Fractional Cosmology with conformal and nonminimal couplings a possible resolution to H_0 tension?

Genly Leon *

Associate Professor
Departamento de Matemáticas, Universidad Católica del Norte, Avenida Angamos 0610, Casilla 1280 Antofagasta, Chile
Institute of Systems Science, Durban University of Technology, P.O. Box 1334, Durban 4000, Republic of South Africa

Tensions in Cosmology 2023 - 6 Sept 2023 - 12 Sept 2023, Mon-Repos, Corfu, Greece

* Funded by Vicerrectoría de Investigación y Desarrollo Tecnológico (VRIDT) at Universidad Católica del Norte through Resolución VRIDT No. 040/2022, Resolución VRIDT No. 054/2022, Resolución VRIDT No. 026/2023 and Resolución VRIDT No. 027/2023.

Overview

Introduction

Introduction I

- Fractional Cosmology show potential for resolving the H_0 tension finding alignment with Supernova H_0 and Planck's values for $z < 1.5$, but a discrepancy remains for $1.5 < z < 2.5$ [García-Aspeitia et al., 2022, González et al., 2023, Leon et al., 2023].
- We can expand the Einstein-Hilbert action and scalar field theory by incorporating nonminimal coupling with gravity and the scalar field. The coupling constant, ξ, can be minimal ($\xi = 0$) or conformal ($\xi = 1/6$), with any other value indicating nonminimal coupling.
- *Fractional calculus is used to modify the Friedmann and Klein-Gordon equations in the conformal and nonminimal coupling theory. The μ fractional parameter and age of the Universe, represented by t_0, affect the evolution of cosmic species densities.*
- Fractional cosmology can potentially solve cosmological problems, such as the H_0 tension.

Introduction II

- The variational approach with fractional Action was developed, e.g., by [El-Nabulsi, 2005, El-Nabulsi, 2007a, El-Nabulsi, 2007b, El-Nabulsi, 2008, Roberts, 2014, Frederico, 2008]. Given the fractional Action integral,

$$S = \frac{1}{\Gamma(\mu)} \int_0^{\tau} \mathcal{L}\left(\theta, q_i(\theta), \dot{q}_i(\theta), \ddot{q}_i(\theta)\right) (\tau - \theta)^{\mu-1} d\theta, \quad (1)$$

where $\Gamma(\mu)$ is the Gamma function, $\mathcal{L}$ is the lagrangian, μ is the constant fractional parameter, τ and θ are physical and intrinsic time respectively.

Introduction III

- Variation of (1) with respect to q_i leads to the Euler-Poisson equations [Frederico, 2008],

$$\frac{\partial \mathcal{L}\left(\theta, q_i(\theta), \dot{q}_i(\theta), \ddot{q}_i(\theta)\right)}{\partial q_i} - \frac{d}{d\theta}\frac{\partial \mathcal{L}\left(\theta, q_i(\theta), \dot{q}_i(\theta), \ddot{q}_i(\theta)\right)}{\partial \dot{q}_i} + \frac{d^2}{d\theta^2}\frac{\partial \mathcal{L}\left(\theta, q_i(\theta), \dot{q}_i(\theta), \ddot{q}_i(\theta)\right)}{\partial \ddot{q}_i}$$
$$= \frac{1-\mu}{\tau-\theta}\left(\frac{\partial \mathcal{L}\left(\theta, q_i(\theta), \dot{q}_i(\theta), \ddot{q}_i(\theta)\right)}{\partial \dot{q}_i} - 2\frac{d}{d\theta}\frac{\partial \mathcal{L}\left(\theta, q_i(\theta), \dot{q}_i(\theta), \ddot{q}_i(\theta)\right)}{\partial \ddot{q}_i}\right)$$
$$- \frac{(1-\mu)(2-\mu)}{(\tau-\theta)^2}\frac{\partial \mathcal{L}\left(\theta, q_i(\theta), \dot{q}_i(\theta), \ddot{q}_i(\theta)\right)}{\partial \ddot{q}_i}. \quad (2)$$

Cosmological model in the fractional formulation

Modified Friedmann equations I

- In cosmology, it is assumed that the flat Friedmann provides the geometry of spacetime Lemaître–Robertson–Walker (FLRW) metric:

$$ds^2 = -N^2(t)dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{3}$$

where $a(t)$ denotes the scale factor and $N(t)$ is the lapse function. This result is based on Planck's observations [Aghanim et al., 2020].
For the metric (3), the Ricci' scalar depends on the second derivatives of a and first derivatives of N and reads

$$R(t) = 6\left(\frac{\ddot{a}(t)}{a(t)N^2(t)} + \frac{\dot{a}^2(t)}{a^2(t)N^2(t)} - \frac{\dot{a}(t)\dot{N}(t)}{a(t)N^3(t)}\right). \tag{4}$$

Modified Friedmann equations II

- Consider the point-like action integral

$$S(\tau) = \int_0^\tau \left[\frac{R(\theta)}{2} + \frac{\dot{\phi}^2(\theta)}{2N^2(\theta)} - V(\phi(\theta)) + \xi R(\theta)\phi^2(\theta) + L_{\text{matter}}(\theta)\right] a^3(\theta)N(\theta)d\theta, \quad (5)$$

where $R(\theta)$ is the Ricci scalar (4).

Now, we use fractional variational calculus with classical and Caputo derivatives.

$$S = \frac{1}{\Gamma(\mu)} \int_0^\tau \left[\frac{R(\theta)}{2} + \frac{\dot{\phi}^2(\theta)}{2N^2(\theta)} - V(\phi(\theta)) + \xi R(\theta)\phi^2(\theta) + L_{\text{matter}}(\theta)\right] a^3(\theta)N(\theta)(\tau-\theta)^{\mu-1}d\theta, \quad (6)$$

where $\Gamma(\mu)$ is the Gamma function, $\mathcal{L}$ is the lagrangian, μ is the constant fractional parameter, τ and θ are physical and intrinsic time respectively.

