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"BRST Formalism of Weyl Invariant Gravity"

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P. Saake (Max-Planck-Institute, Heidelberg) and I.O., Phys. Rev. D 106 (2022) 106007; I.O., Phys. Rev. D 105 (2022) 126018; I.O.,Phys. Rev. D 105 (2022) 066001; M. Ohta (OIST) and I.O., to appear (2023).

1. Introduction

Without any sign of new particles (except Higgs) at LHC,
we are now in an era of paradigm-shift from the old view that

"The standard model (SM) is incomplete so it should be replaced with a completely new theory such as SUSY, GUT or string theory"

to the new view that

"The overall structure of the SM is true and we should look for its minimal extension which largely preserves the SM".

We are watching a mounting evidence that its minimal extension could survive to the Planck scale where quantum gravity (QG) would unify gravity with the other interactions, i.e.,"Great desert scenario".

From the SM point of view, it seems that the Planck scale M_{Pl} is a special point in the sense that

(1) Scalar self-coupling is zero; $\lambda(M_{Pl}) \approx 0$
(2) Its beta function is zero; $\dot{\lambda}(M_{Pl}) \approx 0$
(3) Higgs bare mass is zero; $m^2(M_{Pl}) \approx 0$

Thus, it is conceivable that the SM is the low-energy limit of a distinct theory with a global scale symmetry at the Planck scale.

Indeed, in 1995, Bardeen has already advocated the idea that instead of SUSY, the global scale symmetry might be a fundamental symmetry and play an important role, in particular, in naturalness problems, and afterwards various interesting models based on the global scale symmetry were proposed.

Here, related to scale invariance, let us ask an elementary question:

"The global scale symmetry makes sense in QG?"

Answer: "Perhaps, no sense!"

The key observation: No-hair theorem of quantum black holes and string theory

Global additive conservation laws, such as baryon and lepton number conservation, cannot hold in any QG. Moreover, in string theory, we never get any additive conservation laws, and at least in known string vacua, the additive global symmetries are either gauge symmetries or explicitly violated. By contrast, gauge symmetries such as $U(1)$ electric charge conservation law cause no trouble for black hole physics.

To put it differently, in the great desert scenario, global scale symmetry must be replaced by local scale symmetry, i.e., "Weyl symmetry"!

Our strategy is first to construct a Weyl invariant quantum gravity and then to construct a theory beyond the standard model.

2. Classical theory

We wish to consider Einstein gravity, i.e., Einstein-Hilbert term, since general relativity (GR) can account for many astrophysical and cosmological phenomena without any conflict with observations and experiments.

However, Einstein-Hilbert Lagrangian is not invariant under Weyl transformation or equivalently, local scale transformation. We therefore introduce a scalar field ϕ and construct a Weyl invariant scalar-tensor gravity:

$$\mathcal{L}_c = \sqrt{-g}\left(\frac{1}{12}\phi^2 R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)$$

The key point: In unitary gauge for Weyl invariance, $\phi = \sqrt{\frac{3}{4\pi G}}$, $\mathcal{L}_c$ becomes the Einstein-Hilbert Lagrangian.

If we ignore the problem of massive ghost in higher-derivative gravity, we can incorporate conformal gravity in addition to Weyl invariant scalar-tensor gravity:

$$\mathcal{L}_c = \sqrt{-g}\left(\frac{1}{12}\phi^2 R + \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) \underline{- \sqrt{-g}\,\alpha_c C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}}$$

It is straightforward to apply our formalism to this Lagrangian as well (cf. M. Ohta and I. O., to appear). Furthermore, we can also apply the present formalism to a Weyl invariant scalar-tensor gravity in Weyl geometry (P. Saake and I.O, Phys. Rev. D 106 (2022) 106007).

Owing to lack of time and simplicity of presentation, I will discuss only the Weyl-invariant scalar-tensor gravity in Riemann geometry in this talk.

3. Quantum theory

$\mathcal{L}_c$ is invariant under general coordinate transf. (GCT) and Weyl transf.

