Cosmological Implication of $f(T)$ Gravity Models Through Phase Space Analysis

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Outline of Presentation

- Introduction.
- Mathematical formalism.
- Dynamical framework in $f(T)$ gravity at background and perturbed level.
- Outcome of the study

Introduction

- Einstein published his General Theory of Relativity (GR) in 1915, and redefines Newton's laws of gravitation.
- Three main cosmological phenomena which GR unables to describe

  (i) Dark Matter

  (ii) Inflation

  (iii) Dark Energy
- During the inflationary phase, the Universe grew exponentially, expanded rapidly and in a short span of time attained an immense size.

Introduction

• The accelerating expansion of the Universe was first discovered in 1998 by the observations of Type Ia supernovae (SNe Ia)[1,2].

• There are essentially two approaches one could take when attempting to solve the dark matter, dark energy and inflation problems, that is either to modify matter part (by adding additional dark matter, dark energy component) or modify geometric part (to study modified theory of gravity).

• This new gravity is known as the teleparallel equivalent of GR (TEGR), which can be generalized to the commonly called the $f(T)$[3], gravity by taking a nonlinear modification of the TEGR Lagrangian.

• The TEGR is formulated in terms of the tetrad field and of the corresponding torsion tensor, which is the antisymmetric part of the Weitzenböck connection.

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[1] A.G Riess, *et al.*, *Astrons. J.*, **116**, 1009 (1998).
[2] S. Perlmutter, *Astrophys. J.* , **517**, 565 (1999).
[3] R. Ferraro, F. Fiorini, Phys. Rev. D, **78**, 124019, (2008).

Dynamical systems framework:

• What is a dynamical system analysis in cosmology?

• Dynamical system approach [4] is an effective tool to examine the entire asymptotic behavior of the cosmological model and it allows us to avoid the challenge of solving non-linear cosmological equations. Through the careful choice of the dynamical variables, a given cosmological model can be written as an autonomous system of differential equations.

• The dynamical systems technique offers a crucial approach in the toolkit of probes of background cosmology. It offers an avenue to explore what critical points a model has associated with it, and what are the natures of each of these points. These points can then be correlated with the evolution of the Universe as evidenced from observational cosmology, which can be a compelling first test of any proposed model stemming from modified gravity.

• This technique also describes the overall dynamics of the Universe by analyzing the local asymptotic behavior of critical points of the system and connecting them to the major cosmological epochs of the Universe.

• For example, the radiation and matter-dominated periods correlate to saddle points (unstable), but late-time (the dark energy sector) dominance normally corresponds to a stable point.

[4] J. Wainwright and G. Ellis, *Dynamical Systems in Cosmology*, **Cambridge University Press**, (1997).

Mathematical Formalism :

We consider a flat isotropic and homogeneous Friedmann–Lemaître–Robertson–Walker (FLRW) metric.

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)\,, \tag{1}$$

Where $a(t)$ is the scale factor and the tetrad field can be described as follow,

$$e_\mu^A = (1, a(t), a(t), a(t))\,, \tag{2}$$

the tetrad e_μ^A (and its inverses E_A^μ) relate to the metric as the fundamental variable of theory through the relations,

$$g_{\mu\nu} = e_\mu^A e_\nu^B \eta_{AB}\,, \qquad \eta_{AB} = E_A^\mu E_B^\nu g_{\mu\nu}\,, \tag{3}$$

The tetrads must satisfy orthogonality conditions which take of the form,

$$e_\mu^A E_B^\mu = \delta_B^A\,, \qquad e_\mu^A E_A^\nu = \delta_\mu^\nu\,, \tag{4}$$

The Weitzenböck connection can be defined as,

$$\Gamma^\sigma{}_{\nu\mu} := E_A^\sigma \left(\partial_\mu e_\nu^A + \omega^A_{B\mu} e_\nu^B\right)\,, \tag{5}$$

Mathematical Formalism :

The torsion tensor can be described as follow,

$$T^{\sigma}_{\mu\nu} := 2\Gamma^{\sigma}_{[\nu\mu]}\,, \tag{6}$$

By an appropriate combination of contractions of torsion tensors, a torsion scalar can be written as follow,

$$T := \frac{1}{4}T^{\alpha}{}_{\mu\nu}T_{\alpha}{}^{\mu\nu} + \frac{1}{2}T^{\alpha}{}_{\mu\nu}T^{\nu\mu}_{\ \ \ \alpha} - T^{\alpha}{}_{\mu\alpha}T^{\beta\mu}_{\ \ \ \beta}\,, \tag{7}$$