Modified Friedmann equations III

- We consider the transition to the effective fractional action used in [García-Aspeitia et al., 2022], given by (1) with $q_i \in \{N, a, \phi\}$ and

$$
\begin{aligned}
S = \frac{1}{\Gamma(\mu)} \int_0^{\tau} \Bigg[& \frac{3a(\theta)\left(N(\theta)\left(a(\theta)\ddot{a}(\theta) + \dot{a}^2(\theta)\right) - a(\theta)\dot{a}(\theta)\dot{N}(\theta)\right)}{N^2(\theta)} \\
& - \frac{3\xi\phi^2(\theta)a(\theta)\left(N(\theta)\left(a(\theta)\ddot{a}(\theta) + \dot{a}^2(\theta)\right) - a(\theta)\dot{a}(\theta)\dot{N}(\theta)\right)}{N^2(\theta)} \\
& + a^3(\theta)N(\theta)\left(\frac{\dot{\phi}^2(\theta)}{2N^2(\theta)} - V(\phi(\theta))\right) - \rho_0 N(\theta)a(\theta)^{-3w}(\tau-\theta)^{-(\mu-1)(w+1)}\Bigg](\tau-\theta)^{\mu-1}d\theta,
\end{aligned}
\tag{7}
$$

where $w = p/\rho$ is a constant EoS for matter, ϕ is the scalar field, V is the potential which depends on the scalar field.

Modified Friedmann equations IV

- Under the rescaling $(\tau, \theta) \mapsto (2t, t)$, where new cosmological time t [Shchigolev, 2011] is used, the Euler–Poisson equations (2) become

$$\left(1-\xi\phi^2(t)\right)\left(\dot{H}(t)+\frac{(1-\mu)H(t)}{t}+\frac{3H^2(t)}{2}+\frac{(\mu-2)(\mu-1)}{2t^2}\right)$$

$$= -\frac{1}{2}p(t)-\frac{1}{4}\dot{\phi}(t)^2+\frac{1}{2}V(\phi(t))+\xi\phi(t)\left(2H(t)\dot{\phi}(t)-\frac{2(\mu-1)\dot{\phi}(t)}{t}+\ddot{\phi}(t)+\dot{\phi}(t)\right), \quad (8)$$

$$\left(1-\xi\phi^2(t)\right)\left(H^2(t)+\frac{(1-\mu)H(t)}{t}\right)=\frac{1}{3}\rho(t)+\frac{1}{6}\dot{\phi}^2(t)+\frac{1}{3}V(\phi(t))+2\xi H(t)\phi(t)\dot{\phi}(t), \quad (9)$$

$$\ddot{\phi}(t)+3H(t)\dot{\phi}(t)-\frac{(\mu-1)\dot{\phi}(t)}{t}+V'(\phi(t))+\xi\phi(t)\left(6\dot{H}(t)+12H(t)^2\right)=0, \quad (10)$$

$$\dot{\rho}(t)=-3\left(H(t)+\frac{(1-\mu)}{3t}\right)(p(t)+\rho(t)). \quad (11)$$

Modified Friedmann equations V

- By demanding that (9) is conserved in time for $\mu \neq 1$, we acquire the equation of state of matter given by

$$\rho = \frac{3H(-tH+\mu-1)\left(\xi\phi^2-1\right)}{t} - 6\xi H\phi\dot{\phi} - V(\phi) - \frac{1}{2}\dot{\phi}^2, \tag{12}$$

$$\begin{aligned} p = &-\frac{1}{2t^2\left(\xi(12\xi+1)\phi^2-1\right)}\Big[6t^2H^2\left(\xi\phi^2-1\right)^2 \\ &-t\left(\xi\phi^2-1\right)\Big(-12H\Big(-\mu+\xi\phi\Big((\mu+12\xi-3)\phi+t\dot{\phi}\Big)+3\Big) \\ &-12\xi\phi\Big((\mu-1)\dot{\phi}+tV'(\phi)\Big)-t\dot{\phi}^2\Big) \\ &+12\xi t^2\dot{\phi}^2-2t^2\left(\xi(12\xi+1)\phi^2-1\right)V(\phi)-6(\mu-2)(\mu-1)\left(\xi\phi^2-1\right)^2\Big]. \end{aligned} \tag{13}$$

Modified Friedmann equations VI

- Replacing the expression of p defined by (13) into (8) and (10), we obtain

$$\dot{H} = (\xi\phi^2 - 1)\left(\frac{(\mu-2)(\mu-1)}{t^2\left(\xi(12\xi+1)\phi^2-1\right)} - \frac{2(\mu-4)H}{t\left(\xi(12\xi+1)\phi^2-1\right)}\right) - \frac{2\xi H\phi\dot{\phi}}{\xi(12\xi+1)\phi^2-1}$$
$$+ \frac{H^2\left(3-3\xi(8\xi+1)\phi^2\right)}{\xi(12\xi+1)\phi^2-1} + \frac{2\xi t\dot{\phi}^2 - 2\xi\phi\left((\mu-1)\dot{\phi}+tV'(\phi)\right)}{t\left(\xi(12\xi+1)\phi^2-1\right)}, \tag{14}$$

$$\ddot{\phi} = (\xi\phi^2-1)\left(\frac{12(\mu-4)\xi H\phi}{t\left(\xi(12\xi+1)\phi^2-1\right)} + \frac{6\xi H^2\phi}{\xi(12\xi+1)\phi^2-1} - \frac{t^2V'(\phi)+6(\mu-2)(\mu-1)\xi\phi}{t^2\left(\xi(12\xi+1)\phi^2-1\right)}\right)$$
$$- \frac{3H\left(\xi(8\xi+1)\phi^2-1\right)\dot{\phi}}{\xi(12\xi+1)\phi^2-1} + \frac{\dot{\phi}\left(-\mu+\xi\phi\left((\mu-1)(24\xi+1)\phi-12\xi t\dot{\phi}\right)+1\right)}{t\left(\xi(12\xi+1)\phi^2-1\right)}. \tag{15}$$