GCT BRST transformation

$$\delta_B^{(1)} g_{\mu\nu} = -(\nabla_\mu c_\nu + \nabla_\nu c_\mu), \quad \delta_B^{(1)} \phi = -c^\lambda \partial_\lambda \phi,$$

$$\delta_B^{(1)} c^\rho = -c^\lambda \partial_\lambda c^\rho,$$

$$\delta_B^{(1)} \bar{c}_\mu = i B_\mu, \quad \delta_B^{(1)} B_\mu = 0, \quad \delta_B^{(1)} b_\mu = -c^\lambda \partial_\lambda b_\mu.$$

Here, new NL field $b_\mu = B_\mu - i\, c^\lambda \partial_\lambda \bar{c_\mu}$ is introduced.

Weyl BRST transformation

$$\delta_B^{(2)} g_{\mu\nu} = 2c g_{\mu\nu}, \quad \delta_B^{(2)} \phi = -c\phi,$$

$$\delta_B^{(2)} \bar{c} = iB, \quad \delta_B^{(2)} c = \delta_B^{(2)} B = 0.$$

To assure the relation $\{\delta_B^{(1)}, \delta_B^{(2)}\} = \delta_B^{(1)}\ \delta_B^{(2)} + \delta_B^{(2)}\ \delta_B^{(1)} = 0$ as well as nilpotency $(\delta_B^{(1)})^2 = (\delta_B^{(2)})^2 = 0$, we further set up BRST transformation:

$$\delta_B^{(1)} B = -c^\lambda \partial_\lambda B, \quad \delta_B^{(1)} c = -c^\lambda \partial_\lambda c, \quad \delta_B^{(1)} \bar{c} = -c^\lambda \partial_\lambda \bar{c},$$
$$\delta_B^{(2)} b_\mu = \delta_B^{(2)} c^\mu = \delta_B^{(2)} \bar{c}_\mu = 0.$$

<u>Gauge-fixing conditions</u>

For GCT, "Extended de Donder gauge": $\partial_\mu \left(\widetilde{g^{\mu\nu}}\ \phi^2 \right) = 0$

For Weyl gauge symmetry, "Scalar gauge": $\partial_\mu \left(\widetilde{g^{\mu\nu}}\ \phi \partial_\nu \phi \right) = 0 \Leftrightarrow$ Massless dilaton

Here, $\widetilde{g^{\mu\nu}} \equiv \sqrt{-g}\ g^{\mu\nu}$.

The gauge-fixed and BRST invariant Lagrangian:

$$\mathcal{L}_q = \mathcal{L}_c + i\delta_B^{(1)}(\tilde{g}^{\mu\nu}\phi^2\partial_\mu\bar{c}_\nu) + i\delta_B^{(2)}[\bar{c}\partial_\mu(\tilde{g}^{\mu\nu}\phi\partial_\nu\phi)]$$
$$= \sqrt{-g}\left(\frac{1}{12}\phi^2 R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)$$
$$-\tilde{g}^{\mu\nu}\phi^2(\partial_\mu b_\nu + i\partial_\mu\bar{c}_\lambda\partial_\nu c^\lambda) + \tilde{g}^{\mu\nu}\phi\partial_\mu B\partial_\nu\phi - i\tilde{g}^{\mu\nu}\phi^2\partial_\mu\bar{c}\partial_\nu c.$$

From this quantum Lagrangian, we can derive field equations.
The bottom line is that we can obtain the d’Alembert type of field equations:

$$\boxed{g^{\mu\nu}\partial_\mu\partial_\nu X^M = 0}$$

Here, $X^M = \{x^\mu, b_\mu, \sigma, B, c^\mu, \bar{c_\mu}, c, \bar{c}\}$ where $\phi = e^\sigma$ (σ: “Dilaton”)

Together with the gauge condition $\partial_\mu\left(\widetilde{g^{\mu\nu}}\ \phi^2\right) = 0$, this equation gives us two kinds of conserved currents:

$$\mathcal{P}^{\mu M} = \widetilde{g^{\mu\nu}}\ \phi^2 \partial_\nu X^M \ \ and \ \ \mathcal{M}^{\mu MN} = \widetilde{g^{\mu\nu}}\ \phi^2 (\, X^M \overleftrightarrow{\partial_\nu} Y^N)$$

where $X^M \overleftrightarrow{\partial_\nu} Y^N = \ X^M \partial_\nu\ Y^N$- $(\partial_\nu\ X^M)\ Y^N$.