Further, the superpotential and contortion tensor can be expressed as,,

$$S_{\theta}{}^{\mu\nu} \equiv \frac{1}{2}(K^{\mu\nu}{}_{\theta} + \delta^{\mu}_{\theta}T^{\alpha\nu}{}_{\alpha} - \delta^{\nu}_{\theta}T^{\alpha\mu}{}_{\alpha})\,, \tag{8}$$

$$K^{\mu\nu}{}_{\theta} \equiv \frac{1}{2}(T^{\nu\mu}{}_{\theta} + T_{\theta}{}^{\mu\nu} - T^{\mu\nu}{}_{\theta}) \tag{9}$$

Mathematical Formalism of $f(T)$ Gravity:

The action of $f(T)$ gravity,

$$S = \frac{1}{16\pi G}\int d^4x\, e\,[T + f(T) + \mathcal{L}_m], \tag{10}$$

The field equations of $f(T)$ gravity,

$$e^{-1}\partial_\mu(eE_A^\rho S_\rho{}^{\mu\nu})[1+f_T] + E_A^\rho S_\rho{}^{\mu\nu}\partial_\mu(T)f_{TT} - E_A^\lambda T^\rho{}_{\mu\lambda}S_\rho{}^{\nu\mu}[1+f_T] + \frac{1}{4}E_A^\nu[T+f(T)] = 4\pi G E_A^\rho T_\rho{}^\nu \tag{11}$$

$$T = 6H^2 \tag{12}$$

The field equations,

$$3H^2 = 8\pi G\rho_m - \frac{f}{2} + Tf_T \tag{13}$$

$$\dot{H} = -\frac{4\pi G(\rho_m + p_m)}{1 + f_T + 2Tf_{TT}} \tag{14}$$

Mathematical Formalism of $f(T)$ Gravity:

Comparing the Einstein GR field equation to (13), (14) respectively,

$$\rho_{de} \equiv \frac{1}{16\pi G}\left[-f + 2Tf_T\right] \tag{15}$$

$$p_{de} \equiv -\frac{1}{16\pi G}\left[\frac{-f + Tf_T - 2T^2 f_{TT}}{1 + f_T + 2Tf_{TT}}\right] \tag{16}$$

We also obtain the dark energy and total EOS parameter and deceleration parameter as,

$$\omega_{de} = -1 + \frac{(f_T + 2Tf_{TT})(-f + T + 2Tf_T)}{(1 + f_T + 2Tf_{TT})(-f + 2Tf_T)} \equiv \frac{p_{de}}{\rho_{de}} \tag{17}$$

$$\omega_{tot.} = -1 - \frac{2\dot{H}}{3H^2} \equiv \frac{p_m + p_{de}}{\rho_m + \rho_{de}} \tag{18}$$

$$q = -1 - \frac{\dot{H}}{H^2} \tag{19}$$

We can also write the first Friedmann equation in terms of density parameters as,

$$\Omega_{de} + \Omega_m = 1 \tag{20}$$

Dynamical system framework

In one of our previous works [5], we studied dynamical system analysis at the background level. In this work, we are interested to broaden it further by including the impact of perturbations. To do this, the equation governing the growth of matter perturbations on sub-horizon scales can be invoked in the form [6]

$$\ddot{\delta} + 2H\dot{\delta} = \frac{4\pi G\rho\delta}{1+f_T} \tag{21}$$

where $\delta = \frac{\delta\rho}{\rho}$ is the matter over density . Referring Eqn. (13), Eqn.(14) and (21), initially we set up the dynamical system of the background and perturbed equations for a general function of f(T) as

$$x = -\frac{f}{6H^2}, \qquad y = -2f_T, \qquad \sigma = \frac{d(ln\delta)}{d(lna)} \tag{22}$$

[5] L. K. Duchaniya and Santosh V. Lohakare and B. Mishra and S. K. Tripathy, EPJC, **82**, 448, (2022).

[6] Fotios K. Anagnostopoulos and Spyros Basilakos and Emmanuel N. Saridakis, Physical Review D, **08**, 100, (2019).