Minimal Coupling

Minimal coupling I

- By introducing the logarithmic independent variable $\tau = -\ln(1+z) = \ln a$, with $\tau \to -\infty$ as $z \to \infty$, $\tau \to 0$ as $z \to 0$ and $\tau \to \infty$ as $z \to -1$, and defining the age parameter as $\alpha = tH$, we obtain the initial value problem

$$\alpha'(\tau) = 9 - 2\mu - 3\alpha(\tau) + \frac{(\mu-2)(\mu-1)}{\alpha(\tau)}, \tag{16}$$

$$t'(\tau) = t(\tau)/\alpha(\tau), \tag{17}$$

$$\alpha(0) = t_0 H_0, t(0) = t_0. \tag{18}$$

Minimal coupling II

- The exact solution is

$$t(\tau) = t_0 \exp\left(\frac{2\left(\tan^{-1}\left(\frac{6H_0t_0+2\mu-9}{\sqrt{8(9-2\mu)\mu-105}}\right) - \tan^{-1}\left(\frac{6\alpha(\tau)+2\mu-9}{\sqrt{8(9-2\mu)\mu-105}}\right)\right)}{\sqrt{8(9-2\mu)\mu-105}}\right), \tag{19}$$

where α is obtained in implicit form through

$$\frac{1}{6}\log\left(-2\mu\alpha(\tau) - 3\alpha(\tau)^2 + 9\alpha(\tau) + \mu^2 - 3\mu + 2\right) - \frac{(2\mu-9)\tan^{-1}\left(\frac{6\alpha(\tau)+2\mu-9}{\sqrt{-16\mu^2+72\mu-105}}\right)}{3\sqrt{-16\mu^2+72\mu-105}}$$
$$= \frac{1}{6}\left(\log\left(-2H_0\mu t_0 - 3H_0t_0(H_0t_0-3) + \mu^2 - 3\mu + 2\right) + \frac{2(9-2\mu)\tan^{-1}\left(\frac{6H_0t_0+2\mu-9}{\sqrt{8(9-2\mu)\mu-105}}\right)}{\sqrt{8(9-2\mu)\mu-105}} - 6\tau\right). \tag{20}$$

Minimal coupling III

- We obtain a numerical solution of $E(z)$ by integrating the initial value problem numerically

$$E'(z) = \frac{E(z)\tau(z)(3E(z)\tau(z) + 2\mu - 8) - (\mu - 2)(\mu - 1)}{(z+1)E(z)\tau(z)^2}, \; E(0) = 1, \tag{21}$$

$$\tau'(z) = -\frac{1}{(1+z)E(z)}, \tag{22}$$

$$\tau(0) = \frac{1}{6}(2\epsilon_0 + 1)(9 - 2\mu + r), \; \epsilon_0 = \frac{1}{2}\lim_{t\to\infty}\left(\frac{t_0 H_0 - tH}{tH}\right). \tag{23}$$

Minimal coupling IV

- For further comparison, we also constrain the free parameters of the ΛCDM model, whose respective Hubble parameter as a function of the redshift is given by

$$H(z) = H_0\sqrt{\Omega_{m,0}(1+z)^3 + 1 - \Omega_{m,0}}. \tag{24}$$

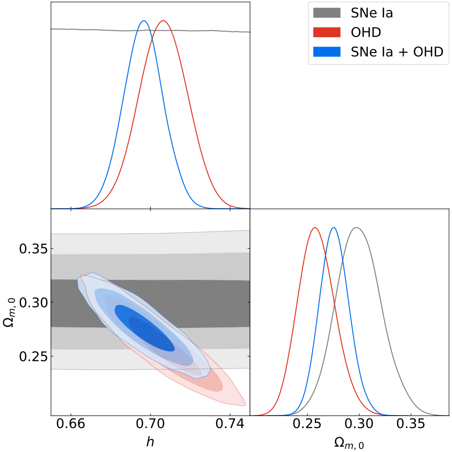
- The free parameters of the fractional cosmological model are $\theta = \{h, \mu, \epsilon_0\}$ and the free parameters of the ΛCDM model are $\theta = \{h, \Omega_{m,0}\}$ (as a reminder, $H_0 = 100\frac{\text{km/s}}{\text{Mpc}}h$). For the free parameters μ, ϵ_0 and $\Omega_{m,0}$, we consider the following flat priors: $\mu \in F(1,4)$, $\epsilon_0 \in F(-0.1, 0.1)$ and $\Omega_{m,0} \in F(0,1)$.
- ϵ_0 *is a measure of the limiting value of the relative error in the age parameter* tH *when it is approximated by* t_0H_0 *as given by Equation* (23). *For the mean value* $\epsilon_0 = 0$, *we acquire* $\alpha_0 = \frac{1}{6}(-2\mu + r + 9)$, *which implies* $c = 0$.