In fact, for instance,

$$\partial_\mu \mathcal{P}^{\mu M} = \partial_\mu\left(\widetilde{g^{\mu\nu}}\ \phi^2 \partial_\nu X^M\right) = \widetilde{g^{\mu\nu}}\ \phi^2 \partial_\mu \partial_\nu X^M = \sqrt{-g}\phi^2 \underbrace{g^{\mu\nu}\partial_\mu\partial_\nu X^M}_{=0} = 0.$$

After performing canonical quantization, it will turn out that charges corresponding to these currents generate a very huge IOSp(10|10) Poincare-like algebra, which is a global symmetry in the present theory.

4. Canonical quantization and equal-time (anti)commutation relations

① Set up the Poisson brackets: e.g.

$$\{g_{\mu\nu}(t,\vec{x}), \pi_g^{\rho\sigma}(t,\vec{x}')\}_P = \frac{1}{2}(\delta_\mu^\rho \delta_\nu^\sigma + \delta_\mu^\sigma \delta_\mu^\rho)\delta(\vec{x}-\vec{x}'), \quad \{\phi(t,\vec{x}), \pi_\phi(t,\vec{x}')\}_P = \delta(\vec{x}-\vec{x}'), \ldots$$

② Calculate the canonical conjugate momenta: e.g.

$$\pi_g^{\mu\nu} \equiv \frac{\partial \mathcal{L}_q}{\partial \dot{g}_{\mu\nu}} = -\frac{1}{24}\sqrt{-g}\,\phi^2(-g^{0\lambda}g^{\mu\nu}g^{\sigma\tau} - g^{0\tau}g^{\mu\lambda}g^{\nu\sigma} - \ldots)\partial_\lambda g_{\sigma\tau}$$
$$-\frac{1}{12}\sqrt{-g}(g^{0\mu}g^{\rho\nu} + g^{0\nu}g^{\rho\mu} + \ldots)\phi\partial_\rho\phi - \frac{1}{2}\sqrt{-g}(g^{0\mu}g^{\rho\nu} + g^{0\nu}g^{\rho\mu} + \ldots)\phi^2 b_\rho$$

③ Calculate all the equal-time (anti)commutation relations: e.g.

$$[\dot{g}_{\rho\sigma}, g'_{\mu\nu}] = -12i\tilde{f}\phi^{-2}[g_{\rho\sigma}g_{\mu\nu} - g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu} + \sqrt{-g}\tilde{f}(\delta_\rho^0\delta_\mu^0 g_{\sigma\nu} + \delta_\rho^0\delta_\nu^0 g_{\sigma\mu}$$
$$+\delta_\sigma^0\delta_\mu^0 g_{\rho\nu} + \delta_\sigma^0\delta_\nu^0 g_{\rho\mu})]\delta^3$$

Here, $\tilde{f} = \frac{1}{\tilde{g^{00}}} = \frac{1}{\sqrt{-g}g^{00}}$, $\delta^3 = \delta(\vec{x}-\vec{x}')$.

5. IOSp(10|10) Poincare-like symmetry

Let us consider an OSp(10|10) rotation, which is rewritten as

$$\delta X^M = \eta^{ML} \varepsilon_{LN} X^N \equiv \varepsilon^M{}_N X^N$$

Here, η_{MN} is an OSp(10|10) metric given by ($\epsilon \equiv \frac{1}{6}$)

$$\eta_{NM} = \tilde{\eta}_{MN} = \begin{array}{cc} \begin{array}{c} x^\nu \\ b_\nu \\ \sigma \\ B \\ c^\nu \\ \bar{c}_\nu \\ c \\ \bar{c} \end{array} & \left(\begin{array}{cc|cc|cc|cc} & \delta^\nu_\mu & & & & & & \\ \delta^\mu_\nu & & & & & & & \\ \hline & & \epsilon & -1 & & & & \\ & & -1 & 0 & & & & \\ \hline & & & & & -i\delta^\nu_\mu & & \\ & & & & i\delta^\mu_\nu & & & \\ \hline & & & & & & & -i \\ & & & & & & i & \end{array} \right) . \\ & \begin{array}{cccccccc} x^\mu & b_\mu & \sigma & B & c^\mu & \bar{c}_\mu & c & \bar{c} \end{array} \end{array}$$

Moreover, ε_{MN} is an infinitesimal parameter.