Dynamical system framework

The background cosmological parameters Ω_m, Ω_{de}, ω_{de}, ω_{tot} and q can expressed as

$$\Omega_m = 1 - x - y \tag{23}$$

$$\Omega_{de} = x + y \tag{24}$$

$$\omega_{de} = \frac{-2x - y + 4Tf_{TT}}{2(x+y)(1 + f_T + 2Tf_{TT})}, \tag{25}$$

$$\omega_{tot} = -1 - \frac{(x + y - 1)}{(1 + f_T + 2Tf_{TT})}, \tag{26}$$

$$q = -1 - \frac{3(x + y - 1)}{2(1 + f_T + 2Tf_{TT})}. \tag{27}$$

Dynamical system framework

In term of the dynamical variables of equation, the cosmological equations can be written as an autonomous system as below

$$\frac{dx}{dN} = -\frac{\dot{H}}{H^2}(y+2x)\,, \tag{28}$$

$$\frac{dy}{dN} = -4\frac{\dot{H}}{H^2}(Tf_{TT})\,, \tag{29}$$

$$\frac{d\sigma}{dN} = -\sigma(\sigma+2) - \frac{3(x+y-1)}{(2-y)} - \frac{\dot{H}}{H^2}\sigma\,, \tag{30}$$

$\frac{\dot{H}}{H^2} = \frac{3(x+y-1)}{2(1+f_T+2Tf_{TT})}\,.$

Model-I

We choose the logarithmic form of $f(T)$ [7],

$$f(T) = \beta T \ln\left(\frac{T}{T_0}\right) \tag{31}$$

$$\frac{dx}{dN} = -\frac{3(x+y-1)(2x+y)}{(2+4\beta-y)}$$

$$\frac{dy}{dN} = -\frac{12\beta(x+y-1)}{(2+4\beta-y)}$$

$$\frac{d\sigma}{dN} = -\sigma(\sigma+2) - 3(x+y-1)\left(\frac{1}{(2-y)} + \frac{\sigma}{(2+4\beta-y)}\right)$$

$$\omega_{de} = \frac{-4\beta+2x+y}{(x+y)(-4\beta+y-2)}$$

$$\omega_{tot} = \frac{4\beta+2x+y}{-4\beta+y-2}$$

$$q = -1 + \frac{3(x+y-1)}{-4\beta+y-2}$$

[7] Y. Zhang, H. Li, Y. Gong, and Z.-H. Zhu, JCAP, **07**, 2011, (2011).

Model-I

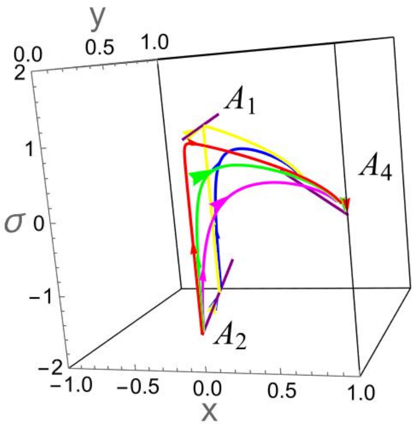
Table 1: Critical points for the dynamical system.

C.P.	x_c	y_c	σ_c	ω_{de}	ω_{tot}	q	Ω_{de}	Ω_m	Exists for
A_1	x	$-2x$	1	0	0	$\frac{1}{2}$	$-x$	$1+x$	Always
A_2	x	$-2x$	$-\frac{3}{2}$	0	0	$\frac{1}{2}$	$-x$	$1+x$	Always
A_3	x	$1-x$	-2	$-1+\frac{8\beta}{1+4\beta+x}$	-1	-1	1	0	Always
A_4	x	$1-x$	0	$-1+\frac{8\beta}{1+4\beta+x}$	-1	-1	1	0	Always

Table 2: Eigenvalues and stability condition.

C.P.	Stability Conditions	λ_1	λ_2	λ_3
A_1	Saddle Unstable	0	$-\frac{5}{2}$	3
A_2	Node Unstable	0	$\frac{5}{2}$	3
A_3	Saddle Unstable	0	-3	2
A_4	Node Stable	0	-3	-2

Model-I



The selected trajectory moved from matter dominated to dark-energy-dominated critical points. we can easily observe the transition of the trajectory like A_2 (node unstable) $\rightarrow A_1$ (saddle unstable) $\rightarrow A_4$ (node stable).