Minimal coupling V

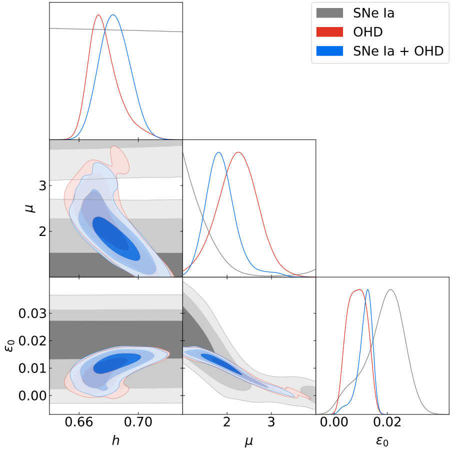
- The analysis from the SNe Ia data, OHD and the joint analysis with data from SNe Ia + OHD leads respectively to $h = 0.696^{+0.302}_{-0.295}$, $\mu = 1.340^{+2.651}_{-0.339}$ and $\epsilon_0 = \left(1.976^{+1.709}_{-2.067}\right) \times 10^{-2}$, $h = 0.675^{+0.041}_{-0.021}$, $\mu = 2.239^{+1.386}_{-1.190}$ and $\epsilon_0 = \left(0.865^{+0.793}_{-0.773}\right) \times 10^{-2}$, and $h = 0.684^{+0.031}_{-0.027}$, $\mu = 1.840^{+1.446}_{-0.773}$ and $\epsilon_0 = \left(1.213^{+0.482}_{-1.057}\right) \times 10^{-2}$, where the best-fit values are calculated at 3σ CL.
- On the other hand, these best-fit values lead to an age of the Universe with a value of $t_0 = \alpha_0/H_0 = 25.62^{+6.89}_{-4.46}$ Gyrs, a current deceleration parameter of $q_0 = -0.37^{+0.08}_{-0.11}$, both at 3σ CL, and a current matter density parameter of $\Omega_{m,0} = 0.531^{+0.195}_{-0.260}$ at 1σ CL.

Table: Best-fit values and χ^2_{min} criteria for the fractional cosmological model with free parameters h, μ and ϵ_0 and for the ΛCDM model with free parameters h and $\Omega_{m,0}$.

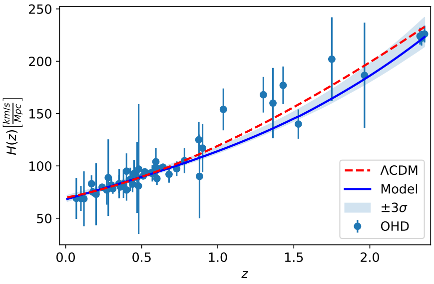
Data	Best-Fit Values				
	h	$\Omega_{m,0}$	μ	$\epsilon_0 \times 10^2$	χ^2_{min}
ΛCDM model					
SNe Ia	$0.692^{+0.209\ +0.296\ +0.307}_{-0.120\ -0.278\ -0.292}$	$0.299^{+0.022\ +0.046\ +0.068}_{-0.021\ -0.042\ -0.059}$	...	...	1026.9
OHD	$0.706^{+0.012\ +0.024\ +0.035}_{-0.012\ -0.024\ -0.036}$	$0.259^{+0.018\ +0.038\ +0.059}_{-0.017\ -0.033\ -0.047}$	...	...	27.5
SNe Ia + OHD	$0.696^{+0.010\ +0.020\ +0.029}_{-0.010\ -0.020\ -0.029}$	$0.276^{+0.014\ +0.030\ +0.043}_{-0.014\ -0.027\ -0.040}$	...	...	1056.3
Fractional cosmological model					
SNe Ia	$0.696^{+0.215\ +0.293\ +0.302}_{-0.204\ -0.284\ -0.295}$	...	$1.340^{+0.492\ +2.447\ +2.651}_{-0.245\ -0.328\ -0.339}$	$1.976^{+0.599\ +1.133\ +1.709}_{-0.905\ -1.848\ -2.067}$	1028.1
OHD	$0.675^{+0.013\ +0.029\ +0.041}_{-0.008\ -0.015\ -0.021}$	...	$2.239^{+0.449\ +0.908\ +1.386}_{-0.457\ -0.960\ -1.190}$	$0.865^{+0.395\ +0.650\ +0.793}_{-0.407\ -0.657\ -0.773}$	29.7
SNe Ia + OHD	$0.684^{+0.011\ +0.021\ +0.031}_{-0.010\ -0.020\ -0.027}$	...	$1.840^{+0.343\ +1.030\ +1.446}_{-0.298\ -0.586\ -0.773}$	$1.213^{+0.216\ +0.383\ +0.482}_{-0.310\ -0.880\ -1.057}$	1061.1



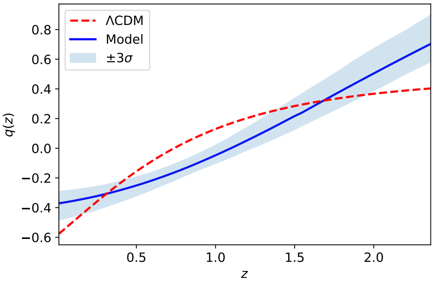
(a) Posterior distribution and joint admissible regions of the free parameters h and $\Omega_{m,0}$ for the ΛCDM model, obtained in the MCMC analysis.



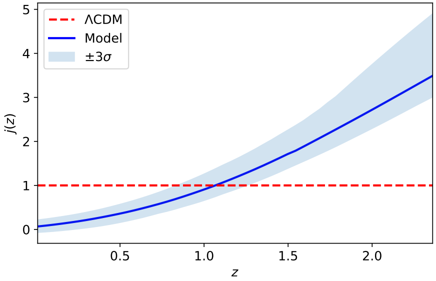
(b) Posterior distribution and joint admissible regions of the free parameters h, μ, and ϵ_0 for the Fractional cosmological model, obtained in the MCMC analysis.