To find the conserved currents, let us assume that ε_{MN} is a local infinitesimal parameter, i.e., $\varepsilon_{MN} = \varepsilon_{MN}(x)$. Then, after some calculations, the quantum Lagrangian transforms as

$$\delta\mathcal{L}_q = -\frac{1}{2}\partial_\mu \varepsilon_{NM} \mathcal{M}^{\mu MN}.$$

Thus, $\mathcal{M}^{\mu MN}$ is the conserved current for OSp(10|10) rotation.

Similarly, for local infinitesimal translation $\delta X^M = \varepsilon^M(x)$, we have

$$\delta\mathcal{L}_q = -\frac{1}{2}\partial_\mu \varepsilon_M \mathcal{P}^{\mu M}.$$

Thus, $\mathcal{P}^{\mu M}$ is the conserved current for translation.

The corresponding charges:

$$M^{MN} = \int d^3x\, \mathcal{M}^{0MN} = \int d^3x\, \tilde{g}^{0\nu} \phi^2 X^M \overleftrightarrow{\partial_\nu} X^N,$$

$$P^M = \int d^3x\, \mathcal{P}^{0M} = \int d^3x\, \tilde{g}^{0\nu} \phi^2 \partial_\nu X^M.$$

For instance, M^{MN} includes GCT and Weyl BRST charges:

$$Q_B^{(1)} = M(b_\rho, c^\rho) = \int d^3x\, \tilde{g}^{0\nu} \phi^2 b_\rho \overleftrightarrow{\partial}_\nu c^\rho,$$
$$Q_B^{(2)} = M(B, c) = \int d^3x\, \tilde{g}^{0\nu} \phi^2 B \overleftrightarrow{\partial}_\nu c.$$

Using equal-time commutation relations (ETCRs) obtained above, one can prove that these generators constitute the IOSp(10|10) algebra:

$$[P^M, P^N\} = 0, \quad [M^{MN}, P^R\} = i[P^M \eta^{NR} - (-)^{|N||R|} P^N \eta^{MR}],$$

$$[M^{MN}, M^{RS}\} = i[M^{MS}\eta^{NR} - (-)^{|N||R|} M^{MR}\eta^{NS} - (-)^{|N||R|} M^{NS}\eta^{MR} + (-)^{|M||R|+|N||S|} M^{NR}\eta^{MS}].$$

The global IOSp(10|10) symmetry is purely of quantum nature, so it survives even in the presence of gravity!

6. Gravitational conformal symmetry

A huge global IOSp(10|10) symmetry includes "gravitational conformal symmetry" as a subgroup.

For instance, by definition, translation generator P^μ must obey the equation for a generic field Φ:

$$[iP_\mu, \Phi(x)] = \partial_\mu \Phi(x).$$

We can construct such the generator from IOSp generators like

$$P_\mu = P_\mu(b) = \int d^3x \, \tilde{g}^{0\nu} \phi^2 \partial_\nu b_\mu.$$

Similarly, by definition, GL(4) generator $G^\mu{}_\nu$ must satisfy the equations:

$$[iG^\mu{}_\nu, \phi] = x^\mu \partial_\nu \phi,$$

$$[iG^\mu{}_\nu, g_{\rho\sigma}] = x^\mu \partial_\nu g_{\rho\sigma} + \delta^\mu_\rho g_{\nu\sigma} + \delta^\mu_\sigma g_{\nu\rho}.$$