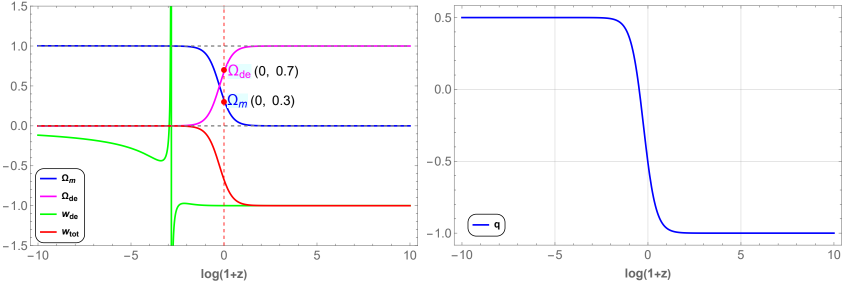


Figure 1: Plots for model-I. Initial conditions are $x = 10^{-2}$, $y = 10^{-6}$ and $\beta = 0.0001$.

The evolutionary behavior of the density parameters in redshift $\left(N = ln(\frac{1}{1+z})\right)$. At present (vertical dashed line), as the observation revealed, dark matter and dark energy predominate. We obtain $\Omega_{de} \approx 0.7$ and $\Omega_m \approx 0.3$.

Evolution behavior of the EoS and deceleration parameters

• The total EoS parameter begins with a matter dominated value of 0, and finally approaches -1 as the role of dark energy becomes more significant.

• We also notice the dark energy EoS parameter and at present, $\omega_{de} \approx -1$. Which is compatible with the present Planck Collaboration result $[\omega_{de}(z=0) = -1.028 \pm 0.032]$. [8]

• The deceleration parameter shows a transition from deceleration to acceleration with the transition at $z = 0.59$, which is consistent with the observational constraint $[z_{trans.} = 0.7679^{+0.1831}_{-0.1829}]$. [9]

• The present value of the deceleration parameter can be obtained as, $q(z=0) \approx -0.57$, consistent with the visualized cosmological observations. [10]

[8] Planck Collaboration, N. Aghanim *et al.*, *Astron. Astrophys.*, **641**, A6 (2020).
[9] S. Capozziello, O. Farooq, O. Luongo, and B. Ratra, *Phys. Rev, D*, **90**, 044016 (2014).
[10] D. Camarena and V. Marra, *Rev. Res.*, **2**, 013028 (2020).

Model-II

We consider the power law form of $f(T)$ [11] as,

$$f(T) = f_0(-T)^m, \tag{32}$$

$$\frac{dx}{dN} = -\frac{3(x+y-1)(2x+y)}{(2+(1-2m)y)},$$

$$\frac{dy}{dN} = \frac{6y(m-1)(x+y-1)}{(2+(1-2m)y)},$$

$$\frac{d\sigma}{dN} = -\sigma(\sigma+2) - \frac{3(x+y-1)}{(2-y)} - \frac{3\sigma(x+y-1)}{(2+(1-2m)y)}.$$

$$\omega_{de} = \frac{(2m-1)y+2x}{((2m-1)y-2)(x+y)},$$

$$\omega_{tot} = -1 + \frac{2(x+y-1)}{(2m-1)y-2},$$

$$q = -1 + \frac{3(x+y-1)}{(2m-1)y-2}.$$

[11] G. R. Bengochea and R. Ferraro, phys. Rev. D, **86**, 124019, (2012).

Model-II

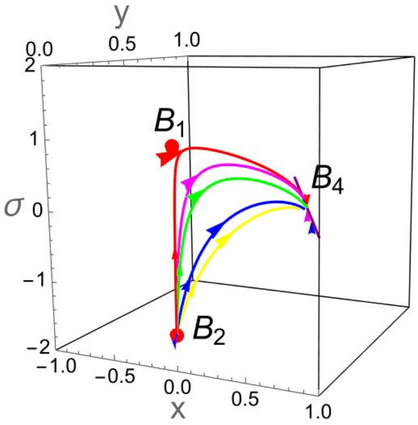
Table 3: Critical points for the dynamical system.