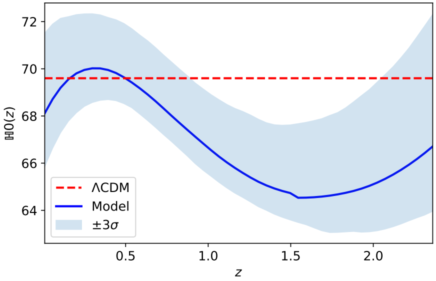
(a) Reconstruction of the $H(z)$



(b) Reconstruction of the $q(z)$

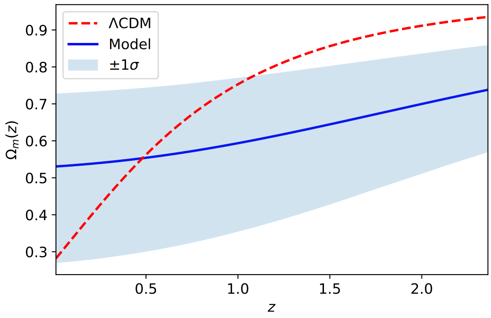


(c) Reconstruction of the $j(z)$

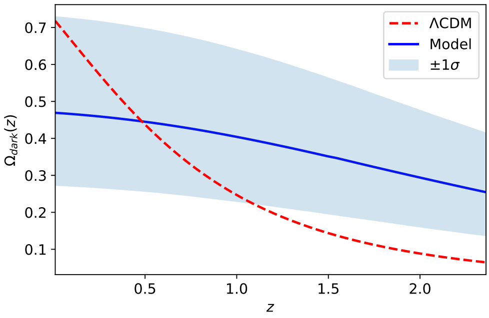


(d) $\mathbb{H}0(z)$ diagnostic

Figure: Theoretical Hubble parameter, Deceleration parameter, Jerk, and $\mathbb{H}0$ diagnostic (solid blue line) as a function of the redshift z for the Fractional cosmological model. The shaded curve represents the confidence region at $3\sigma(99.7\%)$ CL. Each model is compared with the ΛCDM model (red dashed line). Fig. 2a is contrasted with the OHD sample. The rest of the figure is obtained using the best-fit values from the joint analysis in Table 1 (see Ref. [González et al., 2023]).



(a) Matter density parameter for the ΛCDM model (red dashed line) and the Fractional cosmological model (solid blue line) as a function of the redshift z.



(b) Dark energy density parameter for the ΛCDM model (red dashed line) and the Fractional cosmological model (solid blue line) as a function of the redshift z.

Dynamical system formulation I

To present the formulation of a dynamical system, we introduce the new variables

$$x_1 = \frac{t(\tau)\phi'(\tau)}{\sqrt{6}\alpha(\tau)}, \quad x_2 = t(\tau)\phi(\tau), \quad x_3 = \alpha(\tau), \tag{25}$$

which satisfies

$$\rho + \Lambda = -\frac{3x_2^2 x_3 \left(-\mu\xi + \xi + x_3 \left(\xi + x_1^2 + 2\sqrt{6}\xi x_1\right)\right)}{t(\tau)^4} + \frac{3x_3(-\mu + x_3 + 1)}{t(\tau)^2}, \tag{26}$$

and the new time

$$f' \equiv \alpha^2 \frac{df}{d\tau}, \tag{27}$$

Dynamical system formulation II

we obtain the dynamical system

$$x_1' = -2\mu^2 x_1 + \mu x_1(x_2 + 4x_3 + 6) - x_1(3x_2x_3 + x_2 + 2(8 - 3x_3)x_3 + 4), \tag{28}$$

$$x_2' = x_3\left(\sqrt{6}x_1x_3^2 + x_2\right), \tag{29}$$

$$x_3' = x_3\left(\mu^2 - 3\mu - 2\mu x_3 - 3(x_3 - 3)x_3 + 2\right). \tag{30}$$

The equilibrium points are the following

1. $P_1 : (x_1, x_2, x_3) := (0, x_{2c}, 0)$. It is normally hyperbolic with a 2D stable manifold for $1 < \mu < 2, x_2 < 2\mu - 4$, a 2D unstable manifold for $\mu < 1, x_2 < 2\mu - 4$, or $\mu > 2, x_2 > 2\mu - 4$. It is a saddle for $\mu < 1, x_2 > 2\mu - 4$, or $1 < \mu < 2, x_2 > 2\mu - 4$, or $\mu > 2, x_2 < 2\mu - 4$.

Dynamical system formulation III

2. $P_2 : (x_1, x_2, x_3) := (x_{1c}, 2(\mu - 2), 0)$. The eigenvalues are $0, 0, (-2+\mu)(-1+\mu)$. It is nonhyperbolic.
3. $P_3 : (x_1, x_2, x_3) := \left(0, 0, \frac{1}{6}\left(-2\mu - \sqrt{8\mu(2\mu-9)+105}+9\right)\right)$. It is a sink for $\mu \notin [1, 2]$. It is a source for $\mu \in (1, 2)$.
4. $P_4 : (x_1, x_2, x_3) := \left(\frac{\mu\left(-12\mu+3\sqrt{8\mu(2\mu-9)+105}+65\right)-2\left(5\sqrt{8\mu(2\mu-9)+105}+51\right)}{4\sqrt{6}(\mu-2)(\mu-1)^2},\right.$ $\left.\frac{1}{12}\left(4\mu - \sqrt{8\mu(2\mu-9)+105}-3\right), \frac{1}{6}\left(-2\mu - \sqrt{8\mu(2\mu-9)+105}+9\right)\right)$. It is a saddle.
5. $P_5 : (x_1, x_2, x_3) := \left(0, 0, \frac{1}{6}\left(-2\mu + \sqrt{8\mu(2\mu-9)+105}+9\right)\right)$. It is a saddle.

Dynamical system formulation IV

6. $P_6:(x_1,x_2,x_3):=\left(\frac{\mu\left(-12\mu-3\sqrt{8\mu(2\mu-9)+105}+65\right)+2\left(5\sqrt{8\mu(2\mu-9)+105}-51\right)}{4\sqrt{6}(\mu-2)(\mu-1)^2},\right.$ $\left.\frac{1}{12}\left(4\mu+\sqrt{8\mu(2\mu-9)+105}-3\right),\frac{1}{6}\left(-2\mu+\sqrt{8\mu(2\mu-9)+105}+9\right)\right)$. It is a saddle.