$G^\mu{}_\nu$ can be constructed as

$$G^\mu{}_\nu = M^\mu{}_\nu(x,b) - iM^\mu{}_\nu(c^\tau, \bar{c}_\tau) = \int d^3x\, \tilde{g}^{0\lambda} \phi^2 (x^\mu \overleftrightarrow{\partial_\lambda} b_\nu - ic^\mu \overleftrightarrow{\partial_\lambda} \bar{c}_\nu).$$

Similarly, special conformal generator K^μ and dilatation generator D can be made to

$$K^\mu = 2M^\mu(x,b) = 2\int d^3x\, \tilde{g}^{0\nu} \phi^2 x^\mu \overleftrightarrow{\partial_\nu} B.$$

$$D = G^\mu{}_\mu + P(B) = \int d^3x\, \tilde{g}^{0\lambda} \phi^2 (x^\mu \overleftrightarrow{\partial_\lambda} b_\mu - ic^\mu \overleftrightarrow{\partial_\lambda} \bar{c}_\mu + \partial_\lambda B).$$

We can check that generators { P_μ, $G^\mu{}_\nu$, K^μ, D } make a closed algebra.

$$[P_\mu, P_\nu] = 0, \quad [P_\mu, G^\rho{}_\sigma] = iP_\sigma \delta^\rho_\mu, \quad [P_\mu, K^\nu] = -2i(G^\rho{}_\rho - D)\delta^\nu_\mu, \quad [P_\mu, D] = iP_\mu,$$
$$[G^\mu{}_\nu, G^\rho{}_\sigma] = i(G^\mu{}_\sigma \delta^\rho_\nu - G^\rho{}_\nu \delta^\mu_\sigma), \quad [G^\mu{}_\nu, K^\rho] = iK^\mu \delta^\rho_\nu, \quad [G^\mu{}_\nu, D] = 0,$$
$$[K^\mu, K^\nu] = 0, \quad [K^\mu, D] = -iK^\mu, \quad [D, D] = 0.$$

To construct an analog of conformal algebra in flat Minkowski space-time, we need to introduce the Minkowski metric $\eta_{\mu\nu}$ and make the Lorentz generator $M_{\mu\nu}$:

$$M_{\mu\nu} \equiv -\eta_{\mu\rho} G^\rho{}_\nu + \eta_{\nu\rho} G^\rho{}_\mu.$$

The generators { P_μ, $M_{\mu\nu}$, K^μ, D } make a closed algebra, which I call, "Gravitational conformal algebra (GCA)".

$$\boxed{\begin{aligned} &[P_\mu, P_\nu] = 0, \quad [P_\mu, M_{\rho\sigma}] = i(P_\rho \eta_{\mu\sigma} - P_\sigma \eta_{\mu\rho}), \quad [P_\mu, K^\nu] = -2i(G^\rho{}_\rho - D)\delta^\nu_\mu, \\ &[P_\mu, D] = iP_\mu, \quad [M_{\mu\nu}, M_{\rho\sigma}] = -i(M_{\mu\sigma}\eta_{\nu\rho} - M_{\nu\sigma}\eta_{\mu\rho} + M_{\rho\mu}\eta_{\sigma\nu} - M_{\rho\nu}\eta_{\sigma\mu}), \\ &[M_{\mu\nu}, K^\rho] = i(-K_\mu \delta^\rho_\nu + K_\nu \delta^\rho_\mu), \quad [M_{\mu\nu}, D] = [K^\mu, K^\nu] = [D, D] = 0, \quad [K^\mu, D] = -iK^\mu. \end{aligned}}$$

7. Spontaneous symmetry breakdown (SSB)

Physical state conditions: $Q_B^{(1)}\,|phys\rangle = Q_B^{(2)}\,|phys\rangle = 0.$

From the BRST quartet mechanism, we can show that physical modes are only massless graviton while the other modes including dilaton belong to unphysical sector.

Remark

I wish to stress that the Nambu-Goldstone theorem never tells us if the NG boson is physical or unphysical. In the present situation, the dilaton is the NB boson associated with Weyl invariance, which is an unphysical mode.