C.P.	x_c	y_c	σ_c	ω_{de}	ω_{tot}	q	Ω_{de}	Ω_m	Exists for
B_1	0	0	1	$-$	0	$\frac{1}{2}$	0	1	Always
B_2	0	0	$-\frac{3}{2}$	$-$	0	$\frac{1}{2}$	0	1	Always
B_3	x	$1-x$	-2	$\frac{x(2m-3)-2m+1}{x(2m-1)-2m+3}$	-1	-1	1	0	Always
B_4	x	$1-x$	0	$\frac{x(2m-3)-2m+1}{x(2m-1)-2m+3}$	-1	-1	1	0	Always

Table 4: Eigenvalues and stability condition.

C.P.	Stability Conditions	λ_1	λ_2	λ_3
B_1	Unstable	3	$-\frac{5}{2}$	$-3(m-1)$
B_2	Unstable	3	$\frac{5}{2}$	$-3(m-1)$
B_3	Unstable	0	2	$-\frac{3(3-2m+2x-x^2+2mx^2)}{(1+x)(3-2m-x+2mx)}$
B_4	Stable for $(x\|\, m) \in \mathbb{R}$	0	-2	$-\frac{3(3-2m+2x-x^2+2mx^2)}{(1+x)(3-2m-x+2mx)}$

Model-II



The trajectories show a path from the matter-dominated unstable critical points B_1 and B_2 to the stable dark energy-dominated critical point B_4 ($B_2 \to B_1 \to B_4$).

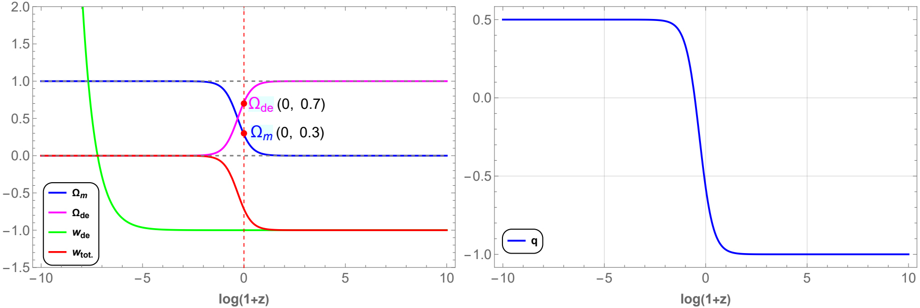


Figure 2: Plots for model-II. Initial conditions are $x = 10^{-3}$, $y = 10^{-6}$ and $m = 0.5$.

• We also notice the dark energy EoS parameter and at present, $\omega_{de} \approx -1$. Which is compatible with the present Planck Collaboration result [$\omega_{de}(z=0) = -1.028 \pm 0.032$]. [12]

• The deceleration parameter shows a transition from deceleration to acceleration with the transition at $z = 0.64$, which is consistent with the observational constraint [$z_{trans.} = 0.7679^{+0.1831}_{-0.1829}$]. [13]

• The present value of the deceleration parameter can be obtained as, $q(z=0) \approx -0.62$, consistent with the visualized cosmological observations. [14]

[12] Planck Collaboration, N. Aghanim *et al.*, *Astron. Astrophys.*, **641**, A6 (2020).
[13] S. Capozziello, O. Farooq, O. Luongo, and B. Ratra, *Phys. Rev, D*, **90**, 044016 (2014).
[14] D. Camarena and V. Marra, *Rev. Res.*, **2**, 013028 (2020).

Outcome of Study

- In the present work we performed a combined dynamical system analysis of both background and perturbation equations.
- We utilize the dynamical variables together with the equations of motion, are then used to derive the system of autonomous equations which express the behaviour of the model in phase space. These first order equations of motion of the dynamical variables are represented as derivatives with respect to $N = \ln a$, which shows the behaviour of the system in a more direct way.
- For both cases, we obtain a matter-dominated saddle point characterized by the correct growth rate of matter perturbations, followed by the transition to a stable dark-energy dominated accelerated universe in which matter perturbations remain constant.
- For the critical points in the de-Sitter phase, both the values of the dark energy and total EoS parameter and deceleration parameter are -1, which confirms the accelerating model with the ΛCDM-like behavior.
- We have obtained the present value of the matter and dark energy density parameters are $\Omega_m \approx 0.3$ and $\Omega_{de} \approx 0.7$ and it fits the recent suggestions from cosmological observations.


Thank you