Label	λ_1	λ_2	λ_3
P_1	0	$(\mu-1)(-2\mu+x_2+4)$	$(\mu-2)(\mu-1)$
P_2	0	0	$(\mu-2)(\mu-1)$
P_3	$\frac{1}{3}(-2\mu-r+9)$	$\frac{1}{6}(-2\mu-r+9)$	$\frac{1}{6}(-2\mu(8\mu+r-36)+9r-105)$
P_4	$\frac{1}{6}(-2\mu(8\mu+r-36)+9r-105)$	$\frac{1}{12}\left(-2\mu-3\sqrt{2}\sqrt{2\mu(5\mu-27)+(2\mu-9)r+93}-r+9\right)$	$\frac{1}{12}\left(-2\mu+3\sqrt{2}\sqrt{2\mu(5\mu-27)+(2\mu-9)r+93}-r+9\right)$
P_5	$\frac{1}{6}(-2\mu+r+9)$	$\frac{1}{3}(-2\mu+r+9)$	$\frac{1}{6}(2\mu(-8\mu+r+36)-3(3r+35))$
P_6	$\frac{1}{6}(2\mu(-8\mu+r+36)-3(3r+35))$	$\frac{1}{12}\left(-2\mu-3\sqrt{20\mu^2-4\mu(r+27)+6(3r+31)}+r+9\right)$	$\frac{1}{12}\left(-2\mu+3\sqrt{20\mu^2-4\mu(r+27)+6(3r+31)}+r+9\right)$

Table: Critical points of system (28), (29) and (30) and their eigenvalues.

Nonminimal coupling

Dynamical system formulation I

We consider the more general case $\xi \neq 0$.

- Defining $\alpha = tH$, and using the rules

$$\frac{d}{dt} = H\frac{d}{d\tau}, \quad \frac{d^2}{dt^2} = H^2\left(\frac{d^2}{d\tau^2} - (1+q)\frac{d}{d\tau}\right), \quad q := -1 - \frac{\dot{H}}{H^2}, \tag{31}$$

the system (14)–(15) becomes

$$\alpha'(\tau) = \frac{(\mu-2)(\mu-1)\left(\xi\phi^2-1\right)}{\alpha\left(\xi(12\xi+1)\phi^2-1\right)} + \frac{\alpha\left(3-3\xi(8\xi+1)\phi^2\right)}{\xi(12\xi+1)\phi^2-1} + \frac{\xi(-2\mu+12\xi+9)\phi^2+2\mu-9}{\xi(12\xi+1)\phi^2-1}$$
$$+\frac{2\xi t^2\phi^2\phi'^2}{\alpha\left(\xi(12\xi+1)\phi^2-1\right)} + \phi'\left(-\frac{2(\mu-1)\xi t\phi^2}{\alpha\left(\xi(12\xi+1)\phi^2-1\right)} - \frac{2\xi t\phi^2}{\xi(12\xi+1)\phi^2-1}\right), \tag{32}$$

Dynamical system formulation II

$$
\begin{aligned}
\phi''(\tau) = {} & \frac{12(\mu-4)\xi\phi\left(\xi\phi^2-1\right)}{\alpha\left(\xi(12\xi+1)\phi^2-1\right)} - \frac{6(\mu-2)(\mu-1)\xi\phi\left(\xi\phi^2-1\right)}{\alpha^2\left(\xi(12\xi+1)\phi^2-1\right)} + \frac{6\xi\phi\left(\xi\phi^2-1\right)}{\xi(12\xi+1)\phi^2-1} \\
& + \phi'^2\left(\frac{\frac{2(\mu-1)\xi t\phi^2}{\xi(12\xi+1)\phi^2-1} - \frac{12\xi^2t^2\phi^3}{\xi(12\xi+1)\phi^2-1}}{\alpha^2} + \frac{2\xi t\phi^2}{\alpha\left(\xi(12\xi+1)\phi^2-1\right)}\right) - \frac{2\xi t^2\phi^2\phi'^3}{\alpha^2\left(\xi(12\xi+1)\phi^2-1\right)} \\
& + \phi'\left(\frac{3\left(\xi(8\xi+1)\phi^2-1\right)}{\xi(12\xi+1)\phi^2-1} + \frac{\frac{(\mu-1)t\phi\left(\xi(24\xi+1)\phi^2-1\right)}{\xi(12\xi+1)\phi^2-1} - \frac{(\mu-2)(\mu-1)\left(\xi\phi^2-1\right)}{\xi(12\xi+1)\phi^2-1}}{\alpha^2} + \frac{\frac{2(\mu-4)\left(\xi\phi^2-1\right)}{\xi(12\xi+1)\phi^2-1} + \frac{t\left(3\phi-3\xi(8\xi+1)\phi^3\right)}{\xi(12\xi+1)\phi^2-1}}{\alpha}\right),
\end{aligned}
\tag{33}
$$

$$t'(\tau) = t(\tau)/\alpha(\tau), \tag{34}$$

$$\alpha(0) = t_0H_0, \quad \phi(0) = \phi_0, \quad \phi'(0) = \phi'_0, \quad t(0) = t_0, \tag{35}$$

