct transformation properties, corresponding to that of a Dirac operator.

Indeed, as was found by Parisi and Sourlas, the Langevin equations with the correct properties are

$$\eta_1 = \partial_x \phi_2 + \partial_y \phi_1 + \frac{\partial W}{\partial \phi_1}$$
$$\eta_2 = \partial_x \phi_1 - \partial_y \phi_2 + \frac{\partial W}{\partial \phi_2}$$

And we notice that two flavors appear naturally!

Emergence of the Dirac matrices

We can write these equations as

$$\eta_I = \sigma^\mu_{IJ}\partial_\mu\phi_J + \frac{\partial W}{\partial\phi_I}$$

where the matrices σ^μ can be identified with the Pauli matrices, namely,

$$\sigma^1 = \sigma^x \quad \text{and} \quad \sigma^2 = \sigma^z$$

In $D = 2$ spacetime dimensions, in Euclidian signature, the generators of the Clifford algebra can be chosen to belong to a Majorana representation.
The notation is misleading, because the I, J are flavor indices.

Emergence of the Dirac matrices

Nevertheless, the Jacobian,

$$\left|\det\frac{\delta\eta_I}{\delta\phi_J}\right| = \left|\det(\sigma^x\partial_x + \sigma^z\partial_y + \frac{\partial^2 W}{\partial\phi_I\partial\phi_J})\right| =$$
$$e^{-\mathrm{i}\theta_{\mathrm{det}}}\int[\mathscr{D}\psi_I][\mathscr{D}\chi_I]\,e^{\int\, d^2x\,\psi_I\left(\sigma^x_{IJ}\partial_x+\sigma^z_{IJ}\partial_y+\frac{\partial^2 W}{\partial\phi_I\partial\phi_J}\right)\chi_J}$$

upon being introduced into the exponent, as a local contribution, using anticommuting fields, can be recognized as describing 2 flavors of target space fermions. The notation simply expresses the non-trivial mixing between internal and spacetime symmetries (that supersymmetry allows).
Of course what's really meant, by the condensed notation, is that the ψ_I and χ_J are the single components of two-component Majorana spinors, that carry I and J as flavor indices.

The "problematic" issues of one flavor

Can we write a Langevin equation

$$\eta = (a\partial_x + b\partial_y)\phi + \frac{\partial W}{\partial \phi}$$

with just one scalar? And use it to define a useful change of variables in the partition function, in order to obtain (up to surface terms) the usual kinetic and potential terms for the scalar?

It seems that we can.

We would need $a^2 = 1 = b^2$, which reflect the "holomorphic factorization" of the kinetic term of the fermionic action, in two dimensions–which can break down for non-trivial superpotentials. This expresses the subtleties of $\mathcal{N} = 1$ SUSY, which, indeed, is described by one commuting and two anticommuting degrees of freedom (i.e. one target space spinor). This deserves a fuller study. (E.g. along the lines of I. Antoniadis, C. Bachas, C. Kounnas and P. Windey, Phys. Lett. B 171 (1986) 51).

The least number of flavors in $D = 2$

This means that the necessity for the appearance of flavors in $D = 2$ can be described by the Langevin equations (for the case of the cubic superpotential)

$$\begin{aligned} \eta_1 &= \partial_x\phi_2 + \partial_y\phi_1 + g(\phi_1^2 - \phi_2^2) + \kappa\phi_1 \\ \eta_2 &= \partial_x\phi_1 - \partial_y\phi_2 + 2g\phi_1\phi_2 + \kappa\phi_2 \end{aligned}$$

in the sense that the consistent dynamics requires *at least* two flavors and doesn't "factorize", locally, into two theories of one flavor: We can have more than two flavors–but we can't have, only, one. Supersymmetry appears through the identities satisfied by the noise fields, namely,

$$\left\langle(\eta_I(x) - \langle\eta_I(x)\rangle)(\eta_J(x') - \langle\eta_J(x')\rangle)\right\rangle = \sigma^2\delta_{IJ}\delta(x - x')$$

with the higher order connected correlation functions given by Wick's theorem, where the noise fields are expressed in terms of the scalars.

The SO(2) flavor symmetry can be broken by the terms proportional to κ and supersymmetry (even anomalously broken) controls the breaking.