Based on the gravitational conformal algebra, we can show that GL(4), special conformal symmetry and dilatation are spontaneously broken to the Poincare symmetry. As a result, we can prove that the graviton and dilaton are exactly massless at non-perturbative level.

Let us assume the translational invariant vacuum $P_\mu|0\rangle = 0$ where $\langle 0|0\rangle = 1$, and non-zero VEVs:

$$\langle 0|g_{\mu\nu}|0\rangle = \eta_{\mu\nu}, \qquad \langle 0|\phi|0\rangle = \phi_0 \neq 0.$$

From GL(4), $[iG^\mu{}_\nu, g_{\rho\sigma}] = x^\mu \partial_\nu g_{\rho\sigma} + \delta^\mu_\rho g_{\nu\sigma} + \delta^\mu_\sigma g_{\nu\rho}.$

Taking its VEV, $\langle 0|[iG^\mu{}_\nu, g_{\rho\sigma}]|0\rangle = \delta^\mu_\rho \eta_{\nu\sigma} + \delta^\mu_\sigma \eta_{\nu\rho}.$

Then, $\langle 0|[iM_{\mu\nu}, g_{\rho\sigma}]|0\rangle = 0, \quad \langle 0|[i\bar{M}_{\mu\nu}, g_{\rho\sigma}]|0\rangle = 2(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}).$

Here, $\bar{M}_{\mu\nu} \equiv \eta_{\mu\rho} G^\rho{}_\nu + \eta_{\nu\rho} G^\rho{}_\mu.$

SSB of GL(4) $\longrightarrow$ SO(1, 3) : Massless graviton with 16 – 6 = 10 components

As for dilatation, from $[iD, \phi(x)] = x^\mu \partial_\mu \phi(x) + \phi(x),$

Taking VEV and recalling $\phi(x) = e^{\sigma(x)}$,

$$\langle 0|[iD, \sigma]|0\rangle = 1.$$

Thus, the dilatation is also spontaneously broken and its NG-boson is the dilaton, which must be exactly massless at non-perturbative level.

How about special conformal transformation (SCT)? From $[i\,K^\mu, \phi\,] = 2x^\mu \phi,$

$$\langle 0|[iK^\mu, \partial_\nu \sigma]|0\rangle = 2\delta^\mu_\nu.$$

This means that the SCT certainly receives SSB and its NG-boson is the derivative of the dilaton. This is also verified by gravitational CA as follows:

Jacobi identity: $\left[\,[P_\mu, K^\nu], \sigma\right] + \left[\,[K_\nu, \sigma], P_\mu\right] + \left[\,[\sigma, P_\mu], K^\nu\right] = 0.$

Using the GCA, $[P_\mu, K^\nu] = -2i\left(G^\rho_\rho - D\right)\delta^\nu_\mu$ and $[P_\mu, \sigma] = -i\partial_\mu \sigma,$ the VEV of the Jacobi identity produces

$$\langle 0|[K^\nu, \partial_\mu \sigma]|0\rangle = -2\delta^\nu_\mu \langle 0|[G^\rho{}_\rho - D, \sigma]|0\rangle = -2i\delta^\nu_\mu.$$

8. Conclusion

What we have explained in this talk:

① We have constructed a BRST formalism of Weyl invariant scalar-tensor gravity.
② We have shown the existence of the global Poincare-like IOSp(10|10) symmetry.
③ The gravitational conformal algebra exists as a subgroup of the IOSp(10|10) symmetry.
④ We can show that GL(4) spontaneously breaks down to Poincare symmetry and its NG-boson is the graviton. Dilatation and special conformal symmetry is also done so and its NG-boson is dilaton. Thus, the graviton and dilaton must be exactly massless.
⑤ The dilaton belongs to unphysical sector like ghosts in YM theory, so it causes no trouble in the fifth-force problem.

What we have not explained in this talk:

In particular, we have not discussed the issue of Weyl anomaly. If we introduced an additional scalar field playing the role of the renormalization mass μ, we might not have anomaly (?), but the price we have to pay is the non-renormalizability.
Cf. F. Englert et.al., NPB117(1976)407; A. Codello et.al., CQG30(2013)115015.