Dynamical system formulation III

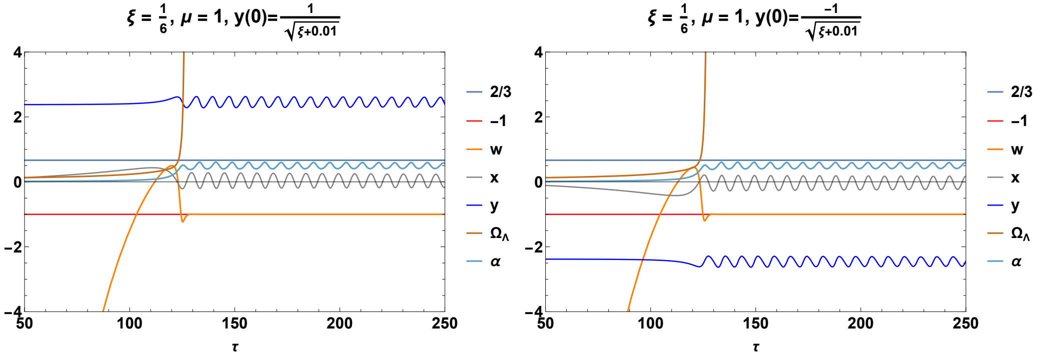
- In [Rami, 2015] was explored the assumption $H = H(\phi, \dot{\phi}) = \varepsilon\dot{\phi}/\phi$, which corresponds to $\phi'(\tau)/\phi(\tau) = \varepsilon$, and $\alpha = t\varepsilon\dot{\phi}/\phi$. In this paper, we consider a more general case.
- We define the variables

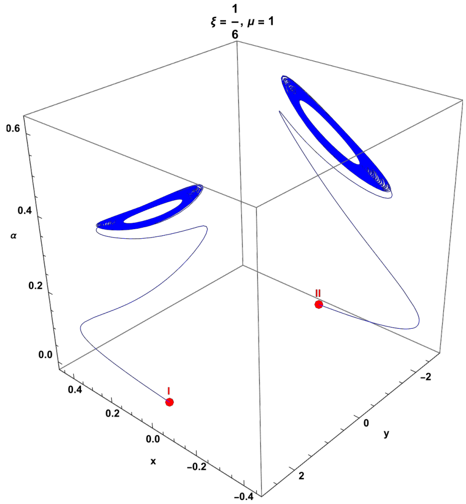
$$x = \frac{\dot{\phi}(t)}{\sqrt{6}H(t)}, \quad y = \phi(t), \quad \Omega_\Lambda = \frac{\Lambda}{3H^2(t)}, \quad \Omega_m = \frac{\rho_m}{3H^2} \tag{36}$$

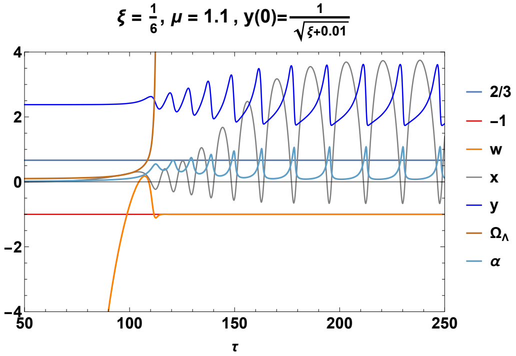
which satisfies

$$(x + \sqrt{6}\xi y)^2 + \xi y^2(1 - 6\xi) + \frac{(\mu - 1)\left(1 - \xi y^2\right)}{\alpha} + \Omega_\Lambda + \Omega_m = 1 \tag{37}$$

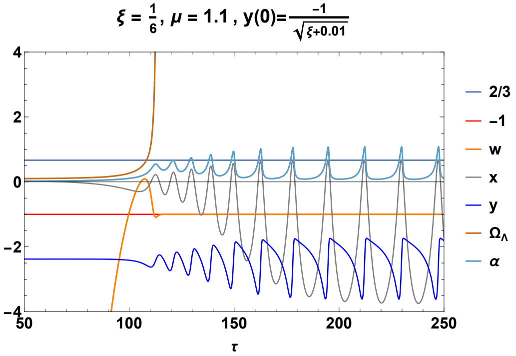
where Ω_m is the dimensionless energy density of matter, and Ω_Λ is interpreted as the energy density of the cosmological constant.


ξ = 1/6, μ = 1, y(0)= 1/√(ξ+0.01)
4
2
0
−2
−4
50
100
150
200
250
τ
2/3
−1
w
x
y
Ω_Λ
α
ξ = 1/6, μ = 1, y(0)= −1/√(ξ+0.01)

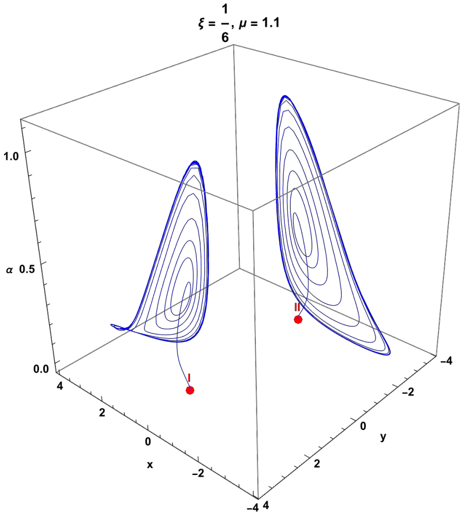

$\xi = \frac{1}{6}$, $\mu = 1$
0.6
0.4
α
0.2
0.0
0.4
0.2
0.0
x
−0.2
−0.4
−2
0
y
2
I
II



$\xi = \frac{1}{6}, \mu = 1.1, y(0) = \frac{1}{\sqrt{\xi + 0.01}}$
4
2
0
−2
−4
50
100
150
200
250
τ
2/3
−1
w
x
y
Ω_Λ
α



$\xi = \frac{1}{6}, \mu = 1.1, y(0) = \frac{-1}{\sqrt{\xi + 0.01}}$
4
2
0
−2
−4
50
100
150
200
250
τ
2/3
−1
w
x
y
Ω_Λ
α