Doubling

If $D \not\equiv 2 \bmod 8$, (e.g. $D = 3$ or $D = 4$ spacetime dimensions) then the σ^μ_{IJ}, necessarily, have imaginary entries, therefore, in the Nicolai map(s),

$$\eta_I = \sigma^\mu_{IJ}\partial_\mu\phi_J + \frac{\partial W}{\partial\phi_I}$$

the RHS is complex, so the LHS must be, too. Therefore, we must introduce the complex conjugate:

$$\eta_I^\dagger = \sigma^\mu_{JI}\partial_\mu\phi_J^\dagger + \left(\frac{\partial W}{\partial\phi_I}\right)^\dagger$$

(since the σ^μ can be taken as Hermitian) and the partition function for the noise fields is, now,

$$Z = 1 = \int [\mathscr{D}\eta_I][\mathscr{D}\eta_I^\dagger]\, e^{-\int\, d^Dx\, \frac{\eta_I(x)\eta_J(x)^\dagger \delta^{IJ}}{\sigma^2}}$$

Hidden SUSY

These remarks can be summarized in the following way: The first non-trivial statement is that the classical action

$$S[\phi_I] - \int d^D x\, \psi_I \left\{ \sigma^{\mu}_{IJ}\partial_{\mu} + \frac{\partial^2 W}{\partial\phi_I \partial\phi_J} \right\} \chi_J$$

is invariant under $\mathcal{N} = 2$ SUSY transformations. This was stressed by Parisi and Sourlas. Here the I, J are flavor indices and the anticommuting fields, ψ_I and χ_J are individual components of target space spinors.

Hidden SUSY

The next, even more remarkable, statement is that this statement is equivalent to the statement that the correlation functions of the noise fields,

$$\eta_I(x) \equiv \sigma^{\mu}_{IJ}\partial_\mu\phi_J + \frac{\partial W}{\partial\phi_I}$$

defined by the RHS of this equation, when the scalars are sampled by the, full, classical action, do satisfy the identities of Gaussian fields, with ultra–local 2-point function:

$$\Big\langle(\eta_I(x) - \langle\eta_I(x)\rangle)(\eta_J(x') - \big\langle\eta_J(x')\big\rangle)\Big\rangle = \sigma^2\delta_{IJ}\delta(x - x')$$

with the higher-point correlation functions given by Wick's theorem.

This means that, while the scalars can have non-trivial interactions, the fields $\eta_I(x)$ don't–they're free fields. But what's, also, important is that the 2–point function is ultra–local.

Hidden SUSY

For it is this property that ensures that the action of the scalars is local. So the Nicolai map isn't, simply, a transformation from interacting fields to free fields; it's a transformation from interacting local fields to non-interacting ultra–local fields.
The physics behind the mathematical transformation is the resolution of the bath of the fluctuations.

How about gauge theories?

Gauge theories, with compact gauge group, can be described by scalar fields, taking values on the group manifold. The "natural" noise distribution isn't a Gaussian, with ultra–local 2–point function, but uniform over the group manifold.
This has been studied on the lattice, through the so-called "trivializing maps", introduced by Lüscher. These are, indeed, the avatars of the Nicolai map for the group manifolds.
However their construction is, still, work in progress.

Abelian gauge theories

For abelian gauge fields it's possible to take a shortcut (in Lorenz–Feynman gauge):

$$\begin{aligned}
\eta_I &= \sigma^\mu_{IJ}\partial_\mu\phi_J \\
\xi_I &= \sigma^\mu_{IJ}\nabla_\mu\varphi_J + \frac{\partial W}{\partial\varphi_I} \\
\xi_I^\dagger &= \sigma^\mu_{JI}[\nabla_\mu\varphi_J]^\dagger + \left(\frac{\partial W}{\partial\varphi_I}\right)^\dagger \\
\nabla_\mu &\equiv \partial_\mu - \mathrm{i}qA_\mu \equiv \partial_\mu - \mathrm{i}q\phi_\mu
\end{aligned}$$

where

$$\phi_\mu \equiv \phi_I \equiv A_\mu$$

and q is the charge of the matter fields under the gauge field. Here φ_I are the scalar superpartners of the fermions of the hypermultiplet(s).

The partition function for (S)QED

$$Z = \int \underbrace{[\mathscr{D}\eta_I][\mathscr{D}\xi_I][\mathscr{D}\xi_I^\dagger]}_{[\mathscr{D}h_I]} e^{-\int d^Dx\left\{\frac{1}{2}\eta_I\eta_J\delta^{IJ}+\xi_I\xi_J^\dagger\delta^{IJ}\right\}} = 1 =$$

$$\int \underbrace{[\mathscr{D}\phi_I][\mathscr{D}\varphi_I][\mathscr{D}\varphi_I^\dagger]}_{[\mathscr{D}\Phi_I]} \left|\det\frac{\delta h_I}{\delta\Phi_J}\right| e^{-S[\phi_I,\varphi_I,\varphi_I^\dagger]}$$

The fermions are “hidden” in the determinant and “emerge” upon introducing it in the exponent.
For $D = 4$, we must double the degrees of freedom correspondingly. It is in this way that the dual photon naturally appears.

Emergent flavor

This framework allows us to deduce that there must be, *at least*, 4 complex scalars in $d = 4$ and, therefore, *at least*, $N_f = 4$ families.
So the question isn't,
"why is there more than *one* family?"
but,
"where's the *fourth* family?"
The number of families is defined by the requirement that the system, *along with the qantum fluctuations* be consistently closed.
And we can, actually, see why it would be hard to detect it, since it depends on the factorization of the fermionic determinant into familes.

Conclusions and Outlook

Any field theory (and that includes particle models, in the path integral formalism), whose fields take all possible real values, can be understood as providing a mapping between white noise fields and commuting fields; the anticommuting fields "emerge" from the Jacobian. The relation between the commuting and anticommuting fields is that they are superpartners. This is extended supersymmetry.
The superpartners may be thought of as "BSM" particles; but, in fact, they are part of the SM, since they resolve the quantum fluctuations of the fields of the SM!
That's the essence of the proposal of Parisi and Sourlas; and the way to understand it, in practice, is by computing the identities that should be satisfied by the Nicolai map.

Conclusions and Outlook

Consequences for the Standard Model:

- One scalar field is a semi-classical property, relevant within perturbation theory; in a relativistic field theory, it's not possible to describe, fully, the fluctuations of just one scalar field; there are, inevitably, more-in $D = 4$ the least number is 8, which implies the existence of, at least, a fourth family.
- Flavor (non-)universality can be straightforwardly accommodated, since the fermion determinant doesn't, inevitably, "factorize" over the flavors (and the gaugini). How it does is of interest to spell out.
- Chiral fermions can be described using partial SUSY breaking $\mathcal{N} = 2 \to \mathcal{N} = 1$; one way of realizing this is the domain–wall construction, that leads to "partial" SUSY breaking to $\mathcal{N} = 1$ on the brane, from the $\mathcal{N} = 2$ in the "bulk".

Conclusions and Outlook

Another issue of practical significance is that, insofar as the absolute value of the determinant–that describes the contribution of the fermions–is generated by the fluctuations, this means that it is possible, in principle, to express fermionic correlators in terms of the correlators of their bosonic superpartners, sampled using the bosonic action, which is much easier to do, than the fermionic action. This remains to be spelled out for practical applications.

Conclusions and Outlook

There's a "natural" way to understand the relevance of SUSY for any field theory and the SM, in particular. There's, still, considerable work to be done to understand how this approach can be realized for non-abelian gauge theories and how this can lead to search strategies in real experiments. However SUSY isn't an optional property of Nature (or of the SM) but an inevitable part of it.
It's necessary to learn how to see it. How it can be realized can be quite unexpected (recall that the quarks cancel the gauge anomalies of the leptons and vice versa!)

Conclusions and outlook

- In $D \not\equiv 2 \bmod 8$, what can, typically, occur is that (extended) SUSY is *anomalously* broken–and this can be probed by studying the identities (more precisely, *not*) satisfied by the noise fields–this is, in fact, the "take home message" of the work of Nicolai, Parisi and Sourlas.
- Nevertheless, it is possible to use it to understand how flavor can be understood as the consequence of global Lorentz invariance and equilibrium with quantum fluctuations. This leads to the *minimal* number of flavors–and, in $D = 4$, implies the existence of, at least, *four* families.
- SUSY breaking, indeed, implies that the system is "open"; anomalous breaking implies that the degrees of freedom, that are needed to "close" the system, are "non-local" wrt the degrees of freedom already present.

Conclusions and Outlook

All theories are supersymmetric.
Some theories are born supersymmetric;
some become supersymmetric;
some have supersymmetry thrust upon them...