ξ = 1/6, μ = 1.1
1.0
α 0.5
0.0
4
2
0
x
−2
−4
4
2
0
y
−2
−4
I
II

Conclusions

Conclusions I

- Figure 2a shows the Hubble parameter for the ΛCDM model and the Fractional cosmological model as a function of redshift z, compared to the OHD sample. The shaded curve indicates the confidence region for the Fractional cosmological model at 3σ CL. The deceleration parameter is calculated using specific values for $\alpha(s)$ and μ, as shown in Equation 38. Best-fit values can be found in Table 1.

$$q(\alpha(s)) = 2 + \frac{2(\mu-4)}{\alpha(s)} - \frac{(\mu-2)(\mu-1)}{\alpha^2(s)}. \tag{38}$$

- In in Figure 2b we compared the deceleration parameter of the Fractional cosmological model to the ΛCDM model, using best-fit estimates from Table 1. Our results show a transition at $z_t \gtrapprox 1$, with a higher transition redshift than the ΛCDM model. The current deceleration parameter for the Fractional cosmological model is $q_0 = -0.37^{+0.08}_{-0.11}$ at 3σ CL.

Conclusions II

- We use the Jerk to determine the type of dark energy in the Fractional cosmological model. Its formula is based on the value of q in (38) through equation $j(s) = q(s)(2q(s)+1) - \frac{dq(s)}{ds}$. By plugging in $\alpha(s)$, we obtain

$$j(\alpha(s)) = \frac{12(\mu-4)}{\alpha(s)} + \frac{(\mu-21)\mu+50}{\alpha(s)^2} - \frac{2(\mu-3)(\mu-2)(\mu-1)}{\alpha(s)^3} + 10. \quad (39)$$

- Figure 2c compares the Fractional and ΛCDM models for Jerk. The shaded region shows the 3σ CL error band using best-fit values from Table 1. Deviating from this may suggest a different cosmology with a dynamic equation of state during late times.

Conclusions III

- We display the $\mathbb{H}0$ diagnostic for the ΛCDM and Fractional cosmological models in Figure 2d. The figure was created using the best-fit joint analysis values in Table 1 and includes an error band at 3σ CL. Figures 2c and 2d also show the Jerk and $\mathbb{H}0$ diagnostic for the ΛCDM model as a reference.
- We've displayed matter density and fractional density parameters for the Fractional cosmological model, using best-fit values from Table 1 and an error band at 1σ CL. Figures 3a and 3b illustrate how these parameters change with redshift z. The figures also include corresponding parameters for the ΛCDM model.
- The matter density parameter in the Fractional cosmological model has significant uncertainties due to the absence of EoS in the Hubble parameter used for reconstruction. At 1σ CL, the current value of this parameter is $\Omega_{m,0} = 0.531^{+0.195}_{-0.260}$, which aligns with the asymptotic value $\Omega_{m,t\to\infty} = 0.519^{+0.199}_{-0.262}$ determined at the same confidence level through joint analysis.

Conclusions IV

- The Fractional cosmological model suggests that a higher $\Omega_{m,0}$ could explain the lower deceleration parameter q_0 and excess of matter in ρ_{frac}. The current value of $\Omega_{\text{frac},0}$ is 0.469, satisfying $\Omega_{m,0} + \Omega_{\text{frac},0} = 1$ and potentially alleviating the Coincidence Problem.
- Currently we are analyzing data with cosmological constraints similar to $\xi = 0$ for the non-minimal coupled case.
- Fractional Cosmology with conformal and nonminimal couplings may resolve the Hubble constant tension. Is research on Fractional Cosmology worthwhile?

Thanks for your attention!

Are there questions?

References

References I

Aghanim, N. et al. (2020).
Planck 2018 results. VI. Cosmological parameters.
Astron. Astrophys., 641:A6.
[Erratum: Astron.Astrophys. 652, C4 (2021)].

El-Nabulsi, R. A. (2005).
Fractional unstable Euclidean universe.
Electron. J. Theor. Phys., 2(8):1–11.

El-Nabulsi, R. A. (2007a).
Accelerated universe dominated by holographic dark energy, supergravity inflationary potential.
Rom. J. Phys., 52:163–170.

References II

El-Nabulsi, R. A. (2007b).
Cosmology with a fractional action principle.
Rom. Rep. Phys., 59:763–771.

El-Nabulsi, R. A. (2008).
Increasing effective gravitational constant in fractional ADD brane cosmology.
Electron. J. Theor. Phys., 5(17):0103–106.

Frederico, G.A.S.F.; Torres, D. (2008).
Necessary optimality conditions for fractional action-like problems with intrinsic and observer times.
WSEAS Trans. Math., 7:6–11.

References III

García-Aspeitia, M. A., Fernandez-Anaya, G., Hernández-Almada, A., Leon, G., and Magaña, J. (2022).
Cosmology under the fractional calculus approach.
Mon. Not. Roy. Astron. Soc., 517(4):4813–4826.

González, E., Leon, G., and Fernandez-Anaya, G. (2023).
Exact solutions and cosmological constraints in fractional cosmology.
Fractal Fract., 7:368.

Leon, G., García-Aspeitia, M. A., Fernandez-Anaya, G., Hernández-Almada, A., Magaña, J., and González, E. (2023).
Cosmology under the fractional calculus approach: a possible H_0 tension resolution?

References IV

Rami, E.-N. A. (2015).
Fractional action oscillating phantom cosmology with conformal coupling.
Eur. Phys. J. Plus, 130(6):102.

Roberts, M. D. (2014).
Fractional Derivative Cosmology.
SOP Trans. Theor. Phys., 1:310.

Shchigolev, V. K. (2011).
Cosmological Models with Fractional Derivatives and Fractional Action Functional.
Commun. Theor. Phys., 56:389–396.